On the Role of Basic Production Terms in an Activator-Inhibitor System Modeling Biological Pattern Formation

By

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Dedicated to the memory of the late Professor Tetsuro Miyakawa

Abstract. Considered is the asymptotic behavior of solutions to a class of reaction-diffusion systems comprised of an activator and an inhibitor, which includes the system proposed by Gierer and Meinhardt as a model of biological pattern formation. By the basic production terms we mean those independent of the unknown functions. We prove that, when the basic production term for the activator is absent, some solutions with large initial data converge to the trivial state, i.e., the activator vanishes identically. Also, we demonstrate that there exist solutions which start from large initial data and converge to a small stationary solution in the case where the basic production term for the inhibitor is nontrivial and that for the activator is sufficiently small.

Key Words and Phrases. Activator-inhibitor system, Pattern formation, Collapse of patterns, Steady-state patterns.

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1. Introduction and statement of results

1.1. Problem

In the celebrated paper [19], A. M. Turing found that the reaction between two chemicals with different diffusion rates may cause the destabilization of the spatially homogeneous state, thus leading to the formation of nontrivial spatial structure. Developing Turing's idea, A. Gierer and H. Meinhardt [1] proposed several reaction-diffusion systems to model biological pattern formation. Among them is the following system of equations consisting of a slowly diffusing activator and a rapidly diffusing inhibitor:

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$$(GM) \quad \begin{cases} a_t = D_a a_{xx} - \mu a + \rho \left(c \frac{a^2}{h} + \rho_0 \right) & (0 < x < l, t > 0), \\ h_t = D_h h_{xx} - \nu h + c' \rho' a^2 & (0 < x < l, t > 0), \\ a_x = h_x = 0 & (x = 0, l, t > 0), \end{cases}$$

where $a = a(x, t)$ and $h = h(x, t)$ denotes the respective concentrations of the activator and the inhibitor; $D_a, D_h, \mu, \nu, c, c', \rho_0$ are assumed to be positive constants; whereas $\rho(x)$ and $\rho'(x)$ are positive functions. They assumed that a change in cells or tissue takes place in the region where the activator concentration is high, and used this system to simulate the regeneration and transplantation experiments on *hydra*. In their computer simulations the spatial dependence of ρ and ρ' reflects the “polarity” which is believed to exist along the body column of hydra to point the direction of the head. This system of equations is called an activator-inhibitor system with different sources, and has been widely used to model various biological pattern formation. A naive question raised immediately is why they introduced the basic production term $\rho\rho_0$ only for the activator. The purpose of the present paper is to answer this question.

The biological pattern formation often happens in nonhomogeneous environments and sometimes the spatial inhomogeneity plays an essential rôle in selecting, e.g., the orientation. Therefore, it is natural to study reaction-diffusion systems with coefficients depending on spatial variables.

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Suppose $(d_{i,j}^{(a)}(x))$ and $(d_{i,j}^{(h)}(x))$ are $N \times N$ real symmetric matrices for which there exist positive constants d_a and d_h such that

$$|\xi|^2 \leq \sum_{i,j=1}^N d_{i,j}^{(a)} \xi_i \xi_j \leq d_a |\xi|^2, \quad |\xi|^2 \leq \sum_{i,j=1}^N d_{i,j}^{(h)} \xi_i \xi_j \leq d_h |\xi|^2$$

for all $\xi \in \mathbf{R}^N$ and $x \in \Omega$. We assume that the function $d_{i,j}^{(a)}(x)$ and $d_{i,j}^{(h)}(x)$ and their first order derivatives $\nabla d_{i,j}^{(a)}(x)$, $\nabla d_{i,j}^{(h)}(x)$ are Hölder continuous in $\bar{\Omega}$. We define two elliptic partial differential operators A_a and A_h by

$$A_a = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(d_{i,j}^{(a)}(x) \frac{\partial}{\partial x_j} \right), \quad A_h = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(d_{i,j}^{(h)}(x) \frac{\partial}{\partial x_j} \right)$$

and associated boundary operators B_a and B_h by

$$B_a = \sum_{i,j=1}^N v_i d_{i,j}^{(a)} \frac{\partial}{\partial x_j}, \quad B_h = \sum_{i,j=1}^N v_i d_{i,j}^{(h)} \frac{\partial}{\partial x_j},$$

where $\nu = (\nu_1, \dots, \nu_N)$ denotes the unit outer normal vector to $\partial\Omega$. The following is a slightly generalized form of the activator-inhibitor system proposed by Gierer and Meinhardt:

$$(1.1) \quad \begin{cases} \frac{\partial A}{\partial t} = \varepsilon^2 A_a A - \mu_a(x)A + \rho_a(A, H, x) \frac{A^p}{H^q} + \sigma_a(x) & \text{for } x \in \Omega, t > 0, \\ \tau \frac{\partial H}{\partial t} = D A_h H - \mu_h(x)H + \rho_h(A, H, x) \frac{A^r}{H^s} + \sigma_h(x) & \text{for } x \in \Omega, t > 0. \end{cases}$$

