The Rate of Convergence to the Asymptotics for the Wave Equation in an Exterior Domain

By

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Dedicated to Professor Tatsuo Nishitani on the occasion of his 60th birthday

Abstract. In this paper we consider the mixed problem for the wave equation exterior to a non-trapping obstacle in odd space dimensions. We derive a rate of the convergence of the solution for the mixed problem to its asymptotic profile, which is written as a solution for the Cauchy problem. As a by-product, we are able to find out the radiation field of solutions to the mixed problem in terms of the scattering data.

Key Words and Phrases. Wave equation, Exterior problem, Asymptotic behavior, Convergence rate.

2000 Mathematics Subject Classification Numbers. 35L20; 35B40.

1. Introduction

This paper is concerned with the global behavior of solutions to the mixed problem for the wave equation in an exterior domain:

$$(1.1) \quad (\partial_t^2 - \Delta)u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \Omega,$$

$$(1.2) \quad u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega,$$

$$(1.3) \quad u(0, x) = f_0(x), \quad (\partial_t u)(0, x) = f_1(x), \quad x \in \Omega,$$

where $\Omega = \mathbf{R}^n \setminus \bar{\mathcal{O}}$, and $\mathcal{O}$ is a bounded open set in $\mathbf{R}^n$ with smooth boundary. Throughout this paper, we suppose that n is an odd integer with $n \geq 3$. We assume that Ω is connected and that the initial data $\vec{f} = (f_0, f_1)$ belongs to the associated energy space $\mathcal{H}_D(\Omega)$. Here and in the following, for an open set $Y \subset \mathbf{R}^n$, $\mathcal{H}_D(Y)$ stands for the completion of $(C_0^\infty(Y))^2$ with respect to $\|\vec{f}\|_{\mathcal{H}_D(Y)} = \|\nabla f_0\|_{L^2(Y)} + \|f_1\|_{L^2(Y)}$. $U(t)$ denotes the propagator of the mixed problem (1.1) with (1.2) and (1.3); in other words, for each $t \in \mathbf{R}$ we define a unitary operator $U(t)$ on $\mathcal{H}_D(\Omega)$ by

$$U(t)\vec{f} = (u(t, \cdot), \partial_t u(t, \cdot))$$

for $\vec{f} \in \mathcal{H}_D(\Omega)$, where u is the solution to (1.1)–(1.3).

It is well known that the solution to the above problem asymptotically approaches a solution to the Cauchy problem. More precisely, for a given initial data $\vec{f} \in \mathcal{H}_D(\Omega)$ there exists uniquely a scattering data $\vec{f}_+ \in \mathcal{H}_D(\mathbf{R}^n)$ such that

$$(1.4) \quad \|U(t)\vec{f} - (U_0(t)\vec{f}_+)|_\Omega\|_{\mathcal{H}_D(\Omega)} \rightarrow 0 \quad (t \rightarrow \infty),$$

where $U_0(t)$ is a unitary operator on $\mathcal{H}_D(\mathbf{R}^n)$ given by

$$U_0(t)\vec{g} = (u_0(t, \cdot), \partial_t u_0(t, \cdot)), \quad \vec{g} = (g_0, g_1) \in \mathcal{H}_D(\mathbf{R}^n)$$

with u_0 being the solution to the Cauchy problem

$$(1.5) \quad (\partial_t^2 - \Delta)u_0(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbf{R}^n,$$

$$(1.6) \quad u_0(0, x) = g_0(x), \quad (\partial_t u_0)(0, x) = g_1(x), \quad x \in \mathbf{R}^n.$$

It is also known that the local energy of $U(t)\vec{f}$ decays to zero as t tends to infinity. Namely, for any $R > 0$ and any $\vec{f} \in \mathcal{H}_D(\Omega)$, we have

$$(1.7) \quad \lim_{t \rightarrow \infty} \int_{\{x \in \Omega; |x| < R\}} \{|\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2\} dx = 0,$$

where u is the solution to (1.1)–(1.3).

In view of these facts, we see that the main part of the perturbed wave $U(t)\vec{f}$ escapes from any ball with a fixed radius as $t \rightarrow \infty$ and that it approaches some unperturbed wave $U_0(t)\vec{f}_+$ in the sense of the energy. However, to our knowledge, the rate of the convergence in (1.4) is not found explicitly in the literature. Therefore, it is natural to ask at which rate the perturbed wave tends to an unperturbed wave. In addition, we are interested in the regularity and decay properties of the scattering data $\vec{f}_+$. Namely, we wish to know whether the scattering data becomes smoother and decays faster at the spatial infinity or not, if the initial data is smooth and compactly supported. This consideration might be useful for the application to the nonlinear wave equation in an exterior domain. For instance, we are able to obtain a precise lower bound of the lifespan in our forthcoming paper [13].

Here we introduce notation in order to state our main result. Let m be a nonnegative integer and Y be an open set in $\mathbf{R}^n$. We set $\mathcal{H}^m(Y) = H^{m+1}(Y) \times H^m(Y)$ and $\|\vec{f}\|_{\mathcal{H}^m(Y)} = \|f_0\|_{H^{m+1}(Y)} + \|f_1\|_{H^m(Y)}$ for $\vec{f} = (f_0, f_1) \in \mathcal{H}^m(Y)$. Similarly, we put $\mathcal{W}^{m, \infty}(Y) = W^{m+1, \infty}(Y) \times W^{m, \infty}(Y)$ and $\|\vec{f}\|_{\mathcal{W}^{m, \infty}(Y)} = \|f_0\|_{W^{m+1, \infty}(Y)} + \|f_1\|_{W^{m, \infty}(Y)}$ for $\vec{f} \in \mathcal{W}^{m, \infty}(Y)$. Here $H^m(Y)$ (resp. $W^{m, \infty}(Y)$) stands for the Sobolev space based on $L^2(Y)$ (resp. $L^\infty(Y)$). In addition, we denote by $X^m(\Omega)$ the set of all $\vec{f} = (f_0, f_1) \in \mathcal{H}^m(\Omega)$ satisfying

the compatibility condition of the m -th order for the problem (1.1)–(1.3), that is to say, $f_j = 0$ on $\partial\Omega$ for any $j = 0, \dots, m$, where we have set

$$(1.8) \quad f_j(x) \equiv Af_{j-2}(x) \quad \text{for } x \in \bar{\Omega} \text{ and } j \geq 2.$$

Besides, we put $\mathcal{H}^\infty(Y) = \bigcap_{m=0}^\infty \mathcal{H}^m(Y)$ and $X^\infty(\Omega) = \bigcap_{m=0}^\infty X^m(\Omega)$.

We will use the notation $\partial_j = \partial_{x_j}$ for $1 \leq j \leq n$, and $\partial_x^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. We set

$$Z = (Z_0, Z_1, \dots, Z_N) = (\partial_t, \partial_1, \dots, \partial_n, (O_{ij})_{1 \leq i < j \leq n})$$

with $N = n(n+1)/2$, and $Z^\beta = Z_0^{\beta_0} Z_1^{\beta_1} \dots Z_N^{\beta_N}$ for a multi-index $\beta = (\beta_0, \beta_1, \dots, \beta_N)$, where O_{ij} for $1 \leq i, j \leq n$ is given by $O_{ij} = x_i \partial_j - x_j \partial_i$.

For $r > 0$ and $y \in \mathbf{R}^n$, $B_r(y)$ stands for an open ball of radius r centered at y . We write B_r for $B_r(0)$. Besides, we set $\Omega_r = \Omega \cap B_r$.

In what follows, for a constant C , when we write $C = C(p_1, \dots, p_m)$ with $p_1, \dots, p_m$ being some given constants or functions, it means that C is a constant depending only on $p_1, \dots, p_m$ if the space dimension n and the obstacle $\mathcal{O}$ are fixed (thus C may depend also on n and $\mathcal{O}$ actually).

Our main result reads as follows.

Theorem 1.1. *Let the space dimension n be odd, and $n \geq 3$. Assume that $\mathcal{O}$ is non-trapping, and $\bar{\mathcal{O}} \subset B_1$. Let $a (> 1)$ be a fixed number. Then for any $\vec{f} \in X^\infty(\Omega)$ with $\text{supp } \vec{f} \subset \bar{\Omega}_a$, there exists uniquely $\vec{f}_+ \in \mathcal{H}^\infty(\mathbf{R}^n)$ satisfying (1.4). Moreover, there exists a positive constant $\mu = \mu(a)$ having the following property: For any nonnegative integer k , there exists a positive constant $C = C(k, a)$ such that*

$$(1.9) \quad \begin{aligned} & \|\exp(\mu \langle \cdot \rangle) (U(t)\vec{f} - (U_0(t)\vec{f}_+)|_\Omega)\|_{\mathcal{H}^k(\Omega)} \\ & \leq C \exp(-\mu t) \|\vec{f}\|_{\mathcal{H}^k(\Omega)} \quad \text{for } t \geq 0, \end{aligned}$$

$$(1.10) \quad \|\exp(2\mu \langle \cdot \rangle) \vec{f}_+\|_{\mathcal{H}^{-k, \infty}(\mathbf{R}^n)} \leq C \|\vec{f}\|_{\mathcal{H}^{k+[n/2]+1}(\Omega)},$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$ for $x \in \mathbf{R}^n$, and $[n/2]$ denotes the largest integer not exceeding $n/2$.

Theorem 1.1 will be proved in Section 3. Our proof of Theorem 1.1 relies on the exponential decay of the local energy (see Lemma 2.5 below), and this is the reason why $n(\geq 3)$ is assumed to be odd and the obstacle $\mathcal{O}$ to be non-trapping. For the notion of the non-trapping obstacle, we refer to Melrose [10] for instance (see also Shibata–Tsutsumi [11, 12]). For example, star-shaped obstacles are known to be non-trapping.

Note that (1.10) implies that each component of $\vec{f}_+$ belongs to the Schwartz class $\mathcal{S}$, the class of rapidly decreasing functions.

Now we turn our attention to the asymptotic pointwise behavior of the perturbed wave $U(t)\vec{f}$. To describe the result, we define the Friedlander radiation field $\mathcal{F}_0[\vec{g}]$ by

$$(1.11) \quad \mathcal{F}_0[\vec{g}](s, \eta) = \frac{1}{2(2\pi)^{(n-1)/2}} \sum_{j=0}^1 (-\partial_s)^{(n-1)/2-j} \mathcal{R}[g_j](s, \eta)$$

for $\vec{g} = (g_0, g_1) \in (\mathcal{S}(\mathbf{R}^n))^2$. Here $\mathcal{R}[\varphi]$ denotes the Radon transform of $\varphi = \varphi(x)$, that is

$$\mathcal{R}[\varphi](s, \eta) = \int_{y \cdot \eta = s} \varphi(y) dS_y,$$

where dS_y denotes the area element on the hyperplane $\{y; y \cdot \eta = s\}$. The radiation field $\mathcal{F}_0[\vec{g}]$ is introduced to describe the main part of the unperturbed wave $U_0(t)\vec{g}$ for $\vec{g} \in (C_0^\infty(\mathbf{R}^3))^2$ in Friedlander [2]. The Friedlander radiation field is also closely connected to the translation representation for the wave equation (see Lax–Phillips [9], Friedlander [5] for instance). By Theorem 1.1 and the Sobolev imbedding theorem, we immediately see that the main part of $U(t)\vec{f}$ can be described by $\mathcal{F}_0[\vec{f}_+]$, and we would like to investigate its convergence rate. Our result is the following.

