Generating Systems for Finite Irreducible Complex Reflection Groups

By
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Abstract. For each finite irreducible complex reflection group $G$ in $\text{GL}(n, \mathbb{C})$, we construct a system $E_G(z)$ of differential equations on $\mathbb{Z} \setminus \mathbb{C}_0$ of rank $n$ with the monodromy group $G$, and with the following generating property: If a system $E'(z)$ on $\mathbb{Z}$ of rank $n$ has a finite monodromy group and a projective monodromy group which is a subgroup of $\text{P}(G)$, there is an algebraic transformation

$$E'(z) = \theta(z)^{1/k} E_G(\sigma(z)),$$

where $k$ is an integer, $\theta(z)$ a rational function on $\mathbb{Z}$, and $\sigma(z)$ a rational map of $\mathbb{Z}$ to $\mathbb{Z}$. For $n = 2, 3$, we give explicit forms of $E_G(z)$. Several examples of the above algebraic transformation are also given.

Key Words and Phrases. Generating system, Complex reflection group, Hypergeometric function, Schwarz map, Monodromy group.

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1. Introduction

There are five types of finite groups in $\text{PGL}(2, \mathbb{C})$, referred to as cyclic, dihedral, tetrahedral, octahedral, and icosahedral groups. For each group $G$, there are Gauss hypergeometric differential equations $\, _2E_1(a,b;c;z)$ defined by

$$z(1 - z) \frac{d^2u}{dz^2} + (c - (a + b + 1)z) \frac{du}{dz} - abu = 0$$

whose projective monodromy groups are isomorphic to $G$. Among them, there are special differential equations $E(z)$ having the following “generating property”: For any second order Fuchsian differential equation $E'(z)$ on $\mathbb{P}^1$ with a finite monodromy group and with the projective monodromy group $\text{P}(G)$, there is an algebraic transformation

$$E'(z) = \theta(z)^{1/k} E(\sigma(z)),$$

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where $\sigma(z)$ and $\theta(z)$ are rational functions, and $k$ is an integer. Such differential equation $E(z)$ can also be characterized as an equation whose Schwarz (multi-valued) map

$$P^1 \setminus \{0, 1, \infty\} \to P^1$$

$$z \mapsto [u_1(z) : u_2(z)]$$

has a single-valued inverse map, where $u_1, u_2$ are solutions of $E(z)$. The inverse map of (1.1) is given by

$$[u_1, u_2] \mapsto \frac{P(u_1, u_2)}{Q(u_1, u_2)},$$

where $P(u)$ and $Q(u)$ are homogeneous polynomials of degree $|P(G)|$. For example, the differential equation $E_1(-1/60, 29/60; 4/5; z)$ is an equation having the above property for the icosahedral group $G_{60} \subset \text{PGL}(2, \mathbb{C})$ ([K1]).

In this paper we show similar results for all the finite irreducible complex reflection groups in $\text{GL}(n, \mathbb{C})$. An element $M$ of $\text{GL}(n, \mathbb{C})$ is called a complex reflection if all but one eigenvalues of $M$ are 1, and $M^k = I$ for some integer $k$. A group generated by complex reflections is called a complex reflection group. For each finite irreducible complex reflection group $G$ in $\text{GL}(n, \mathbb{C})$, we prove in a constructive way the existence of a completely integrable Fuchsian system $E_G(z)$ of rank $n$ on $\mathbb{C}P^n$ with the following two properties (see Theorem 2.9):

1. The monodromy group of $E_G(z)$ is $G$.
2. The Schwarz (multi-valued) map $s_p(z) = [\varphi_1(z) : \varphi_2(z) : \ldots : \varphi_n(z)]$ defined by a ratio of linearly independent solutions of $E_G(z)$ has a single-valued inverse map. That is, there is a rational map $\pi$ of $\mathbb{P}^{n-1}$ to $Z$ satisfying $\pi(s_p(z)) = z$ for all $z$ in an open dense subset of $Z$.

Such a system $E_G(z)$ has the generating property: If a completely integrable Fuchsian system $E'(z)$ of rank $n$ on $Z$ with a non-degenerate Schwarz map (see Definition 2.2) has the same monodromy group $G$, there is an algebraic transformation

$$E'(z) = \theta(z)^{1/m}E_G(\sigma(z)),$$

where $\sigma(z)$ is a rational transformation of $Z$, $\theta(z)$ a rational function on $Z$, and $m$ the order of the center $C(G)$ of $G$. Furthermore, if $E'(z)$ has the property (2) above, $\sigma(z)$ in (1.2) becomes birational. The system $E_G(z)$ is unique in this sense. We call $E_G(z)$ a generating system for $G$. In Sections 3 and 4, we give an explicit form of $E_G(z)$ for each $G$ in $\text{GL}(2, \mathbb{C})$ and $\text{GL}(3, \mathbb{C})$ (Theorems 3.1, 4.8, 4.10 and 4.11).
In Section 6, we illustrate the examples for generating systems, useful for the study of relations among various systems of differential equations. More than a hundred years ago, Boulanger [Bl] obtained a system of differential equations with the projective monodromy group $P(G_{1296})$, where $G_{1296}$ is a complex reflection group in $GL(3, C)$ of order 1296. We can express this system by our generating system for $G_{1296}$ (Corollary 6.4). In addition, the Lamé equation in the list of Beukers and van der Waall [BW] with the monodromy group $G_{48}$, which is a complex reflection group in $GL(2, C)$ of order 48, can be expressed by using the Gauss hypergeometric equation which is a generating system for $G_{48}$ (Proposition 6.1). Finally, the Jordan-Pochhammer equation of rank 3 with the monodromy group $G(3,3,3)$ has two distinct prolongations: Appell’s hypergeometric system $E_1$ and the generating system for $G(3,3,3)$. We establish the relation between these prolongations in Section 6.3.

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2. Existence of generating systems

We put $U_n = \{u = (u_1, u_2, \ldots, u_n) | u_j \in C\}$, and $P(U_n) = \{[u] = [u_1 : u_2 : \ldots : u_n] | u \neq 0\}$. Let $G$ be a finite irreducible complex reflection group in $GL(U_n)$ with the center $C(G)$ of order $m$. Any $A \in GL(n, C)$ acts on $U_n$ by $u \rightarrow uA$. In this way we identify $GL(U_n)$ with $GL(n, C)$. We denote by $P(G)$ the transformation group of $P(U_n)$ defined by the actions of $G$ on $P(U_n)$. We have $P(G) \approx G/C(G)$. Let $F_{d_j}(u)$, $1 \leq j \leq n$ be homogeneous polynomials of degree $d_j$ generating the ring $C[u_1, u_2, \ldots, u_n]^G$ of $G$-invariant polynomials. In Table VII, ST, we see the values $|G|, |C(G)|, d_j$ and other data for finite irreducible complex reflection groups $G$. We also find that $|C(G)| = \gcd(d_1, d_2)$ for all primitive groups, that is, the groups not of No. 2 in the table.

In the following we shall construct the generating system for $G$ utilizing the invariant polynomials $F_{d_j}(u)$.

Lemma 2.1. If two points $\alpha = (x_1, x_2, \ldots, x_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ of $U_n$ belong to different $G$-orbits, there is a $G$-invariant homogeneous polynomial $F(u) = F(u_1, u_2, \ldots, u_n)$ such that $F(\alpha) \neq 0$ and $F(\beta) = 0$.

Proof. Let $l(u)$ be a homogeneous linear function such that $l(\beta) = 0$ and $l(u) \neq 0$ at any point in the $G$-orbit of $\alpha$. Then, $F(u) = \prod_{A \in G} l(uA)$ has the desired property. \hfill $\square$

Corollary 2.2. (1) $F_{d_j}(u)$, $1 \leq j \leq n$ do not have a common zero in $P(U_n)$.

(2) Two points $\alpha, \beta$ of $U_n$ belong to the same $G$-orbit if and only if $F_{d_j}(\alpha) = F_{d_j}(\beta)$, $1 \leq j \leq n$. 

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Proof. Proof of (1): For any point \([x_1 : x_2 : \ldots : x_n]\) in \(P(U_n)\), there is a non constant \(G\)-invariant homogeneous polynomial \(F(u)\) such that \(F(x_1, x_2, \ldots, x_n) \neq 0\). Since \(F(u)\) is a polynomial in \(F_d(u)\), \(1 \leq j \leq n\), \([x_1 : x_2 : \ldots : x_n]\) is not a common zero of \(F_d(u)\), \(1 \leq j \leq n\).

Proof of (2): “If part” is clear. “Only if part” follows from the previous lemma.

Lemma 2.3. Let \(G'\) be a finite group in \(GL(U_n)\) satisfying \(P(G') \subset P(G)\). Let \(F(u)\) be a \(G\)-invariant homogeneous polynomial. Then, \(F(u)'^{[G']}\) is \(G'\)-invariant.

Proof. Put \(d = \deg F(u)\). By the assumption, for each \(A' \in G'\), there is a non-zero constant \(c(A')\) such that \(c(A')A' \in G\). Then, we have

\[
F(u)^{[G']} = \prod_{A' \in G'} F(u) = \prod_{A' \in G'} F(u \cdot (c(A')A')) = \left( \prod_{A' \in G'} (c(A'))^d \right) \prod_{A' \in G'} F(uA'),
\]

which is \(G'\)-invariant.

The \(G\)-orbit (= \(P(G)\)-orbit) \(G \cdot [u] = \{ [uA] \mid A \in G \}\) of “generic” \([u] \in P(U_n)\) consists of \(|P(G)|\) points by virtue of the following lemma.

Lemma 2.4. Let \(S_G = \{ [u] \in P(U_n) \mid |(G \cdot [u])| < |P(G)| \}\). Then, \(S_G\) consists of a finite number of the sets \(P(V)\), where \(V\) is a linear subspace of \(U_n\) of dimension less than \(n\).

Proof. Let \([x] = [x_1 : x_2 : \ldots : x_n]\) be a point of \(P(U_n)\) such that \(|(G \cdot [x])| < |P(G)|\). Then, there is a non-scalar matrix \(A\) in \(G\) which fixes \([x]\). This means that the point \((x_1, x_2, \ldots, x_n)\) in \(U_n\) is an eigenvector of \(A\) with an eigenvalue, say \(\lambda\). The eigenspace of \(A\) corresponding to the eigenvalue \(\lambda\) is a linear subspace of \(U_n\) of dimension less than \(n\). This proves the lemma.

Proposition 2.5. For each finite irreducible complex reflection group \(G\) in \(GL(U_n)\), there exist rational functions \(R_j(u)\) \((1 \leq j \leq n - 1)\) satisfying the following properties:

1. \(R_j(u) = R_{j1}(u)/R_{j2}(u)\), where \(R_{j1}(u)\) and \(R_{j2}(u)\) are \(G\)-invariant homogeneous polynomials of the same degree \(r_j\) with \(\prod_{j=1}^{n-1} r_j = |P(G)|\).

2. Define a rational map \(\pi_R : P(U_n) \rightarrow P^{n-1}\) by

\[
\pi_R([u]) = [R_1(u) : R_2(u) : \ldots : R_{n-1}(u) : 1],
\]

and let \(Sing(\pi_R)\) denote the union of \(S_G\) (see Lemma 2.4) and the zeros of \(R_{j2}(u)\). Then, the restriction of \(\pi_R\) to \(P(U_n)\) \(\setminus Sing(\pi_R)\) is a \(|P(G)| : 1\) (holomorphic) covering map and is a (set theoretically) \(G\)-quotient map of \(P(U_n)\) \(\setminus Sing(\pi_R)\).
Proof. First we assume that \( G \) is primitive and that the \( G \)-invariant polynomials \( F_d(u) = F_d(u_1, u_2, \ldots, u_n) \) are chosen so that \( d_1 < d_2 < \cdots < d_n \). Then, \( (d_1, d_2) = m \) for all \( G \). Put \( k_1 = d_1/m, \ k_2 = d_2/m \). For \( j \geq 3 \), there are non-negative integers \( p_j, q_j \) such that \( p_j d_1 + q_j d_2 = d_j \). Put \( Q_j(u) = F_{d_1}(u)^{p_j}F_{d_2}(u)^{q_j} \), and

\[
R_1(u) = \frac{F_{d_1}(u)}{Q_n(u)}, R_2(u) = \frac{F_{d_1}(u)}{Q_{n-1}(u)}, \ldots, R_{n-2}(u) = \frac{F_{d_1}(u)}{Q_3(u)}, R_{n-1}(u) = \frac{F_{d_1}(u)^{k_1}}{F_{d_1}(u)^{k_2}}.
\]

Then, because

\[
r_1 r_2 \ldots r_{n-2} \cdot r_{n-1} = d_1 d_{n-1} \ldots d_3 \cdot (d_2 k_1) = (d_1 d_{n-1} \ldots d_1)/m = |P(G)|,
\]

\( R_j(u) \) \( (1 \leq j \leq n-1) \) satisfy the property (1).

We will show that they satisfy the property (2). Let \( [x] = [x_1 : x_2 : \ldots : x_n] \) be a point in \( P(U_n) \setminus \text{Sing}(\pi_R) \). Since \( |G : [x]| = |P(G)| \) by Lemma 2.4, the equality \( G \cdot [x] = \pi_R^{-1}(\pi_R([x])) \) implies that the restriction of \( \pi_R \) to \( P(U_n) \setminus \text{Sing}(\pi_R) \) is a holomorphic covering map.

Because \( \pi_R \) is \( G \)-invariant, we have \( G \cdot [x] \subset \pi_R^{-1}(\pi_R([x])) \). Put \( \zeta_j = R_j([x]), \ 1 \leq j \leq n-1 \). Then, \( \pi_R^{-1}(\pi_R([x])) \) is equal to

\[
\{ \{u \} \in P(U_n) | F_{d_n+1}(u) - \zeta_j F_{d_{n+2}}(u) = 0, 1 \leq j \leq n-2, \}
\]

\[
F_{d_1}(u)^{k_1} - \zeta_{n-1} F_{d_1}(u)^{k_2} = 0 \}
\]

which is zero dimensional, because otherwise \( F_{d_1}(u), 1 \leq j \leq n \) have a common zero in \( P(U_n) \), contrary to (1) of Corollary 2.2. Consequently, \( |\pi_R^{-1}(\pi_R([x]))| \leq |P(G)| \) by Bezout's theorem. This implies \( G \cdot [x] = \pi_R^{-1}([\zeta_1 : \zeta_2 : \ldots : \zeta_{n-1} : 1]) \).

Assume \( G \) is imprimitive, \( G = G(r, p, n) \) with \( r = pq, \ d = (p, n) \). In this case \( G \) has the basic \( G \)-invariant polynomials \( F_r(u_1, u_2, \ldots, u_n) = \sigma_j(u_1, u_2, \ldots, u_n), 1 \leq j \leq n-1 \) and \( \sigma_n(u_1, u_2, \ldots, u_n)^q = \prod_{k=1}^n u_k^q \), where \( \sigma_j(u) \) denotes the \( j \)-th elementary symmetric function. Then, we can prove that the rational functions

\[
R_j(u) = \frac{F_{(n-j)r}(u)}{F_r(u)^{n-j}}, \quad 1 \leq j \leq n-2, \quad \text{and} \quad R_{n-1}(u) = \frac{\sigma_n(u)^{r/d}}{F_r(u)^{n/d}},
\]

satisfy (1) and (2) by the same way as in the primitive case. This proves the proposition. \( \square \)

**Definition 2.1.** We call \( \pi_R \) a rational \( G \)-quotient map.

For example, let \( G \) be the group in \( \text{GL}(4, C) \) of Shephard-Todd No. 32. There are generators \( F_d(u), \ d = 12, 18, 24, 30 \) of \( G \)-invariant polynomials. Then, \( \pi_R(u) = [R_3(u) : R_2(u) : R_3(u) : R_3(u) : 1] = [F_{18}/(F_{12}F_{18}) : F_{24}/F_{12}^2 : F_{18}^2/F_{12}^3 : 1] \) defines a rational \( G \)-quotient map of \( P(U_4) \).
Corollary 2.6. The subgroup \(\{A \in \text{GL}(U_n) \mid F_d(uA) = F_d(u), 1 \leq j \leq n\}\) of \(\text{GL}(U_n)\) is equal to \(G\).

Proof. Put \(G' = \{A \in \text{GL}(U_n) \mid F_d(uA) = F_d(u), 1 \leq j \leq n\}\). It is clear that \(G \subset G'\), and hence \(P(G) \subset P(G')\). For generic \([u] \in P(U_n)\), we have
\[
\pi_R^{-1}(\pi_R([u])) = P(G) \cdot [u] = P(G') \cdot [u],
\]
which proves \(|P(G)| = |P(G')|\). Assume \(cI_n \in G'\) for \(c \in C\). Then we have \(c^j = 1, 1 \leq j \leq n\), which implies \(c^m = 1\), where \(m = |C(G)|\). This concludes that \(C(G') = C(G)\). Consequently, we have \(G' = G\). \(\square\)

Definition 2.2. Let \(E\) be a completely integrable system on \(P^{n-1}\) of rank \(n\). Take a simply connected open subset \(V\) of \(P^{n-1}\backslash \text{Sing}(E)\), and let \(\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)\) be a set of linearly independent holomorphic solutions of \(E\) in \(V\). The holomorphic map
\[
V \to P(U_n)
\]
\[
P \mapsto [\varphi_1(P) : \varphi_2(P) : \ldots : \varphi_n(P)]
\]
can be continued analytically to a multi-valued map of \(P^{n-1}\backslash \text{Sing}(E)\) to \(P(U_n)\). We denote it by \(s_\varphi\) and call a Schwarz map of \(E\). The Schwarz map \(s_\varphi\) is said to be non-degenerate if the Jacobian of \(s_\varphi\) does not identically vanish. This property is independent of the choice of solutions \(\varphi_j\). A rational map
\[
\pi : P(U_n) \to P^{n-1}
\]
satisfying
\[
\pi(s_\varphi(P)) = P
\]
for all \(P\) in an open dense subset of \(P^{n-1}\backslash \text{Sing}(E)\) is called a rational inverse map of \(s_\varphi\).

Let \(M_\varphi\) be the monodromy group of \(E\) with respect to \(\varphi\). By definition, the set of values of \(s_\varphi\) at a point in \(P^{n-1}\backslash \text{Sing}(E)\) is a \(M_\varphi\)-orbit (or \(P(M_\varphi)\)-orbit) in \(P(U_n)\). Thus, a Schwarz map \(s_\varphi\) of \(E\) induces a single-valued map of \(P^{n-1}\backslash \text{Sing}(E)\) to the orbit space \(P(U_n)/M_\varphi\) of \(M_\varphi\). If this map is injective, we say that the Schwarz map \(s_\varphi\) of \(E\) induces an injection to the orbit space \(P(U_n)/M_\varphi\) of \(M_\varphi\), or simply, that the Schwarz map \(s_\varphi\) of \(E\) has a single-valued inverse. It is clear that the injectivity does not depend on the choice of solutions \(\varphi_1, \varphi_2, \ldots, \varphi_n\) of \(E\). If \(s_\varphi\) has a rational inverse map, then \(s_\varphi\) is non-degenerate and induces an injection to the orbit space \(P(U_n)/M_\varphi\) of \(M_\varphi\).

Lemma 2.7. Let \(\pi_R([u]) = [R_1(u) : R_2(u) : \ldots : R_{n-1}(u) : 1]\) be a rational \(G\)-quotient map of \(P(U_n)\) to \(Z = \{[z] = [z_1 : z_2 : \ldots : z_{n-1} : 1]\} \cong P^{n-1}\). Let \(E'(x)\)
be a completely integrable Fuchsian system of rank $n$ defined on $X = \{ [x] = [x_1 : x_2 : \ldots : x_{n-1} : 1] \} \cong \mathbb{P}^{n-1}$ with a non-degenerate Schwarz map. Let $M_u$ be the monodromy group of $E'(x)$ with respect to a system of linearly independent holomorphic solutions $u_j(x), 1 \leq j \leq n$ of $E'(x)$ in a simply connected open set in $X \setminus \text{Sing}(E')$. We assume $M_u$ is finite.

1. If $P(M_u) \subset P(G)$, then $z_j(x) = R_j(u_1(x), u_2(x), \ldots, u_n(x))$ are rational functions on $X$.

2. If $P(M_u) = P(G)$ and the Schwarz map $s_u$ induces an injection to the orbit space $P(U_n)/G$ of $G$, then $x \mapsto z(x) = \pi_R(u_1(x), u_2(x), \ldots, u_n(x))$ is a birational map, and the Schwarz map of $E'(x)$ has a rational inverse map.

Proof. Proof of (1): Put $N' = |M_u|$, and $R_j(u) = R_{j1}(u)/R_{j2}(u)$, where $R_{j1}(u)$ and $R_{j2}(u)$ are $G$-invariant homogeneous polynomials of the same degree. By the assumption, we know that, for each $A \in M_u$, there is a non-zero constant $c(A)$ such that $c(A)A \in G$. Then, we have

$$R_j(A \cdot u) = \frac{R_{j1}(A \cdot u)}{R_{j2}(A \cdot u)} = \frac{R_{j1}(c(A)A \cdot u)}{R_{j2}(c(A)A \cdot u)} = \frac{R_{j1}(u)}{R_{j2}(u)} = R_j(u),$$

which implies $R_j(u)$ is $M_u$-invariant. Note that $R_{j2}(u(x)) \neq 0$ because the Schwarz map of $E'(x)$ is non-degenerate. Consequently, $R_j(u(x))$ is single-valued in an open dense subset of $X$. On the other hand, $R_j(u(x))^{N'}$ is a rational function on $X$ by virtue of Lemma 2.3. This implies that $R_j(u(x))$ is itself a rational function on $X$.