We impose the non-flux boundary condition and the initial condition

$$(1.2) \quad \mathbf{B}_a A = 0 \quad \text{and} \quad \mathbf{B}_h H = 0 \quad \text{for } x \in \partial\Omega, t > 0,$$

$$(1.3) \quad A(x, 0) = A_0(x), \quad H(x, 0) = H_0(x) \quad \text{for } x \in \Omega.$$

Here ε , D and τ are positive constants; the removal rates $\mu_a(x)$, $\mu_h(x)$ are Hölder continuous functions on $\bar{\Omega}$ such that

$$(1.4) \quad 0 < k_1^{(a)} \leq \mu_a(x) \leq k_2^{(a)}, \quad 0 < k_1^{(h)} \leq \mu_h(x) \leq k_2^{(h)} \quad \text{for all } x \in \bar{\Omega};$$

the cross-reaction coefficients $\rho_a(A, H, x)$ and $\rho_h(A, H, x)$ are positive functions defined for $-\infty < A < +\infty$, $-\infty < H < +\infty$, $x \in \bar{\Omega}$ which are continuously differentiable in (A, H) and Hölder continuous in $x \in \bar{\Omega}$. Moreover, there are positive constants c_a , C_a , c_h , C_h such that

$$(1.5) \quad 0 < c_a < \rho_a(A, H, x) \leq C_a, \quad \left| \frac{\partial \rho_a}{\partial A}(A, H, x) \right|, \left| \frac{\partial \rho_a}{\partial H}(A, H, x) \right| \leq C_a,$$

$$(1.6) \quad 0 < c_h < \rho_h(A, H, x) \leq C_h, \quad \left| \frac{\partial \rho_h}{\partial A}(A, H, x) \right|, \left| \frac{\partial \rho_h}{\partial H}(A, H, x) \right| \leq C_h$$

for $A \geq 0$, $H \geq 0$, $x \in \bar{\Omega}$. Concerning the basic production terms σ_a and σ_h , we assume

$$(1.7) \quad \sigma_a, \sigma_h \in C^\gamma(\bar{\Omega}), \quad \text{and} \quad \sigma_a(x) \geq 0, \sigma_h(x) \geq 0 \quad \text{on } \bar{\Omega}$$

and as to the initial data, we assume

$$(1.8) \quad A_0, H_0 \in C^{2+\gamma}(\bar{\Omega}), \quad \mathbf{B}_a A_0|_{\partial\Omega} = \mathbf{B}_h H_0|_{\partial\Omega} = 0 \quad \text{and} \\ A_0(x) > 0, H_0(x) > 0 \quad \text{on } \bar{\Omega},$$

where $0 < \gamma < 1$. Furthermore, the exponents $p > 0$, $q > 0$, $r > 0$, $s \geq 0$ satisfy

$$(1.9) \quad 0 < \frac{p-1}{r} < \frac{q}{s+1}.$$

We remark here that we follow [3] for the names of the coefficients μ_a , μ_h , ρ_a , ρ_h .

1.2. Quick review of known results on the existence and boundedness of solutions

From a mathematical point of view, one of the fundamental questions is whether the initial-boundary value problem (1.1)–(1.3) has a solution for all $t > 0$ or not. When A_a and A_h are the Laplace operator $A = \sum_{j=1}^N \partial^2 / \partial x_j^2$ and μ_a, μ_h are constants, there have appeared several results on this question (see, e.g., [14], [7], [21], [4], [2]). In particular, under the assumption that $\min_{x \in \bar{\Omega}} \sigma_a(x) > 0$, $\sigma_h(x) \equiv 0$ and $(p-1)/r < 2/(N+2)$, Masuda and Takahashi [7] proved not only that the solution exists for all $t > 0$ but also that, as $t \rightarrow +\infty$, the set $\{(A(x, t), H(x, t)) \in \mathbf{R}^2 \mid x \in \Omega\}$ is confined in a fixed rectangle which is independent of the initial data. Meanwhile, Li, Chen and Qin [4] assumed $p-1 < r$ and $\min_{x \in \bar{\Omega}} \sigma_a(x) > 0$ to prove that the solution exists for all $t > 0$ in the case where $\rho_a(A, H, x) \equiv \rho_h(A, H, x) \equiv 1$ and $\rho_h(x) \equiv 0$. Jiang [2] and the authors independently obtained similar results on the existence and boundedness of solutions including the case where $\min_{x \in \bar{\Omega}} \sigma_a(x) = 0$. The method of proving these results carries over the case of nonconstant coefficients without additional difficulties. Putting the known results together, we can state the following Theorems A–C which complement the results by [7], [4], [2] and [15], and give us a complete understanding of the existence and the boundedness of solutions to the generalized system (1.1)–(1.3) in the case $p-1 < r$.