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold. Then for any $\vec{f} \in X^\infty(\Omega)$ with $\text{supp } \vec{f} \subset \overline{\Omega}_a$, there exists $\vec{f}_+ \in (\mathcal{S}(\mathbf{R}^n))^2$ satisfying the following property: For any nonnegative integer k , there exists a positive constant $C = C(k, a, \vec{f})$ such that, writing $x = r\omega$ with $r = |x|$ and $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in S^{n-1}$, we have*

$$(1.12) \quad \sum_{|z| \leq k} |Z^z \{u(t, x) - r^{-(n-1)/2} \mathcal{F}_0[\vec{f}_+](r - t, \omega)\}| \\ \leq C(1 + t + r)^{-(n+1)/2} \exp\left(-\frac{\mu}{2}|r - t|\right),$$

$$(1.13) \quad \sum_{|z| \leq k} |Z^z \{\partial_t u(t, x) - (-1)r^{-(n-1)/2} (\partial_s \mathcal{F}_0[\vec{f}_+])(r - t, \omega)\}| \\ + \sum_{|z| \leq k} \sum_{j=1}^n |Z^z \{\partial_j u(t, x) - \omega_j r^{-(n-1)/2} (\partial_s \mathcal{F}_0[\vec{f}_+])(r - t, \omega)\}| \\ \leq C(1 + t + r)^{-(n+1)/2} \exp\left(-\frac{\mu}{2}|r - t|\right)$$

for $(t, x) \in (0, \infty) \times \Omega$ satisfying $r \geq t/2 \geq 1$, where $u(t, x)$ is the solution to (1.1)–(1.3), and $\mu = \mu(a)$ is the positive constant from Theorem 1.1.

The proof of Theorem 1.2 will be given in Section 5, after obtaining the detailed convergence rate for the Cauchy problem in Section 4 (see Proposition 4.1).

We underline that the decaying factor $\exp(-\mu|r-t|/2)$ in the above estimates is quite meaningful even if the initial data is compactly supported, say $\text{supp } \vec{f} \subset \overline{\Omega_a}$, unlike the case of the Cauchy problem. In fact, the solution $u(t, x)$ for the mixed problem is not expected to vanish for $r-t \leq -a$ in general, because of the presence of the obstacle, while it is identically zero for $r-t \geq a$ and $t \geq 0$, in view of the domain of dependence (see Lemma 2.2 below). Accordingly, the radiation field $\mathcal{F}_0[\vec{f}_+](s, \omega)$ for the solution to the mixed problem is not supposed to be zero for $s \leq -a$ and $\omega \in S^{n-1}$ in general, while it vanishes for $s \geq a$ and $\omega \in S^{n-1}$ due to (1.12). In contrast to this, if there is no obstacle, it is known that the radiation field for compactly supported data vanishes not only for $s \geq a$, but also for $s \leq -a$ (this property is closely connected to the Huygens principle; see Lemma 2.3 below). In conclusion, it is essential to extract the decay factor with respect to $|r-t|$, in order to describe the behavior for the mixed problem in the region $r-t \leq -a$.

2. Preliminaries

Let Y be an open subset of $\mathbf{R}^n$, and Ω be as in the previous section. For the notational convenience, we put

$$(2.1) \quad \mathcal{H}_a^\infty(Y) = \{\vec{f} = (f_0, f_1) \in \mathcal{H}^\infty(Y); \text{supp } \vec{f} \subset \overline{Y \cap B_a}\},$$

$$(2.2) \quad X_a^\infty(\Omega) = \{\vec{f} = (f_0, f_1) \in X^\infty(\Omega); \text{supp } \vec{f} \subset \overline{\Omega_a}\}$$

for $a > 0$.

$U_0(t)$ and $U(t)$ are unitary in $\mathcal{H}_D(\mathbf{R}^n)$ and $\mathcal{H}_D(\Omega)$, respectively, and the energy inequalities for higher derivatives are also available: For a nonnegative integer k , there exists a positive constant C_k such that

$$(2.3) \quad \sum_{|z| \leq k} \|\partial_x^z U_0(t) \vec{g}\|_{\mathcal{H}_D(\mathbf{R}^n)} \leq C_k \sum_{|z| \leq k} \|\partial_x^z \vec{g}\|_{\mathcal{H}_D(\mathbf{R}^n)}, \quad t \in \mathbf{R}$$

for any $\vec{g} \in \mathcal{H}^k(\mathbf{R}^n)$, and

$$(2.4) \quad \sum_{|z| \leq k} \|\partial_x^z U(t) \vec{f}\|_{\mathcal{H}_D(\Omega)} \leq C_k \sum_{|z| \leq k} \|\partial_x^z \vec{f}\|_{\mathcal{H}_D(\Omega)}, \quad t \in \mathbf{R}$$

for any $\vec{f} \in X^k(\Omega)$, where $\partial_x = (\partial_1, \dots, \partial_n)$ and α is a multi-index. Combining (2.3) or (2.4) with the elementary inequality

$$(2.5) \quad \|h(t)\|_{L^2(Y)} \leq \|h(t_0)\|_{L^2(Y)} + \int_{t_0}^t \|\partial_t h(\tau)\|_{L^2(Y)} d\tau, \quad t \geq t_0,$$

which is valid for any smooth function h and any open subset Y of $\mathbf{R}^n$, we obtain the following estimates.

Lemma 2.1. *Let k be a nonnegative integer. Then we have*

$$(2.6) \quad \|U_0(t)\vec{g}\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq C_k(1+|t|)\|\vec{g}\|_{\mathcal{H}^k(\mathbf{R}^n)}, \quad t \in \mathbf{R}$$

for any $\vec{g} \in \mathcal{H}^k(\mathbf{R}^n)$. We also have

$$(2.7) \quad \|U(t)\vec{f}\|_{\mathcal{H}^k(\Omega)} \leq C_k(1+|t|)\|\vec{f}\|_{\mathcal{H}^k(\Omega)}, \quad t \in \mathbf{R}$$

for any $\vec{f} \in X^k(\Omega)$.

The following property is well known (see [9, Chapter V] for instance).

Lemma 2.2 (Domain of dependence). *Let n be a positive integer. Let $\tau, t_0 \in \mathbf{R}$ with $\tau < t_0$, and let $x_0 \in \mathbf{R}^n$. We define*

$$A(t_0, x_0, \tau) = \{(t, x) \in (\tau, t_0) \times \mathbf{R}^n; |x - x_0| < t_0 - t\}.$$

Suppose that $\psi = \psi(t, x)$ satisfies

$$(\partial_t^2 - \Delta)\psi(t, x) = 0, \quad (t, x) \in A(t_0, x_0, \tau).$$

Then we have

$$(2.8) \quad \|\partial\psi(t)\|_{L^2(B_{t_0-t}(x_0))} \leq \|\partial\psi(\tau)\|_{L^2(B_{t_0-\tau}(x_0))}, \quad t \in (\tau, t_0),$$

where $\partial\psi = (\partial_t\psi, \nabla\psi)$. As a consequence, if we also assume

$$\psi(\tau, x) = (\partial_t\psi)(\tau, x) = 0, \quad x \in B_{t_0-\tau}(x_0),$$

then we have $\psi(t, x) = 0$ for any $(t, x) \in A(t_0, x_0, \tau)$.

The above assertions are also valid if we replace $A(t_0, x_0, \tau)$ by

$$A^*(t_0, x_0, \tau) = \{(t, x) \in (2\tau - t_0, \tau) \times \mathbf{R}^n; |x - x_0| < t + t_0 - 2\tau\},$$

and (2.8) by

$$\|\partial\psi(t)\|_{L^2(B_{t+t_0-2\tau}(x_0))} \leq \|\partial\psi(\tau)\|_{L^2(B_{t_0-\tau}(x_0))}, \quad t \in (2\tau - t_0, \tau).$$

From the lemma above, we see that $\vec{f} \in X_a^\infty(\Omega)$ (resp. $\vec{g} \in \mathcal{H}_a^\infty(\mathbf{R}^n)$) implies $\text{supp}(U(t)\vec{f}) \subset \overline{\mathcal{O}_{|t|+a}}$ (resp. $\text{supp}(U_0(t)\vec{g}) \subset \overline{B_{|t|+a}}$).

In odd space dimensions, we have a stronger result.

Lemma 2.3 (The Huygens principle). *Let n be an odd integer with $n \geq 3$. Then $\vec{g} \in \mathcal{H}_a^\infty(\mathbf{R}^n)$ implies*

$$\text{supp}(U_0(t)\vec{g}) \subset \{x \in \mathbf{R}^n; |t| - a \leq |x| \leq |t| + a\}, \quad t \in \mathbf{R}.$$

Lemma 2.3 follows immediately from the explicit expression of $U_0(t)\vec{g}$ (see (4.30) below).

For the later usage, we mention an immediate consequence of Lemma 2.3 for inhomogeneous wave equations:

Corollary 2.4. *Let φ be a solution to*

$$\begin{cases} (\partial_t^2 - \Delta)\varphi(t, x) = F(t, x), & (t, x) \in (0, \infty) \times \mathbf{R}^n, \\ (\varphi(0, x), (\partial_t \varphi)(0, x)) = \vec{g}(x), & x \in \mathbf{R}^n, \end{cases}$$

where $\vec{g} \in \mathcal{H}^\infty(\mathbf{R}^n)$, and $F \in C^\infty([0, \infty) \times \mathbf{R}^n)$. Suppose that

$$(2.9) \quad \text{supp } \vec{g} \subset \overline{B_a}, \quad \text{and} \quad \text{supp } F \subset [0, T_0] \times \overline{B_b}$$

with some positive constants a, b , and T_0 . Then for $t \geq T_0$ we have

$$\text{supp } \varphi(t, \cdot) \subset D(\min\{t - a, (t - T_0) - b\}, \max\{t + a, t + b\}),$$

where $D(r_1, r_2) = \{x \in \mathbf{R}^n; r_1 \leq |x| \leq r_2\}$ for $0 \leq r_1 < r_2$.

Proof. By the Duhamel principle, $(\varphi, \partial_t \varphi)$ can be written as

$$(2.10) \quad (\varphi(t, x), \partial_t \varphi(t, x)) = (U_0(t)\vec{g})(x) + \int_0^t (U_0(t - \tau)\vec{F}(\tau, \cdot))(x) d\tau,$$

where $\vec{F}(t, x) = (0, F(t, x))$. For $t \geq T_0$, it follows from (2.9), (2.10), and Lemma 2.3 that

$$\text{supp } \varphi(t, \cdot) \subset D(t - a, t + a) \cup \bigcup_{\tau \in [0, T_0]} D((t - \tau) - b, (t - \tau) + b),$$

which implies the desired result immediately. $\square$

Next we introduce the local energy decay of the perturbed wave at exponential rate (for the proof, see for instance Melrose [10]; see also Shibata–Tsutsumi [11]).