Proof of (2): If $P(M_u) = P(G)$, and if the Schwarz map $s_u$ induces an injection to the orbit space $P(U_n)/M_u$ of $M_u$, then the rational map $z(x) = (\pi_R \circ s_u)(x)$ is also an injective holomorphic map (= biholomorphic map) on an open dense subset of $X$. This implies that $x \mapsto z(x)$ is birational. Let $x(z)$ be the inverse map of $z(x)$. Then $x(\pi_R([u]))$ is a rational inverse map of $s_u(x)$.

\begin{lemma}
(1) $m(= |C(G)|) = (d_1, d_2, \ldots, d_n)(= \gcd(d_1, d_2, \ldots, d_n))$.

(2) There are $G$-invariant homogeneous polynomials $H_1(u), H_2(u)$, such that $\deg H_1(u) - \deg H_2(u) = m$ and that there is no rational function $Q(u)$ satisfying $H_1(u)/H_2(u) = Q(u)^k$ for a factor $k > 1$ of $m$.

(3) There is a closed curve $\Gamma$ in $P(U_n)/\{ \pi_R \circ s_u \} \cup \{ H_1(u)H_2(u) = 0 \}$ such that the multi-valued function $u_j/\sqrt[n]{H_m(u)}$ is multiplied by $e^{2\pi \sqrt{-1}/m}$ after the analytic continuation along $\Gamma$.

Proof. Proof of (1): Put $m' = (d_1, d_2, \ldots, d_n)$ and $e(x) = e^{2\pi \sqrt{-1}x}$. Because $m'$ is a factor of $d_j$, $e(k/m')I_n, k \in \mathbb{Z}$ keeps invariant $F_{d_j}(u)$, $1 \leq j \leq n$. Then, Corollary 2.6 implies $e(k/m')I_n \in G$. Consequently, $m' \leq m$ holds. Because $e(1/m)I_n \in C(G)$ and $F_{d_j}(e(1/m)u) = e(d_j/m)F_{d_j}(u)$,
e(d_j/m) = 1 for 1 ≤ j ≤ n. This means that m is a common factor of
d_1, d_2, ..., d_n, which implies m ≤ m'.

Proof of (2): From (1), there are integers a_j, 1 ≤ j ≤ n such that
\[ \sum_j a_j d_j = m. \]
Let \( \hat{H}_1(u) = \prod \{ F_d(u)^{a_j} \mid a_j > 0 \}, \)
\( \hat{H}_2(u) = \prod \{ F_d(u)^{-a_j} \mid a_j < 0 \}, \)
\( H_m(u) = \hat{H}_1(u)/\hat{H}_2(u). \) Then, \( \hat{H}_1(u), \hat{H}_2(u) \) are G-invariant and homogeneous
satisfying \( \deg \hat{H}_1(u) - \deg \hat{H}_2(u) = m. \) Let \( P_1(u), P_2(u) \) be relatively prime G-
invariant homogeneous polynomials. Let \( \lambda \in C. \) Put \( H_1(\lambda; u) = \hat{H}_1(u)(P_1(u) - \lambda P_2(u)), \)
\( H_2(u) = \hat{H}_2(u)P_2(u) \) and \( H_m(\lambda; u) = H_1(\lambda; u)/H_2(u). \) If there is no
rational function \( Q(u) \) such that \( H_m(\lambda; u) = Q(u)^k \) for a factor \( k > 1 \) of \( m, \) then
the proof is completed. Assume this does not hold. That is, assume that
for each \( \lambda \in C, \) there are rational function \( Q(\lambda; u) \) and a factor \( k(\lambda) > 1 \) of \( m \)
such that \( H_m(\lambda; u) = Q(\lambda; u)^{k(\lambda)} \) holds. Then, except finite number of \( \lambda, \)
the polynomial \( P_1(u) - \lambda P_2(u) \) has only multiple zeros. This implies that the zero
set \( V(P_1(u) - \lambda P_2(u)) \) of \( P_1(u) - \lambda P_2(u) \) in \( U_n \) coincides with its singular locus.
This contradicts Bertini's theorem (see Lemma 5.4 below). This completes the
proof of (2).

Proof of (3): Put \( U' = P(U_n) \cup \{ \text{Sing}(\pi_R) \cup \{ H_1(u)H_2(u) = 0 \} \}, \)
and let \( u^0 \) be a point in \( U'. \) Let \( f_j(u) \) be a branch of \( u_j/\sqrt{H_m(u)} \) at \( u^0, \) and \( \Gamma, f_j(u) \) the
analytic continuation of \( f_j(u) \) along a loop \( \Gamma \) in \( \pi_1(U', u^0). \) It is clear that
\( \Gamma^* f_j(u) = c_j(\Gamma)f_j(u) \) for a m-th root of unity. Note that \( c_j(\Gamma) \) does not depend
on \( j. \) Put \( M = \{ c_j(\Gamma) \mid \Gamma \in \pi_1(U', u^0) \}. \)

If \( |M| = m, \) then (3) holds. Assume \( |M| = p \) for a proper factor \( p \) of \( m. \)
Then, \( Q(u) := f_j(u)^p \) is single-valued, and consequently a rational function on
\( P(U_n). \) This means that \( H_m(u) = (u^p/Q(u))^k, k = m/p. \) This contradicts the
property of \( H_m(u) \) proved in (2).

\textbf{Definition 2.3.} Let \( G \) be a finite irreducible complex reflection group in
\( GL(U_n) \) with the center \( C(G) \) of order \( m. \) Let \( \pi_R \) be a rational G-quotient
map, and \( H_m(u) = H_1(u)/H_2(u) \) be a G-invariant rational function satisfying the
property (2) of the previous lemma. Set
\[ Z' = Z \setminus \pi_R(\text{Sing}(\pi_R) \cup \{ H_1(u)H_2(u) = 0 \}). \]
We define \( \varphi(z) \) to be the \( |G| \)-valued map of \( Z' \) to \( U_n \) given by
\[ (2.1) \quad \varphi(z) = (\varphi_1(z), \varphi_2(z), \ldots, \varphi_n(z)) = \frac{\pi_R^{-1}(z)}{\sqrt[n]{H_m(\pi_R^{-1}(z))}}. \]

\textbf{Theorem 2.9.} Let the notation be the same as Definition 2.3. Then, the
multi-valued functions \( \varphi_j(u), 1 \leq j \leq n \) are solutions of a completely integrable
Fuchsian system on \( Z \) of rank \( n, \) which we denote by \( E(\pi_R, H_m; z). \) The system
\( E(\pi_R, H_m; z) \) satisfies the following properties:
(1) \( E(\pi_R, H_m; z) \) has the monodromy group \( G \).
(2) \( \pi_R \) is a rational inverse map of a Schwarz map of \( E(\pi_R, H_m; z) \).
(3) Let \( E'(x) \) be a completely integrable Fuchsian system on \( X = \{ [x] = [x_1 : x_2 : \ldots : x_{n-1} : 1] \} \cong \mathbb{P}^{n-1} \) of rank \( n \) with a non-degenerate Schwarz map. Then, the following two facts hold.

(3-i) If the monodromy group of \( E'(x) \) is finite, and the projective monodromy group of \( E'(x) \) is a subgroup of \( \mathcal{P}(G) \), then there is an algebraic transformation

\[
E'(x) = \theta(x)^{1/k} E(\pi_R, H_m; z(x)),
\]

where \( \theta(x) \) is a rational function on \( X \), \( k \) a positive integer, and \( x \mapsto z(x) \) a rational map of \( X \) to \( Z \). If \( E'(x) \) has the projective monodromy group \( \mathcal{P}(G) \), and its Schwarz map induces an injection to the orbit space \( \mathcal{P}(U_n)/G \) of \( G \), then \( x \mapsto z(x) \) is birational, and the Schwarz map of \( E'(x) \) has a rational inverse map.

(3-ii) If the monodromy group of \( E'(x) \) is a subgroup of \( G \), then there is an algebraic transformation

\[
E'(x) = \theta(x)^{1/m} E(\pi_R, H_m; z(x)),
\]

where \( \theta(x) \) is a rational function on \( X \), and \( x \mapsto z(x) \) a rational map of \( X \) to \( Z \). If \( E'(x) \) has the monodromy group \( G \), and its Schwarz map induces an injection to the orbit space \( \mathcal{P}(U_n)/G \) of \( G \), then the rational map \( x \mapsto z(x) \) is birational, and the Schwarz map of \( E'(x) \) has a rational inverse map.

**Proof.** Proof of the property (1): For \( \alpha \in Z' \), let \( \{ \varphi(z) \} \) denote the set of values of \( \varphi(z) \) at \( z = \alpha \). By the definition (2.1) of \( \varphi(z) \), we have \( |\{ \varphi(z) \}| = m \). Take \( \beta, \beta' \in \{ \varphi(z) \} \). Then, because

\[
F_d(\beta) = (F_d/H_m^{d/m})(\pi_R^{-1}(z)) = F_d(\beta')
\]

holds for every \( j \), it follows from (2) of Corollary 2.2 that \( \beta' = A \cdot \beta \) for some \( A \in G \). This proves that the set \( \{ \varphi(z) \} \) is a \( G \)-orbit in \( U_n \).

We next prove the “connectedness” of the \( G \)-orbit \( \{ \varphi(z) \} \). That is, there is a closed curve \( \gamma \) in \( Z' \) starting and ending at \( \alpha \) such that a branch of \( \varphi(x) \) is a curve in \( U_n \) starting at \( \beta \) and ending at \( \beta' \). First, let \( \Gamma_1 \) be a curve in \( \pi_R^{-1}(Z') \) starting at \( [\beta] \) and ending at \( [\beta'] \). Put \( \gamma_1 = \pi_R(\Gamma_1) \). Let \( \varphi^0(z) \) be a branch of \( \varphi(z) \) at \( z = \alpha \) satisfying \( \varphi^0(\alpha) = \beta \). The value \( \beta'' \) of \( \gamma_1 \varphi^0(z) \) at \( z = \alpha \) is in \( \{ \varphi(z) \} \), and satisfies \( \beta'' = [\beta] \). This implies \( \beta'' = \beta' \), for an \( m \)-th root of unity. Then, from (3) of Lemma 2.8, there is a closed curve \( \Gamma_2 \) in \( \pi_R^{-1}(Z') \) starting and ending at \( [\beta'] = [\beta''] \) such that the value of \( \gamma_1 \varphi^0(z) \) at \( z = \alpha \) is \( \beta' \). This means the “connectedness” of the \( G \)-orbit \( \{ \varphi(z) \} \).
Thus, we have proved both that \( \phi_j(z) \), \( 1 \leq j \leq n \) are solutions of an integrable system (denoted by \( E(\pi_R, H_m; z) \)) of rank \( n \) and that the monodromy group of the system is \( G \). It is clear that \( E(\pi_R, H_m; z) \) is a Fuchsian system on \( Z \) singular at most along \( \pi_R(\text{Sing}(\pi_R) \cup \{ H_1(u)H_2(u) = 0 \}) \).

Proof of the property (2): For \( z \in Z' \), (2.1) implies \( \pi_R([\phi_1(z) : \phi_2(z) : \ldots : \phi_n(z)]) = z \). This proves (2).

Proof of the property (3-i): Let \( u_j(x) \), \( j = 1, 2, \ldots, n \) be holomorphic solutions of \( E'(x) \) in a simply connected open subset of \( X' \setminus \text{Sing}(E'(x)) \), such that the projective monodromy group \( P(M_u) \) with respect to \( u(x) = (u_1(x), u_2(x), \ldots, u_n(x)) \) is a subgroup of \( P(G) \).

Put \( N' = |M_u| \), and

\[
\pi_R(u(x)), \quad \theta(x) = H_m(u(x))^{N'}.
\]

Then, by virtue of Lemma 2.3, \( \theta(x) \) is \( M_u \)-invariant, and hence a rational function on \( X \). By Lemma 2.7, \( x \mapsto z(x) \) is a rational map of \( X \) to \( Z \). Under these notations the following equalities hold:

\[
u(x) = \theta(x)^{1/(mN')} \cdot \frac{u(x)}{\sqrt[1/mN']{H_m(u(x))}} = \theta(x)^{1/(mN')} \cdot \frac{\pi_R^{-1}(z(x))}{\sqrt[n]{H_m(\pi_R^{-1}(z(x)))}} = \theta(x)^{1/(mN') \phi(z(x))}.
\]

This implies the equality (2.2) with \( k = mN' \).

If \( P(M_u) = P(G) \), and if the Schwarz map \( s_u \) induces an injection to the orbit space \( P(U_n)/M_u \) of \( M_u \), then the rational map \( x \mapsto z(x) \) is birational, and the Schwarz map of \( E'(x) \) has a rational inverse map by virtue of Lemma 2.7.

Proof of the property (3-ii): All we must show is that the equation (2.2) changes to (2.3) in this case. Other statements in (3-ii) have already been proved in (3-i).

Let \( u_j(x) \), \( j = 1, 2, \ldots, n \) be holomorphic solutions of \( E'(x) \) in a simply connected open subset of \( X' \setminus \text{Sing}(E'(x)) \), such that the monodromy group \( M_u \) with respect to \( u(x) = (u_1(x), u_2(x), \ldots, u_n(x)) \) is a subgroup of \( G \). Put

\[
z(x) = \pi_R(u(x)), \quad \theta(x) = H_m(u(x)).
\]

Since \( \theta(x) \) is \( M_u \)-invariant, \( \theta(x) \) is a rational function on \( X \). We get (2.3) by the same way as we derived (2.2) in the proof of (3-i).

**Corollary 2.10.** Let \( G \) be a finite irreducible complex reflection group in \( \text{GL}(U_n) \), and we put \( m = |C(G)| \). Let \( E(z) \) (resp. \( E'(x) \)) be a completely integrable Fuchsian system on \( Z \cong P^{n-1} \) (resp. \( X \cong P^{n-1} \)) of rank \( n \) with a non-degenerate Schwarz map. Assume that the Schwarz map of \( E(z) \) has a rational inverse map.
(1) If the monodromy group of $E(z)$ is $G$ and that of $E'(x)$ is a subgroup of $G$, there is an algebraic transformation

$$E'(x) = \theta(x)^{1/m} E(\sigma(x)), \quad (2.5)$$

where $x \mapsto \sigma(x)$ is a rational map of $X$ to $Z$, and $\theta(x)$ is a rational function on $X$. If, in addition, the monodromy group of $E'(x)$ coincides with $G$ and the Schwarz map of $E'(x)$ has a rational inverse map, then $\sigma(x)$ is a birational map of $X$ to $Z$.

(2) Assume that the monodromy groups of $E(z)$ and $E'(x)$ are finite. If the projective monodromy group of $E(z)$ is $P(G)$ and that of $E'(x)$ is a subgroup of $P(G)$, there is an algebraic transformation

$$E'(x) = \theta(x)^{1/k} E(\sigma(x)),$$

where $k$ is an integer, $x \mapsto \sigma(x)$ is a rational map of $X$ to $Z$, and $\theta(x)$ is a rational function on $X$. If, in addition, the projective monodromy group of $E'(x)$ coincides with $P(G)$ and the Schwarz map of $E'(x)$ has a rational inverse map, then $\sigma(x)$ is a birational map of $X$ to $Z$.

**Proof.** Proof of (1): By virtue of (3) of Theorem 2.9, there are a rational functions $y_1(x)$ on $X$, $y_2(z)$ on $Z$, a rational map $x \mapsto \sigma_1(x)$ of $X$ to $Z$, and a birational transformation $\sigma_2(z)$ of $Z$, such that

$$E'(x) = \theta_1(x)^{1/m} E(\pi_R, H_m; \sigma_1(x)), \quad E(z) = \theta_2(z)^{1/m} E(\pi_R, H_m; \sigma_2(z))$$

hold. From these equalities, we obtain (2.5), where $\sigma(x) = \sigma_2^{-1}(\sigma_1(x))$, and $\theta(x) = \theta_1(x)/\theta_2(\sigma(x))$.

If the monodromy group of $E'(x)$ is $G$ and the Schwarz map of $E'(x)$ has a rational inverse map, then $x \mapsto \sigma_1(x)$ is birational by virtue of (3) of Theorem 2.9. Hence $x \mapsto \sigma(x)$ is also birational.

(2) is proved in the same way. 

Taking the above result into account, we give the following definition.

**Definition 2.4.** Let $G$ be a finite irreducible complex reflection group in $GL(U_n)$. A completely integrable Fuchsian system $E$ of rank $n$ on $P^{n-1}$ is called a generating system for $G$ if $E$ has the monodromy group $G$ and if the Schwarz map of $E$ has a rational inverse map. $E$ is called a generating system for $P(G)$, if the monodromy group is finite, the projective monodromy group is $P(G)$ and if the Schwarz map of $E$ has a rational inverse map.

Theorem 2.9 shows that the system $E(\pi_R, H_m; z)$ is a generating system for $G$. 

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**Finite Irreducible Complex Reflection Groups**
3. Generating systems for reflection groups in $GL(2, \mathbb{C})$

In this section we give generating systems for all the finite irreducible complex reflection groups in $GL(2, \mathbb{C})$. These groups are divided into primitive ones and imprimitive ones. The imprimitive ones are the groups $G(r, p, 2)$, which are classically called dihedral groups. The primitive ones are classified by their projective form into three types: tetrahedral, octahedral and icosahedral groups. In Table VII in [ST], the groups of No. 4–7 are tetrahedral, those of No. 8–15 are octahedral, and those of No. 16–22 are icosahedral.

**Theorem 3.1.** (1) For each imprimitive reflection group $G = G(r, p, 2)$ with $r = pq \geq 2$, a generating differential equation is given by

\[
z^{-p/(2r)}E_1(-1/(2r), 1/(2r); 1/2; z) \quad \text{if } p \text{ is odd},
\]

\[
z^{-p/r}E_1(-1/r, 1/r; 1/2; z) \quad \text{if } p \text{ is even and } q \text{ is odd},
\]

\[
(z - c)^{-p/(2r)}E_1(-1/r, 1/r; 1/2; z), \quad c \neq 0, 1, \infty, \quad \text{if } p, q \text{ are even}.
\]

(2) For each primitive reflection group of No. 4–22 in Table VII in [ST], a generating differential equation is given by the following table.

| No. | $|P(G)|$ | $|C(G)|$ | differential equation |
|-----|---------|---------|-----------------------|
| 4   | 12      | 2       | $z^{-1/4}E_1(-1/12, 1/4; 1/2; z)$ |
| 5   | 12      | 6       | $2E_1(-1/12, 5/12; 2/3; z)$ |
| 6   | 12      | 4       | $2E_1(-1/12, 1/4; 1/2; z)$ |
| 7   | 12      | 12      | $z^{-1/6}E_1(-1/12, 1/4; 1/2; z)$ |
| 8   | 24      | 4       | $z^{-1/8}E_1(-1/24, 7/24; 1/2; z)$ |
| 9   | 24      | 8       | $2E_1(-1/24, 7/24; 1/2; z)$ |
| 10  | 24      | 12      | $2E_1(-1/24, 11/24; 2/3; z)$ |
| 11  | 24      | 24      | $z^{1/24}E_1(-1/24, 11/24; 2/3; z)$, |
| 12  | 24      | 2       | $z^{-1/6}E_1(-1/24, 5/24; 2/3; z)$ |
| 13  | 24      | 4       | $z^{-1/8}E_1(-1/24, 7/24; 3/4; z)$ |
| 14  | 24      | 6       | $2E_1(-1/24, 5/24; 1/2; z)$ |
| 15  | 24      | 12      | $z^{-1/12}E_1(-1/24, 5/24; 2/3; z)$ |
| 16  | 60      | 10      | $z^{-1/20}E_1(-1/60, 19/60; 1/2; z)$ |
| 17  | 60      | 20      | $2E_1(-1/60, 19/60; 1/2; z)$ |
| 18  | 60      | 30      | $2E_1(-1/60, 29/60; 2/3; z)$ |
| 19  | 60      | 60      | $z^{1/60}E_1(-1/60, 29/60; 2/3; z)$ |
| 20  | 60      | 6       | $z^{-1/12}E_1(-1/60, 11/60; 1/2; z)$ |
| 21  | 60      | 12      | $2E_1(-1/60, 11/60; 1/2; z)$ |
| 22  | 60      | 4       | $z^{-1/10}E_1(-1/60, 19/60; 4/5; z)$ |
Proof. Proof of (1): The ring of $G(r, p, 2)$-invariant polynomials is generated by
\[ F_r(u_1, u_2) = u_1^p + u_2^p, \quad f_2^q(u_1, u_2) = (u_1 u_2)^q. \]

Assume $p$ is odd. Put $p = 2p_1 + 1$. Then, $r = 2p_1 q + q$ and $m = q$. Put
\[ z = \pi_R(u_1, u_2) = \frac{F_r(u_1, u_2)^2}{4f_2(u_1, u_2)^{pq}}, \quad H_m(u_1, u_2) = \frac{F_r(u_1, u_2)}{f_2(u_1, u_2)^{pq}}. \]

Recall that $\phi_j(u) = u_j / \sqrt{H_m(u)}$, $j = 1, 2$ are solutions of $E(H_m; u)$.