Theorem A. *Assume, in addition to (1.9), that*

$$(1.10) \quad p-1 < r$$

and, in addition to (1.7), that

$$\max_{x \in \bar{\Omega}} \sigma_a(x) > 0.$$

Then the initial-boundary value problem (1.1)–(1.3) has a unique solution for all $t > 0$. Moreover, there exist positive constants r_a, r_h, R_a, R_h , independent of the initial data $(A_0(x), H_0(x))$, such that

$$\begin{aligned} r_a &\leq \liminf_{t \rightarrow +\infty} \min_{x \in \bar{\Omega}} A(x, t) \leq \limsup_{t \rightarrow +\infty} \max_{x \in \bar{\Omega}} A(x, t) \leq R_a, \\ r_h &\leq \liminf_{t \rightarrow +\infty} \min_{x \in \bar{\Omega}} H(x, t) \leq \limsup_{t \rightarrow +\infty} \max_{x \in \bar{\Omega}} H(x, t) \leq R_h. \end{aligned}$$

Theorem B. *Assume that (1.10) is satisfied in addition to (1.9). Suppose that*

$$\sigma_a(x) \equiv 0 \quad \text{and} \quad \max_{x \in \bar{\Omega}} \sigma_h(x) > 0.$$

Then the initial-boundary value problem (1.1)–(1.3) has a unique solution for all $t > 0$. Moreover, there exist positive constants r_h , R_a , R_h which are independent of the initial data $(A_0(x), H_0(x))$ such that

$$e^{-k_2^{(a)}t} \min_{x \in \bar{\Omega}} A_0(x) \leq A(x, t) \quad \text{for all } x \in \bar{\Omega}, t > 0,$$

and

$$\limsup_{t \rightarrow +\infty} \max_{x \in \bar{\Omega}} A(x, t) \leq R_a,$$

$$r_h \leq \liminf_{t \rightarrow +\infty} \min_{x \in \bar{\Omega}} H(x, t) \leq \limsup_{t \rightarrow +\infty} \max_{x \in \bar{\Omega}} H(x, t) \leq R_h.$$

Theorem C. Assume that (1.10) is satisfied in addition to (1.9). Suppose that

$$\sigma_a(x) \equiv 0 \quad \text{and} \quad \sigma_h(x) \equiv 0.$$

Then the initial-boundary value problem (1.1)–(1.3) has a unique solution for all $t > 0$. Moreover, there are positive constants λ and μ which are dependent only on $p, q, r, s, \tau, k_1^{(a)}, k_2^{(h)}, C_a$ and c_h , and a positive constant C depending on initial data $(A_0(x), H_0(x))$ such that

$$e^{-k_2^{(a)}t} \min_{x \in \bar{\Omega}} A_0(x) \leq A(x, t) \leq Ce^{\lambda t}, \quad e^{-k_2^{(h)}t/\tau} \min_{x \in \bar{\Omega}} H_0(x) \leq H(x, t) \leq Ce^{\mu t}$$

for all $t > 0$ and $x \in \bar{\Omega}$.

Some remarks on Theorems A–C are in order. First, all the three theorems assume that $p - 1 < r$, which is important to rule out the occurrence of finite time blow-up of solutions. Indeed, [4] and [8] showed in the case of a homogeneous medium that if $p - 1 > r$, then there exist solutions of (1.1)–(1.3) with $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$ which blow up in finite time. Here and in what follows by the *a homogeneous medium* we mean that $A_a = A_h = A$, $\mu_a \equiv \mu_h \equiv \rho_a \equiv \rho_h \equiv 1$, and both of σ_a and σ_h are nonnegative constants.