Lemma 2.5. *Let n be odd and $n \geq 3$. Assume that $\mathcal{O}$ is non-trapping, and $\overline{\mathcal{O}} \subset B_1$. Suppose that $a, b > 1$, and k is a nonnegative integer. Then there exist two positive constants $C = C(k, a, b)$ and $\sigma = \sigma(a, b)$ such that for any $\vec{f} \in X_a^\infty(\Omega)$ we have*

$$(2.11) \quad \|U(t)\vec{f}\|_{\mathcal{H}^k(\Omega_b)} \leq C \exp(-\sigma t) \|\vec{f}\|_{\mathcal{H}^k(\Omega)} \quad \text{for } t \geq 0.$$

The following lemma, motivated by the arguments in Ikawa [7], tells us that the perturbed wave can be decomposed into the unperturbed wave and the correction term. The former is the main part of the perturbed wave, while the latter takes care of the effect from the boundary and its size can be small compared with the initial energy. This lemma is crucial for proving Theorem 1.1.

Lemma 2.6. *Let n , $\mathcal{O}$, and a be as in Theorem 1.1. Then, for any $\vec{f} \in X_a^\infty(\Omega)$ and $T(\geq a+2)$, there exist $\vec{g}_1 \in \mathcal{H}_{T+a}^\infty(\mathbf{R}^n)$ and $\vec{f}_1 \in X_3^\infty(\Omega)$ satisfying*

$$(2.12) \quad U(t)\vec{f} = (U_0(t-T)\vec{g}_1)|_\Omega + U(t-T)\vec{f}_1, \quad t \geq T,$$

$$(2.13) \quad \text{supp } U_0(t-T)\vec{g}_1 \subset D(t-T+1, t+a), \quad t \geq T,$$

$$(2.14) \quad \|\vec{g}_1\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq C_0(1+T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)},$$

$$(2.15) \quad \|\vec{f}_1\|_{\mathcal{H}^k(\Omega)} \leq C_0 \exp(-\sigma T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)}$$

for any nonnegative integer k with some positive constants $C_0 = C_0(k, a)$ and $\sigma = \sigma(a)$. Here $D(r_1, r_2)$ is defined in Corollary 2.4.

Remark. By (2.13), we see that $U_0(t-T)\vec{g}_1$ vanishes on $\bar{\mathcal{O}}(\subset B_1)$ for $t \geq T$. Thus the uniqueness of the solution to the mixed problem (1.1)–(1.3) implies

$$(2.16) \quad (U_0(t-T)\vec{g}_1)|_\Omega = U(t-T)(\vec{g}_1|_\Omega), \quad t \geq T.$$

Proof. In this proof, various positive constants depending only on k will be indicated by the same letter C_k . Let $\vec{f} \in X_a^\infty(\Omega)$ be a given function.

Step 1: Construction of $\vec{g}_1$ and proof of (2.14). Let $\chi_1 = \chi_1(x)$ be a smooth and nonnegative function satisfying $\chi_1(x) = 1$ for $|x| \geq 1$, and $\chi_1(x) = 0$ for $x \in \bar{\mathcal{O}}$. For a function F on Ω , $\mathcal{E}[F]$ denotes the extension of F by 0 outside Ω . We put

$$(2.17) \quad \vec{\psi} := \chi_1 \mathcal{E}[U(T-2)\vec{f}] \quad \text{in } \mathbf{R}^n,$$

and we define

$$(2.18) \quad \vec{g}_1 := U_0(2)\vec{\psi} \quad \text{in } \mathbf{R}^n.$$

Then we have $\vec{g}_1 \in \mathcal{H}_{T+a}^\infty(\mathbf{R}^n)$. Indeed, since $\vec{f} \in X_a^\infty(\Omega)$, Lemma 2.2 implies $U(T-2)\vec{f} \in X_{T-2+a}^\infty(\Omega)$; consequently we get $\vec{\psi} \in \mathcal{H}_{T-2+a}^\infty(\mathbf{R}^n)$, which implies $\vec{g}_1 \in \mathcal{H}_{T+a}^\infty(\mathbf{R}^n)$ by Lemma 2.2 again.

Next we show (2.14). By (2.17) and (2.7), we obtain

$$(2.19) \quad \|\vec{\psi}\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq C_k \|U(T-2)\vec{f}\|_{\mathcal{H}^k(\Omega)} \leq C_k(T-1)\|\vec{f}\|_{\mathcal{H}^k(\Omega)}.$$

Therefore, from (2.18) and (2.6), we obtain

$$\|\vec{g}_1\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq C_k \|\vec{\psi}\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq C_k(T-1)\|\vec{f}\|_{\mathcal{H}^k(\Omega)},$$

which implies (2.14).

Step 2: Proof of (2.13). We prove (2.13) by applying Corollary 2.4 in the following way: We define a function z on $[0, \infty) \times \mathbf{R}^n$ by

$$z(t, x) = \begin{cases} u(t, x) & \text{for } (t, x) \in [0, T-2] \times \Omega, \\ 0 & \text{for } (t, x) \in [0, T-2] \times \bar{\Omega}, \\ v(t, x) & \text{for } (t, x) \in (T-2, \infty) \times \mathbf{R}^n, \end{cases}$$

where u and v are given by

$$\begin{aligned} (u(t, x), (\partial_t u)(t, x)) &:= (U(t)\vec{f})(x), & t \geq 0, x \in \Omega, \\ (v(t, x), (\partial_t v)(t, x)) &:= (U_0(t-T)\vec{g}_1)(x) \\ &= (U_0(t-T+2)\vec{\psi})(x), & t \geq T-2, x \in \mathbf{R}^n. \end{aligned}$$

Noting that we have

$$(u(t, x), (\partial_t u)(t, x)) = \vec{\psi}(x) = (v(t, x), (\partial_t v)(t, x))$$

for $t = T-2$ and $|x| \geq 1$ (cf. (2.17)), we see that z is smooth in $[0, \infty) \times (\mathbf{R}^n \setminus B_1)$, and $(\partial_t^2 - \Delta)z = 0$ in $[0, \infty) \times (\mathbf{R}^n \setminus B_1)$. We cut off z to make it smooth in $[0, \infty) \times \mathbf{R}^n$: For $\varepsilon > 0$, let $\chi_{2,\varepsilon}$ be a smooth function in $[0, \infty) \times \mathbf{R}^n$ such that

$$\chi_{2,\varepsilon}(t, x) = \begin{cases} 1 & \text{for } |x| \geq 1 + \varepsilon \text{ or } t \geq T-2 + \varepsilon, \\ 0 & \text{for } (t, x) \in [0, T-2] \times B_1. \end{cases}$$

For $(t, x) \in [0, \infty) \times \mathbf{R}^n$ we define

$$z_\varepsilon(t, x) = \chi_{2,\varepsilon}(t, x)z(t, x).$$

Note that we have

$$(2.20) \quad U_0(t-T)\vec{g}_1 = (z_\varepsilon(t), \partial_t z_\varepsilon(t)), \quad t \geq T-2 + \varepsilon$$

by definition. Let ε be a positive and small constant satisfying $1 + \varepsilon \leq a$. We are going to show

$$(2.21) \quad \text{supp } z_\varepsilon(t, \cdot) \subset D(t-T+1-2\varepsilon, t+a)$$

for $t \geq T-2 + \varepsilon$. Note that (2.13) follows from (2.20) and (2.21): Indeed it follows from (2.20) and (2.21) that

$$(2.22) \quad \text{supp } U_0(t-T)\vec{g}_1 \subset D(t-T+1-2\varepsilon, t+a)$$

for $t \geq T(> T-2 + \varepsilon)$, provided that ε satisfies $0 < \varepsilon \leq a-1$ and is small; from (2.22) we obtain (2.13) because we have

$$\bigcap_{0 < \varepsilon \leq \varepsilon_0} D(t-T+1-2\varepsilon, t+a) = D(t-T+1, t+a)$$

for any $\varepsilon_0 > 0$. We start the proof of (2.21). In view of Corollary 2.4, our task is to investigate the support of $(z_\varepsilon(0), \partial_t z_\varepsilon(0))$ and $(\partial_t^2 - \Delta)z_\varepsilon$. Since $\vec{f} \in X_a^\infty(\Omega)$, it is easy to see

$$(2.23) \quad \text{supp } z_\varepsilon(0, \cdot) \cup \text{supp } \partial_t z_\varepsilon(0, \cdot) \subset \overline{B_a}.$$

Since $(\partial_t^2 - \Delta)z(t, x) = 0$ for $(t, x) \in \text{supp } \chi_{2, \varepsilon}$, we get

$$(\partial_t^2 - \Delta)z_\varepsilon = (\partial_t^2 \chi_{2, \varepsilon} - \Delta \chi_{2, \varepsilon})z + 2\{(\partial_t \chi_{2, \varepsilon})(\partial_t z) - \nabla \chi_{2, \varepsilon} \cdot \nabla z\}.$$

Hence we obtain

$$(2.24) \quad \text{supp}(\partial_t^2 - \Delta)z_\varepsilon \subset [0, T - 2 + \varepsilon] \times \overline{B_{1+\varepsilon}},$$

because $\text{supp } \partial_{t,x}^\alpha \chi_{2, \varepsilon} \subset [0, T - 2 + \varepsilon] \times \overline{B_{1+\varepsilon}}$ for $|\alpha| = 1, 2$. By (2.23) and (2.24) we can apply Corollary 2.4 (with $\varphi = z_\varepsilon$, $\vec{g} = (z_\varepsilon(0), \partial_t z_\varepsilon(0))$, $F = (\partial_t^2 - \Delta)z_\varepsilon$, $T_0 = T - 2 + \varepsilon$, and $b = 1 + \varepsilon$) to obtain (2.21) for $t \geq T_0$ (note that we have $\min\{t - a, t - (T - 2 + \varepsilon) - (1 + \varepsilon)\} = t - T + 1 - 2\varepsilon$ and $\max\{t + a, t + (1 + \varepsilon)\} = t + a$ because $1 + \varepsilon \leq a \leq T - 2$).

Step 3: Construction of f_1 , and proof of (2.12) and (2.15). We define

$$(2.25) \quad \vec{f}_1 := U(T)\vec{f} - \vec{g}_1|_\Omega \quad \text{in } \Omega.$$

Since (2.13) implies (2.16), we get

$$\begin{aligned} & (U_0(t - T)\vec{g}_1)|_\Omega + U(t - T)\vec{f}_1 \\ &= U(t - T)(\vec{g}_1|_\Omega + \vec{f}_1) = U(t - T)U(T)\vec{f} = U(t)\vec{f} \end{aligned}$$

for $t \geq T$, which shows (2.12).