We find the local exponents of the differential equation $E(\pi_R, H_m; z)$. The map $\pi_R : P(U_2) \rightarrow Z$ ramifies over $z = 0, 1, \infty$, with the ramification index 2, 2, $pq$ respectively. Let $[z] = [z_1 : z_2]$ be a point in $\pi_R^{-1}(0)$. This means that $F_r(z) = 0$. Then, $E(H_m; u)$ has the local exponents $\{-1/m, 1 - 1/m\}$ at this point. Since the ramification index of $\pi_R$ at $[z]$ is 2, $E(\pi_R, H_m; z)$ has the local exponents $\{-1/2m, 1/2 - 1/2m\}$ at $z = 0$. Similarly, $E(\pi_R, H_m; z)$ has the local exponents $\{0, 1/2\}$ at $z = 1$, and $\{p_1/r, (p_1 + 1)/r\}$ at $z = \infty$ because $E(H_m; u)$ has the local exponents $\{0, 1\}$ at a point in $\pi_R^{-1}(1)$, and $\{p_1, p_1 + 1\}$ at a point in $\pi_R^{-1}(\infty)$. $E(\pi_R, H_m; z)$ has the local exponents $\{0, 1\}$ at any other point. This concludes that $E(\pi_R, H_m; z) = z^{-p/2r}E_1(-1/(2r), 1/(2r); 1/2; z)$.

Assume $p = 2p_1$ is even, and $q$ is odd. Then, $r = 2p_1 q$ and $m = 2q$. Put
\[ z = \pi_R(u_1, u_2) = \frac{F_r(u_1, u_2) + 2f_2(u_1, u_2)^{pq}}{4f_2(u_1, u_2)^{pq}}, \]
\[ H_m(u_1, u_2) = \frac{(F_r(u_1, u_2) + 2f_2(u_1, u_2)^{pq})^2}{f_2(u_1, u_2)^{(p-1)q}}. \]

Then, the equality $E(\pi_R, H_m; z) = z^{-p/2r}E_1(-1/r, 1/r; 1/2; z)$ follows as in the previous case.

Assume $p = 2p_1$ and $q$ are even. Then $r = 2p_1 q$ and $m = 2q$. Put
\[ z = \pi_R(u_1, u_2) = \frac{F_r(u_1, u_2) + 2f_2(u_1, u_2)^{pq}}{4f_2(u_1, u_2)^{pq}}, \]
\[ H_m(u_1, u_2) = f_2(u_1, u_2)^q(z - c), \quad c \neq 0, 1, \infty. \]

Then, the equality $E(\pi_R, H_m; z) = (z - c)^{-p/(2r)}E_1(-1/r, 1/r; 1/2; z)$ follows as in the previous cases.

Proof of (2): For each case we quote generators of invariant polynomials from [ST], and define $\pi_R(u_1, u_2)$ and $H_m(u_1, u_2)$ using these generators. Then, in a similar way as in the proof of (1), we can get the generating differential equation.
No. 4–7: Cases of tetrahedral type.
In these cases, there are semi-invariant (homogeneous) polynomials

\[ F_{4,1}(u_1, u_2) = u_1^4 + 2\sqrt{-3}u_1^2u_2^2 + u_2^4, \]
\[ F_{4,2}(u_1, u_2) = -(u_1^4 - 2\sqrt{-3}u_1^2u_2^2 + u_2^4), \]
\[ F_6(u_1, u_2) = \sqrt{-432}u_1u_2(u_1^4 - u_2^4), \]
satisfying \( F_{4,1}^3 + F_{4,2}^3 = F_6^2 \) ([Kl2]).

<table>
<thead>
<tr>
<th>No.</th>
<th>generators of invariant polynomials</th>
<th>( \pi_R )</th>
<th>( H_m )</th>
</tr>
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<td>( F_6^2/F_{4,1}^3 )</td>
<td>( F_6^4/F_{4,1}^3 )</td>
</tr>
</tbody>
</table>

No. 8–15: Cases of octahedral type.
In these cases, there are semi-invariant (homogeneous) polynomials

\[ F_6(u_1, u_2) = \sqrt{-108}u_1u_2(u_1^4 - u_2^4), \quad F_8(u_1, u_2) = u_1^8 + 14u_1^4u_2^4 + u_2^8, \]
\[ F_{12}(u_1, u_2) = u_1^{12} - 33(u_1^8u_2^4 + u_1^4u_2^8) + u_2^{12}, \]
satisfying \( F_6^4 + F_8^3 = F_{12}^2 \) ([Kl2]).

<table>
<thead>
<tr>
<th>No.</th>
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<th>( H_m )</th>
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No. 16–22: Cases of icosahedral type.

In these cases, there are semi-invariant (homogeneous) polynomials

\[ F_{12}(u_1, u_2) = \sqrt[5]{1728} u_1 u_2 (u_1^{10} + 11u_1^5 u_2^5 - u_2^{10}), \]
\[ F_{20}(u_1, u_2) = (u_1^{20} + u_2^{20}) - 228(u_1^{15} u_2^5 - u_1^5 u_2^{15}) + 494u_1^{10} u_2^{10}, \]
\[ F_{30}(u_1, u_2) = (u_1^{30} + u_2^{30}) + 522(u_1^{25} u_2^5 - u_1^5 u_2^{25}) - 10005(u_1^{20} u_2^{10} + u_1^{10} u_2^{20}), \]

satisfying \( F_{12}^5 + F_{20}^3 = F_{30}^2 \) ([Kl2]).

<table>
<thead>
<tr>
<th>No.</th>
<th>generators of invariant polynomials</th>
<th>( \pi_R )</th>
<th>( H_m )</th>
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</tr>
<tr>
<td>22</td>
<td>( F_{12}, F_{20} )</td>
<td>( -F_{12}^5 / F_{20}^3 )</td>
<td>( F_{12}^2 / F_{20} )</td>
</tr>
</tbody>
</table>

4. Generating systems for reflection groups in \( \text{GL}(3, \mathbb{C}) \)

In this section we give explicit forms of the generating systems for all the finite irreducible reflection groups in \( \text{GL}(3, \mathbb{C}) \).

Let \( G \) be one of such groups. Take homogeneous polynomials \( f(u), g(u), h(u) \) which generate \( C[u_1, u_2, u_3]^G \). We can always take one polynomial without multiple factor from the three generators. Thus, we assume that \( h(u) \) has no multiple factor. Put

\[ S = \{ u \in U_3 \mid h(u) = 1 \}. \]

We first consider a completely integrable Fuchsian system \( E(x_1, x_2) \) on \( X = \{ [x_1 : x_2 : 1] \} \cong \mathbb{P}^2 \) of rank 3 whose monodromy group is \( G \) and whose image of an “affine Schwarz map” is \( S \). Then, we obtain from \( E(x_1, x_2) \) the generating system \( E(\pi_R, H_m; z) \) for \( G \).
4.1. The system $E(x_1, x_2)$

Notation being as above. We put

$$J(u) = \frac{\partial(f, g, h)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix}
    \frac{\partial f}{\partial u_1} & \frac{\partial g}{\partial u_1} & \frac{\partial h}{\partial u_1} \\
    \frac{\partial f}{\partial u_2} & \frac{\partial g}{\partial u_2} & \frac{\partial h}{\partial u_2} \\
    \frac{\partial f}{\partial u_3} & \frac{\partial g}{\partial u_3} & \frac{\partial h}{\partial u_3}
\end{vmatrix}.$$

Lemma 4.1. There is an integer $N$ such that $J(u)^N$ is $G$-invariant.

Proof. Let $R_j$, $1 \leq j \leq k$ be all the reflections in $G$. Let $N$ be the least common multiple of the orders of $R_j$. Then, $J(u)^N$ is $G$-invariant. 

Lemma 4.2. $S$ is a connected smooth surface in $U_3$.

Proof. We put $d_h = \deg h$. Since \(\sum u_i \partial h / \partial u_i = d_h h\), there is no common zero of $\partial h / \partial u_i$ on $S$. This proves that $S$ is smooth.

Put $S = \{[u_1 : u_2 : u_3 : w] \in P^3 | h(u) = w^{d_h}\}$. Since $h(u)$ has no multiple factor, the singular set of $S$ is of codimension $\geq 2$ in $S$. Consequently, it is irreducible. This means that the regular part of $S$ is connected. This implies that $S$ is also connected.

Put $X = P^2$. We identify $(x_1, x_2) \in C^2$ with $[x_1 : x_2 : 1] \in X$. Let $\varpi_S$ be a $|G| : 1$ ramified map of $S$ to $X$ defined by

$$\varpi_S(u) = (f(u), g(u)).$$

Lemma 4.3. (1) For any $(x_1, x_2) \in C^2$, $\varpi_S^{-1}(x_1, x_2)$ is a $G$-orbit.

(2) For $x = (x_1, x_2, x_3) \in S$, we have $|\varpi_S^{-1}(\varpi_S(x))| = |G|$ if and only if $J(x) \neq 0$.

(3) The group of the covering transformations of $\varpi_S$ is $G$.

Proof. It is clear that $\varpi_S^{-1}(x_1, x_2)$ is a union of $G$-orbits. Assume $G \cdot x \neq G \cdot \beta$, for $x, \beta \in S$. Then, in the same way as the proof of Lemma 2.1, there is a $G$-invariant homogeneous polynomial $F(u) = F(u_1, u_2, u_3)$ such that $F(x) \neq 0$ and $F(\beta) = 0$. Consequently, we have $\varpi_S(x) \neq \varpi_S(\beta)$. This proves (1).

The intersection multiplicity \(\{u | f(u) = f(x)\} \cap \{u | g(u) = g(x)\} \cap \{u | h(u) = 1\}\) at $u = x$ is one if and only if $J(x) \neq 0$. This proves (2).

Since $S$ is connected, (1) implies (3).

Put

$$\varpi_S^{-1}(x_1, x_2) = u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2), u_3(x_1, x_2)),$$
which is a $|G|$-valued function on $X$. Put
\[ X' = \{(x_1, x_2) \in X \mid |\mathcal{S}^{-1}(x_1, x_2)| = |G|\}. \]

Then, $\mathcal{S} : \mathcal{S}^{-1}(X') \to X'$ is a holomorphic covering map. We define $E(x_1, x_2)$ to be the system satisfied by $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ and $u_3(x_1, x_2)$.

**Lemma 4.4.** We have
\[
\begin{pmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\
\frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2}
\end{pmatrix}
\frac{1}{J}
\begin{pmatrix}
\frac{\partial (g, h)}{\partial (u_2, u_3)} & \frac{\partial (g, h)}{\partial (u_3, u_1)} & \frac{\partial (g, h)}{\partial (u_1, u_2)} \\
\frac{\partial (h, f)}{\partial (u_2, u_3)} & \frac{\partial (h, f)}{\partial (u_3, u_1)} & \frac{\partial (h, f)}{\partial (u_1, u_2)}
\end{pmatrix}
\circ \mathcal{S}^{-1}.
\]

**Proof.** Since $f(u(x_1, x_2)) = x_1$, $g(u(x_1, x_2)) = x_2$, $h(u(x_1, x_2)) = 1$, we have
\[
\begin{pmatrix}
u_1 & u_2 & u_3 \\
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\
\frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f}{\partial u_1} & \frac{\partial g}{\partial u_1} & \frac{\partial h}{\partial u_1} \\
\frac{\partial f}{\partial u_2} & \frac{\partial g}{\partial u_2} & \frac{\partial h}{\partial u_2} \\
\frac{\partial f}{\partial u_3} & \frac{\partial g}{\partial u_3} & \frac{\partial h}{\partial u_3}
\end{pmatrix}
\begin{pmatrix}
(deg f) \cdot f & (deg g) \cdot g & (deg h) \cdot h \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\]
which proves the lemma. \qed

In the following, we say that a multi-valued function $p(x_1, x_2)$ on $X'$ is a polynomial (resp. a rational function) in $u = (u_1, u_2, u_3)$, if there is a polynomial (resp. rational function) $P(u)$ in $u$ such that $p(x_1, x_2) = P(u) \circ \mathcal{S}^{-1}(x_1, x_2) = P(u(x_1, x_2))$ holds. For example, $\frac{\partial u_1}{\partial x_1}(x_1, x_2)/\partial x_1$ and $\frac{\partial u_2}{\partial x_2}(x_1, x_2)/\partial x_2$ are rational functions in $u$ by virtue of (4.3).

**Corollary 4.5.** The first partial derivatives of $u_j(x_1, x_2)$ are rational functions in $u$ with the denominator $J$. The second partial derivatives of $u_j(x_1, x_2)$ are rational functions in $u$ with the denominator $J^3$.

**Proof.** For example, we have $\frac{\partial^2 u_1}{\partial x_1^2} = \sum_j \frac{1}{\partial x_j} \left( \frac{1}{J} \frac{\partial (g, h)}{\partial (u_2, u_3)} \right) \cdot \frac{\partial u_j}{\partial x_1}$. \qed

**Proposition 4.6.** The system $E(x_1, x_2)$ can be written in the form
\[
\frac{\partial^2 \phi}{\partial x_i \partial x_j} + p^2_i \frac{\partial \phi}{\partial x_1} + p^2_j \frac{\partial \phi}{\partial x_2} + p^2_0 \phi = 0, \quad 1 \leq i \leq j \leq 2,
\]
where $p^l_{ij}$ are rational functions of $(x_1, x_2)$ depending on the choice of $f(u)$, $g(u)$, $h(u)$. The monodromy group with respect to $u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2), u_3(x_1, x_2))$ is $G$.

Proof. Expanding the determinant

$$
\begin{vmatrix}
  u_1 & u_2 & u_3 & \varphi \\
  (u_1)_{x_1} & (u_2)_{x_1} & (u_3)_{x_1} & \varphi_{x_1} \\
  (u_1)_{x_2} & (u_2)_{x_2} & (u_3)_{x_2} & \varphi_{x_2} \\
  (u_1)_{x_i} (u_2)_{x_j} & (u_3)_{x_i} (u_3)_{x_j} & \varphi_{x_i x_j}
\end{vmatrix}
$$

in the fourth column, we have

$$
p^l_{ij} = \frac{1}{W} 
\begin{vmatrix}
  u_1 & u_2 & u_3 \\
  (u_1)_{x_2} & (u_2)_{x_2} & (u_3)_{x_2} \\
  (u_1)_{x_i} (u_2)_{x_j} & (u_3)_{x_i} (u_3)_{x_j}
\end{vmatrix},
$$

$$
p^0_{ij} = -\frac{1}{W} 
\begin{vmatrix}
  u_1 & u_2 & u_3 \\
  (u_1)_{x_1} & (u_2)_{x_1} & (u_3)_{x_1} \\
  (u_1)_{x_2} & (u_2)_{x_2} & (u_3)_{x_2}
\end{vmatrix},
$$

where

$$
W = W(u_1, u_2, u_3) = \begin{vmatrix}
  u_1 & u_2 & u_3 \\
  (u_1)_{x_1} & (u_2)_{x_1} & (u_3)_{x_1} \\
  (u_1)_{x_2} & (u_2)_{x_2} & (u_3)_{x_2}
\end{vmatrix}.
$$

Let $\gamma$ be a closed curve in $X'$. Then, by (1) of Lemma 4.3, there is a matrix $M(\gamma) \in G$ such that the analytic continuation $\gamma_* u(x_1, x_2)$ of $u(x_1, x_2)$ along $\gamma$ is equal to $u(x_1, x_2) M(\gamma)$. Then, we have

$$
\gamma_* ((u_1)_{x_1}, (u_2)_{x_1}, (u_3)_{x_1}) = ((u_1)_{x_1}, (u_2)_{x_1}, (u_3)_{x_1}) M(\gamma),
$$

$$
\gamma_* ((u_1)_{x_2}, (u_2)_{x_2}, (u_3)_{x_2}) = ((u_1)_{x_2}, (u_2)_{x_2}, (u_3)_{x_2}) M(\gamma),
$$

$$
\gamma_* ((u_1)_{x_i}, (u_2)_{x_i}, (u_3)_{x_i}) = ((u_1)_{x_i}, (u_2)_{x_i}, (u_3)_{x_i}) M(\gamma),
$$

and hence

$$
\gamma_* W = W \det M(\gamma),
$$

$$
\gamma_* W = W \det M(\gamma).
$$
This means that $p_j^q$ is single-valued in $X'$. In the same way, we can show that $p_j^2$ and $p_j^3$ are also single-valued in $X'$.

By (4.4), we have $W = \text{deg}(h)/J$. Corollary 4.5 shows that $p_j^qJ^4$ are polynomials in $u$. By Lemma 4.1, there is an integer $N$ with $N \geq 4$ such that $J(u)^N$ is a $G$-invariant polynomial in $u$, and hence a polynomial in $f(u)$, $g(u)$, $h(u)$. Then, we both that $p_j^qJ(u)^N$ are polynomials in $u$, and that $p_j^qJ(u(x_1, x_2))^N$ are single-valued in $X'$. This means $p_j^qJ(u(x_1, x_2))^N$ are $G$-invariant polynomials in $u$, that is, polynomials in $(x_1, x_2)$. Since $J(u(x_1, x_2))^N$ is also a polynomial in $(x_1, x_2)$, $p_j^q$ are rational functions in $(x_1, x_2)$.

Since $S$ is connected, the monodromy group of (4.5) with respect to the system of solutions $u_1(x_1, x_2)$, $u_2(x_1, x_2)$, $u_3(x_1, x_2)$ is the covering transformation group of $\varpi_S$, which is equal to $G$ by virtue of Lemma 4.3.

\[\square\]

### 4.2. Generating systems for primitive reflection groups

Let $G$ be a finite irreducible primitive complex reflection group in $GL(3, \mathbb{C}) = GL(U_3)$, that is, $G$ is one of $G_{120}$, $G_{336}$, $G_{648}$, $G_{1296}$, $G_{2160}$, where the sub-indices denote the orders of the groups. In the following we fix the generators $F_j(u_j), \ j = 1, 2, 3$ with $d_1 < d_2 < d_3$ of $C[u]^G$ as follows.

For $G = G_{120}$,

\[
F_2 = u_2^2 - u_1u_3,
\]

\[
F_6 = u_2(u_1^5 - u_3^5 + u_2^5 + 5u_1u_2u_3(u_2^2 + u_1u_3)),
\]

\[
F_{10} = -u_1^{10} - u_3^{10} + (u_1^5 - u_3^5)u_2^3(108u_2^2 + 120u_1u_3) - 14u_2^{10} - 90u_2^8u_1u_3
\]

\[-180u_2^6(u_1u_3)^2 - 120u_2^4(u_1u_3)^3 - 90u_2^2(u_1u_3)^4.
\]

For $G = G_{336}$,

\[
F_4 = u_1^3u_2 + u_2^3u_3 + u_3^3u_1,
\]

\[
F_6 = \frac{1}{54} \det \begin{pmatrix}
\partial^2 F_4 / \partial u_1 \partial u_1 & \partial^2 F_4 / \partial u_1 \partial u_2 & \partial^2 F_4 / \partial u_1 \partial u_3 & \partial F_6 / \partial u_1 \\
\partial^2 F_4 / \partial u_2 \partial u_1 & \partial^2 F_4 / \partial u_2 \partial u_2 & \partial^2 F_4 / \partial u_2 \partial u_3 & \partial F_6 / \partial u_2 \\
\partial^2 F_4 / \partial u_3 \partial u_1 & \partial^2 F_4 / \partial u_3 \partial u_2 & \partial^2 F_4 / \partial u_3 \partial u_3 & \partial F_6 / \partial u_3 \\
\partial F_6 / \partial u_1 & \partial F_6 / \partial u_2 & \partial F_6 / \partial u_3 & 0
\end{pmatrix}
= 5(u_1u_2u_3)^2 - u_1^5u_3 - u_3^5u_2 - u_2^5u_1,
\]

\[
F_{14} = \frac{1}{9} \det \begin{pmatrix}
\partial^2 F_4 / \partial u_1 \partial u_1 & \partial^2 F_4 / \partial u_1 \partial u_2 & \partial^2 F_4 / \partial u_1 \partial u_3 & \partial F_6 / \partial u_1 \\
\partial^2 F_4 / \partial u_2 \partial u_1 & \partial^2 F_4 / \partial u_2 \partial u_2 & \partial^2 F_4 / \partial u_2 \partial u_3 & \partial F_6 / \partial u_2 \\
\partial^2 F_4 / \partial u_3 \partial u_1 & \partial^2 F_4 / \partial u_3 \partial u_2 & \partial^2 F_4 / \partial u_3 \partial u_3 & \partial F_6 / \partial u_3 \\
\partial F_6 / \partial u_1 & \partial F_6 / \partial u_2 & \partial F_6 / \partial u_3 & 0
\end{pmatrix}
= u_1^{14} + u_2^{14} + u_3^{14} + \cdots.
\]
For $G = G_{648}$,
\[
F_6 = (u_1^3 + u_2^3 + u_3^3)^2 - 12((u_1 u_2)^3 + (u_2 u_3)^3 + (u_3 u_1)^3),
\]
\[
F_9 = (u_2^3 - u_3^3)(u_3^3 - u_1^3)(u_1^3 - u_2^3),
\]
\[
F_{12} = (u_1^3 + u_2^3 + u_3^3)((u_1^3 + u_2^3 + u_3^3)^3 + 216(u_1 u_2 u_3)^3).
\]

For $G = G_{1296}$, we choose $F_6, F_{12}, F_9^2$ as generators, where $F_6, F_9$ and $F_{12}$ are the generators for $G_{648}$. For later use, we define two functions.
\[
F'_{12} = u_1 u_2 u_3 \prod_{0 \leq i, j \leq 2} (u_1 + \omega^i u_2 + \omega^j u_3),
\]
\[
F_{36} = 4F_{12}^3 - (F_6^3 - 3F_6 F_{12} - 432F_9^2)^2,
\]
where $\omega = e^{2\pi i/3}$. It is known that
\[
(F'_{12})^3 = \frac{1}{6912} F_{36},
\]
which is $G$-invariant for both $G = G_{648}$ and $G = G_{1296}$.