Second, in [8] it is proved in the case of a homogeneous medium that if $p - 1 \leq r$, $q \geq s + 1$ and $s + 1 < (p - 1)\tau \leq q$, then some solutions of (1.1)–(1.3) with $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$ exist for all $t > 0$, but they are unbounded. By virtue of Theorem A, all solutions are bounded if σ_a is nontrivial.

1.3. Main results

As is seen from the known results stated in the previous subsection, if $\sigma_a(x) \not\equiv 0$ then we are in a nice situation in the sense that all the solutions of the initial-boundary value problem (1.1)–(1.3) exist for all $t > 0$ and will be confined in a fixed rectangle eventually, provided that $p - 1 < r$. If $\sigma_a(x) \equiv 0$ and $p - 1 < r$, then solutions still exist for all $t > 0$ but may become unbounded when $\sigma_h(x) \equiv 0$. This may be a partial answer to the naive question raised

in the introduction. However, in this paper we report a new phenomenon, called the collapse of patterns, which we suspect is another reason for adding the basic production term only to the activator. In some numerical simulations, we encounter a solution starting from an almost uniform initial value that develops localization in the activator concentration for a while, but it begins to oscillate and eventually converges uniformly to the trivial state $A \equiv 0$. Indeed, for the homogeneous medium with $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$, Wu and Li [21] found that there are solutions of (1.1)–(1.3) with large initial data such that $(A(x, t), H(x, t)) \rightarrow (0, 0)$ uniformly on $\bar{\Omega}$ as $t \rightarrow +\infty$. We call vaguely this phenomenon the *collapse of patterns*. As a matter of fact, Yanagihara [22] proved the existence of solutions which converge to $(0, 0)$ for (GM) with $\rho_0 = 0$ as early as in the late 1970's, assuming that the initial data is sufficiently small. One of the purposes of the present paper is to study this phenomenon in detail and know the structure of kinetics which leads to the collapse of patterns.

For lack of uniform lower bounds for the activator in the case $\sigma_a(x) \equiv 0$, Theorems B and C suggest that patterns in the activator can collapse even though $\sigma_h(x) > 0$. We prove that this is indeed the case. To begin with we introduce two functions $\Sigma_{a,\varepsilon}(x)$ and $\Sigma_{h,D}(x)$ which are respective solutions of the following linear boundary value problems:

$$(1.11) \quad \begin{cases} \varepsilon^2 A_a \Sigma_{a,\varepsilon} - \mu_a(x) \Sigma_{a,\varepsilon} + \sigma_a(x) = 0 & \text{for } x \in \Omega, \\ B_a \Sigma_{a,\varepsilon} = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

and

$$(1.12) \quad \begin{cases} D A_h \Sigma_{h,D} - \mu_h(x) \Sigma_{h,D} + \sigma_h(x) = 0 & \text{for } x \in \Omega, \\ B_h \Sigma_{h,D} = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

By the standard theory of linear elliptic partial differential operators, each of these boundary value problems has a unique classical solution. The solution $\Sigma_{a,\varepsilon}$ of (1.11) is strictly positive whenever $\max_{x \in \bar{\Omega}} \sigma_a(x) > 0$, while $\Sigma_{a,\varepsilon}(x) \equiv 0$ if $\sigma_a(x) \equiv 0$. The same is true for the solution $\Sigma_{h,D}$ of (1.12).

It is convenient to classify the basic production terms as

Case I: $\sigma_a(x) \equiv 0$ and $\sigma_h(x) \equiv 0$; *Case II:* $\sigma_a(x) \equiv 0$ and $\sigma_h(x) \not\equiv 0$;
Case III: $\sigma_a(x) \not\equiv 0$ and $\sigma_h(x) \not\equiv 0$; *Case IV:* $\sigma_a(x) \not\equiv 0$ and $\sigma_h(x) \equiv 0$.

The following theorem slightly generalizes the result by Wu and Li [21] to inhomogeneous media.

Theorem 1.1 (Final phase of collapse of patterns, Cases I and II). *Let $\sigma_a(x) \equiv 0$. Assume that τ satisfies $\tau > qk_2^{(h)}/(p-1)k_1^{(a)}$ and the initial data satisfy*

$$(1.13) \quad \left(\min_{x \in \bar{\Omega}} H_0(x) \right)^q > \frac{C_a(p-1)}{k_1^{(a)}(p-1) - qk_2^{(h)}/\tau} \left(\max_{x \in \bar{\Omega}} A_0(x) \right)^{p-1}.$$

Then the solution $(A(x, t), H(x, t))$ of (1.1)–(1.3) satisfies

$$0 < \max_{x \in \bar{\Omega}} A(x, t) \leq C e^{-k_1^{(a)} t}, \quad \max_{x \in \bar{\Omega}} |H(x, t) - \Sigma_{h,D}(x)| \leq C e^{-k_1^{(h)} t / \tau},$$

in which C is a positive constant depending on $(A_0(x), H_0(x))$, and $\Sigma_{h,D}(x)$ is the solution of (1.12).