Next we will show that $\vec{f}_1 \in X_3^\infty(\Omega)$. From (2.13), we see that $\vec{g}_1$ vanishes in an open set containing $\partial\Omega$. Thus the compatibility condition of the infinite order is satisfied for $\vec{g}_1|_\Omega$, and we find

$$(2.26) \quad \vec{g}_1|_\Omega \in X_{T+a}^\infty(\Omega).$$

Since $U(T)\vec{f} \in X^\infty(\Omega)$, from (2.25) and (2.26) we get $\vec{f}_1 \in X^\infty(\Omega)$. Hence it remains to prove

$$(2.27) \quad \text{supp } \vec{f}_1 \subset \overline{\Omega_3}$$

in order to show $\vec{f}_1 \in X_3^\infty(\Omega)$. For the proof of (2.27), we put

$$(w(t, x), (\partial_t w)(t, x)) := (U(t)\vec{f})(x) - (U_0(t - T + 2)\vec{\psi})(x)$$

for $(t, x) \in [T - 2, \infty) \times \Omega$. Then we have

$$(2.28) \quad \square w(t, x) = 0, \quad (t, x) \in (T - 2, \infty) \times \Omega.$$

From (2.17) we also get

$$(2.29) \quad (w(T-2, x), (\partial_t w)(T-2, x)) = (U(T-2)\vec{f})(x) - \vec{\psi}(x) = (0, 0)$$

for $x \in \mathbf{R}^n \setminus B_1$. If $t \geq T-2$ and $|x| \geq t - T + 3$, then we have

$$A(t, x, T-2) \subset [T-2, \infty) \times (\mathbf{R}^n \setminus B_1),$$

where $A(t_0, x_0, \tau)$ is from Lemma 2.2. Hence (2.28), (2.29), and Lemma 2.2 imply

$$(2.30) \quad w(t, x) = 0 \quad \text{for } |x| \geq t - T + 3, t \geq T-2,$$

which shows (2.27) because we have $\vec{f}_1 = (w(T), (\partial_t w)(T))$.

Finally we are going to prove (2.15). From (2.18), (2.25), and (2.27), we get

$$(2.31) \quad \|\vec{f}_1\|_{\mathcal{H}^k(\Omega)} \leq \|U(T)\vec{f}\|_{\mathcal{H}^k(\Omega_3)} + \|U_0(2)\vec{\psi}\|_{\mathcal{H}^k(B_3)}.$$

It follows from (2.8), (2.5) and (2.17) that

$$(2.32) \quad \|U_0(2)\vec{\psi}\|_{\mathcal{H}^k(B_3)} \leq C_k \|\vec{\psi}\|_{\mathcal{H}^k(B_5)} \leq C_k \|U(T-2)\vec{f}\|_{\mathcal{H}^k(\Omega_5)}.$$

Using Lemma 2.5, from (2.31) and (2.32) we obtain

$$\begin{aligned} \|\vec{f}_1\|_{\mathcal{H}^k(\Omega)} &\leq \|U(T)\vec{f}\|_{\mathcal{H}^k(\Omega_3)} + C_k \|U(T-2)\vec{f}\|_{\mathcal{H}^k(\Omega_5)} \\ &\leq C \exp(-\sigma T) \|\vec{f}\|_{\mathcal{H}^k(\Omega)}, \end{aligned}$$

where $C = C(k, a)$ and $\sigma = \sigma(a)$ are positive constants. This completes the proof. $\square$

We conclude this section by showing the following lemma concerning the uniqueness of the scattering data.

Lemma 2.7. *Let $\vec{f} \in X^\infty(\Omega)$. If $\vec{f}_+ \in \mathcal{H}^0(\mathbf{R}^n)$ satisfies*

$$(2.33) \quad \lim_{t \rightarrow \infty} \|U(t)\vec{f} - (U_0(t)\vec{f}_+)|_\Omega\|_{\mathcal{H}_D(\Omega)} = 0,$$

then $\vec{f}_+$ is determined uniquely.

Proof. Let $\vec{f}_+ \in \mathcal{H}^0(\mathbf{R}^n)$ be as in the assertion and suppose that $\vec{g}_+ \in \mathcal{H}^0(\mathbf{R}^n)$ satisfies $\lim_{t \rightarrow \infty} \|U(t)\vec{f} - (U_0(t)\vec{g}_+)|_\Omega\|_{\mathcal{H}_D(\Omega)} = 0$. Then we have

$$(2.34) \quad \lim_{t \rightarrow \infty} \|U_0(t)(\vec{f}_+ - \vec{g}_+)\|_{\mathcal{H}_D(\Omega)} = 0.$$

Therefore it suffices to show

$$(2.35) \quad \lim_{t \rightarrow \infty} \|U_0(t)(\vec{f}_+ - \vec{g}_+)\|_{\mathcal{H}_D(B_1)} = 0.$$

In fact, once we obtain (2.35), then we see from (2.34) and the unitarity of $U_0(t)$ on $\mathcal{H}_D(\mathbf{R}^n)$ that $\vec{f}_+ = \vec{g}_+$ in $\mathcal{H}_D(\mathbf{R}^n)$. Moreover, the Hölder inequality and the Sobolev imbedding theorem imply that, for any $R > 0$, there exists a positive constant C_R such that we have

$$\|v\|_{L^2(B_R)} \leq C_R \|v\|_{L^{2n/(n-2)}(\mathbf{R}^n)} \leq C_R \|\nabla v\|_{L^2(\mathbf{R}^n)}$$

for any $v \in \dot{H}^1(\mathbf{R}^n)$. We thus find $\vec{f}_+ = \vec{g}_+$ in $\mathcal{H}^0(\mathbf{R}^n)$.

Now we show (2.35). Since $\vec{f}_+, \vec{g}_+ \in \mathcal{H}^0(\mathbf{R}^n)$, for any $\varepsilon > 0$ there exists $\vec{h} \in (C_0^\infty(\mathbf{R}^n))^2$ such that $\|(\vec{f}_+ - \vec{g}_+) - \vec{h}\|_{\mathcal{H}_D(\mathbf{R}^n)} < \varepsilon$. Let $\text{supp } \vec{h} \subset \overline{B_M}$. Since the Huygens principle implies $U_0(t)\vec{h} = 0$ for $|x| \leq 1$ and $t \geq M + 1$, we obtain

$$\begin{aligned} \|U_0(t)(\vec{f}_+ - \vec{g}_+)\|_{\mathcal{H}_D(B_1)} &= \|U_0(t)(\vec{f}_+ - \vec{g}_+ - \vec{h})\|_{\mathcal{H}_D(B_1)} \\ &\leq \|\vec{f}_+ - \vec{g}_+ - \vec{h}\|_{\mathcal{H}_D(\mathbf{R}^n)} < \varepsilon \end{aligned}$$

for $t \geq M + 1$, which leads to (2.35). This completes the proof. $\square$

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by dividing the argument into four steps. We would like to construct f_+ satisfying (1.9) and (1.10) for all k , but first we fix k and construct f_+ having the desired property for this fixed k . Then we will show that f_+ is independent of k in fact.

Step 1: Construction of $\vec{f}_+$ with fixed k . We fix a nonnegative integer k and set

$$a_* = \max\{a, 3\}, \quad \mu = \sigma(a_*)/4, \quad \text{and} \quad C_1 = C_0(k, a_*),$$

where σ and C_0 are from Lemma 2.6. We choose $T(\geq a_* + 2)$ to be so large that $C_1 \exp(-\mu T) \leq 1$, and we put

$$C_{2,T} := C_1(1 + T).$$

Then we see from Lemma 2.6 that for $\vec{f} \in X_a^\infty(\Omega)$, there exist $\vec{g}_1 \in \mathcal{H}_{T+a_*}^\infty(\mathbf{R}^n)$ and $\vec{f}_1 \in X_3^\infty(\Omega)$ satisfying (2.12),

$$\|\vec{g}_1\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq C_1(1 + T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)} = C_{2,T}\|\vec{f}\|_{\mathcal{H}^k(\Omega)},$$

and

$$\|\vec{f}_1\|_{\mathcal{H}^k(\Omega)} \leq C_1 \exp(-4\mu T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)} \leq \exp(-3\mu T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)}.$$

We apply Lemma 2.6 to $\vec{f}_1$ again to find $\vec{g}_2 \in \mathcal{H}_{T+a_*}^\infty(\mathbf{R}^n)$ and $\vec{f}_2 \in X_3^\infty(\Omega)$ for which we have

$$U(t-T)\vec{f}_1 = (U_0(t-2T)\vec{g}_2)|_\Omega + U(t-2T)\vec{f}_2 \quad \text{for } t \geq 2T,$$

$$\|\vec{g}_2\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq C_1(1+T)\|\vec{f}_1\|_{\mathcal{H}^k(\Omega)} \leq C_{2,T} \exp(-3\mu T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)},$$

and

$$\|\vec{f}_2\|_{\mathcal{H}^k(\Omega)} \leq \exp(-3\mu T)\|\vec{f}_1\|_{\mathcal{H}^k(\Omega)} \leq \exp(-6\mu T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)}.$$

Repeating the same procedure, we can construct sequences $\{\vec{g}_j\}_{j=1}^\infty \subset \mathcal{H}_{T+a_*}^\infty(\mathbf{R}^n)$ and $\{\vec{f}_j\}_{j=1}^\infty \subset X_3^\infty(\Omega)$ in such a way that

$$(3.1) \quad U(t-(j-1)T)\vec{f}_{j-1} = (U_0(t-jT)\vec{g}_j)|_\Omega + U(t-jT)\vec{f}_j, \quad t \geq jT,$$

$$(3.2) \quad \|\vec{g}_j\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq C_{2,T} \exp(-3\mu(j-1)T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)},$$

and

$$(3.3) \quad \|\vec{f}_j\|_{\mathcal{H}^k(\Omega)} \leq \exp(-3\mu jT)\|\vec{f}\|_{\mathcal{H}^k(\Omega)}$$

for $j \geq 1$, where we have put $\vec{f}_0 = \vec{f}$.

Now we define

$$(3.4) \quad \vec{f}_+ = \sum_{j=1}^{\infty} U_0(-jT)\vec{g}_j.$$

Then it belongs to $\mathcal{H}^\infty(\mathbf{R}^n)$. In fact, (2.6) and (3.2) lead to

$$(3.5) \quad \begin{aligned} \|U_0(-jT)\vec{g}_j\|_{\mathcal{H}^k(\mathbf{R}^n)} &\leq C(1+jT) \exp(-3\mu(j-1)T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)} \\ &\leq C \exp(-2\mu(j-1)T)\|\vec{f}\|_{\mathcal{H}^k(\Omega)}, \end{aligned}$$

where C is a constant depending on k and T , but is independent of j . Here we have used the following elementary equality:

$$\max_{y \in \mathbf{R}} (1+T+y) \exp(-\mu y) = \mu^{-1} \exp(\mu(1+T)-1).$$

Therefore we have

$$(3.6) \quad \|\vec{f}_+\|_{\mathcal{H}^k(\mathbf{R}^n)} \leq \sum_{j=1}^{\infty} C(\exp(-2\mu T))^{j-1} \|\vec{f}\|_{\mathcal{H}^k(\Omega)} \leq C\|\vec{f}\|_{\mathcal{H}^k(\Omega)}.$$

Step 2: Proof of (1.10) for fixed k . First we observe that the Huygens principle leads to

$$(3.7) \quad \text{supp}(U_0(t - jT)\vec{g}_j) \subset \{x \in \mathbf{R}^n; |x| - |jT - t| \leq T + a_*\}$$

for any natural number j and $t \in \mathbf{R}$, because we have $\text{supp } \vec{g}_j \subset \overline{B_{T+a_*}}$.