For $G = G_{2160}$,
\[
F_6 = 10u_1^3 u_2^3 + 9u_3(u_1^5 + u_2^5) - 45(u_1 u_2 u_3)^2 - 135u_1 u_2 u_3^4 + 27u_3^6,
\]
\[
F_{12} = \frac{1}{121500} \det \begin{pmatrix}
\frac{\partial^2 F_6}{\partial u_1 \partial u_1}
& \frac{\partial^2 F_6}{\partial u_1 \partial u_2}
& \frac{\partial^2 F_6}{\partial u_1 \partial u_3}
& \frac{\partial F_6}{\partial u_1}
+ \frac{1}{6} F_6^2

\frac{\partial^2 F_6}{\partial u_2 \partial u_1}
& \frac{\partial^2 F_6}{\partial u_2 \partial u_2}
& \frac{\partial^2 F_6}{\partial u_2 \partial u_3}
& \frac{\partial F_6}{\partial u_2}

\frac{\partial^2 F_6}{\partial u_3 \partial u_1}
& \frac{\partial^2 F_6}{\partial u_3 \partial u_2}
& \frac{\partial^2 F_6}{\partial u_3 \partial u_3}
& \frac{\partial F_6}{\partial u_3}

\frac{\partial F_{12}}{\partial u_1}
& \frac{\partial F_{12}}{\partial u_2}
& \frac{\partial F_{12}}{\partial u_3}
& 0
\end{pmatrix}
\]

For every case, we put
\[
f(u) = F_{d_1}(u), \quad g(u) = F_{d_2}(u), \quad h(u) = F_{d_4}(u).
\]
Then, $h(u)$ does not have a multiple factor.

The following lemma is found in [ST] and [Y]. Recall that $J$ is defined by (4.1).

**Lemma 4.7.** (1) The zeros of $J$ are the union of mirrors of complex reflections in $G$.

(2) Except for $G_{648}$ and $G_{1296}$, all the complex reflections in $G$ are of order 2, and $J^2$ is $G$-invariant.
For $G = G_{648}$, we have $J = 2592(F_{12}')^2$, whence $J^3$ is $G$-invariant. For $G = G_{1296}$, we have $J = 5184F_9(F_{12}')^2$, whence $J^6$ is $G$-invariant. $G_{648}$ and $G_{1296}$ have the same set of reflections of order 3. The zeros of $F_{12}'$ are the union of mirrors of reflections of order 3, and those of $F_9$ are the union of mirrors of reflections of order 2 in $G_{1296}$.

For each $G$, we choose a rational $G$-quotient map $\pi_R([u]) = [R_1(u) : R_2(u) : 1]$ and a $G$-invariant rational function $H_m(u)$ as in the following table.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$H_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{120}$</td>
<td>$F_{10}/F_2^5$</td>
<td>$F_6/F_2^3$</td>
<td>$F_2$</td>
</tr>
<tr>
<td>$G_{336}$</td>
<td>$F_{14}/(F_6F_3^2)$</td>
<td>$F_6^2/F_3^3$</td>
<td>$F_6/F_4$</td>
</tr>
<tr>
<td>$G_{648}$</td>
<td>$F_{12}/F_6^2$</td>
<td>$F_9^2/F_6^3$</td>
<td>$F_9/F_6$</td>
</tr>
<tr>
<td>$G_{1296}$</td>
<td>$F_6^2/F_6^3$</td>
<td>$F_{12}/F_6^2$</td>
<td>$F_6$</td>
</tr>
<tr>
<td>$G_{2160}$</td>
<td>$F_{36}/F_6^6$</td>
<td>$F_{12}/F_6^2$</td>
<td>$F_6$</td>
</tr>
</tbody>
</table>

Table 1. $\pi_R$ and $H_m$ for primitive case.

**Theorem 4.8.** We define $G$-invariant polynomials $\tilde{D}(u)$ by

$$
\tilde{D}(u) = \begin{cases}
J(u)^2 & \text{for } G = G_{120}, G_{336}, G_{2160}, \\
F(u) & \text{for } G = G_{648}, \\
F_9(u)^2F_{36}(u) & \text{for } G = G_{1296}.
\end{cases}
$$

Then $\tilde{D}(u)$ becomes a polynomial in $f(u)$, $g(u)$, $h(u)$, which we denote by $P_D(f(u), g(u), h(u))$. We define $D(z_1, z_2)$ by

$$
D(z_1, z_2) = \begin{cases}
P_D(z_1, z_2, 1) & \text{for } G = G_{120}, G_{1296}, G_{2160}, \\
z_2^{3/2}P_D(z_1\sqrt{z_2}, \sqrt{z_2}, 1) & \text{for } G = G_{336}, \\
z_2^2P_D(z_1, \sqrt{z_2}, 1) & \text{for } G = G_{648}.
\end{cases}
$$

Explicit forms of $D(z_1, z_2)$ are given in the following.

Then, for each finite irreducible primitive complex reflection group $G$ in $\text{GL}(3, \mathbb{C})$, the generating system $E(\pi_R, H_m; z_1, z_2)$ is given by

$$
D(z_1, z_2) \frac{\partial^3 \varphi}{\partial z_i \partial z_j} + a_{ij}(z_1, z_2) \frac{\partial \varphi}{\partial z_1} + a_{ij}(z_1, z_2) \frac{\partial \varphi}{\partial z_2} + a_{ij}(z_1, z_2) \varphi = 0 \quad (1 \leq i \leq j \leq 2),
$$

where $a_{ij}(z_1, z_2)$ are polynomials in $(z_1, z_2)$ given in the following.
(1) \[ G = G_{120} \]
\[ D = 100(-z_1^3 + 20z_1^2z_2 + 2z_1^2 + 720z_1z_2^3 + 480z_1z_2^2 + 4z_1 \\
+ 1728z_2^3 + 7360z_2^4 + 160z_2^3 + 640z_2^2 - 80z_2 - 8), \]
\[ a_{11}^1 = 20(-5z_1^2 - 18z_1z_2^2 + 100z_1z_2 + 10z_1 + 1512z_2^3 + 756z_2^2), \]
\[ a_{11}^2 = 8z_2(3z_1 - 27z_2^2 + 50z_2 + 19), \]
\[ a_{11}^0 = z_1 + 36z_2^2 - 20z_2 - 2, \]
\[ a_{12}^1 = 200z_2(3z_1^2 + 280z_1z_2 + 140z_1 + 1296z_2^3 + 3888z_2^2 + 108), \]
\[ a_{12}^2 = 10(-5z_1^2 + 36z_1z_2^2 + 576z_2^3 + 888z_2^2 + 20), \]
\[ a_{12}^0 = 20z_2(-3z_1 - 28z_2 - 14), \]
\[ a_{22}^1 = 1000(-z_1^3 - 72z_1^2z_2 - 18z_1^2 - 384z_1z_2^3 - 720z_1z_2^2 + 4z_1 \\
- 576z_2^4 + 1152z_2^3 - 1440z_2^2 + 288z_2 + 72), \]
\[ a_{22}^2 = 200(-3z_1^2z_2 + 5z_1^2 + 260z_1z_2^2 + 100z_1z_2 + 864z_2^4 \\
+ 3472z_2^3 + 120z_2^2 + 212z_2 - 20), \]
\[ a_{22}^0 = 100(z_1^2 - 48z_2^3 - 14z_2^2 - 4). \]

(2) \[ G = G_{336} \]
\[ D = z_2^2(z_1^3z_2 - 88z_1^2z_2 + 1008z_1z_2^2 + 1088z_1z_2 - 256z_1 \\
+ 1728z_2^3 - 60032z_2^2 + 22016z_2 - 2048), \]
\[ a_{11}^1 = \frac{z_2^2}{49}(59z_1^2z_2 + 108z_1z_2^2 - 3436z_1z_2 + 80z_1 + 13800z_2^2 + 27872z_2 - 5632), \]
\[ a_{11}^2 = \frac{2z_2^3}{49}(-3z_1z_2 + 81z_2^2 + 520z_2 - 80), \]
\[ a_{11}^0 = \frac{z_2^2}{196}(-9z_1z_2 + 108z_2^2 + 1040z_2 - 160), \]
\[ a_{12}^1 = \frac{z_2}{98}(17z_1^3 - 144z_1^2z_2^2 - 1340z_1z_2^2 + 80z_1^2 + 64336z_1z_2^2 + 8896z_1z_2 \\
+ 1280z_1 + 141120z_2^3 - 3214208z_2^2 + 516864z_2 + 5120), \]
\[ a_{12}^2 = \frac{z_2^2}{98}(29z_1^2z_2 - 216z_1z_2^2 - 1752z_1z_2 - 160z_1 + 21792z_2^2 - 2432z_2 - 1280), \]
\[ a_{12}^0 = \frac{z_2}{196} (13z_1^2 z_2 - 72z_1 z_2^2 - 804z_1 z_2 - 80z_1 + 7824z_2^2 - 544z_2 - 640), \]
\[ a_{12}^1 = \frac{2}{49} (3z_1^4 z_2 + 24z_1^3 z_2^2 - 357z_1^3 z_2 + 10z_1^3 - 2952z_1^2 z_2^2 + 9516z_1^2 z_2 \]
\[ - 544z_1^2 - 9408z_1 z_2^2 + 61056z_1 z_2^2 + 95232z_1 z_2 - 10624z_1 \]
\[ + 605248z_2^3 + 2533632z_2^2 - 42240z_2 - 45056), \]
\[ a_{22}^0 = \frac{z_2}{49} (65z_1^3 z_2 + 72z_1^2 z_2^2 - 5798z_1^2 z_2 - 40z_1^2 + 66616z_1 z_2^2 + 75520z_1 z_2 \]
\[ - 13184z_1 + 141120z_2^3 - 4276032z_2^2 + 1359744z_2 - 102912), \]
\[ a_{22}^0 = \frac{1}{196} (3z_1^3 z_2 + 48z_1^2 z_2^2 - 336z_1 z_2^2 - 40z_1^2 + 5248z_1 z_2^2 + 7824z_1 z_2 \]
\[ + 2496z_1 + 28224z_2^3 - 528704z_2^2 - 2176z_2 + 22528). \]

(3) \[ G = G_{648} \]

\[ D = z_2^2 (4z_1^3 - (3z_1 + 43z_2) - 1)^2), \]
\[ a_{12}^1 = \frac{2z_2^2}{3} (8z_1^2 + 1080z_1 x_2 - 13z_1 - 2160 z_2 + 5), \]
\[ a_{12}^2 = \frac{z_2^3}{2} (-3z_1 + 2160z_2 - 5), \]
\[ a_{12}^0 = \frac{z_2^3}{18} (-7z_1 + 2160z_2 - 5), \]
\[ a_{12}^1 = \frac{z_2^2}{6} (4z_1^3 - 2880z_1^2 x_2 - 9z_1^2 - 4752z_1 z_2 + 6z_1 - 870912z_2^2 + 2448z_2 - 1), \]
\[ a_{12}^2 = \frac{2z_2^2}{3} (4z_1^2 - 1080z_1 z_2 - 5z_1 - 432z_2 + 1), \]
\[ a_{12}^0 = \frac{z_2^2}{9} (4z_1^2 - 720z_2 z_2 - 5z_1 - 432z_2 + 1), \]
\[ a_{22}^1 = 16z_2 (20z_1^3 - 21z_1^2 + 2592z_1 z_2 - 432z_2 + 1), \]
\[ a_{22}^2 = \frac{z_2^2}{6} (20z_1^3 + 2880z_1^2 z_2 - 45z_1^2 - 21168z_1 z_2 + 30z_1 \]
\[ - 1741824z_2^3 + 6192z_2 - 5), \]
\[ a_{22}^0 = \frac{1}{18} (-4z_1^3 + 960z_1^2 z_2 + 9z_1^2 - 912z_1 z_2 - 6z_1 - 165888z_2^2 - 48z_2 + 1). \]
(4) \[ G = G_{1296} \]

\[ D = z_1(4z_2^3 - (1 - 3z_2 - 432z_1)^2), \]

\[ a_{11}^1 = \frac{1}{2}(-456192z_1^2 + 960z_1z_2^2 - 5328z_1z_2 + 1488z_1 + 4z_2^3 - 9z_2^2 + 6z_2 - 1), \]

\[ a_{11}^2 = 16(20z_2^3 - 21z_2^2 + 2592z_2z_1 - 432z_1 + 1), \]

\[ a_{11}^0 = \frac{20}{3}(-4z_2^2 + 5z_2 + 432z_1 - 1), \]

\[ a_{12}^1 = \frac{2z_1}{3}(4z_2^2 - 1080z_2z_1 - 5z_2 - 432z_1 + 1), \]

\[ a_{12}^2 = 24z_1(-20z_2^2 - 15z_2 - 4752z_1 + 11), \]

\[ a_{12}^0 = 40z_2z_1, \]

\[ a_{22}^1 = \frac{z_1^2}{2}(-3z_2 + 2160z_1 - 5), \]

\[ a_{22}^2 = \frac{2z_1}{3}(8z_2^2 + 1080z_2z_1 - 13z_2 - 2160z_1 + 5), \]

\[ a_{22}^0 = \frac{5z_1}{36}(-z_2 - 432z_1 + 1). \]

(5) \[ G = G_{2160} \]

\[ D = 2700(9z_1^3 + 7128z_1^2z_2^2 - 1584z_1^2z_2 + 64z_1^2 + 186624z_1z_2^5 \]

\[ + 1622592z_1z_2^4 - 707952z_1z_2^3 + 96640z_1z_2^2 - 4096z_1z_2 \]

\[ + 34338816z_2^7 + 107371008z_2^6 - 73887360z_2^5 + 17406016z_2^4 \]

\[ - 1765376z_2^3 + 65536z_2^2), \]

\[ a_{11}^1 = 36(765z_1^2 + 97200z_1z_2^2 + 390852z_1z_2 - 93568z_1z_2 + 4160z_1 \]

\[ + 10575360z_2^5 + 50463648z_2^4 - 23093816z_2^3 \]

\[ + 3271488z_2^2 - 145408z_2), \]

\[ a_{11}^2 = 72(-27z_1z_2 + 5z_1 + 19440z_2^4 - 17928z_2^3 + 4549z_2^2 - 352z_2), \]

\[ a_{11}^0 = 135(-z_1 - 864z_2^3 + 144z_2^2), \]

\[ a_{12}^1 = 72(-121500z_1^2z_2^2 + 171945z_1^2z_2 - 17280z_1^2 - 777600z_1z_2^4 \]

\[ + 71424288z_1z_2^3 - 23021864z_1z_2^2 + 2157248z_1z_2 - 51200z_1 \]
As is shown in Proposition 2.5, the following way.

Thus, from (2.3), we obtain

\[ E = a_1^2 = 18(495z_1^2 - 194400z_1z_2^3 + 287496z_1z_2^2 - 50464z_1z_2 + 1280z_1 \]

\[ - 7153920z_2^5 + 20767104z_2^4 - 6908768z_2 + 705024z_2^2 - 16384z_2) , \]

\[ a_1^0 = 540(540z_1z_2^2 - 307z_1z_2 + 28z_1 + 19872z_2^4 - 36600z_2^3 + 9388z_2^2 - 640z_2) , \]

\[ a_2^1 = 72(303750z_1^3z_2 - 222075z_1z_2^3 + 20412000z_1^2z_2^3 - 91027260z_1^2z_2^2 \]

\[ + 17524080z_1z_2^2 - 288960z_1^2 - 3594844800z_1z_2^5 - 7652624256z_1z_2^4 \]

\[ + 316237232z_1z_2^3 - 250015488z_1z_2^2 - 14954496z_1z_2 + 1146880z_1 \]

\[ + 44754923520z_2^7 + 258854420736z_2^6 - 221701400320z_2^5 \]

\[ + 787122130608z_2^4 - 14385156096z_2^3 + 1282310144z_2^2 - 42991616z_2) , \]

\[ a_2^2 = 72(121500z_1^2z_2^2 + 95355z_1^2z_2 - 12420z_1^2 + 18273600z_1z_2^4 \]

\[ + 50270112z_1z_2^3 - 16800436z_1z_2^2 + 1466752z_1z_2 - 25600z_1 \]

\[ + 1881480960z_2^6 + 4651449408z_2^5 - 2578086368z_2^4 \]

\[ + 458582272z_2^3 - 31544320z_2^2 + 622592z_2) , \]

\[ a_2^0 = 540(-1350z_1^2z_2 + 225z_1^2 - 131760z_1z_2^3 - 49756z_1z_2^2 \]

\[ + 19568z_1z_2 - 1536z_1 - 778924z_2^5 - 12211104z_2^4 \]

\[ + 5970224z_2^3 - 865536z_2^2 + 40960z_2) . \]

**Proof.** Following the proof of Proposition 4.6, we can construct the system \( E(x_1, x_2) \) for each \( G \) by using \( f(u) \), \( g(u) \) and \( h(u) \) given in (4.7). Then, the generating systems \( E(\pi_R, H_m; z) \) can be obtained by using these \( E(x_1, x_2) \) in the following way.

Put \( u(x_1, x_2) = \pi_K^{-1}(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2), u_3(x_1, x_2)) \) (see (4.2)). As is shown in Proposition 2.5, \( \pi_R \) gives a rational \( G \)-quotient map.

For \( G = G_{120}, G_{1296}, G_{2160}, z(x_1, x_2)(= \pi_R(u(x_1, x_2))) \) and \( \theta(x_1, x_2)(= H_m(u(x_1, x_2))) \) in (2.4) are given by

\[ z(x_1, x_2) = [x_1 : x_2 : 1], \quad \theta(x_1, x_2) = 1. \]

Then, from (2.3), we obtain \( E(\pi_R, H_m; z) = E(x_1, x_2)|_{x_1=x_2} \).
For $G = G_{336}$, $z(x_1, x_2)$ and $\theta(x_1, x_2)$ in (2.4) are given by

$$z(x_1, x_2) = \left[ \frac{x_1}{x_2} : x_2^2 : 1 \right], \quad \theta(x_1, x_2) = x_2.$$ 

Then, from (2.3), we obtain

$$E(\pi_R, H_m; z) = z_2^{-1/4} E(x_1, x_2)|_{x_1 = z_1 \sqrt{z_2}, x_2 = \sqrt{z_2}},$$

from which we obtain the explicit form of $E(\pi_R, H_m; z)$.

For $G = G_{648}$, $z(x_1, x_2)$ and $\theta(x_1, x_2)$ in (2.4) are given by

$$z(x_1, x_2) = \left[ x_1 : x_2^2 : 1 \right], \quad \theta(x_1, x_2) = x_2.$$ 

Hence, we have

$$E(\pi_R, H_m; z) = z_2^{-1/6} E(x_1, x_2)|_{x_1 = z_1, x_2 = \sqrt{z_2}},$$

from which we obtain the explicit form of $E(\pi_R, H_m; z)$. \hfill \Box

Remark. (1) We denote by $E_{G_{648}}(z_1, z_2)$ and $E_{G_{1296}}(z_1, z_2)$ the generating systems $E(\pi_R, H_m; z_1, z_2)$ for $G_{648}$ and $G_{1296}$, respectively. Since $G_{648}$ is a subgroup of $G_{1296}$, $E_{G_{648}}(z_1, z_2)$ is represented by $E_{G_{1296}}(z_1, z_2)$ by the algebraic transformation (2.3). In fact, we have

$$E_{G_{648}}(z_1, z_2) = z_2^{-1/6} E_{G_{1296}}(z_2, z_1).$$

(2) For $G = G_{336}$, $G_{648}$, the Schwarz map of (4.5) defined by $(x_1, x_2) \mapsto [u_1(x_1, x_2) : u_2(x_1, x_2) : u_3(x_1, x_2)]$ does not induce an injection to the set of $G$-orbits. For $G = G_{336}$, we have $u(-x_1, -x_2) = \sqrt{-1} u(x_1, x_2)$. Hence, $(x_1, x_2)$ and $(-x_1, -x_2)$ have the common $G$-orbit as the image of the Schwarz map. For $G = G_{648}$, we have $u(x_1, -x_2) = -u(x_1, x_2)$. In this case, $(x_1, x_2)$ and $(x_1, -x_2)$ have the common $G$-orbit as the image of the Schwarz map.

(3) Let $G$ be $G_{336}$. Then the restriction $E(\pi_R, H_m, z_1, z_2)|_{z_2 = -\mu}$ of $E(\pi_R, H_m, z_1, z_2)$ to the line $z_2 = -\mu$ is equal to the third order differential equation (3.9) in [Kato2]. We shall discuss such a restriction more generally in the next section.