We emphasize that Theorem 1.1 is a “semi-global” result in the sense that the initial data $(A_0(x), H_0(x))$ may be taken very far from the limit $(0, \Sigma_{h,D}(x))$. In Case II, we can prove the following:

Theorem 1.2 (Final phase of collapse of patterns, Case II). *Assume that $\sigma_a(x) \equiv 0$ and $\max_{x \in \bar{\Omega}} \sigma_h(x) > 0$. Let*

$$(1.14) \quad \delta_h = \min_{x \in \bar{\Omega}} \Sigma_{h,D}(x) \quad \text{and} \quad \Gamma_h = \max_{x \in \bar{\Omega}} \Sigma_{h,D}(x).$$

If the initial data $(A_0(x), H_0(x))$ satisfies

$$(1.15) \quad \min \left\{ \left((\delta_h / \Gamma_h)^{k_2^{(h)} / k_1^{(h)}} \min_{x \in \bar{\Omega}} H_0(x) \right)^q, (\delta_h k_1^{(h)} / k_2^{(h)})^q \right\} \\ > \frac{C_a}{k_1^{(a)}} \left(\max_{x \in \bar{\Omega}} A_0(x) \right)^{p-1},$$

then

$$0 < \max_{x \in \bar{\Omega}} A(x, t) \leq C e^{-k_1^{(a)} t}, \quad \max_{x \in \bar{\Omega}} |H(x, t) - \Sigma_{h,D}(x)| \leq C e^{-k_1^{(h)} t / \tau}.$$

Here C is a positive constant depending also on $(A_0(x), H_0(x))$, and $\Sigma_{h,D}(x)$ is a solution of (1.12).

We note that, in Case I, the largeness of τ is necessary for the collapse of pattern to occur, as the following proposition shows.

Proposition 1.3. *Let $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$. If there exists a solution $(A(x, t), H(x, t))$ of (1.1)–(1.3) satisfying (i) $H(x, t)^q > \rho_a(A, H, x) A(x, t)^{p-1} / \mu_a(x)$ for all $x \in \bar{\Omega}$ and $t > 0$ and (ii) $(A(x, t), H(x, t)) \rightarrow (0, 0)$ as $t \rightarrow +\infty$, then $\tau \geq q k_1^{(h)} / (k_2^{(a)} (p - 1))$.*

By Theorem A it is tempting to guess that patterns never collapse as long as $\sigma_a(x) \not\equiv 0$. However, in Case III with $\max_{x \in \bar{\Omega}} \sigma_a(x)$ being sufficiently small, Theorem 1.5 below gives a *stable* stationary solution of (1.1)–(1.2) in the neighborhood of $(A, H) = (\Sigma_{a,e}, \Sigma_{h,D})$ which we call an *almost decoupled pattern*. This leaves room for the occurrence of collapse in Case III, too.

First we give a precise definition of an almost decoupled pattern.

Definition 1.4. A stationary solution $(A(x), H(x))$ of (1.1)–(1.2) is called an almost decoupled pattern if

$$-\mu_a(x) + \rho_a(A(x), H(x), x) \frac{A(x)^{p-1}}{H(x)^q} < 0$$

and

$$-\mu_h(x) + \rho_h(A(x), H(x), x) \frac{A(x)^r}{H(x)^{s+1}} < 0$$

for all $x \in \bar{\Omega}$.

Note that if $\sigma_h(x) \equiv 0$, then there is no almost decoupled pattern, since

$$(1.16) \quad 0 = \int_{\Omega} \left\{ D A_h H - \left(\mu_h - \rho_h \frac{A^r}{H^{s+1}} \right) H \right\} dx = \int_{\Omega} - \left(\mu_h - \rho_h \frac{A^r}{H^{s+1}} \right) H \, dx.$$