For $\vec{h} = (h_0, h_1)$, we write

$$(3.8) \quad |\vec{h}(x)|_k = \sum_{|\alpha| \leq k+1} |\partial_x^\alpha h_0(x)| + \sum_{|\alpha| \leq k} |\partial_x^\alpha h_1(x)|$$

in what follows. Using the Sobolev imbedding theorem, from (3.5) and (3.7) with $t = 0$, we get

$$(3.9) \quad |(U_0(-jT)\vec{g}_j)(x)|_k \leq C \exp(-2\mu|x|) \|\vec{f}\|_{\mathcal{H}^{k+[n/2]+1}(\Omega)},$$

where C is a constant depending on k , a and T , but is independent of j and x . (1.10) follows from (3.9). Indeed, for each fixed $x \in \Omega$, the number of j for which we have $x \in \text{supp}(U_0(-jT)\vec{g}_j)$ is at most $[2(T + a_*)/T] + 1$ because of (3.7). Therefore, remembering (3.4), we can sum up (3.9) to obtain

$$|\vec{f}_+(x)|_k \leq C \exp(-2\mu|x|) \|\vec{f}\|_{\mathcal{H}^{k+[n/2]+1}(\Omega)},$$

which shows (1.10).

Step 3: Proof of (1.9) for fixed k . For $t \geq T$, we are able to find a positive integer J such that $t \in [JT, (J+1)T)$. By (3.1) with $j = 1, \dots, J$ we have

$$U(t)\vec{f} = \sum_{j=1}^J (U_0(t - jT)\vec{g}_j)|_\Omega + U(t - JT)\vec{f}_J.$$

By (3.4) we get

$$(3.10) \quad \|e^{\mu\langle \cdot \rangle} (U(t)\vec{f} - (U_0(t)\vec{f}_+)|_\Omega)\|_{\mathcal{H}^k(\Omega)} \\ \leq \sum_{j=J+1}^{\infty} \|e^{\mu\langle \cdot \rangle} U_0(t - jT)\vec{g}_j\|_{\mathcal{H}^k(\Omega)} + \|e^{\mu\langle \cdot \rangle} U(t - JT)\vec{f}_J\|_{\mathcal{H}^k(\Omega)}.$$

Note that (3.10) is also valid for $0 \leq t < T$, by regarding $J = 0$ and $\vec{f}_0 = \vec{f}$. So we assume $J \geq 0$ and $t \in [JT, (J+1)T)$ in the following.

First we evaluate the second term on the right-hand side of (3.10). Since $\text{supp } \vec{f}_J \subset \overline{B_3}$ and $|t - JT| \leq T$ for $t \in [JT, (J+1)T)$, we have $|x| \leq T + 3$ for $x \in \text{supp}(U(t - JT)\vec{f}_J)$, and hence $\sum_{|\alpha| \leq k+1} |\partial_x^\alpha e^{\mu\langle x \rangle}| \leq C$ with some positive constant $C = C(k, a, T)$. Therefore, using (2.7) and (3.3), we get

$$(3.11) \quad \|e^{\mu\langle \cdot \rangle} U(t - JT)\vec{f}_J\|_{\mathcal{H}^k(\Omega)} \leq C(1 + T) \exp(-3\mu JT) \|\vec{f}\|_{\mathcal{H}^k(\Omega)}.$$

Next we estimate the first term on the right-hand side of (3.10). Let $j \geq J+1$. From (3.7) we have $|x| \leq T + a_* + jT - t$ for $x \in \text{supp}(U_0(t - jT)\vec{g}_j)$, so that $\sum_{|\alpha| \leq k+1} |\partial_x^\alpha e^{\mu\langle x \rangle}| \leq Ce^{\mu jT}$ with some positive constant $C = C(k, a, T)$. Hence, it follows from (2.6) and (3.2) that

$$\begin{aligned} \|e^{\mu\langle \cdot \rangle} U_0(t - jT)\vec{g}_j\|_{\mathcal{H}^k(\Omega)} &\leq Ce^{\mu jT} (1 + |t - jT|) \|\vec{g}_j\|_{\mathcal{H}^k(\mathbf{R}^n)} \\ &\leq C(1 + jT) e^{\mu jT - 3\mu(j-1)T} \|\vec{f}\|_{\mathcal{H}^k(\Omega)} \\ &\leq Ce^{-\mu(j-1)T} \|\vec{f}\|_{\mathcal{H}^k(\Omega)}, \end{aligned}$$

where C is a constant independent of j and J . Thus we get

$$(3.12) \quad \sum_{j=J+1}^{\infty} \|e^{\mu\langle \cdot \rangle} U_0(t - jT)\vec{g}_j\|_{\mathcal{H}^k(\Omega)} \leq C \exp(-\mu JT) \|\vec{f}\|_{\mathcal{H}^k(\Omega)},$$

where C is a constant independent of J . Finally by (3.11) and (3.12) we see that (1.9) holds for $t \in [JT, (J+1)T)$ with $J \geq 0$, and hence for all $t \geq 0$.

Step 4: Conclusion. For each fixed k , we have constructed f_+ satisfying (1.9) and (1.10) for this fixed k . Observe that the construction itself depends on k through the choice of T . However, by using Lemma 2.7, we can show that $\vec{f}_+$ being constructed in the above is independent of k . In fact, let $\vec{f}_+^{(1)}$ and $\vec{f}_+^{(2)}$ denote $\vec{f}_+$ constructed in the above with the choice of $k = k_1$ and $k = k_2$, respectively, where k_1 and k_2 are nonnegative integers. Then, from (1.9) we see that (2.33) is valid for $\vec{f}_+ = \vec{f}_+^{(1)}$ and $\vec{f}_+ = \vec{f}_+^{(2)}$. Hence Lemma 2.7 implies $\vec{f}_+^{(1)} = \vec{f}_+^{(2)}$.

Now we find that the same $\vec{f}_+$ enjoys the estimates (1.9) and (1.10) for *all* k . This completes the proof of Theorem 1.1. $\square$

4. The Friedlander radiation field for rapidly decreasing data

Our aim in this section is to discuss the Friedlander radiation field for the Cauchy problem with rapidly decreasing data. The case of compactly supported data is well known (see Friedlander [2, 3, 4]; see also Hörmander [6] and John [8]). Indeed, for any $\vec{g} \in (C_0^\infty(\mathbf{R}^n))^2$ and any multi-index α , there exists a positive constant $C = C(\alpha, \vec{g})$ such that

$$(4.1) \quad \begin{aligned} &|Z^\alpha \{u_0(t, x) - r^{-(n-1)/2} \mathcal{F}_0[\vec{g}](r - t, \omega)\}| \\ &\leq C(1 + t + r)^{-(n+1)/2} (1 + |r - t|)^{-(3-n)/2}, \end{aligned}$$

holds for $r \geq t/2 \geq 1$ with $r = |x|$ and $\omega = r^{-1}x$, where $u_0(t, \cdot)$ is the first component of $U_0(t)\vec{g}$, and the radiation field $\mathcal{F}_0[\vec{g}](s, \eta)$ is given by (1.11) (here

n can be either odd or even). The case of rapidly decreasing data was also treated in [6] through the conformal compactification of the Minkowski space, and (4.1) is valid also for this case. However the decay away from the light cone in (4.1) seems not to be sharp. For this reason, we would like to extend (4.1) in the two directions described below, restricting our attention to the odd space dimensional case.

The one is to give an argument to treat the case of rapidly decreasing data based on the explicit representation of u_0 . This requires us to evaluate the spherical means over the whole sphere. In contrast, it is enough to estimate the integral over the portion of the sphere if the support of the data is compact.

The other is to extract a more detailed decay property away from the cone, in accordance with the stronger decay property of the scattering data $\vec{f}_+$ obtained in Theorem 1.1 than general functions in $\mathcal{S}(\mathbf{R}^n)$, where $\mathcal{S}(\mathbf{R}^n)$ is the Schwartz class, the set of rapidly decreasing functions. To this end, we introduce the following class of the data. Throughout this section, $\chi = \chi(s)$ is some given non-decreasing function of $s \geq 0$, satisfying $\chi(s) \geq 1$ for all $s \geq 0$. For $\varphi \in C^\infty(\mathbf{R}^n)$, $m \geq 0$ and a nonnegative integer k , we define

$$\|\varphi\|_{\chi, k, m} = \left(\sup_{x \in \mathbf{R}^n} \sum_{|x| \leq k} (1 + |x|^2)^m \chi^2(|x|) |\partial_x^\alpha \varphi(x)|^2 \right)^{1/2},$$

and let $\mathcal{S}_\chi(\mathbf{R}^n)$ be the set of all $\varphi \in C^\infty(\mathbf{R}^n)$ satisfying $\|\varphi\|_{\chi, m, k} < \infty$ for any nonnegative integers m and k . Apparently we have $\mathcal{S}_\chi(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$. Note that $\mathcal{S}_\chi(\mathbf{R}^n) = \mathcal{S}(\mathbf{R}^n)$ if χ is identically equal to 1.

Our main result in this section is the following.

Proposition 4.1. *Let n be an odd integer with $n \geq 3$, and let $v \geq 0$. For any $\vec{g} \in (\mathcal{S}_\chi(\mathbf{R}^n))^2$ and any multi-index α , there exists a positive constant $C = C(\alpha, v, \vec{g})$ such that we have*

$$(4.2) \quad |Z^\alpha \{u_0(t, x) - r^{-(n-1)/2} \mathcal{F}_0[\vec{g}](r - t, \omega)\}| \\ \leq C(1 + t + r)^{-(n+1)/2} (1 + |r - t|)^{-v} \chi^{-1}(|r - t|),$$

$$(4.3) \quad |Z^\alpha \{\partial_t u_0(t, x) - (-1)r^{-(n-1)/2} (\partial_s \mathcal{F}_0[\vec{g}])(r - t, \omega)\}| \\ + \sum_{j=1}^n |Z^\alpha \{\partial_j u_0(t, x) - \omega_j r^{-(n-1)/2} (\partial_s \mathcal{F}_0[\vec{g}])(r - t, \omega)\}| \\ \leq C(1 + t + r)^{-(n+1)/2} (1 + |r - t|)^{-v} \chi^{-1}(|r - t|)$$

for $r \geq t/2 \geq 1$ with $r = |x|$ and $\omega = (\omega_1, \dots, \omega_n) = r^{-1}x$, where $u_0(t, \cdot)$ is the first component of $U_0(t)\vec{g}$.

Before proceeding to prove Proposition 4.1, we give preliminaries about the Radon transform and spherical means in subsections 4.1 and 4.2, respectively, because the radiation field $\mathcal{F}_0[\vec{g}]$ and the solution $U_0(t)\vec{g}$ are closely related to the Radon transform and the spherical means, respectively.