4.3. Generating systems for imprimitive reflection groups

In this subsection, we consider the imprimitive reflection groups $G = G(r, p, 3)$ with $r \geq p$. We put $r = pq$. The ring $C[u_1, u_2, u_3]^G$ of $G$-invariant polynomials is generated by

$$F_{2r}(u_1, u_2, u_3) = \sum_{1 \leq i < j \leq 3} u_i^r u_j^r, \quad F_{r}(u_1, u_2, u_3) = \sum_{j=1}^{3} u_j^r, \quad f_3(u_1, u_2, u_3)^q = (u_1 u_2 u_3)^q.$$

First, we give a lemma which is proved in the end of Appendix.
Lemma 4.9. Let \( \varphi(x_1, x_2) \) be a solution of the algebraic equation
\[
T^{3r} - x_2 T^{2r} + x_1 T^r - 1 = 0.
\]
Then, \( \varphi(x_1, x_2) \) satisfies the following system of differential equations:

\[
\begin{align*}
D(x_1, x_2) \frac{\partial^2 \varphi}{\partial x_1^2} + ((4 - 6z)x_1^2 + (z - 1)x_1 x_2^2 - 3(1 - 3z)x_2) \frac{\partial \varphi}{\partial x_1} \\
+ (2zx^3 - (1 + 9z)x_1 x_2 + 9(1 + 3z)) \frac{\partial \varphi}{\partial x_2} + 6z^2(x_2^2 - 3x_1) \varphi = 0,
\end{align*}
\]

\[
\begin{align*}
D(x_1, x_2) \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} - (-zx_1^2 x_2 + (6z - 2)x_2^2 + (6 - 9z)x_1) \frac{\partial \varphi}{\partial x_1} \\
-(zx_1 x_2^2 - (6z + 2)x_1^2 + (6 + 9z)x_2) \frac{\partial \varphi}{\partial x_2} + 3z^2(9 - x_1 x_2) \varphi = 0,
\end{align*}
\]

\[
\begin{align*}
D(x_1, x_2) \frac{\partial^2 \varphi}{\partial x_2^2} + ((4 + 6z)x_2^2 + (-z - 1)x_1^2 x_2 - 3(1 + 3z)x_1) \frac{\partial \varphi}{\partial x_2} \\
+ (-2zx_1^3 - (1 - 9z)x_1 x_2 + 9(1 - 3z)) \frac{\partial \varphi}{\partial x_1} + 6z^2(x_1^2 - 3x_2) \varphi = 0,
\end{align*}
\]

where \( D(x_1, x_2) = 4x_1^3 + 4x_2^3 - x_1^2 x_2^2 - 18x_1 x_2 + 27 \), and \( z = -1/(3r) \).

4.3.1. The case \( p = r \)

We consider the imprimitive groups \( G = G(r, r, 3) \). We put
\[
\begin{align*}
f(u) &= F_2(u), & g(u) &= F_r(u), & h(u) &= f_3(u).
\end{align*}
\]

Note that \( h(u) \) has no multiple factor. Put \( u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2), u_3(x_1, x_2)) = \omega_s^{-1}(x_1, x_2) \) as in (4.2). By definition \( u_j(x_1, x_2) \) are solutions of \( E(x_1, x_2) \) whose monodromy group is \( G \). On the other hand \( u_j(x_1, x_2) \) are solutions of (4.8) because \( u(x_1, x_2) \) satisfies
\[
(4.10) \quad F_2(u(x_1, x_2)) = x_1, \quad F_r(u(x_1, x_2)) = x_2, \quad f_3(u(x_1, x_2)) = 1.
\]

Consequently, the system (4.9) is the system \( E(x_1, x_2) \) in this case.

According to the congruence class of \( r \) modulo 3, we choose a rational \( G \)-quotient map \( \pi_R([u]) = [R_1(u) : R_2(u) : 1] \) and a \( G \)-invariant rational function \( H_m(u) \) as in the following table.

| \( r \) | \( R_1 \) | \( R_2 \) | \( m = |C(G)| \) | \( H_m \) |
|-----|-----|-----|-----|-----|
| \( r = 3r_1 \) | \( F_2/f_3^{2r_1} \) | \( F_r/f_3^{r_1} \) | 3 | \( f_3 \) |
| \( r = 3r_1 + 1 \) | \( F_2/F_r^2 \) | \( F_r^{1/r_1} \) | 1 | \( F_r/F_3^{r_1} \) |
| \( r = 3r_1 - 1 \) | \( F_2/F_r^2 \) | \( F_r^{1/r_1} \) | 1 | \( f_3^{r_1}/F_r \) |

Table 2. \( \pi_R \) and \( H_m \) for imprimitive case with \( p = r \).
Theorem 4.10. Let $E(x_1, x_2)$ be the system (4.9). The generating systems $E(p_R, H_m; z_1, z_2)$ for imprimitive groups $G = G(r, r, 3)$ are obtained from $E(x_1, x_2)$ in the following way:

1. If $r \equiv 0 \pmod{3}$,

$$E(p_R, H_3; z_1, z_2) = E(x_1, x_2)|_{x_1 = z_1, x_2 = z_2}.$$  

2. If $r \equiv 1 \pmod{3}$,

$$E(p_R, H_3; z_1, z_2) = x_2^{-1} E(x_1, x_2)|_{x_1 = z_1^{2/3}, x_2 = z_2^{1/3}},$$

and the explicit form is given by

$$D(z_1, z_2) \frac{\partial^2 \phi}{\partial z_1^2} + z_2^3((12\alpha + 6)z_1^2z_2 - (3\alpha + 1)z_1z_2 - (54\alpha + 18)z_1 + 9\alpha - 3) \frac{\partial \phi}{\partial z_1}$$

$$+ 3z_2^4(-(9\alpha + 1)z_1z_2 + 24z_2 + 27\alpha + 9) \frac{\partial \phi}{\partial z_2}$$

$$- (3\alpha + 1)z_2^3(z_1z_2(6\alpha + 1) - 2z_2\alpha - 9)\phi = 0,$$

$$D(z_1, z_2) \frac{\partial^2 \phi}{\partial z_1 \partial z_2} + z_2((3\alpha + 1)((4z_1^3z_2^2 - z_1^2z_2^2)/3$$

$$- 12z_1^2z_2 + (-2z_2)/3) + 3z_1z_2(5\alpha + 2) - 9) \frac{\partial \phi}{\partial z_1}$$

$$- 3z_2^3((4z_1^2z_2 - z_1z_2)\alpha + 6z_1(-3\alpha - 1) + 3\alpha + 2) \frac{\partial \phi}{\partial z_2}$$

$$+ (3\alpha + 1)z_2^2((-4z_1^2z_2 + z_1z_2)\alpha + 6z_1 + 3\alpha - 2)\phi = 0,$$

$$D(z_1, z_2) \frac{\partial^2 \phi}{\partial z_2^2} - ((3\alpha + 1)(8z_1^3z_2 - 6z_1^2z_2 + z_1z_2) + 6z_1 + 3\alpha - 1) \frac{\partial \phi}{\partial z_1}$$

$$+ z_2((3\alpha - 5)(-4z_1^3z_2^2 + z_1^2z_2^2)/3 + 12z_1^2z_2(3\alpha + 1)$$

$$+ 3z_1z_2(-5\alpha - 11) + 2z_2(3\alpha + 10)/3 + 36) \frac{\partial \phi}{\partial z_2}$$

$$+ (3\alpha + 1)z_2((6\alpha - 1)(-4z_1^3z_2 + z_1^2z_2)/9 + 4z_1^2$$

$$+ z_1(4\alpha - 3) + 2(-3\alpha + 2)/9)\phi = 0,$$
where

\[ D(z_1, z_2) = z_2^2(4z_1^3z_2^2 - z_1^2z_2^2 - 18z_1z_2 + 4z_2 + 27), \quad \alpha = -\frac{1}{3r}. \]

(3) If \( r \equiv 2 \pmod{3}, \)

\[ (4.13) \quad E(\pi_R, H_5; z_1, z_2) = x_2E(x_1, x_2)|_{x_1=z_1z_2^{2/3}, x_2=z_2^{1/3}}, \]

and the explicit form is given by

\[
D(z_1, z_2) \frac{\partial^2 \phi}{\partial z_1^2} + z_2^3(6z_1^2z_2(2x + 1) + z_1z_2(-3x - 1)
+ 18z_1(-3x - 1) + 3(3x - 1) \frac{\partial \phi}{\partial z_1}
+ 3z_2^4(z_1z_2(-9x - 1) + 2z_2x + 9(3x + 1)) \frac{\partial \phi}{\partial z_2}
+ z_2^3(z_1z_2(-18x^2 + 9x + 1) + 2z_2x(3x - 1) + 9(-3x - 1)) \phi = 0,
\]

\[
D(z_1, z_2) \frac{\partial^2 \phi}{\partial z_1 \partial z_2} + z_2 \left( \left( \frac{x - 1}{3} \right) (4z_1^3z_2^2 - z_1^2z_2^2) + 12z_1^2z_2(-3x - 1)
+ 3z_1z_2(5x + 6) - 2z_2 \left( \frac{x - 5}{3} - 27 \right) \frac{\partial \phi}{\partial z_1}
- 3z_2^3(4z_1^2z_2x - z_1z_2x + 6z_1(-3x - 1) + 3x + 2) \frac{\partial \phi}{\partial z_2}
+ z_2^2(x(3x - 1)(-4z_1^2z_2 + z_1z_2) + 6z_1(-3x - 1) + 9x^2 + 3x + 2) \phi = 0,
\]

\[
D(z_1, z_2) \frac{\partial^2 \phi}{\partial z_2^2} + \left( -(3x + 1)z_1z_2(4z_1 - 1)(2z_1 - 1) - 6z_1 - 3x + 1 \right) \frac{\partial \phi}{\partial z_1}
+ z_2^2((3x - 1)(-4z_1^3z_2 + z_1^2z_2)/3 + 12z_1^2(3x + 1) + 3z_1(-5x - 3)
+ 2(3x + 2)/3) \frac{\partial \phi}{\partial z_2}
+ ((3x - 1)(6x + 1)(-4z_1^3z_2^2 + z_1^2z_2^2)
- 2(3x + 2)z_2)/9 - 4z_1^2z_2(3x + 1) + z_1z_2(12x^2 + 5x - 1) + 6) \phi = 0,
\]

where

\[ D(z_1, z_2) = z_2^2(4z_1^3z_2^2 - z_1^2z_2^2 - 18z_1z_2 + 4z_2 + 27), \quad \alpha = -\frac{1}{3r}. \]
Proof. For the case $r = 3r_1$, we have

$$z(x_1, x_2) = \pi_R(u(x_1, x_2)) = \left[ \frac{x_1}{x_2} : x_2^3 : 1 \right], \quad H_3(u(x_1, x_2)) = 1.$$  

For the case $r = 3r_1 + 1$, we have

$$z(x_1, x_2) = \pi_R(u(x_1, x_2)) = \left[ \frac{x_1}{x_2^2} : x_2^3 : 1 \right], \quad H_1(u(x_1, x_2)) = x_2.$$  

For the case $r = 3r_1 - 1$, we have

$$z(x_1, x_2) = \pi_R(u(x_1, x_2)) = \left[ \frac{x_1}{x_2^2} : x_2^2 : 1 \right], \quad H_1(u(x_1, x_2)) = x_2^{-1}.$$  

Then, the assertions in the theorem follow immediately. \hfill \Box

4.3.2. The case $p < r$

We consider the imprimitive groups $G = G(r, p, 3)$ with $r = pq$, $q > 1$. We put

$$f(u) = f_3(u)^q, \quad g(u) = F_2(u), \quad h(u) = F_r(u).$$

Put $u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2), u_3(x_1, x_2)) = \varpi_S^{-1}(x_1, x_2)$ as before. Then, $u_j(x_1, x_2)$ are solutions of the system $E(x_1, x_2)$, whose monodromy group is $G(r, p, 3)$. Put

$$v_j(x_1, x_2) = x_2^{1/j}u_j(x_2^{-3/j}, x_1x_2^{-2}), \quad j = 1, 2, 3.$$  

Then, $v(x_1, x_2) = (v_1(x_1, x_2), v_2(x_1, x_2), v_3(x_1, x_2))$ satisfies (4.10), so that $v_j(x_1, x_2)$, $j = 1, 2, 3$ are solutions of (4.9). Hence, $E(x_1, x_2)$ is given by the following:

\begin{equation}
(4.14) \quad x_1^{2-2p}D(x_1, x_2) \frac{\partial^2 \varphi}{\partial x_1^2} + x_1^{1-p}(9(6xp + p + 3)x_1^{2p}

- 12p(3x + 1)x_1^p x_2^2 - 3(7xp - p + 6)x_1^p x_2

+ 2(3xp + 2)x_1^p + 4(3xp + 1)x_2^3 - x_2^2(3xp + 1)\frac{\partial \varphi}{\partial x_1}$$

+ p^2(-6x_1^p x_2 - (3x - 1)x_1^p - (3x + 1)(8x_2^3 - 6x_2^2 + x_2)\frac{\partial \varphi}{\partial x_2}$$

+ 3xp^2(3x + 1)(3x_1^p - 4x_2^2 + x_2)\varphi = 0,
\end{equation}
\[ x_1^{1-p}D(x_1, x_2) \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} + 3x_1(3x + 1)x_1^p x_2 - (3x + 2)x_1^p \\
+ z_2(-4x_2 + 1) \frac{\partial \varphi}{\partial x_1} + p x_1^p (9(3x + 2)x_1^p + 12(3x + 1)x_2^2) \\
- 3(11x + 4)x_2 + 2(3x + 1) \frac{\partial \varphi}{\partial x_2} + 6zp(3x + 1)x_1^p(3x_2 - 1)\varphi = 0, \\
D(x_1, x_2) \frac{\partial^2 \varphi}{\partial x_2^2} + (3/p)x_1^{p+1}(-9(3x + 1)x_1^p + (9x + 1)x_2 - 2z) \frac{\partial \varphi}{\partial x_1} \\
+ x_1^p(-18(3x + 1)x_1^px_2 + 3(3x - 1)x_1^p + 6(2x + 1)x_2^2) \\
- x_2(3x + 1) \frac{\partial \varphi}{\partial x_2} + 3x(3x + 1)x_1^p(-9x_1^p + x_2)\varphi = 0, \\
\text{where} \\
D(x_1, x_2) = x_1^p(27x_1^{2p} - 18x_1^px_2 + 4x_1^p + 4x_2^3 - x_2^2), \quad z = -\frac{1}{3r}. \\
\]

According to the congruence class of \( p \) modulo 3, we choose a rational \( G \)-quotient map \( \pi_R([u]) = [R_1(u) : R_2(u) : 1] \) and a \( G \)-invariant rational function \( H_m(u) \) as in Table 3.

| \( p \) | \( R_1 \) | \( R_2 \) | \( m = |\mathcal{C}(G)| \) | \( H_m \) |
|-------|------|------|--------|------|
| \( p = 3p_1 \) | \( f_3^{p_1q}/F_r \) | \( F_2^2/F_r^2 \) | \( 3q \) | \( F_r/f_3^{q(p-1)} \) |
| \( p = 3p_1 + 1 \) | \( f_3^{1}/F_r \) | \( F_2^2/F_r^2 \) | \( q \) | \( F_r/f_3^{q} \) |
| \( p = 3p_1 - 1 \) | \( f_3^{1}/F_r \) | \( F_2^2/F_r^2 \) | \( q \) | \( f_3^{p_1q}/F_r \) |

Table 3. \( \pi_R \) and \( H_m \) for imprimitive case with \( p < r \).

**Theorem 4.11.** Let \( E(x_1, x_2) \) be the system (4.14). The generating systems \( E(\pi_R, H_m; z_1, z_2) \) for imprimitive groups \( G = G(r, p, 3) \) with \( p < r \) are obtained from \( E(x_1, x_2) \) in the following way.

1. If \( p \equiv 0 \pmod{3} \),
\[ E(\pi_R, H_{3q}; z_1, z_2) = z_1^{(p_1-1)/r} E(x_1, x_2)|_{x_1 = z_1, x_2 = z_2}. \]
2. If \( p \equiv 1 \pmod{3} \),
\[ E(\pi_R, H_q; z_1, z_2) = z_1^{p_1/r} E(x_1, x_2)|_{x_1 = z_1, x_2 = z_2}. \]

In particular, for \( G = G(r, 1, 3) \), we have
\[ E(\pi_R, H_q; z_1, z_2) = E(x_1, x_2)|_{x_1 = z_1, x_2 = z_2}. \]
(3) If \( p \equiv 2 \pmod{3} \),
\[
E(\pi_R, H_q; z_1, z_2) = z_1^{-p_1/r} E(x_1, x_2) \big|_{x_1 = z_1^{1/p}, x_2 = z_2^r}.
\]

Proof. For the case \( p = 3p_1 \), we have
\[
z(x_1, x_2) = \pi_R(u(x_1, x_2)) = [x_1^{p_1} : x_2 : 1], \quad H_q(u(x_1, x_2)) = x_1^{-p_1 + 1}.
\]

For the case \( p = 3p_1 + 1 \), we have
\[
z(x_1, x_2) = \pi_R(u(x_1, x_2)) = [x_1^p : x_2 : 1], \quad H_q(u(x_1, x_2)) = x_1^{-p_1}.
\]

For the case \( p = 3p_1 - 1 \), we have
\[
z(x_1, x_2) = \pi_R(u(x_1, x_2)) = [x_1^p : x_2 : 1], \quad H_q(u(x_1, x_2)) = x_1^{p_1}.
\]

Then, the assertions in the theorem follow immediately. \( \square \)

5. Restrictions of generating systems to a line

In this section we assume that \( G \) is an irreducible primitive reflection group in \( GL(3, \mathbb{C}) \), or an imprimitive reflection group \( G(r, r, 3) \) or \( G(r, 1, 3) \) with \( r \geq 2 \).

Let \( E_G(z) = E(\pi_R, H_m; z) \) be the generating system for \( G \) defined on \( Z \cong P^2 \) given in the previous section. We prove that the restriction of \( E_G(z) \) to a line \( z_2 = \lambda \) has also the monodromy group \( G \) for generic \( \lambda \).

Let \( u = (u_1, u_2, u_3) \) and \([u] = [u_1 : u_2 : u_3]\) denote the coordinates in \( U_3 \) and \( P(U_3) \), respectively. Let
\[
(5.1) \quad f(u) = F_{d_1}(u), \quad g(u) = F_{d_2}(u), \quad h(u) = F_{d_3}(u)
\]
be the generators of \( G \)-invariant polynomials as in the previous section, and \( J(u) \) the Jacobian defined by (4.1). For each \( G \), we have \( (d_1, d_2) = m \), where \( m = |C(G)| \). Put \( k_1 = d_1/m, \; k_2 = d_2/m \). Then, we have
\[
(k_1, k_2) = 1, \quad \text{and} \quad k_2d_1 = k_1d_2 = d_1d_2/m.
\]

Lemma 5.1. Let \( z = (z_1, z_2, z_3) \) be a common zero of \( g(u) \) and \( h(u) \). If \( z_j \neq 0 \), then we have
\[
(5.2) \quad J(z) = d_3 f(z) \frac{1}{z_j} \left| \begin{array}{cc} \partial g/\partial u_{j+1} & \partial h/\partial u_{j+1} \\ \partial g/\partial u_{j+2} & \partial h/\partial u_{j+2} \end{array} \right|(z),
\]
where \( j + 1 \) and \( j + 2 \) take values in \( \{1, 2, 3\} \) modulo 3.

Proof. Assume \( z_3 \neq 0 \). By Euler’s identity, we have
\[
\frac{\partial F_{d_1}}{\partial u_3} = \frac{1}{u_3} \left( d_1 F_{d_1} - u_1 \frac{\partial F_{d_1}}{\partial u_1} - u_2 \frac{\partial F_{d_1}}{\partial u_2} \right).
\]
Consequently, we have

$$J(u) = \begin{vmatrix}
\partial f/\partial u_1 & \partial f/\partial u_2 & \partial f/\partial u_3 \\
\partial g/\partial u_1 & \partial g/\partial u_2 & \partial g/\partial u_3 \\
\partial h/\partial u_1 & \partial h/\partial u_2 & \partial h/\partial u_3
\end{vmatrix} = \frac{1}{u_3} \begin{vmatrix}
\partial f/\partial u_1 & \partial f/\partial u_2 & \partial f/\partial u_3 \\
\partial g/\partial u_1 & \partial g/\partial u_2 & \partial g/\partial u_3 \\
\partial h/\partial u_1 & \partial h/\partial u_2 & \partial h/\partial u_3
\end{vmatrix}.
$$

This implies the equality (5.2). □

**Lemma 5.2.** Let $I(g, h)$ be the ideal $(g(u), h(u))C[u_1, u_2, u_3]$ of $C[u_1, u_2, u_3]$ generated by $g(u)$ and $h(u)$. If $G$ is one of $G_{120}$, $G_{336}$, $G_{2160}$, then $J(u)^2 - cf(u)^3 \in I(g, h)$ for some $c \in C^\times$. If $G$ is $G_{648}$, then $J(u)^3 - cf(u)^6 \in I(g, h)$ for some $c \in C^\times$. If $G$ is $G_{1296}$, then $J(u)^6 - cf(u)^{11} \in I(g, h)$ for some $c \in C^\times$.