We recall here the definition of the stability of stationary solutions. A stationary solution $(A_*(x), H_*(x))$ of (1.1)–(1.3) is said to be *stable* if for any $\eta > 0$ there exists $\delta > 0$ such that $\|A_0 - A_*\|_{W^{2,p}(\Omega)} + \|H_0 - H_*\|_{W^{2,p}(\Omega)} < \delta$ implies $\|A(\cdot, t) - A_*\|_{W^{2,p}(\Omega)} + \|H(\cdot, t) - H_*\|_{W^{2,p}(\Omega)} < \eta$ for all $t > 0$, in which $P > N$. Here, $W^{2,p}(\Omega)$, with $P > N$, denotes the Banach space consisting of all functions in $L^P(\Omega)$ whose partial derivatives of order up to two are also in $L^P(\Omega)$. We say that a stationary solution $(A_*(x), H_*(x))$ is *asymptotically stable* if it is stable and moreover $\|A(\cdot, t) - A_*\|_{L^\infty(\Omega)} + \|H(\cdot, t) - H_*\|_{L^\infty(\Omega)} \rightarrow 0$ as $t \rightarrow +\infty$. Note that if it is asymptotically stable then we actually have $\|A(\cdot, t) - A_*\|_{W^{2,p}(\Omega)} + \|H(\cdot, t) - H_*\|_{W^{2,p}(\Omega)} \rightarrow 0$ as $t \rightarrow +\infty$ by the standard boot-strap method. Finally, a stationary solution (A_*, H_*) is said to be *unstable* if it is not stable.

Theorem 1.5 (Almost decoupled pattern). *Let $\max_{x \in \bar{\Omega}} \sigma_a(x) > 0$. Moreover, assume that $\min_{x \in \bar{\Omega}} \sigma_a(x) \geq \gamma_a (\max_{x \in \bar{\Omega}} \sigma_a(x))^p$ if $0 < r < 1$, where γ_a is a positive constant independent of $\sigma_a(x)$. Let $\max_{x \in \bar{\Omega}} \sigma_h(x) > 0$. Then there exists a positive number m_0 such that if $\max_{x \in \bar{\Omega}} \sigma_a(x) \leq m_0$, then (1.1)–(1.2) has a stationary solution $(A_*(x), H_*(x))$ which satisfies*

$$(1.17) \quad \begin{aligned} \|A_* - \Sigma_{a,e}\|_{L^\infty} &\leq C \left(\max_{x \in \bar{\Omega}} \sigma_a(x) \right)^p, \\ \|H_* - \Sigma_{h,D}\|_{L^\infty} &\leq C \left(\max_{x \in \bar{\Omega}} \sigma_a(x) \right)^r, \end{aligned}$$

where C is a positive constant and $\Sigma_{a,e}$, $\Sigma_{h,D}$ are solutions of (1.11), (1.12), respectively. Furthermore, this stationary solution is asymptotically stable.

Remark 1.6. (1) The positive constant γ_a will be specified in the paragraph containing (3.20) in Section 3.1.

(2) The constant m_0 depends on $\mu_a, \rho_a, D, A_h, \mu_h, \rho_h, \sigma_h$ and (p, q, r, s) , but independent of ε, A_a and τ . (See the proof of Theorem 1.5 in Section 3.) In Proposition 3.5 we shall prove that actually the solution converges exponentially as $t \rightarrow +\infty$.

We now introduce important quantities κ_a and K_a to state the result in Case III. Let $0 < \kappa_a < K_a$ be two positive roots of the algebraic equation

$$(1.18) \quad -k_1^{(a)}\xi + \frac{C_a}{\delta_h^q}\xi^p + \max_{x \in \bar{\Omega}} \sigma_a(x) = 0,$$

where δ_h is defined in (1.14), $\max_{x \in \bar{\Omega}} \sigma_a(x) \leq m_0$ and m_0 is the positive constant appearing in Theorem 1.5. It is not difficult to check that, as $\max_{x \in \bar{\Omega}} \sigma_a(x) \rightarrow 0$,

$$(1.19) \quad \kappa_a = \frac{1}{k_1^{(a)}} \max_{x \in \bar{\Omega}} \sigma_a(x) + O\left(\left(\max_{x \in \bar{\Omega}} \sigma_a(x)\right)^p\right),$$

$$(1.20) \quad K_a = \left(\frac{k_1^{(a)}\delta_h^q}{C_a}\right)^{1/(p-1)} - \frac{k_1^{(a)}}{p-1} \max_{x \in \bar{\Omega}} \sigma_a(x) + o\left(\max_{x \in \bar{\Omega}} \sigma_a(x)\right).$$

Theorem 1.7 (Collapse to almost decoupled pattern, Case III). *Under the same assumption as in Theorem 1.5, if the initial data $(A_0(x), H_0(x))$ satisfies*

$$(1.21) \quad \max_{x \in \bar{\Omega}} A_0(x) < K_a, \quad H_0(x) \geq \max_{x \in \bar{\Omega}} \Sigma_{h,D}(x),$$

then

$$\max_{x \in \bar{\Omega}} (|A(x, t) - A_*(x)| + |H(x, t) - H_*(x)|) \leq Ce^{-\gamma t} \quad \text{for all } t \geq 0.$$

Here (A_*, H_*) is the almost decoupled pattern given by Theorem 1.5, and C and γ are positive constants depending also on (A_0, H_0) .