4.1. Basic properties of the Radon transform

The Radon transform $\mathcal{R}[\varphi](s, \eta)$ for $\varphi \in \mathcal{S}(\mathbf{R}^n)$ is defined by

$$\mathcal{R}[\varphi](s, \eta) = \int_{\Pi(s, \eta)} \varphi(y) dS_y, \quad (s, \eta) \in \mathbf{R} \times S^{n-1},$$

where $\Pi(s, \eta) = \{y \in \mathbf{R}^n; y \cdot \eta = s\}$, and dS_y denotes the area element on $\Pi(s, \eta)$. For $\eta \in S^{n-1}$ and a smooth function $\varphi = \varphi(y)$ on $\mathbf{R}^n$, $D_\eta \varphi$ denotes the directional derivative of φ in the direction η ; in other words, we define $(D_\eta \varphi)(y) = \eta \cdot \nabla_y \varphi(y)$. We write

$$o_{ij} = \eta_i \partial_{\eta_j} - \eta_j \partial_{\eta_i}, \quad 1 \leq i, j \leq n.$$

We put $o = (o_1, \dots, o_{n(n-1)/2}) = (o_{ij})_{1 \leq i < j \leq n}$, where o_{ij} 's are regarded to be arranged in dictionary order. We write $o^\alpha = o_1^{\alpha_1} \dots o_d^{\alpha_d}$ with a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, where $d = n(n-1)/2$. O^α is similarly defined using O_{ij} instead of o_{ij} , where $(O_{ij}\varphi)(y) = y_i(\partial_j \varphi)(y) - y_j(\partial_i \varphi)(y)$ as before.

It is easy to check

$$(4.4) \quad \partial_s \mathcal{R}[\varphi](s, \eta) = \mathcal{R}[D_\eta \varphi](s, \eta) \quad \left(= \int_{\Pi(s, \eta)} (D_\eta \varphi)(y) dS_y \right),$$

$$(4.5) \quad o_{ij} \mathcal{R}[\varphi](s, \eta) = \mathcal{R}[O_{ij} \varphi](s, \eta), \quad 1 \leq i < j \leq n$$

for $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Because integrals over $\Pi(s, \eta)$ of directional derivatives of φ in directions proportional to $\Pi(s, \eta)$ vanish, we get

$$(4.6) \quad \mathcal{R}[\partial_i \varphi](s, \eta) = \mathcal{R}[\eta_i D_\eta \varphi](s, \eta) = \eta_i \partial_s \mathcal{R}[\varphi](s, \eta)$$

for $1 \leq i \leq n$. Moreover, we have the following.

Lemma 4.2. *Let $\varphi \in \mathcal{S}_\chi(\mathbf{R}^n)$, $v \geq 0$, k be a nonnegative integer, and α, β be multi-indices. Then we have*

$$(4.7) \quad |\partial_s^k o^\alpha \mathcal{R}[\varphi](s, \eta)| \leq C_{k, \alpha} \|\varphi\|_{\chi, k+|\alpha|, v+n+|\alpha|} (1+s^2)^{-v/2} \chi^{-1}(|s|)$$

for $(s, \eta) \in \mathbf{R} \times S^{n-1}$, and

$$(4.8) \quad \left| \partial_x^\alpha O^\beta \left\{ (\partial_s^k \mathcal{R}[\varphi]) \left(|x| - t, \frac{x}{|x|} \right) \right\} - (\partial_s^k \mathcal{R}[\partial_x^\alpha O^\beta \varphi]) \left(|x| - t, \frac{x}{|x|} \right) \right| \\ \leq C \frac{\|\varphi\|_{\chi, k+|\alpha|+|\beta|, v+n+|\alpha|+|\beta|}}{(1+t+|x|)(1+||x|-t|)^v \chi(|x|-t)}$$

for $|x| \geq t/2 \geq 1$. Here $C = C(k, v, \alpha, \beta)$ is a positive constant.

Proof. First we prove (4.7). Writing $\rho = |y - (y \cdot \eta)\eta|$, we have $|y|^2 = s^2 + \rho^2$ for $y \in \Pi(s, \eta)$. Hence we get

$$(4.9) \quad |(D_\eta^k O^\alpha \varphi)(y)| \leq C_{k, \alpha} (1 + s^2 + \rho^2)^{-v/2} (1 + \rho)^{-n} \chi^{-1}(|s|) \|\varphi\|_{\chi, k+|\alpha|, v+n+|\alpha|}$$

for $y \in \Pi(s, \eta)$. Using (4.4) and (4.5), we obtain (4.7).

Next we prove (4.8). Since $O^\beta \{\psi(|x|^{-1}x)\} = (o^\beta \psi)(|x|^{-1}x)$ for any $\psi \in C^\infty(S^{n-1})$, it suffices to show (4.8) for $\beta = 0$, thanks to (4.5). Besides, it is clear that (4.8) holds for $\alpha = \beta = 0$. By (4.6) we have

$$\begin{aligned} & \partial_i \{ (\partial_s^k \mathcal{R}[\varphi])(|x| - t, |x|^{-1}x) \} \\ &= \left(\eta_i \partial_s^{k+1} \mathcal{R}[\varphi](s, \eta) - |x|^{-1} \sum_{j=1}^n \eta_j o_{ij} \partial_s^k \mathcal{R}[\varphi](s, \eta) \right) \Big|_{(s, \eta) = (|x| - t, |x|^{-1}x)} \\ &= (\partial_s^k \mathcal{R}[\partial_i \varphi])(|x| - t, |x|^{-1}x) - \sum_{j=1}^n \frac{x_j}{|x|^2} (o_{ij} \partial_s^k \mathcal{R}[\varphi])(|x| - t, |x|^{-1}x) \end{aligned}$$

for $1 \leq i \leq n$ and any nonnegative integer k . Therefore, (4.7) implies

$$\begin{aligned} & |\partial_i \{ (\partial_s^k \mathcal{R}[\varphi])(|x| - t, |x|^{-1}x) \} - (\partial_s^k \mathcal{R}[\partial_i \varphi])(|x| - t, |x|^{-1}x)| \\ & \leq C \frac{\|\varphi\|_{\chi, k+1, v+n+1}}{(1+t+|x|)(1+||x|-t|)^v \chi(|x|-t)} \end{aligned}$$

for $|x| \geq t/2 \geq 1$, $1 \leq i \leq n$, and $v \geq 0$, where C is a positive constant depending on k and v . Hence (4.8) holds for $|\alpha| = 1$. We can show (4.8) for general α in a similar manner. This completes the proof. $\square$

4.2. Spherical means

Let $n(\geq 3)$ be an odd integer. We define

$$(4.10) \quad \mathcal{Q}[\varphi](t, x) = \frac{1}{A_n} \int_{\theta \in S^{n-1}} \varphi(x + t\theta) dS'_\theta$$

for $\varphi \in \mathcal{S}(\mathbf{R}^n)$ and $(t, x) \in (0, \infty) \times \mathbf{R}^n$. Here A_n is the total measure of S^{n-1} , i.e., $A_n = 2\pi^{n/2}/\Gamma(n/2)$ with the Gamma function Γ , and dS'_θ is the area element on S^{n-1} . $Q[\varphi]$ is the spherical mean of φ . The aim of this subsection is to prove the following.

Lemma 4.3. *Let ℓ be a nonnegative integer, and $v \geq 0$. Then there exist a positive constant C and a large integer N such that we have*

$$(4.11) \quad |A_n t^{n-1} \partial_t^\ell Q[\varphi](t, x) - ((-\partial_s)^\ell \mathcal{R}[\varphi])(r - t, \omega)| \\ \leq C \|\varphi\|_{\chi, \ell+1, N} (1+t+r)^{-1} (1+|r-t|)^{-v} \chi^{-1}(|r-t|)$$

for $(t, x) \in (0, \infty) \times \mathbf{R}^n$ satisfying $r \geq t/2 \geq 1$ with $r = |x|$ and $\omega = x/|x|$, provided that $\varphi \in \mathcal{S}_\chi(\mathbf{R}^n)$.

To prove Lemma 4.3, we first make some reduction: For $\varphi = \varphi(y)$, let $D_\theta \varphi$ be the directional derivative of φ in the direction θ as before, and let $c_{\ell, \alpha}(\theta)$ be the coefficient for ∂_y^α in the expansion of D_θ^ℓ :

$$D_\theta^\ell = (\theta \cdot \partial_y)^\ell = \sum_{|\alpha|=\ell} c_{\ell, \alpha}(\theta) \partial_y^\alpha,$$

so that $c_{\ell, \alpha}$ is a homogeneous polynomial of degree ℓ . Then it follows from (4.10) that

$$(4.12) \quad \partial_t^\ell Q[\varphi](t, x) = \frac{1}{A_n} \int_{S^{n-1}} (D_\theta^\ell \varphi)(x + t\theta) dS'_\theta \\ = \frac{1}{A_n} \sum_{|\alpha|=\ell} \int_{S^{n-1}} c_{\ell, \alpha}(\theta) (\partial_y^\alpha \varphi)(x + t\theta) dS'_\theta \\ = \frac{1}{A_n t^{n-1}} \sum_{|\alpha|=\ell} \int_{S(t, x)} c_{\ell, \alpha}(t^{-1}(y-x)) \partial_y^\alpha \varphi(y) dS_y^*,$$

where $S(t, x) = \{y \in \mathbf{R}^n; |y-x| = t\}$ and dS_y^* stands for the area element on $S(t, x)$. On the other hand, (4.4) implies that

$$(4.13) \quad (-\partial_s)^\ell \mathcal{R}[\varphi](r-t, \omega) = \sum_{|\alpha|=\ell} c_{\ell, \alpha}(-\omega) \mathcal{R}[\partial_y^\alpha \varphi](r-t, \omega).$$

Thus the proof of (4.11) is reduced to proving

$$(4.14) \quad \left| \int_{S(t, x)} c_{\ell, \alpha}(t^{-1}(y-x)) \partial_y^\alpha \varphi(y) dS_y^* - c_{\ell, \alpha}(-\omega) \mathcal{R}[\partial_y^\alpha \varphi](r-t, \omega) \right| \\ \leq C \|\varphi\|_{\chi, |\alpha|+1, N} (1+t+r)^{-1} (1+|r-t|)^{-v} \chi^{-1}(|r-t|)$$

for $r \geq t/2 \geq 1$. In order to proceed further, we decompose $S(t, x)$ as follows: Let ε be a small and positive constant. For $r > 0$, $t > 0$ and $\omega \in S^{n-1}$, we set

$$A_\varepsilon^1(t, r, \omega) = \{y \in S(t, r\omega); |y| > (t+r)^\varepsilon\},$$

$$A_\varepsilon^2(t, r, \omega) = \{y \in S(t, r\omega); |y| \leq (t+r)^\varepsilon\}.$$

We are going to prove that the integral over A_ε^1 can be regarded as the remainder term (Lemma 4.4), while the integral over A_ε^2 can be approximated by the Radon transform (Lemma 4.5). Using these estimates we will show (4.14) to finish the proof of Lemma 4.3.

Lemma 4.4. *Let c be a bounded function on S^{n-1} , and $\varphi \in \mathcal{S}_\chi(\mathbf{R}^n)$. Let $\varepsilon > 0$ and $\kappa > 0$. Suppose that N_1 is a positive integer satisfying $N_1\varepsilon \geq \kappa + n - 1$. Then there exists a positive constant $C = C(\varepsilon, \kappa, N_1)$ such that we have*

$$\left| \int_{A_\varepsilon^1(t, r, \omega)} c(t^{-1}(y - r\omega))\varphi(y) dS_y^* \right| \leq C(1+t+r)^{-\kappa} \chi^{-1}(|r-t|) \|c\|_{L^\infty(S^{n-1})} \|\varphi\|_{\chi, 0, N_1}$$

for any $(t, r, \omega) \in [0, \infty) \times [0, \infty) \times S^{n-1}$.