**Proof.** Let $G$ be one of $G_{120}$, $G_{336}$, $G_{2160}$. Let $\tilde{D}(u)$ and $D(z_1, z_2)$ be the polynomials given in Theorem 4.8. From (4.1), we find $\deg \tilde{D}(u) = \deg J(u)^2 = 2(d_1 + d_2 + d_3 - 3)$, which is equal to $3d_3$. Consequently, $D(u) = P_D(f(u), g(u), h(u))$ has the form $\sum_{p_d+q_d+r_d=3d} c_{p,q,r} f(u)^p g(u)^q h(u)^r$. Because the degree of $D(z_1, z_2)$ with respect to $z_1$ is three, the coefficient $c_{3,0,0}$ is not zero. This proves the lemma for $G = G_{120}, G_{336}, G_{2160}$.

For $G = G_{648}$, from Lemma 4.7 we obtain

$$J(u)^3 = 2592^3(F_{12}(u))^6 = 2592^4F_{36}(u)^2 \equiv 5832f(u)^6 \mod I(g, h).$$

For $G = G_{1296}$, we have $J(u)^6 \equiv cf(u)^{11}$ for some $c \in C^\times$ by the same way as above. □

For a homogeneous polynomial $F(u) = F(u_1, u_2, u_3)$, we put

$$V(F) = \{ u \in U_3 \mid F(u) = 0 \}.$$

**Lemma 5.3.** Two curves $V(h)$ and $V(g)$ intersect transversely at $d_1d_2$ (distinct) points.

**Proof.** From Bezout’s theorem, it follows that $|V(h) \cap V(g)| = d_1d_2$ if and only if the two curves intersect transversely at all points.

For $G = G(r, r, 3)$ or $G(r, 1, 3)$, the equality $|V(h) \cap V(g)| = d_1d_2$ is verified directly.

Let $G$ be one of $G_{120}$, $G_{336}$, $G_{648}$, $G_{1296}$, $G_{2160}$. Let $[x] = [x_1 : x_2 : x_3]$ be a point in $V(h) \cap V(g)$. By Corollary 2.2, we have $f(x) \neq 0$, and hence $J(x) \neq 0$ by virtue of Lemma 5.2. Without loss of generality we may assume that $x_3 \neq 0$. Then, $(v_1, v_2) \mapsto [v_1 : v_2 : 1] ((x_1/x_3, x_2/x_3) \mapsto [x])$ defines a local coordinate system at $[x]$. The Jacobian $\partial (g, h)/\partial (v_1, v_2)$ does not vanish at $v_1 = x_1/x_3$, $v_2 = x_2/x_3$ by (5.2). Consequently, the two curves $V(h)$ and $V(g)$ are smooth and intersect transversely at $[x]$. □
Now we quote a theorem from [GH].

**Lemma 5.4** (Bertini’s theorem). Let $\tilde{g}(u), \tilde{h}(u)$ be relatively prime homogeneous polynomials of the same degree. Put $C(\lambda) = V(\tilde{g} - \lambda \tilde{h})$, and $B = V(\tilde{h}) \cap V(\tilde{g})$. Then, there is a finite subset $A$ of $P^1$ such that, if $\lambda \notin A$, then $C(\lambda)$ is smooth outside $B$.

**Proof.** We denote $\partial \tilde{g}/\partial u_i = \tilde{g}_i$, $\partial \tilde{h}/\partial u_i = \tilde{h}_i$. Let $S(\tilde{g}, \tilde{h})$ be the closure of

$$\left\{ [u] \in P(U_3) \backslash B \mid \frac{\partial (\tilde{g}, \tilde{h})}{\partial (u_i, u_j)} = 0, (i, j) = (1, 2), (1, 3), (2, 3) \right\},$$

which is an algebraic set in $P(U_3)$. Note that the equality $\frac{\partial (\tilde{g}, \tilde{h})}{\partial (u_i, u_j)} = 0$ implies

$$\tilde{h}_i \tilde{g}_j = \tilde{h}_j \tilde{g}_i.$$

Let $[x]$ be a point in $S(\tilde{g}, \tilde{h}) \backslash B$. Assume $\tilde{h}(x) \neq 0$. Put $\tilde{d} = \text{deg} \tilde{h} = \text{deg} \tilde{g}$.

For each $i$ we have

$$\frac{\partial}{\partial u_i} \left( \frac{\tilde{g}}{\tilde{h}} \right)(x) = \frac{\tilde{g}_i \tilde{h} - \tilde{h}_i \tilde{g}}{\tilde{h}^2} = \frac{1}{\tilde{h}^2} \left( \tilde{g}_i \left( \sum_j u_j \tilde{h}_j \right) / \tilde{d} - \tilde{h} \tilde{g} \right)$$

$$= \frac{1}{\tilde{h}^2} \left( \tilde{h}_i \left( \sum_j u_j \tilde{g}_j \right) / \tilde{d} - \tilde{h} \tilde{g} \right) = \frac{\tilde{h}_i \tilde{g} - \tilde{h} \tilde{g}}{\tilde{h}^2} = 0.$$

This implies that $\tilde{g}/\tilde{h}$ is constant on each irreducible component of $S(\tilde{g}, \tilde{h})$. Let $A$ be the set of values of $\tilde{g}/\tilde{h}$ on $S(\tilde{g}, \tilde{h})$.

If $[x] (\notin B)$ is a singular point of $C(\lambda)$, we have $\tilde{g}_i(x) = \lambda \tilde{h}_i(x)$, $i = 1, 2, 3$, which implies $\frac{\partial (\tilde{g}, \tilde{h})}{\partial (u_i, u_j)}(x) = 0$. Then, $\lambda = \tilde{g}(x)/\tilde{h}(x) \in A$. Consequently, if $\lambda \notin A$, then $C(\lambda)$ is smooth outside $B$. \hfill $\square$

Recall $k_1 = d_1/m$, $k_2 = d_2/m$, where $m$ is the order of the center $C(G)$ of $G$.

**Lemma 5.5.** Let $A$ be the finite subset of $P^1$ in Lemma 5.4 for $\tilde{g}(u) = g(u)^{k_1}$ and $\tilde{h}(u) = h(u)^{k_2}$. Then, the following holds.

1. $\infty \in A$. If $k_1 > 1$, then $0 \in A$. If $k_1 = 1$ and $V(g)$ has a singular point outside $B$ ($= V(h) \cap V(g)$), then $0 \in A$.

2. If $\lambda \notin A$, then $C(\lambda) = V(g^{k_1} - \lambda h^{k_2})$ is irreducible.

3. If $\lambda^{k_1} \notin A$, then the affine curve $C_{\text{affine}}(\lambda) = \{ u \in U_3 \mid F_{d_1}(u) = 1, F_{d_2}(u) = \lambda \}$ is smooth and connected.

**Proof.** Proof of (1): Since $k_2 > 1$, all the points of $V(h)$ are singular points of $V(\tilde{h})$. In particular, $C(\infty)$ has a singular point outside $B$. Hence, $\infty \in A$ by Lemma 5.4. Similarly $0 \in A$ if the conditions in (1) hold.
Proof of (2): Assume $\lambda \notin \Lambda$, and take a point $[x] \in C(\lambda) \cap B$. Assume $x_3 \neq 0$, for example. By Lemma 5.3, $[u] \mapsto (g(u_1/u_3, u_2/u_3, 1), h(u_1/u_3, u_2/u_3, 1))$ is biholomorphic at $[x]$. This implies that $C(\lambda)$ has $(k_1, k_2)$-cusp or $(1, k_2)$-flex at $[x]$ according as $k_1 \geq 2$ or $k_1 = 1$. Hence, $C(\lambda)$ is locally irreducible at $[x]$. By Lemma 5.4, $C(\lambda)$ is smooth, in particular, locally irreducible at every point of $C(\lambda) \setminus B$. Hence, $C(\lambda)$ is locally irreducible at all points. This implies that $C(\lambda)$ itself is irreducible.

Proof of (3): Assume $\lambda^{k_1} \notin \Lambda$. Then, by Lemma 5.4, we see that $C(\lambda^{k_1}) \setminus B$ is smooth. The map $\pi$ of $C^\text{affine}(\lambda)$ to $C(\lambda^{k_1}) \setminus B$ defined by $u \mapsto [u]$ is an $m : 1$ covering map because $\pi(u) = \pi(u')$ if and only if $u' = cu$ for some $m$-th root $c$ of unity. Consequently, $C^\text{affine}(\lambda)$ is smooth.

Take a point $[x] \in C(\lambda^{k_1}) \cap B$, and assume $x_3 \neq 0$. Then, as stated in the proof of (2), the map $\varphi : [v_1 : v_2 : 1] \mapsto (g(v_1, v_1, 1), h(v_1, v_1, 1))$ is biholomorphic in a neighborhood of $[x]$. We have

$$[v_1 : v_2 : 1] \in C(\lambda^{k_1}) \iff \varphi([v_1 : v_2 : 1]) = (\lambda^{k_2}, t^{k_1})$$

for some $t$. Put $[v_1(t) : v_2(t) : 1] = \varphi^{-1}(\lambda^{k_2}, t^{k_1})$. Put

$$u_j(t) = t^{-k_1/d_1} u_j(t) \quad (j = 1, 2), \quad u_3(t) = t^{-k_1/d_3}.$$

Then, $u(t) = (u_1(t), u_2(t), u_3(t)) \in C^\text{affine}(\lambda)$. Because $k_1/d_1 = 1/m$, any connected component of $C^\text{affine}(\lambda)$ is invariant under the multiplications by $m$-th roots of unity. Since $C(\lambda^{k_1})$ is irreducible, any connected component of $C^\text{affine}(\lambda)$ is mapped to $C(\lambda^{k_1}) \setminus B$ surjectively by $\pi$. This concludes that $C^\text{affine}(\lambda)$ is connected.

The proof of (3) of Lemma 5.5 implies the following Lemma.

**Lemma 5.6.** Let the notation be the same as the previous lemma. Assume $C(\lambda^{k_1})$ is reduced and irreducible, that is, the set of smooth points of $C(\lambda^{k_1})$ is non-empty and connected. Then, the set of the smooth points of $C^\text{affine}(\lambda)$ is non-empty and connected.

**Theorem 5.7.** Let $G$ be a finite irreducible primitive reflection group in $\text{GL}(3, \mathbb{C})$, or an imprimitive reflection group $G(r, r, 3)$ or $G(r, 1, 3)$ with $r \geq 2$.

1. Let $E(x_1, x_2)$ be the system for $G$ determined by $f, g, h$ in (5.1) (cf. Section 4.1). If $C(\lambda^{k_1})$ is reduced and irreducible, then the restriction $E(x_1, x_2)|_{x_2 = \lambda}$ of $E(x_1, x_2)$ to the line $x_2 = \lambda$ has the monodromy group $G$.

2. Let $E(\pi_R, H_{m_1}; z_1, z_2)$ be the generating system for $G$. Then, the restriction $E(\pi_R, H_{m_1}; z_1, z_2)|_{z_2 = \lambda}$ of $E(\pi_R, H_{m_1}; z_1, z_2)$ to a line $z_2 = \lambda$ has the monodromy group $G$ except for a finite number of $\lambda$.

**Proof.** Proof of (1): Recall $S = \{u \in U_3 \mid h(u) = 1\}$ and $\varpi_S(u) = [f(u) : g(u) : 1]$ is a $|G| : 1$ map of $S$ to $X = \{[x_1 : x_2 : 1]\}$. For any $[x : \lambda : 1] \neq (x, \lambda, 1)$len
$\varpi_S(V(J))$, $\varpi_S^{-1}([x : \lambda : 1])$ is a $G$-orbit with $|\varpi_S^{-1}([x : \lambda : 1])| = |G|$ by Lemma 4.3. If $C(\lambda^k)$ is reduced and irreducible, then, thanks to Lemma 5.6, the group of the covering transformations of $\varpi_S(u)|_{C\text{finite}(\lambda)}$ is $G$. This proves (1).

Proof of (2): From the proof of Theorem 4.8 for primitive $G$, and from (4.11), (4.12), (4.13) and (4.16) for imprimitive $G$, we obtain

$$E(\pi_R, H_m; z_1, z_2)|_{z_2 = \lambda} = E(x_1, x_2)|_{x_1 = ax_1, x_2 = \lambda'},$$

for some $a \in \mathbb{C}$ and some $\lambda'$. This proves (2).

Corollary 5.8. Let $G$ be as in Theorem 5.7. Then, there are three complex reflections $R_j$, $j = 1, 2, 3$ which generate $G$, such that the product $R_1R_2R_3$ has the eigenvalues $e((d_j - 1)/d_3)$, $j = 1, 2, 3$.

For $G = G_{648}$, all $R_j$ are of order three. For $G = G_{1296}$, one of $R_j$ is of order two and others are of order three. For $G = G(r, 1, 3)$, one of $R_j$ is of order $r$ and others are of order two. For any other $G$, all $R_j$ are of order two.

Proof. Let $E(\pi_R, H_m; z_1, z_2)$ be the generating system for $G$. Theorem 5.7 asserts that $G$ is the monodromy group of $E(\pi_R, H_m; z_1, z_2)|_{z_2 = \lambda}$ for generic $\lambda$. Since we know the explicit form of $E(\pi_R, H_m; z_1, z_2)$, we can calculate the local exponents of the ordinary differential equation $E(\pi_R, H_m; z_1, z_2)|_{z_2 = \lambda}$ at all singular points, which shows the assertion.

6. Examples

6.1. Lamé equation with finite monodromy group $G_{48}$

Lamé equation is a second order differential equation of the form

$$p(x)\frac{d^2u}{dx^2} + \frac{1}{2}p'(x)\frac{du}{dx} - (n(n + 1)x + B)u = 0,$$

where $n, B \in \mathbb{C}$, and

$$p(x) = 4x^3 - g_2x - g_3 \quad (g_2, g_3 \in \mathbb{C})$$

is a polynomial having three distinct zeros. Beukers and van der Waall [BW] gave a list of Lamé equations with finite monodromy groups. As is mentioned in the article, all such equations can be expressed by using Gauss hypergeometric differential equation, however, there is no explicit expression in it.

By applying our results, we can obtain such expression in a constructive way. As an example, here we give an expression for Lamé equation $E'(x)$ in the second row in Table 4 in [BW], whose monodromy group is $G_{48}$ of Shephard-Todd number 12. The equation $E'(x)$ is given by (6.1) with $n = 3/4$, $g_2 = -168$, $g_3 = 622$ and $B = 3/8$. The local exponents at three zeros of $p(x)$ are $\{0, 1/2\}$, and those at $x = \infty$ are $\{-3/8, 7/8\}$.
Let $E(z) = E(\pi_R, H_2; z)$ be the generating system for $G_{48}$. As we have shown in the proof of Theorem 3.1, it is determined by

$$\pi_R(u_1, u_2) = -\frac{F_8(u_1, u_2)^3}{F_6(u_1, u_2)^4}, \quad H_2(u_1, u_2) = \frac{F_8(u_1, u_2)}{F_6(u_1, u_2)},$$

where $F_8(u_1, u_2)$ and $F_6(u_1, u_2)$ are generators of $G_{48}$-invariant polynomials satisfying

$$F_6(u_1, u_2)^4 + F_8(u_1, u_2)^3 = F_{12}(u_1, u_2)^2$$

with a semi invariant polynomial $F_{12}(u_1, u_2)$ of degree 12.

**Proposition 6.1.** The Lamé equation $E'(x)$ in Table 4-II in [BW] with the monodromy group $G_{48}$ can be expressed by an algebraic transformation of Gauss hypergeometric differential equation as

$$E'(x) = (6x - 11)^{1/6} E_1 \left( -\frac{1}{24} \frac{5}{24} \frac{1}{3}, \frac{2}{6} \frac{4x^3 - 12x^2 + 57x - 274}{729(6x - 11)^4} \right).$$

**Proof.** The relation we are going to show is the formula (2.3) in Theorem 2.9, where $m = |C(G_{48})| = 2$ and $E(\pi_R, H_m; z)$ is the generating differential equation for $G_{48}$ given as No. 12 in Theorem 3.1, (2). Then, it is enough to determine $\theta(x)$ and $z(x)$ defined by (2.4).

Let $u_1(x), u_2(x)$ be the solutions of $E'(x)$ such that the monodromy group with respect to $(u_1(x), u_2(x))$ is $G_{48}$. Then, $F_6(u_1(x), u_2(x))$ and $F_8(u_1(x), u_2(x))$ are single-valued function of $x$. Let $v_1(x), v_2(x)$ be solutions of $E'(x)$ at $x = \infty$ such that

$$v_1(x) = x^{3/8}(1 + a_1 x^{-1} + a_2 x^{-2} + \cdots),$$

$$v_2(x) = x^{-7/8}(1 + b_1 x^{-1} + b_2 x^{-2} + \cdots).$$

By using (6.1), we have

$$a_1 = -\frac{3}{8}, \quad a_2 = \frac{165}{128}, \quad a_3 = -\frac{4925}{1024}, \quad b_1 = \frac{1}{24}, \ldots$$

Since $u_j(x)$ are linear combinations of $v_1(x)$ and $v_2(x)$, we have

$$F_6(u_1(x), u_2(x)) = G_6(v_1(x), v_2(x)), \quad F_8(u_1(x), u_2(x)) = G_8(v_1(x), v_2(x)),$$

for some homogeneous polynomials $G_6(v_1, v_2)$ and $G_8(v_1, v_2)$ in $(v_1, v_2)$. Since $G_j(v_1(x), v_2(x)), j = 6, 8$ are single-valued, the polynomials $G_j(v_1, v_2)$ are of the form

$$G_6(v_1, v_2) = p_0 v_1^5 v_2 + p_1 v_1 v_2^5, \quad G_8(v_1, v_2) = q_0 v_1^8 + q_1 v_1^4 v_2^4 + q_2 v_2^8.$$
Since $E'(x)$ has a negative exponent ($= -3/8$) only at $x = \infty$, $G_6(v_1(x), v_2(x))$ has a pole only at $x = \infty$ of order at most one, and $G_6(v_1(x), v_2(x))$ has a pole only at $x = \infty$ of order at most three. This means that $G_6(v_1(x), v_2(x))$ is a polynomial in $x$ of degree at most one, and $G_6(v_1(x), v_2(x))$ is that of degree at most three. Substituting (6.2) in (6.3), we have

$$G_6(v_1(x), v_2(x)) = p_0 \left( x - \frac{11}{6} \right), \quad G_8(v_1(x), v_2(x)) = q_0 \left( x^3 - 3x^2 + \frac{57}{4}x - \frac{137}{2} \right).$$

In order that $1 + G_8(v_1(x), v_2(x))^3 / G_6(v_1(x), v_2(x))^4$ has multiple zeros, we must have

$$z(x) = -\frac{F_8(u_1(x), u_2(x))^3}{F_6(u_1(x), u_2(x))^4} = \frac{-2(4x^3 - 12x^2 + 57x - 274)^3}{729(6x - 11)^4} = 1 - \frac{(2x^3 + 84x - 311)(8x^3 - 36x^2 + 30x + 313)^2}{729(6x - 11)^4}.$$

By (2.4), we have $\theta(x) = F_8(u_1(x), u_2(x))/F_6(u_1(x), u_2(x))$. By Theorem 3.1, we have $E(\pi_R, H_2; z) = z^{-1/6} E_1(-1/24, 5/25; 2/3; z)$. Consequently, we have

$$E'(x) = \theta(x)^{1/2} E(\pi_R, H_2; z(x)) = (6x - 11)^{1/6} E_1 \left( -\frac{1}{24}, \frac{5}{24}; \frac{2}{3}; z(x) \right).$$

6.2. Boulanger’s system of differential equations for $G_{1296}$

A system of differential equations of the form (4.5) is called canonical if the coefficients satisfy

$$p_{11}^1 + p_{12}^2 = 0, \quad p_{12}^2 + p_{12}^1 = 0.$$

Boulanger [B1] obtained a canonical system of differential equations $E_B(z_1, z_2)$ whose projective monodromy group is $P(G_{1296})$. Thanks to the generating property, the system can be expressed by using our generating system for $G_{1296}$. In this subsection we give the explicit correspondence. Moreover, we give a general way to transform generating systems for finite irreducible primitive complex reflection groups $G$ in $GL(3, \mathbb{C})$ into canonical ones.

**Proposition 6.2.** Let $E(z)$ be a system on $Z = \{ [z] = [z_1 : z_2 : 1] \} \simeq \mathbb{P}^2$ of the form

$$\frac{\partial^2 \phi}{\partial z_i \partial z_j} + p_{ij}^1 \frac{\partial \phi}{\partial z_i} + p_{ij}^2 \frac{\partial \phi}{\partial z_j} + p_{ij}^0 \phi = 0, \quad 1 \leq i \leq j \leq 2,$$
and \( \phi_j(z_1, z_2), \ j = 1, 2, 3 \) be its linearly independent solutions. Put

\[
W_{\psi}(z_1, z_2) = \begin{vmatrix}
\varphi_1 & \varphi_2 & \varphi_3 \\
\partial \varphi_1 / \partial z_1 & \partial \varphi_2 / \partial z_1 & \partial \varphi_3 / \partial z_1 \\
\partial \varphi_1 / \partial z_2 & \partial \varphi_2 / \partial z_2 & \partial \varphi_3 / \partial z_2 \\
\end{vmatrix}.
\]

Then, the system \( W_{\psi}^{-1/3} E(z) \) satisfied by \( W_{\psi}(z_1, z_2)^{-1/3} \varphi_j(z_1, z_2) \) is canonical.