Remark 1.8. (1) Proposition 1.3 gives a necessary condition on τ for the existence of a solution $(A(x, t), H(x, t))$ of (1.1)–(1.3) converging to $(0, 0)$ as $t \rightarrow +\infty$ in Case I. Since $k_1^{(h)}/k_2^{(h)} \leq 1 \leq k_2^{(a)}/k_1^{(a)}$, the sufficient condition on τ imposed in Theorem 1.1 is stronger than the necessary condition.

(2) We notice that in Case III we can apply both Theorems 1.1 and 1.2. If τ is large and $\min_{x \in \bar{\Omega}} H_0(x)$ is also large, then the sufficient condition given by Theorem 1.1 is weaker.

Remark 1.9. It is to be noted that in contrast to Theorems A–C, we do not assume any further condition on the exponents other than (1.9) in Theorems 1.1, 1.2 and 1.7, yet we have a bounded solution for all $t > 0$ by restricting initial data.

Here we would like to summarize the results on the collapse of patterns. Theorem 1.1 shows that if $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$, then there is a solution $(A(x, t), H(x, t))$ of (1.1)–(1.3) tending to $(0, 0)$ uniformly on $\bar{\Omega}$ provided that τ is sufficiently large. On the other hand, by Theorem 1.2, the collapse of patterns always occurs when $\max_{x \in \bar{\Omega}} \sigma_h(x) > 0$ under a condition on the initial data which is independent of the magnitude of τ . In Case IV where $\sigma_a(x) \not\equiv 0$ and $\sigma_h(x) \equiv 0$, the collapse of patterns never occurs because $A(x, t)$ cannot approach 0 by virtue of Theorem A and there does not exist any almost decoupled pattern due to (1.16). Therefore, we may conclude that the simplest way to avoid the collapse of patterns is to take $\sigma_a(x) \not\equiv 0$ and $\sigma_h(x) \equiv 0$. For more on the role of basic production terms, see also Concluding Remarks at the end of the paper.

The organization of this paper is as follows. In Section 2, we give a proof of Theorem 1.1 and then we proceed to prove Theorem 1.2 and Proposition 1.3. In Section 3, we prove Theorem 1.5 on the existence and stability of almost decoupled patterns, followed by the proof of Theorem 1.7 which is on the collapse to almost decoupled patterns.

2. Collapse of patterns

In this section we will sketch the proofs of Theorems 1.1 and 1.2, and Proposition 1.3.

2.1. Proof of Theorem 1.1

We follow the approach due to [21]. See also [16] where we gave a proof for a homogeneous media.

Step 1: Lower bounds

First, we estimate $A(x, t)$ and $H(x, t)$ from below. The following lemma can be proved easily by using the maximum principle.

Lemma 2.1. *The solution $(A(x, t), H(x, t))$ of (1.1)–(1.3) with $\sigma_a(x) \equiv 0$ satisfies*

$$(2.1) \quad A(x, t) \geq e^{-k_2^{(a)}t} \min_{x \in \bar{\Omega}} A_0(x), \quad H(x, t) \geq e^{-k_2^{(h)}t/\tau} \min_{x \in \bar{\Omega}} H_0(x)$$

for all $t > 0$, $x \in \Omega$.

We put $\underline{v}(t) = e^{-k_2^{(h)}t/\tau} \min_{x \in \bar{\Omega}} H_0(x)$.

Step 2: Upper bounds

In order to derive an estimate of $A(x, t)$ from above, we need the following lemma.

Lemma 2.2. *Let τ satisfy $\tau > qk_2^{(h)}/((p-1)k_1^{(a)})$ and m_1 be a positive number satisfying*

$$(2.2) \quad m_1^{p-1} < \frac{1}{C_a} \left(k_1^{(a)} - \frac{qk_2^{(h)}}{(p-1)\tau} \right) \underline{v}(0)^q.$$

Then the initial value problem

$$(2.3) \quad \frac{dm}{dt} = -k_1^{(a)}m + C_a \frac{m^p}{\underline{v}^q}, \quad m(0) = m_1$$

has a unique solution for all $t > 0$ and there exists a positive constant C depending on m_1 and $\underline{v}(0)$ such that $m(t) \leq Ce^{-k_1^{(a)}t}$.