Proof. Since we have $|y| \geq \max\{|r-t|, (t+r)^\varepsilon\}$ for $y \in A_\varepsilon^1(t, r, \omega)$, we get

$$(4.15) \quad |\varphi(y)| \leq (1+|t+r|^{2\varepsilon})^{-N_1/2} \chi^{-1}(|r-t|) \|\varphi\|_{\chi, 0, N_1}$$

$$\leq C(1+t+r)^{-\kappa-n+1} \chi^{-1}(|r-t|) \|\varphi\|_{\chi, 0, N_1}$$

for any $y \in A_\varepsilon^1(t, r, \omega)$ with some positive constant C . Observing that the total measure of $S(t, r\omega)$ is bounded by $A_n t^{n-1}$, we obtain the desired result from (4.15). $\square$

Lemma 4.5. *Let $c \in C^1(\overline{B_1})$, and $\varphi \in \mathcal{S}_\chi(\mathbf{R}^n)$. Let $0 < \varepsilon \leq 1/4$, and $v \geq 0$. Suppose that N_2 is a positive integer satisfying $N_2 \geq n + 2 + v + (1/\varepsilon)$. Then there exists a positive constant $C = C(\varepsilon, v, N_2)$ such that we have*

$$(4.16) \quad \left| \int_{A_\varepsilon^2(t, r, \omega)} c(t^{-1}(y - r\omega))\varphi(y) dS_y^* - c(-\omega)\mathcal{R}[\varphi](r-t, \omega) \right|$$

$$\leq C(1+t+r)^{-1} (1+|r-t|)^{-v} \chi^{-1}(|r-t|) \|c\|_{C^1(\overline{B_1})} \|\varphi\|_{\chi, 1, N_2}$$

for any $(t, r, \omega) \in [0, \infty) \times [0, \infty) \times S^{n-1}$ with $r \geq t/2 \geq 1$ and $|r-t| \leq (t+r)^\varepsilon$, where

$$\|\psi\|_{C^1(\overline{B_1})} = \sup_{y \in \overline{B_1}} (|\psi(y)|^2 + |\nabla_y \psi(y)|^2)^{1/2}$$

for $\psi \in C^1(\overline{B_1})$.

Proof. Since the right-hand side of (4.16) is invariant under the orthogonal transforms, we may assume $\omega = e_n$ without loss of generality, where $e_n = (0, \dots, 0, 1)$. In the following, we suppose $r \geq t/2 \geq 1$, $|r - t| \leq (t + r)^\varepsilon$ and $0 < \varepsilon \leq 1/4$.

Since $t + r \geq 3$, we get $(t + r)^{\varepsilon-1} \leq 3^{-3/4} < 1/2$, which implies

$$(4.17) \quad (t + r)^\varepsilon < \frac{t + r}{2}.$$

From (4.17) we also get $r \leq 3t$, because otherwise (namely $r > 3t$) we get $r - t > (t + r)/2 > (t + r)^\varepsilon$, which contradicts the assumption. Summing up, we obtain

$$(4.18) \quad 1 \leq \frac{t}{2} \leq r \leq 3t,$$

and we find that t , r and $1 + t + r$ are equivalent to each other.

First we prove that

$$(4.19) \quad \left| \int_{A_e^2(t, r, e_n)} c(t^{-1}(y - re_n))\varphi(y) dS_y^* - c(-e_n) \int_{A_e^2(t, r, e_n)} \varphi(y) dS_y^* \right| \\ \leq 2^{(n-2)/2} A_{n-1} \|c\|_{C^1(\overline{B_1})} \|\varphi\|_{\chi, 1, N_2} t^{-1} (1 + |r - t|^2)^{-v/2} \chi^{-1}(|r - t|).$$

We put

$$\lambda_0(t, r) = \frac{(t + r)^{2\varepsilon} - (r - t)^2}{2rt}.$$

Note that $0 \leq \lambda_0(t, r) \leq 1$. Writing $t\lambda = y_n - (r - t)$, we find that $A_e^2(t, r, e_n)$ is equal to

$$\{y = (t\sqrt{\lambda(2 - \lambda)}\zeta, r - t + t\lambda); \zeta \in S^{n-2}, 0 \leq \lambda \leq \lambda_0(t, r)\}.$$

For the coordinate system (ζ, λ) in the above, we have

$$(4.20) \quad dS_y^* = t^{n-1} \lambda^{(n-3)/2} (2 - \lambda)^{(n-3)/2} d\lambda dS_\zeta'',$$

where dS_ζ'' denotes the area element on S^{n-2} . We also note that

$$(4.21) \quad |y|^2 = (r - t)^2 + 2rt\lambda.$$

We put $\eta(y, r) = |y - re_n|^{-1}(y - re_n) \in S^{n-1}$. Then we get

$$|\eta(y, r) - (-e_n)| = \sqrt{2 + 2\eta_n(y, r)} = \sqrt{2\lambda}$$

for any $y \in A_e^2(t, r, e_n)$. Hence, by the mean value theorem, we get

$$\left| \int_{A_e^2(t, r, e_n)} \{c(\eta(y, r)) - c(-e_n)\} \varphi(y) dS_y^* \right| \leq \sqrt{2} \|c\|_{C^1(\overline{B_1})} J(t, r),$$

where we put

$$J(t, r) = \int_{A_e^2(t, r, e_n)} \lambda^{1/2} |\varphi(y)| dS_y^*$$

with $t\lambda = y_n - (r - t)$. Note that we have $\eta(y, r) = t^{-1}(y - re_n)$ for $y \in A_e^2(t, r, e_n)$. Recalling (4.20) and (4.21), we obtain

$$\begin{aligned} (4.22) \quad & \chi(|r - t|) J(t, r) \\ & \leq A_{n-1} t^{n-1} \|\varphi\|_{\chi, 0, N_2} \int_0^{\lambda_0(t, r)} \frac{\lambda^{(n-2)/2} (2 - \lambda)^{(n-3)/2}}{(1 + (r - t)^2 + 2rt\lambda)^{N_2/2}} d\lambda \\ & \leq 2^{-1/2} A_{n-1} r \left(\frac{t}{r}\right)^{n/2} \|\varphi\|_{\chi, 0, N_2} \\ & \quad \times \int_0^{\lambda_0(t, r)} \frac{(2rt\lambda)^{(n-2)/2}}{(1 + |r - t|^2)^{v/2} (1 + 2rt\lambda)^{(N_2-v)/2}} d\lambda \\ & \leq 2^{(n-1)/2} A_{n-1} r (1 + |r - t|^2)^{-v/2} \|\varphi\|_{\chi, 0, N_2} \int_0^\infty \frac{1}{(1 + 2rt\lambda)^2} d\lambda \\ & \leq 2^{(n-3)/2} A_{n-1} t^{-1} (1 + |r - t|^2)^{-v/2} \|\varphi\|_{\chi, 0, N_2}. \end{aligned}$$

This estimate yields (4.19) immediately.

By (4.19), we find that, in order to show (4.16), it suffices to prove

$$\begin{aligned} (4.23) \quad & \left| \int_{A_e^2(t, r, e_n)} \varphi(y) dS_y^* - \mathcal{R}[\varphi](r - t, e_n) \right| \\ & \leq Ct^{-1} (1 + |r - t|)^{-v} \chi^{-1}(|r - t|) \|\varphi\|_{\chi, 1, N_2} \end{aligned}$$

with some positive constant C . We observe that

$$\left| \int_{A_e^2(t, r, e_n)} \lambda \varphi(y) dS_y^* \right| \leq C(rt)^{-1} (1 + |r - t|)^{-v} \chi^{-1}(|r - t|) \|\varphi\|_{\chi, 0, N_2},$$

which can be shown similarly to (4.22). Therefore we obtain (4.23), if we can show

$$(4.24) \quad |I_1(r, t) - \mathcal{R}[\varphi](r - t, e_n)| \leq Ct^{-1} (1 + |r - t|)^{-v} \chi^{-1}(|r - t|) \|\varphi\|_{\chi, 1, N_2},$$

where we put

$$I_1(r, t) = \int_{A_e^2(t, r, e_n)} (1 - \lambda) \varphi(y) dS_y^*.$$

Now we are going to prove (4.24). Introducing a new coordinate $\rho = t\sqrt{\lambda(2 - \lambda)}$, we get

$$I_1(t, r) = \int_{S^{n-2}} \left(\int_0^{\rho_0(t, r)} \varphi(\rho \zeta, r - \sqrt{t^2 - \rho^2}) \rho^{n-2} d\rho \right) dS_\zeta'',$$

where $\rho_0(t, r) = t\sqrt{\lambda_0(t, r)(2 - \lambda_0(t, r))}$. We also have

$$(4.25) \quad \mathcal{R}[\varphi](r - t, e_n) = I_2(t, r) + I_3(t, r),$$

where we put

$$I_2(t, r) = \int_{S^{n-2}} \left(\int_0^{\rho_0(t, r)} \varphi(\rho \zeta, r - t) \rho^{n-2} d\rho \right) dS_\zeta'',$$

$$I_3(t, r) = \int_{S^{n-2}} \left(\int_{\rho_0(t, r)}^\infty \varphi(\rho \zeta, r - t) \rho^{n-2} d\rho \right) dS_\zeta''.$$

Since $t - \sqrt{t^2 - \rho^2} = \rho^2(t + \sqrt{t^2 - \rho^2})^{-1}$, we get

$$\begin{aligned} & |\varphi(\rho \zeta, r - \sqrt{t^2 - \rho^2}) - \varphi(\rho \zeta, r - t)| \\ & \leq (t - \sqrt{t^2 - \rho^2}) \int_0^1 |(\partial_n \varphi)(\rho \zeta, r - t + \tau(t - \sqrt{t^2 - \rho^2}))| d\tau \\ & \leq \frac{\rho^2 \|\varphi\|_{\chi, 1, N_2}}{t(1 + |r - t|^2)^{v/2} (1 + \rho^2)^{(n+2)/2} \chi(|r - t|)} \end{aligned}$$

for $0 \leq \rho \leq \rho_0(t, r)$, which yields

$$(4.26) \quad |I_1(t, r) - I_2(t, r)| \leq \frac{A_{n-1} \|\varphi\|_{\chi, 1, N_2}}{t(1 + |r - t|^2)^{v/2} \chi(|r - t|)} \int_0^\infty \frac{\rho}{(1 + \rho^2)^{3/2}} d\rho$$

$$\leq CA_{n-1} \|\varphi\|_{\chi, 1, N_2} t^{-1} (1 + |r - t|)^{-v} \chi^{-1}(|r - t|),$$

where C is a positive constant depending only on v .