**Proof.** For a multi-valued function \( \theta(z_1, z_2), \psi_j(z_1, z_2) := \theta(z_1, z_2)^{-1} \times \varphi_j(z_1, z_2) \) are solutions of

\[
\begin{align*}
&\frac{\partial^2 \psi}{\partial z_1^2} + \left( p_{11}^1 + 2 \frac{\partial \log \theta}{\partial z_1} \right) \frac{\partial \psi}{\partial z_1} + p_{11}^2 \frac{\partial \varphi}{\partial z_2} + q_{11}^0 \psi = 0, \\
&\frac{\partial^2 \psi}{\partial z_1 \partial z_2} + \left( p_{12}^1 + \frac{\partial \log \theta}{\partial z_2} \right) \frac{\partial \psi}{\partial z_2} + \left( p_{12}^2 + \frac{\partial \log \theta}{\partial z_1} \right) \frac{\partial \varphi}{\partial z_1} + q_{12}^0 \psi = 0, \\
&\frac{\partial^2 \psi}{\partial z_2^2} + p_{22}^1 \frac{\partial \varphi}{\partial z_2} + \left( p_{22}^2 + 2 \frac{\partial \log \theta}{\partial z_2} \right) \frac{\partial \psi}{\partial z_2} + q_{22}^0 \psi = 0,
\end{align*}
\]

for some functions \( q_{ij}^0(z_1, z_2) \). This system is canonical if and only if

\[
p_{11}^1 + p_{12}^2 + 3 \frac{\partial \log \theta}{\partial z_1} = 0, \quad p_{22}^1 + p_{12}^2 + 3 \frac{\partial \log \theta}{\partial z_2} = 0.
\]

On the other hand, we have

\[
(p_{11}^1 + p_{12}^2) W_{\psi} = \begin{vmatrix}
\varphi_1 & (\varphi_1)_{z_2} & (\varphi_1)_{z_1 z_1} \\
\varphi_2 & (\varphi_2)_{z_2} & (\varphi_2)_{z_1 z_1} \\
\varphi_3 & (\varphi_3)_{z_2} & (\varphi_3)_{z_1 z_1} \\
\end{vmatrix} - \begin{vmatrix}
\varphi_1 & (\varphi_1)_{z_1} & (\varphi_1)_{z_1 z_2} \\
\varphi_2 & (\varphi_2)_{z_1} & (\varphi_2)_{z_1 z_2} \\
\varphi_3 & (\varphi_3)_{z_1} & (\varphi_3)_{z_1 z_2} \\
\end{vmatrix}
= - \frac{\partial}{\partial z_1} \begin{vmatrix}
\varphi_1 & (\varphi_1)_{z_1} & (\varphi_1)_{z_1 z_2} \\
\varphi_2 & (\varphi_2)_{z_1} & (\varphi_2)_{z_1 z_2} \\
\varphi_3 & (\varphi_3)_{z_1} & (\varphi_3)_{z_1 z_2} \\
\end{vmatrix}
= - \frac{\partial W_{\psi}}{\partial z_1},
\]

which can be written as

\[
p_{11}^1 + p_{12}^2 = - \frac{\partial}{\partial z_1} \log W_{\psi}.
\]

Similarly we have

\[
p_{22}^1 + p_{12}^2 = - \frac{\partial}{\partial z_2} \log W_{\psi},
\]

and then we get the assertion. \( \square \)
Proposition 6.3. Let $E(\pi_R, H_m; z)$ be the generating system for $G$. Let $\phi_j(z_1, z_2)$, $j = 1, 2, 3$ be solutions of $E(\pi_R, H_m; z)$ satisfying (2.1). Let $D(z_1, z_2)$ be given in Theorem 4.8 for each $G$. Then, $W_\phi(z_1, z_2)$ are given as follows.

- $G = G_{120}$, $W_\phi = D^{-1/2}$,
- $G = G_{336}$, $W_\phi = D^{-1/2}$,
- $G = G_{648}$, $W_\phi = z_2^{-1}(4z_1^3 - (1 - 3z_1 - 432z_2)^2)^{-2/3}$,
- $G = G_{1296}$, $W_\phi = z_1^{-1/2}(4z_2^3 - (1 - 3z_2 - 432z_1)^2)^{-2/3}$,
- $G = G_{2160}$, $W_\phi = D^{-1/2}$.

Proof. In all cases, we have $H_m(\phi(z)) = 1$ and $R_j(\phi(z)) = z_j$, $j = 1, 2$. The differentiations of these by $z_1$, $z_2$, and Euler’s identities for $R_j(u)$ and $H_m(u)$ give

\[
\begin{bmatrix}
\varphi_1 & \varphi_2 & \varphi_3 \\
\partial\varphi_1/\partial z_1 & \partial\varphi_2/\partial z_1 & \partial\varphi_3/\partial z_1 \\
\partial\varphi_1/\partial z_2 & \partial\varphi_2/\partial z_2 & \partial\varphi_3/\partial z_2 \\
\end{bmatrix}
\begin{bmatrix}
\partial R_1/\partial u_1 & \partial R_2/\partial u_1 & \partial H_m/\partial u_1 \\
\partial R_1/\partial u_2 & \partial R_2/\partial u_2 & \partial H_m/\partial u_2 \\
\partial R_1/\partial u_3 & \partial R_2/\partial u_3 & \partial H_m/\partial u_3 \\
\end{bmatrix}
= \begin{bmatrix} 0 & 0 & mH_m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = m,
\]

which implies $W_\phi = m[\partial(R_1, R_2, H_m)/\partial(u_1, u_2, u_3)]^{-1}$.

Put $R_1 = f/h^k g^{k_1}$, $R_2 = g^{k_2}/h^{k_1}$, $H_m = g^{m_2}/h^{m_1}$, where $f = F_{d_1}$, $g = F_{d_2}$, $h = F_{d_3}$, as before. Form Table 1 we read $(m_1, m_2) = (1, 1)$ for $G = G_{120}, G_{1296}, G_{2160}$, and $(m_1, m_2) = (1, 1)$ for $G = G_{336}, G_{648}$. Then, we have

\[
\frac{\partial(R_1, R_2, H_m)}{\partial(u_1, u_2, u_3)} = \frac{\partial(R_1, R_2, H_m)}{\partial(f, g, h)} \frac{\partial(f, g, h)}{\partial(u_1, u_2, u_3)}
= \frac{R_1 R_2 H_m}{f g h} J(u_1, u_2, u_3)
= (m_2 k_1 - m_1 k_2) \frac{R_1 R_2 H_m}{f g h} J(u_1, u_2, u_3),
\]

where $m_2 k_1 - m_1 k_2$ does not vanish for any $G$.

In the following of this proof, we use the notation $A \equiv B$ when $A/B$ is a non-zero constant.

For $G = G_{120}$ or $G_{2160}$, we have

\[
\frac{\partial(R_1, R_2, H_m)}{\partial(u_1, u_2, u_3)} \equiv h^{-k_1-l_1} J = [h^{-2k_1-2l_1} J^2]^{1/2},
\]
and hence

\[
\frac{\partial (R_1, R_2, H_m)}{\partial (u_1, u_2, u_3)} \bigg|_{u_j = \phi_j(z_1, z_2)} = D(z_1, z_2)^{1/2}
\]

by Theorem 4.8.

For \( G = G_{1296} \), we have

\[
\frac{\partial (R_1, R_2, H_m)}{\partial (u_1, u_2, u_3)} \equiv h^{-k_1 - l_1} J = \left[ h^{-6k_1 - 6l_1} F_9 J \right]^{1/6}
\]

\[
= \left[ h^{-6k_1 - 6l_1} F_9^{12} (F_{12}')^{1/6} \right]^{1/6} = \left[ h^{-6k_1 - 6l_1} F_9 F_{36}^4 \right]^{1/6}
\]

by using (3) of Lemma 4.7 and the equation (4.6). Consequently, we have

\[
\frac{\partial (R_1, R_2, H_m)}{\partial (u_1, u_2, u_3)} \bigg|_{u_j = \phi_j(z_1, z_2)} = z_1^{1/2} (4z_2^3 - (1 - 3z_2 - 432z_1)^2)^{2/3}.
\]

For \( G = G_{336} \), we have

\[
\frac{\partial (R_1, R_2, H_m)}{\partial (u_1, u_2, u_3)} = \frac{g^2}{h^{14}} J = \left[ \frac{g^2}{h^{14}} J^2 \right]^{1/2}
\]

\[
= \left[ \frac{g^2}{h^{14}} \left( f^3 - 88f^2 gh^2 + 1008f^4 h + \ldots - 2048gh^9 \right) \right]^{1/2},
\]

and hence

\[
\frac{\partial (R_1, R_2, H_m)}{\partial (u_1, u_2, u_3)} \bigg|_{u_j = \phi_j(z_1, z_2)} = D(z_1, z_2)^{1/2}.
\]

For \( G = G_{648} \), we have

\[
\frac{\partial (R_1, R_2, H_m)}{\partial (u_1, u_2, u_3)} = \frac{g^2}{h^{17}} J = \left[ \frac{g^2}{h^{21}} J^3 \right]^{1/3}
\]

\[
= \left[ \frac{g^2}{h^{21}} (F_{12}')^6 \right]^{1/3} = \left[ \frac{g^6}{h^{21}} F_{36}^2 \right]^{1/3}
\]

by using (3) of Lemma 4.7 and the equation (4.6). Consequently, we have

\[
\frac{\partial (R_1, R_2, H_m)}{\partial (u_1, u_2, u_3)} \bigg|_{u_j = \phi_j(z_1, z_2)} = z_1 (4z_1^3 - (1 - 3z_1 - 432z_2)^2)^{2/3}.
\]

This completes the proof. \( \Box \)

As a particular case, we can express Boulanger’s system in terms of our generating system.
6.3. Jordan-Pochhammer equation with monodromy $G(3,3,3)$ and Appell’s $F_1$

Let $E_{G(3,3,3)}(x_1, x_2)$ be the generating system $E(\pi_R, H_m; x_1, x_2)$ for $G_{1296}$ given in Theorem 4.8. Then, Boulanger’s system $E_B(z_1, z_2)$ can be written by using $E_{G(3,3,3)}(z_1, z_2)$ as

$$E_B(z_1, z_2) = z_1^{1/6}(4z_2^3 - (1 - 3z_2 - 432z_1)^2)^{2/9}E_{G(3,3,3)}(z_1, z_2).$$

Corollary 6.4. Let $E_{G_{1296}}(z_1, z_2)$ be the generating system $E(\pi_R, H_m; z_1, z_2)$ for $G_{1296}$ given in Theorem 4.8. Then, Boulanger’s system $E_B(z_1, z_2)$ can be written by using $E_{G_{1296}}(z_1, z_2)$ as

$$E_B(z_1, z_2) = z_1^{1/6}(4z_2^3 - (1 - 3z_2 - 432z_1)^2)^{2/9}E_{G_{1296}}(z_1, z_2).$$

6.3. Jordan-Pochhammer equation with monodromy $G(3,3,3)$ and Appell’s $F_1$

Let $E_{G(3,3,3)}(x_1, x_2)$ be the generating system $E(\pi_R, H_m; x_1, x_2)$ for $G = G(3,3,3)$ given in Theorem 4.10, namely the system (4.9) with $a = -1/9$. As Theorem 5.7 shows, the restriction of $E_{G(3,3,3)}(x_1, x_2)$ to a line $x_2 = x_2^0$ becomes an ordinary differential equation with the same monodromy group $G(3,3,3)$. This ordinary differential equation is of rank 3, and it has four singular points with the multiplicity $(2,1)$ of local exponents at all the singular points. Since there is no logarithmic solution, the system is physically rigid in the sense of Katz [Katz], and hence coincides with some Jordan-Pochhammer equation of rank 3.

It is known that, if we normalize three singular points of the four singular points of a Jordan-Pochhammer equation of rank 3 to 0, 1, $\infty$, the equation can be prolonged to Appell’s hypergeometric system $E_1(a; b, b'; c; y_1, y_2)$ satisfied by the hypergeometric series

$$F_1(a; b, b'; c; y_1, y_2) = \sum_{m,n \geq 0} \frac{(a, m+n)(b, m)(b', n)}{(c, m+n)(1, m)(1, n)} y_1^m y_2^n$$

with some parameters $a$, $b$, $b'$, $c$, where $(a, n) = \Gamma(a+n)/\Gamma(a)$. Sasaki [S] determined the list of the systems $E_1(a; b, b'; c; y_1, y_2)$ which have finite irreducible monodromies. The case 2 in his list in [S, Theorem 2.5] corresponds to the case when the restriction to a line $y_2 = y_2^0$ becomes a Jordan-Pochhammer equation with the monodromy group $G(3,3,3)$. We see that the system $E_1(-1/6; 1/3, 1/6; 2/3; y_1, y_2)$, which we denote simply by $E_1(y_1, y_2)$, realizes the case. It is shown in [S] that the monodromy group of $E_1(y_1, y_2)$ is $G_{1296}$. (We remark that Cohen and Wolfart [CW] also determined the list of $E_1$ with finite monodromies in a way different from Sasaki’s work.)

Thus, the Jordan-Pochhammer equation obtained by the restriction of $E_{G(3,3,3)}$ has two prolongations $E_{G(3,3,3)}$ and $E_1(y_1, y_2)$, which are different because they have different monodromy groups. We are interested in the relation between these two prolongations.

First, we note that $G(3,3,3)$ is a subgroup of $G_{1296}$. 
Lemma 6.5. We have
\[ G(3, 3, 3) \subseteq G(3, 1, 3) \subseteq G_{1296}. \]

Proof. If \( p' \) is a multiple of \( p \), then \( G(r, p', n) \) is a subgroup of \( G(r, p, n) \). Thus, we have \( G(3, 3, 3) \subseteq G(3, 1, 3) \).

Let \( F_6, F_9, F_{12} \) denote the invariant polynomials of \( G_{1296} \) given in Section 4.2, and \( F_{2r}, F_r, f^g \) the invariant polynomials of \( G(r, p, 3) \) given in Section 4.3. When \( r = 3 \), we have
\[
\begin{align*}
F_6(u) &= F_r(u)^2 - 12F_{2r}(u), \\
F_{12}(u) &= F_r(u)(F_r(u)^2 + 216f^3(u)), \\
F_9(u)^2 &= F_r(u)^2F_{2r}(u)^2 - 4F_r(u)^3f^3(u) - 4F_{2r}(u)^3 \\
&\quad + 18F_r(u)F_{2r}(u)f^3(u) - 27f^3(u)^2,
\end{align*}
\]
which proves \( G(3, 1, 3) \subseteq G_{1296} \).

Then, by Theorem 2.9, \( E_{G(3, 3, 3)}(x_1, x_2) \) can be written as an algebraic transformation of a generating system for \( G_{1296} \). Let \( E_{G_{1296}}(z_1, z_2) \) be the generating system \( E(\pi_R, H_{n}; z_1, z_2) \) for \( G_{1296} \) given in Theorem 4.8, (4).

Proposition 6.6. We have
\[
E_{G(3, 3, 3)}(x_1, x_2) = (x_2^2 - 12x_1)^{1/6}E_{G_{1296}}(\sigma_1(x_1, x_2), \sigma_2(x_1, x_2)),
\]
where
\[
\begin{align*}
\sigma_1(x_1, x_2) &= \frac{4x_1^3 + 4x_2^3 - x_1^2x_2^2 - 18x_1x_2 + 27}{(12x_1 - x_2^2)^3}, \\
\sigma_2(x_1, x_2) &= \frac{x_2(x_2^3 + 216)}{(12x_1 - x_2^2)^2}.
\end{align*}
\]

Proof. The equalities (2.4), (4.10), (6.4) imply (6.5) and (6.6).

Since the monodromy group for \( E_1(y_1, y_2) \) is \( G_{1296} \), it can also be written as an algebraic transformation of \( E_{G_{1296}}(z_1, z_2) \).

Proposition 6.7. We have
\[
E_1(y_1, y_2) = (3y_1^3 - y_2 - 1)^{1/6}E_{G_{1296}}(\tau_1(y_1, y_2), \tau_2(y_1, y_2)),
\]
where
\[
\begin{align*}
\tau_1(y_1, y_2) &= \frac{y_1(y_1 - 1)(y_1 - y_2)}{16(3y_1 - y_2 - 1)^3}, \\
\tau_2(y_1, y_2) &= \frac{y_2^2 - y_2 + 1}{(3y_1 - y_2 - 1)^2}.
\end{align*}
\]
Proof. Appell’s hypergeometric series $F_2$ is defined by

$$F_2(a; b, b'; c, c'; y_1, y_2) = \sum_{m,n \geq 0} \frac{(a, m + n) (b, m) (b', n)}{(c, m) (c', m) (1, m) (1, n)} y_1^m y_2^n.$$  

It is known (see [Kato1]) that, at $(y_1, y_2) = (1, 0)$, the hypergeometric system $E_1(a; b, b'; c; y_1, y_2)$ has three linearly independent solutions

$$F_2(a; b, b'; 1 + a + b - c, c - b; 1 - y_1, y_2),$$

$$(1 - y_1)^{c-a-b} F_2(c - b; c - a, b'; 1 - a - b + c, c - b; 1 - y_1, y_2),$$

$$y_2^{1+b-c} F_2(1 + a + b - c; b, 1 + b + b' - c; 1 + a + b - c, 2 + b - c; 1 - y_1, y_2).$$

Thus, the system $E_1(y_1, y_2) = E_1(-1/6; 1/3, 1/6; 2/3; y_1, y_2)$ has the solutions

$$f_1(y_1, y_2) = F_2\left(-\frac{1}{6}; \frac{1}{3}, \frac{1}{3}; 1 - y_1, y_2\right),$$

$$f_2(y_1, y_2) = (1 - y_1)^{1/2} F_2\left(\frac{1}{3}; \frac{5}{6}, \frac{1}{3}; 1 - y_1, y_2\right),$$

$$f_3(y_1, y_2) = y_2^{2/3} F_2\left(\frac{1}{2}; \frac{5}{6}, \frac{5}{3}; 1 - y_1, y_2\right).$$

There are linear combinations $u_j(y_1, y_2), j = 1, 2, 3$ of $f_1(y_1, y_2), f_2(y_1, y_2), f_3(y_1, y_2)$ such that the monodromy group with respect to $u(y_1, y_2) = (u_1(y_1, y_2), u_2(y_1, y_2), u_3(y_1, y_2))$ is $G_{1296}$. Then, there are homogeneous polynomials $G_d(f_1, f_2, f_3)$ in $(f_1, f_2, f_3)$ of degree $d = 6, 9, 12$ such that

$$F_d(u(y_1, y_2)) = G_d(f_1, f_2, f_3) (y_1, y_2),$$

where $F_d, d = 6, 9, 12$ are the invariant polynomials for $G_{648}$ given in Section 4.2. Since $F_d(u(y_1, y_2)), d = 6, 12$ are single-valued, the polynomials $G_d(f_1, f_2, f_3), d = 6, 12$ take the form

$$G_6(f_1, f_2, f_3) = a_0 f_1^6 + a_1 f_1^4 f_2^2 + a_2 f_1^3 f_3^3 + a_3 f_1^2 f_2^4 + a_4 f_1 f_2^3 f_3^3 + a_5 f_2^6 + a_6 f_3^6,$$

$$G_{12}(f_1, f_2, f_3) = b_0 f_1^{12} + b_1 f_1^{10} f_2^2 + b_2 f_1^9 f_3^3 + \cdots.$$  

Since $F_9(u(y_1, y_2))^2$ is single-valued while $F_9(u(y_1, y_2))$ is not, the polynomial $G_9(f_1, f_2, f_3)$ takes the form

$$G_9(f_1, f_2, f_3) = f_2 (c_0 f_1^8 + c_1 f_1^6 f_2^2 + c_2 f_1^5 f_3^3 + \cdots).$$

We know that the system $E_1(y_1, y_2)$ has local exponents $\{0, 0, 1/2\}$ along $\{y_1(y_1 - 1)(y_1 - y_2) = 0\}$, $\{0, 0, 2/3\}$ along $\{y_2(y_2 - 1) = 0\}$, and $\{-1/6,$
values of $S$ isomorphic to the symmetric group $\mathfrak{S}_3$ of $\mathbb{E}_G/\mathbb{C}_0$. Since the singular set $\mathbb{F}_1$ or equivalently with some constants $c$, we get (6.7) as a particular case of (2.3).

Remark. Proposition 6.7 implies that the integrable system (3$y_1 - y_2 - 1)^{-1/6}E_1(y_1, y_2)$ on the projective space $Y = \{ [y_1 : y_2 : 1] \}$ is invariant under a subgroup $G_6$ of $\text{PGL}(3)$ which is isomorphic to the symmetric group $S_3$ of order 6,
(2) a $G_6$-quotient map of $Y$ is given by $\tau : [y_1 : y_2 : 1] \mapsto [\tau_1(y) : \tau_2(y) : 1]$, and

(3) the pull down of $(3y_1 - y_2 - 1)^{-1/6}E_1(y_1, y_2)$ by the map $\tau$ is equal to $E_{G_{1296}}(\tau_1, \tau_2)$.

This fact is more clearly understood if we choose a coordinate of $Y$ as

$$t_1 = y_1/(y_1 - 1), \quad t_2 = (y_1 - y_2)/(y_1 - 1).$$

In this coordinates, the equation (6.7) and the $G_6$-quotient map (6.8) turn out to

$$E_1(-1/6; 1/6, 1/6; 2/3; t_1, t_2) = (t_1 + t_2 + 1)^{1/6}E_{G_{1296}}(\tilde{\tau}_1(t_1, t_2), \tilde{\tau}_2(t_1, t_2),$$

$$\tilde{\tau}_1(t_1, t_2) = \frac{t_1t_2}{16(t_1 + t_2 + 1)}, \quad \tilde{\tau}_2(t_1, t_2) = \frac{t_1^2 + t_2^2 + 1 - t_1t_2 - t_1 - t_2}{(t_1 + t_2 + 1)^2}.$$

The following lemma can be verified directly.

**Lemma 6.8.** Put $z_1 = \tau_1(y_1, y_2), z_2 = \tau_2(y_1, y_2)$, where $\tau_1, \tau_2$ are functions given by (6.8). Then, we have

$$1 - 3z_2 - 432z_1 = \frac{(y_2 + 1)(y_2 - 2)(2y_2 - 1)}{(3y_1 - y_2 - 1)^2},$$

$$(6.9)$$

$$4z_2^3 - (1 - 3z_2 - 432z_1)^2 = \frac{27y_2^2(y_2 - 1)^2}{(3y_1 - y_2 - 1)^6}. \quad (6.10)$$

Now we study the relation between the restrictions $E_{G(3, 3, 3)}(x_1, x_2^0)$ and $E_1(y_1, y_2^0)$. The singular set of $E_{G(3, 3, 3)}(x_1, x_2)$ is given by

$$\{D(x_1, x_2) = 0\} \cup L_\infty,$$

where

$$D(x_1, x_2) = 4x_1^3 + 4x_2^3 - x_1x_2^2 - 18x_1x_2 + 27.$$ 

Take a value $x_2^0$ so that $D(x_1, x_2^0)$ has three distinct zeros $x_1 = \xi_0, \xi_1, \xi_2$. Then, the set of the singular points of the restricted system $E_{G(3, 3, 3)}(x_1, x_2^0)$ is $\{\xi_0, \xi_1, \xi_2, \infty\}$. We consider the fractional linear transformation $x_1 \to y_1$ which sends $(\xi_0, \xi_1, \xi_2, \infty)$ to $(0, 1, \infty)$. If we denote by $y_2^0$ the image of $\xi_2$, we have

$$y_1 = \frac{x_1 - \xi_0}{\xi_1 - \xi_0}, \quad y_2^0 = \frac{\xi_2 - \xi_0}{\xi_1 - \xi_0}.$$ 

The value of $y_2^0$ changes by the $S_3$ action of the permutations of $\{\xi_0, \xi_1, \xi_2\}$, while the value of the $J$-function

$$J(y_2^0) = \frac{4((y_2^0)^2 - y_2^0 + 1)^3}{27(y_2^0)^2(1 - y_2^0)^2}. \quad (6.11)$$
does not. This also implies that $J(y_2^0)$ can be written as a rational function in the elementary symmetric polynomials of $\xi_0$, $\xi_1$, $\xi_2$, and hence in $x_2^0$. In fact, we have

$$J(y_2^0) = \frac{(x_2^0)^3((x_2^0)^3 + 216)^3}{1728((x_2^0)^3 - 27)^3} = j(x_2^0). \quad (6.12)$$

We put

$$X = \{[x_1 : x_2 : 1]\} \cong \mathbb{P}^2, \quad Y = \{[y_1 : y_2 : 1]\} \cong \mathbb{P}^2, \quad Z = \{[z_1 : z_2 : 1]\} \cong \mathbb{P}^2,$$

and define the rational maps $\sigma : X \to Z$ and $\tau : Y \to Z$ by

$$\sigma([x_1 : x_2 : 1]) = [\sigma_1(x_1, x_2) : \sigma_2(x_1, x_2) : 1],$$

$$\tau([y_1 : y_2 : 1]) = [\tau_1(y_1, y_2) : \tau_2(y_1, y_2) : 1],$$

where $\sigma_j$ and $\tau_j$ are given by (6.6) and (6.8). If we set

$$\tau([y_1 : y_2 : 1]) = [z_1 : z_2 : 1],$$

we have (6.9) and (6.10), from which we obtain

$$4z_2^3 \cdot \frac{1}{(1 - 3z_2 - 432z_1)^2} = \frac{4(y_2^2 - y_2 + 1)^3}{(y_2 + 1)^2(y_2 - 2)^2(2y_2 - 1)^2}.$$

On the other hand, we obtain from (6.11)

$$\frac{J(y_2)}{J(y_2) - 1} = \frac{4(y_2^2 - y_2 + 1)^3}{(y_2 + 1)^2(y_2 - 2)^2(2y_2 - 1)^2}.$$

Thus, we get

$$J(y_2) = \frac{4z_2^3}{4z_2^3 - (1 - 3z_2 - 432z_1)^2}.$$

For any $J_0 \in \mathbb{P}^1$, we define the curve

$$V_{J_0} = \left\{[z_1 : z_2 : 1] \middle| \frac{4z_2^3}{4z_2^3 - (1 - 3z_2 - 432z_1)^2} = J_0 \right\} \subset Z.$$

Moreover, for any $x_2^0, y_2^0 \in \mathbb{P}^1$, we define the lines

$$X_{x_2^0} = \{[x_1 : x_2^0 : 1]\} \subset X,$$

$$Y_{y_2^0} = \{[y_1 : y_2^0 : 1]\} \subset Y.$$
Lemma 6.9.  (1) $\sigma$ is a $12:1$ map, and $\tau$ is a $6:1$ map.
(2) Take any $J_0 \in P^1 \setminus \{0, 1, \infty\}$, and choose $x_2^0$ and $y_2^0$ satisfying $J(y_2^0) = j(x_2^0) = J_0$. Then, the restriction of $\sigma$ to the line $X_{x_2}$ and that of $\tau$ to the line $Y_{y_2}$ are both bijection onto $V_{J_0}$. The composite map
\[
(\tau|_{Y_{y_2}})^{-1} \circ \sigma|_{X_{x_2}} : [x_1 : x_2^0 : 1] \to [y_1 : y_2^0 : 1]
\]
is given by
\[
(y_2^0)^2 - y_2^0 + 1 \quad \left(\frac{y_2^0 - 2}{y_2^0 - 1}\right) (3y_1 - y_2^0 - 1)
\]
\[
= \frac{x_2^0((x_2^0)^2 + 216)}{2((x_2^0)^6 - 540(x_2^0)^3 - 5832)} (12y_1 - (x_2^0)^2).
\]
This map sends three zeros of $D(x_1, x_2^0)$ to $\{(y_1 : y_2^0 : 1) | y_1 = 0, 1, y_2^0\}$.
(3) Take $x_2^0$ and $y_2^0$ so that $J(y_2^0) = j(x_2^0) = 0$. Then, the restrictions $\sigma|_{X_{x_2}}$ and $\tau|_{Y_{y_2}}$ are both $3:1$ maps onto the line $V_0 = \{[z_1 : z_2 : 1] | z_2 = 0\}$.
(4) Take $x_2^0$ and $y_2^0$ so that $J(y_2^0) = j(x_2^0) = 1$. Then, the restrictions $\sigma|_{X_{x_2}}$ and $\tau|_{Y_{y_2}}$ are both $2:1$ maps onto the line $V_1 = \{[z_1 : z_2 : 1] | 1 - 3z_2 - 432z_1 = 0\}$.

Proof. Assume that $J(y_2^0) = j(x_2^0) = J_0 \neq 0, 1, \infty$. We set $\tau([y_1 : y_2^0 : 1]) = [z_1 : z_2 : 1]$, which is a point in $V_{J_0}$. From (6.8) and (6.9), we obtain
\[
(6.14) \quad \frac{z_2}{1 - 3z_2 - 432z_1} = \frac{(y_2^0)^2 - y_2^0 + 1}{(y_2^0 - 2)(y_2^0 - 1)} (3y_1 - y_2^0 - 1).
\]
This shows that $y_1$ is uniquely determined by $[z_1 : z_2 : 1] \in V_{J_0}$, and hence $\tau|_{Y_{y_2}}$ is $1:1$. Next we set $\sigma([x_1 : x_2^0 : 1]) = [z_1 : z_2 : 1]$, which is also a point in $V_{J_0}$. We obtain from (6.6)
\[
(6.15) \quad 1 - 3z_2 - 432z_1 = \frac{2((x_2^0)^6 - 540(x_2^0)^3 - 5832)}{(12x_1 - (x_2^0)^2)^3},
\]
and then
\[
(6.16) \quad \frac{z_2}{1 - 3z_2 - 432z_1} = \frac{x_2^0((x_2^0)^2 + 216)}{2((x_2^0)^6 - 540(x_2^0)^3 - 5832)} (12x_1 - (x_2^0)^2).
\]
Thus, $x_1$ is uniquely determined by $[z_1 : z_2 : 1] \in V_{J_0}$, and hence $\sigma|_{X_{x_2}}$ is also $1:1$. The relation (6.13) follows from (6.14) and (6.16). This proves (2).

For generic value $J_0 \in C$, there are twelve $y_2$ satisfying $J(y_2) = J_0$, and six $x_2$ satisfying $j(x_2) = J_0$. Then, the assertion (2) implies (1).
Assume that $J(y_2^0) = j(x_2^0) = J_0 = 0$. Then $\sigma_2(x_1, x_2^0) = \tau_2(y_1, y_2^0) = 0$ by (6.6), (6.8), (6.11) and (6.12). The relation (6.9) (resp. (6.15)) shows that, for any $z_1 \neq 1/432$, $\infty$, there are three $y_1$ (resp. $x_1$) such that $\tau([y_1 : y_2^0 : 1]) = [z_1 : 0 : 1]$ (resp. $\sigma([x_1 : x_2^0 : 1]) = [z_1 : 0 : 1]$). This proves (3).

Assume that $J(y_2^0) = j(x_2^0) = J_0 = 1$. Then,

$$1 - 3\sigma_2(x_1, x_2^0) - 432\sigma_1(x_1, x_2^0) = 1 - 3\tau_2(y_1, y_2^0) - 432\tau_1(y_1, y_2^0) = 0$$

by (6.9), (6.11), (6.12) and (6.15). The relation (6.8) (resp. (6.6)) shows that, for any $z_2 \neq 0$, $\infty$, there are two $y_1$ (resp. $x_1$) such that $\tau([y_1 : y_2^0 : 1]) = [(1 - 3z_2)/432 : z_2 : 1]$ (resp. $\sigma([x_1 : x_2^0 : 1]) = [(1 - 3z_2)/432 : z_2 : 1]$). This proves (4).

Thus, we obtained the relation between the two restrictions via $E_{G_{296}}$.

**Proposition 6.10.** If we take $x_2^0$ and $y_2^0$ such that $j(x_2^0) = J(y_2^0) \neq 0, 1, \infty$, we have

$$E_{G_{296}}(x_1, y_2^0) = E_1 \left( -\frac{1}{6}; \frac{1}{3}; \frac{2}{3}; y_1, y_2^0 \right),$$

where $x_1$ and $y_1$ are related by the relation (6.13).

**Proof.** Let $\lambda : X_{x_0} \rightarrow Y_{x_2^0}$ be the bijection defined by (6.13). Let $i_X : X_{x_2^0} \hookrightarrow X$ and $i_Y : Y_{y_2^0} \hookrightarrow Y$ be the inclusion maps. Then, from the equalities (6.5), (6.7) and Lemma 6.9, we obtain the following parallel commutative diagrams:

This proves the proposition. \qed

**Appendix**

Let $y(x_1, \ldots, x_{n-1})$ be a solution of the algebraic equation

$$y^{mn} + x_{n-1}y^{m(n-1)} + x_{n-2}y^{m(n-2)} + \cdots + x_1y^m - 1 = 0.$$

In this section we construct a system of differential equations satisfied by $y(x_1, \ldots, x_{n-1})$. 

A.1. Generalized binomial functions of several variables.

In [KN] we defined a generalized binomial function of one variable, and showed its fundamental properties. Here we generalize it to a several variables case, and state several properties. These properties can be shown in a similar way as in the case of one variable, and we omit the proofs.

Let $a, s_j, 1 \leq j \leq n - 1$ be complex numbers, and $k_j, 1 \leq j \leq n - 1$ be non negative integers. Put $s = (s_1, s_2, \ldots, s_{n-1})$, $k = (k_1, k_2, \ldots, k_{n-1})$, $\vec{I} = (1, 1, \ldots, 1)$, and $x = (x_1, x_2, \ldots, x_{n-1})$. Put

$$c_0(z, s) = 1,$$
$$c_k(z, s) = z(z + k \cdot s + 1, k \cdot \vec{I} - 1)/k! \quad (k \neq 0),$$

and

$$\psi(z, s, x) = \sum_{k_j \geq 0} c_k(z, s)x^k,$$

where $(a, n) = \Gamma(a + n)/\Gamma(a)$, $k! = \prod_{j=1}^{n-1} k_j!$, and $x^k = \prod_{j=1}^{n-1} x_j^k$.

We call $\psi(z, s, x)$ a generalized binomial function because $\psi(z, 0, x) = (1 - \vec{I} \cdot x)^{-z}$.

Lemma A.1.

$$\psi(z, s, x) = \psi(-z, -s - \vec{I}, -x).$$

Lemma A.2.

$$c_k(z, s) - c_k(z - 1, s) = \sum_{1 \leq j \leq n-1 \atop k_j \geq 0} c_{k-1_j}(z + s_j, s),$$

where $1_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ ($1$ appears only at the $j$-th place).

Proposition A.3. We have the following two equalities.

$$\psi(z, s, x) - \psi(z - 1, s, x) = \sum_{j=1}^{n-1} x_j\psi(z + s_j, s, x),$$
$$\psi(z + \beta, s, x) = \psi(z, s, x)\psi(\beta, s, x).$$

Proposition A.4. Set $e_n = e^{2\pi \sqrt{-1}/n}$ and $s = (-1/n, -2/n, \ldots, -(n-1)/n)$. Then

$$y^n + x_{n-1}y^{n-1} + x_{n-2}y^{n-2} + \cdots + x_1y - 1 = 0.$$
has solutions
\[ f_l(x) := \varepsilon^{l,1}_n(-1/n, s, (\varepsilon^{l}_n x_1, \varepsilon^{2l}_n x_2, \ldots, \varepsilon^{(n-1)l}_n x_{n-1})), \quad 0 \leq l \leq n - 1, \]
in a neighborhood of \( x = 0 \).

**Corollary A.5.** The algebraic equation
\[ y^{mn} + x_{n-1} y^{m(n-1)} + x_{n-2} y^{m(n-2)} + \cdots + x_1 y^m - 1 = 0 \]
has the solutions
\[ \varepsilon^{l,1}_{mn}(\psi(-1/(mn), s, (\varepsilon^{l}_n x_1, \varepsilon^{2l}_n x_2, \ldots, \varepsilon^{(n-1)l}_n x_{n-1})), \quad 0 \leq l \leq mn - 1. \]

**A.2. Differential equations for** \( \psi(x, s, x) \) **with** \( s_j = -j/n \)

Put
\[ \partial_j = \partial/\partial x_j, \quad 1 \leq j \leq n - 1, \quad \partial = (x_1 \partial_1, x_2 \partial_2, \ldots, x_{n-1} \partial_{n-1}). \]

Then we have the following theorem.

**Theorem A.6.** We take \( s = (s_1, s_2, \ldots, s_{n-1}) \) with \( s_j = -j/n \), and set \( s' = s + 1 = s + (1, 1, \ldots, 1) \). Then \( \psi(x) = \psi(x, s, x) \) satisfies the following system of differential equations.

\[ [\partial_i \partial_j - (x + 1 + s_i + s_j + s' \cdot \partial) \partial_{i+j}] \psi(x) = 0, \quad i + j < n, \]
\[ [\partial_i \partial_j - (x + 1 + s_i + s_j + s' \cdot \partial) \partial_{i+j-n}] \psi(x) = 0, \quad i + j > n, \]
\[ [\partial_i \partial_j - (x + s \cdot \partial)(x + s' \cdot \partial)] \psi(x) = 0, \quad i + j = n. \]

**Proof.** Let \( \psi \) denote \( \psi(x, s, x) \). We have
\[ \partial_j \psi = \sum_k \frac{\varepsilon(x + 1 + s_j + s \cdot k, 1 \cdot k)}{k!} x^k, \]
\[ \partial_i \partial_j \psi = \sum_k \frac{\varepsilon(x + 1 + s_i + s_j + s \cdot k, 1 \cdot k + 1)}{k!} x^k. \]

Assume \( i + j < n \). Then, since \( s_i + s_j = s_{i+j} \), we have
\[ \partial_j \partial_j \psi = \sum_k (x + 1 + s_i + s_j + s \cdot k + 1 \cdot k) \frac{\varepsilon(x + 1 + s_{i+j} + s \cdot k, 1 \cdot k)}{k!} x^k \]
\[ = \sum_k (x + 1 + s_i + s_j + s' \cdot k) \frac{\varepsilon(x + 1 + s_{i+j} + s \cdot k, 1 \cdot k)}{k!} x^k \]
\[ = (x + 1 + s_i + s_j + s' \cdot \partial) \partial_{i+j} \psi. \]
Assume \( i + j > n \). Then, since \( s_i + s_j = s_{i+j-n} - 1 \), we have

\[
\partial_i \partial_j \psi = \sum_k \frac{(x+1+s_i+s_j+s\cdot k)\lambda(x+1+s_i+s_j + s\cdot k, \bar{I} \cdot k)}{k!} x^k
\]

\[
= \sum_k \frac{(x+1+s_i+s_j+s\cdot k)\lambda(x+1+s_{i+j-n} + s\cdot k, \bar{I} \cdot k)}{k!} x^k
\]

\[
= (x+1 + s_i + s_j + s\cdot \partial) \partial_{i+j-n} \psi.
\]

Assume \( i + j = n \). Then, since \( s_i + s_j = -1 \), we have

\[
\partial_i \partial_j \psi = \sum_k \frac{\lambda(x+s\cdot k, \bar{I} \cdot k + 1)}{k!} x^k
\]

\[
= \sum_k \frac{(x+s\cdot k)(x+s\cdot k + \bar{I} \cdot k)\lambda(x+1+s\cdot k, \bar{I} \cdot k - 1)}{k!} x^k
\]

\[
= (x+s\cdot \partial)(x+s'\cdot \partial) \psi.
\]

This completes the proof.

We denote by \( E_n(x) \) the system of differential equations defined by (A.3).

**Corollary A.7.** Solutions (A.2) of algebraic equation (A.1) are solutions of \( E_n(-1/(mn)) \).

**Proof.** By Theorem A.6, we see that \( \psi(-1/(mn), s, x) \) is a solution of \( E_n(-1/(mn)) \). Since the system \( E_n(-1/(mn)) \) is invariant under the transformations

\[
(x_1, x_2, \ldots, x_{n-1}) \mapsto (e_n^{-j} x_1, e_n^{-j} x_2, \ldots, e_n^{-j} x_{n-1}),
\]

all solutions (A.2) are solutions of \( E_n(-1/(mn)) \).

By direct calculations, we have the following corollary.

**Corollary A.8.** Set

\[
D(x_1, x_2) := 4x_1^3 - 4x_2^3 - x_1^2 x_2^2 + 18x_1 x_2 + 27 = 0.
\]

Then \( E_3(x) \) is equal to the system

\[
\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_1^1 \frac{\partial u}{\partial x_1} + a_1^2 \frac{\partial u}{\partial x_2} + a_0^2 u = 0, \quad 1 \leq i \leq j \leq 2,
\]
where
\[
\begin{align*}
    a_{11}^1 &= \alpha(-6x_1^2 + x_1x_2^2 - 9x_2) + 4x_1^2 - x_1x_2^2 + 3x_2, \\
    a_{11}^2 &= -\alpha(9x_1x_2 - 2x_2^3 + 27) - x_1x_2 - 9, \\
    a_{22}^1 &= -\alpha(9x_1x_2 + 2x_1^3 + 27) + x_1x_2 + 9, \\
    a_{22}^2 &= \alpha(-6x_2^2 + x_1^2x_2 + 9x_1) - 4x_2^2 - x_1^2x_2 + 3x_1, \\
    a_{12}^1 &= -a_{22}^2 + \frac{1}{2} \frac{\partial D}{\partial x_2}, \\
    a_{12}^2 &= -a_{11}^1 + \frac{1}{2} \frac{\partial D}{\partial x_1}, \\
    a_{11}^0 &= 6\alpha^2(-3x_1 + x_2^2), \\
    a_{22}^0 &= 6\alpha^2(3x_2 + x_1^2), \\
    a_{12}^0 &= -3\alpha^2(x_1x_2 + 9).
\end{align*}
\]

Now we can prove Lemma 4.9 in section 4.3.

**Proof of Lemma 4.9.** Let \( E_3(x; x_1, x_2) \) be the system (A.4). Then, by virtue of Corollaries A.7 and A.8, \( \varphi(x_1, x_2) \) is a solution of \( E_3(-1/(3r); x_1, -x_2) \), which is equal to (4.9). \( \square \)

**References**


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