Next we evaluate $I_3(t, r)$. Notice that

$$(4.27) \quad (\rho_0(t, r))^2 + (r - t)^2 \geq \frac{3}{32} (t + r)^{2\varepsilon}.$$

In fact, it is trivial when $(t+r)^{2\varepsilon}/2 \leq (r-t)^2 \leq (t+r)^{2\varepsilon}$. On the other hand, when $(r-t)^2 \leq (t+r)^{2\varepsilon}/2$, by (4.17) we get

$$(\rho_0(t, r))^2 = \frac{(t+r)^2 - (t+r)^{2\varepsilon}}{4r^2} \{(t+r)^{2\varepsilon} - (r-t)^2\} \geq \frac{3}{32} (t+r)^{2\varepsilon},$$

which shows (4.27). We thus find

$$(4.28) \quad |I_3(t, r)| \leq \frac{A_{n-1} \|\varphi\|_{\chi, 0, N_2}}{\chi(|r-t|)} \int_{\rho_0(t, r)}^{\infty} \frac{\rho}{(1 + \rho^2 + (r-t)^2)^{(v+5+1/\varepsilon)/2}} d\rho \\ \leq C \|\varphi\|_{\chi, 0, N_2} (t+r)^{-1} (1 + |r-t|)^{-v} \chi^{-1}(|r-t|),$$

where C is a constant depending only on v , n and ε .

Finally (4.24) follows from (4.25), (4.26) and (4.28). This completes the proof. $\square$

Now we are in a position to prove (4.14). When $|r-t| \leq (t+r)^\varepsilon$, (4.14) is an immediate consequence of Lemmas 4.4 and 4.5. On the other hand, when $|r-t| > (t+r)^\varepsilon$, we have $S(t, r\omega) = A_\varepsilon^1(t, r, \omega)$. Therefore, using Lemma 4.4 and (4.7) to estimate

$$\int_{S(t, x)} c_{\ell, \alpha}(t^{-1}(y-x)) \partial_y^\alpha \varphi(y) dS_y^* \quad \text{and} \quad c_{\ell, \alpha}(-\omega) \mathcal{R}[\partial_y^\alpha \varphi](r-t, \omega),$$

respectively, we obtain (4.14). This completes the proof of (4.14), and thus Lemma 4.3 is proved.

4.3. Proof of Proposition 4.1

Suppose that all the assumptions of the proposition are fulfilled.

First we recall the explicit representation of $u_0(t, \cdot)$, the first component of $U_0(t)\vec{g}$. We define

$$(4.29) \quad E[\varphi](t, x) := \frac{\sqrt{\pi}}{2\Gamma(n/2)} \left(\frac{1}{2t} \frac{\partial}{\partial t} \right)^{(n-3)/2} (t^{n-2} Q[\varphi](t, x)),$$

where $\Gamma(s)$ is the Gamma function and $Q[\varphi]$ is defined by (4.10). Then it is known that we have

$$(4.30) \quad u_0(t, x) = \partial_t E[g_0](t, x) + E[g_1](t, x)$$

for $\vec{g} = (g_0, g_1) \in (\mathcal{S}(\mathbf{R}^n))^2$ (see, e.g., Courant and Hilbert [1, Chapter VI, Section 12]). Therefore we get

$$(4.31) \quad \partial_t^k \partial_x^\alpha O^\beta u_0(t, x) = \partial_t^{k+1} E[\partial_x^\alpha O^\beta g_0](t, x) + \partial_t^k E[\partial_x^\alpha O^\beta g_1](t, x)$$

for any nonnegative integer k and any multi-indices α, β (note that $\varphi \in \mathcal{S}_\chi(\mathbf{R}^n)$ implies $\partial_x^\alpha \mathcal{O}^\beta \varphi \in \mathcal{S}_\chi(\mathbf{R}^n)$ for any multi-indices α and β). Thus we find that our task is to evaluate $\partial_t^k E[\varphi](t, x)$ for $\varphi \in \mathcal{S}_\chi(\mathbf{R}^n)$ and a nonnegative integer k .

It is easy to see that

$$(4.32) \quad \partial_t^k E[\varphi](t, x) = \frac{\sqrt{\pi}}{2^{(n-1)/2} \Gamma(n/2)} \sum_{\ell=0}^{(n-3)/2+k} a_{k,\ell} t^{1-k+\ell} \partial_t^\ell Q[\varphi](t, x)$$

for $k \geq 0$, where $a_{k,\ell}$'s are suitable constants with $a_{k,\ell} = 1$ for $\ell = (n-3)/2+k$. Let $\varphi \in \mathcal{S}_\chi(\mathbf{R}^n)$, $v \geq 0$, $r (=|x|) \geq t/2 \geq 1$, and $\omega = r^{-1}x$ in the following. Then we find from (4.11) that there exist a large integer N and a positive constant $C = C(k)$ such that

$$(4.33) \quad \left| \partial_t^k E[\varphi](t, x) - \frac{1}{2(2\pi r)^{(n-1)/2}} ((-\partial_s)^{(n-3)/2+k} \mathcal{R}[\varphi])(r-t, \omega) \right| \\ \leq C \|\varphi\|_{\chi, (n-1)/2+k, N} (1+t+r)^{-(n+1)/2} (1+|r-t|)^{-v} \chi^{-1}(|r-t|)$$

holds. In fact, when $2t \geq r \geq t/2 \geq 1$, since we have

$$|t^{-(n-1)/2} - r^{-(n-1)/2}| \leq C(1+t+r)^{-(n+1)/2} (1+|r-t|),$$

from (4.7) we see that it suffices to prove (4.33) with $(2\pi r)^{(n-1)/2}$ in its left-hand side being replaced by $(2\pi t)^{(n-1)/2}$. This replaced estimate can be easily proved by (4.32), (4.11) and (4.7), because t is equivalent to $1+t+r$ in this case. On the other hand, when $r > 2t(\geq 4)$, from (4.7) we get

$$|((-\partial_s)^\ell \mathcal{R}[\varphi])(r-t, \omega)| \leq C \|\varphi\|_{\chi, \ell, \mu+n} (1+t+r)^{-\mu} \chi^{-1}(|r-t|)$$

for any $\mu \geq 0$ (note that $1+|r-t|$ is equivalent to $1+t+r$ in this case). Hence, using (4.32) and (4.11), we find that both $|\partial_t^k E[\varphi](t, x)|$ and $|(-\partial_s)^{(n-3)/2+k} \mathcal{R}[\varphi](r-t, \omega)|$ are bounded from above by the right-hand side of (4.33), provided that N is sufficiently large.

Now, using (1.11), (4.6), (4.7), (4.8), (4.31), and (4.33), we obtain Proposition 4.1. This completes the proof. $\square$

5. Proof of Theorem 1.2

To begin with, we note that, for any $A \in \mathbf{R}$ and any nonnegative integer k , there exists a positive constant C such that we have

$$C^{-1} e^{A\langle x \rangle} \sum_{|x| \leq k} |\partial_x^\alpha \psi(x)| \leq \sum_{|x| \leq k} |\partial_x^\alpha (e^{A\langle x \rangle} \psi(x))| \leq C e^{A\langle x \rangle} \sum_{|x| \leq k} |\partial_x^\alpha \psi(x)|$$

for any $x \in \mathbf{R}^n$ and any $\psi \in C^\infty(\mathbf{R}^n)$. In fact, the latter half is almost apparent, and the first half is nothing but the latter half with A and $\psi(x)$ being replaced by $-A$ and $e^{A\langle x \rangle} \psi(x)$, respectively.

Let the assumptions in Theorem 1.2 be fulfilled. Then, by Theorem 1.1, there exists $f_+ \in \mathcal{H}^\infty(\mathbf{R}^n)$ satisfying (1.9) and (1.10). We write $u(t, \cdot)$ and $u_+(t, \cdot)$ for the first components of $U(t)\vec{f}$ and $(U_0(t)\vec{f}_+)|_\Omega$, respectively.

First we claim that we have

$$(5.1) \quad \begin{aligned} & \|\exp(3\mu\langle \cdot \rangle/4)(Z^\alpha u(t, \cdot) - Z^\alpha u_+(t, \cdot))\|_{H^{[n/2]+1}(\Omega)} \\ & \leq C \exp(-\mu t) \|\vec{f}\|_{\mathcal{H}^{[n/2]+|\alpha|}(\Omega)}, \quad t \geq 0 \end{aligned}$$

for any multi-index α . Let $Z^\alpha = \partial_x^j \partial_x^\beta O^\gamma$ with a nonnegative integer j , and multi-indices β, γ , and let α' be a multi-index satisfying $|\alpha'| \leq [n/2] + 1$. If j is even, then we get

$$\begin{aligned} |\partial_x^{\alpha'} Z^\alpha (u - u_+)(t, x)| &= |A^{j/2} \partial_x^{\alpha'+\beta} O^\gamma (u - u_+)(t, x)| \\ &\leq C(1 + |x|)^{|\gamma|} \sum_{|\beta'| \leq |\alpha| + |\alpha'|} |\partial_x^{\beta'} (u - u_+)(t, x)| \\ &\leq C e^{\mu\langle x \rangle/4} \sum_{|\beta'| \leq |\alpha| + |\alpha'|} |\partial_x^{\beta'} (u - u_+)(t, x)| \end{aligned}$$

with some positive constant $C = C(\alpha, \alpha', \mu)$. Similarly, if j is odd, we get

$$|\partial_x^{\alpha'} Z^\alpha (u - u_+)(t, x)| \leq C e^{\mu\langle x \rangle/4} \sum_{|\beta'| \leq |\alpha| + |\alpha'| - 1} |\partial_x^{\beta'} \partial_t (u - u_+)(t, x)|.$$

Hence the left-hand side of (5.1) is bounded by

$$C \|\exp(\mu\langle \cdot \rangle)(U(t)\vec{f} - (U_0(t)\vec{f}_+)|_\Omega)\|_{\mathcal{H}^{[n/2]+|\alpha|}(\Omega)},$$

and (1.9) implies (5.1).

By (5.1) and the Sobolev imbedding theorem, we get

$$(5.2) \quad \begin{aligned} & |Z^\alpha (u - u_+)(t, x)| \\ & \leq C_k e^{-3\mu\langle x \rangle/4} \|e^{3\mu\langle \cdot \rangle/4} Z^\alpha (u - u_+)(t, \cdot)\|_{H^{[n/2]+1}(\Omega)} \\ & \leq C_k e^{-3\mu(t+\langle x \rangle)/4} \|\vec{f}\|_{\mathcal{H}^{k+[n/2]}(\Omega)} \\ & \leq C_k (1 + t + |x|)^{-(n+1)/2} e^{-\mu(t+|x|)/2} \|\vec{f}\|_{\mathcal{H}^{k+[n/2]}(\Omega)} \end{aligned}$$

for $|x| \geq t/2 \geq 1$ and $|\alpha| \leq k$, where C_k is a positive constant. Hence we find that our task is to show (1.12) and (1.13) with u being replaced by u_+ . Such estimates follow immediately from Proposition 4.1 with $v = 0$ and $\vec{g} = \vec{f}_+$, since

we have $\vec{f}_+ \in (\mathcal{S}_\chi(\mathbf{R}^n))^2$ with $\chi(s) = \exp(\mu s/2)$ from (1.10). This completes the proof. $\square$

Acknowledgement

The authors would like to thank the referee for the useful comment. The research of the first author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 20540211), JSPS. The research of the second author is partially supported by Grant-in-Aid for Science Research (No. 20224013), JSPS.

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(Ricevita la 16-an de majo, 2009)

(Reviziita la 9-an de aprilo, 2010)