Note that C_a is the constant which appeared in (1.5).

Proof. Put $M(t) = 1/m(t)^{p-1}$. Differentiating $M(t)$ in t , we have

$$\frac{dM}{dt} = (p-1)k_1^{(a)}M - (p-1)C_a \frac{1}{\underline{v}^q}.$$

Since $M(0) = 1/m_1^{p-1}$, we obtain that

$$M(t) = \frac{1}{m_1^{p-1}} e^{k_1^{(a)}(p-1)t} - (p-1)C_a \int_0^t \frac{e^{k_1^{(a)}(p-1)(t-\xi)}}{\underline{v}(\xi)^q} d\xi.$$

It follows from the definition of $\underline{v}(t)$ that

$$\begin{aligned} M(t) &= \frac{1}{m_1^{p-1}} e^{k_1^{(a)}(p-1)t} - C_a \frac{p-1}{\underline{v}(0)^q} \int_0^t e^{k_1^{(a)}(p-1)(t-\xi) + qk_2^{(h)}\xi/\tau} d\xi \\ &= \left\{ \frac{1}{m_1^{p-1}} - C_a \frac{p-1}{\underline{v}(0)^q} \cdot \frac{1 - e^{-[k_1^{(a)}(p-1) - qk_2^{(h)}/\tau]t}}{k_1^{(a)}(p-1) - qk_2^{(h)}/\tau} \right\} e^{k_1^{(a)}(p-1)t}. \end{aligned}$$

If

$$\frac{1}{m_1^{p-1}} > C_a \frac{p-1}{\underline{v}(0)^q} \cdot \frac{1}{k_1^{(a)}(p-1) - qk_2^{(h)}/\tau},$$

then $M(t) > 0$ for all $t > 0$. Therefore, $m(t)$ exists for all $t > 0$ and there is a positive constant C such that $m(t) \leq Ce^{-k_1^{(a)}t}$. $\square$

From Lemma 2.2, if $\tau > qk_2^{(h)}/(k_1^{(a)}(p-1))$ and (1.13) is satisfied, then the problem

$$(2.4) \quad \frac{d\bar{u}}{dt} = -k_1^{(a)}\bar{u} + C_a \frac{\bar{u}^p}{\bar{v}^q}, \quad \bar{u}(0) = \max_{x \in \bar{\Omega}} A_0(x)$$

has a solution $\bar{u}(t)$ for all $t > 0$ and $\bar{u}(t) \leq Ce^{-k_1^{(a)}t}$ with some $C > 0$ depending on $\bar{u}(0)$ and $\bar{v}(0)$. Applying the maximum principle, we see that

$$(2.5) \quad A(x, t) \leq \bar{u}(t) \leq Ce^{-k_1^{(a)}t} \quad \text{for all } x \in \bar{\Omega}, t > 0.$$

Next, we consider the initial value problem

$$(2.6) \quad \tau \frac{d\bar{v}}{dt} = -k_1^{(h)}\bar{v} + C_h \frac{\bar{u}^r}{\bar{v}^s} + \|\sigma_h\|_{L^\infty}, \quad \bar{v}(0) = \max_{x \in \bar{\Omega}} H_0(x),$$

where C_h is the constant in (1.5) and we put $\|\sigma_h\|_{L^\infty} = \max_{x \in \bar{\Omega}} \sigma_h(x)$. Because of (2.5), the right-hand side of the first equation of (2.6) is estimated as follows:

$$(2.7) \quad \tau \frac{d\bar{v}}{dt} \leq -k_1^{(h)}\bar{v} + C_h \frac{C^r e^{-rk_1^{(a)}t}}{\bar{v}^s} + \|\sigma_h\|_{L^\infty}.$$

Multiplying the both sides of (2.7) by $\bar{v}^s$ and putting $\bar{v}^{s+1} = W$, we have that

$$(2.8) \quad \frac{dW}{dt} \leq -\frac{k_1^{(h)}(s+1)}{\tau} W + \frac{\|\sigma_h\|_{L^\infty}(s+1)}{\tau} W^{s/(s+1)} + \frac{C_h'(s+1)}{\tau} e^{-rk_1^{(a)}t}.$$

Here $C_h' = C_h C^r$. It follows from (2.8) that $W(t) \leq \kappa$ for all $t > 0$, where κ is the positive root of the algebraic equation $x - \|\sigma_h\|_{L^\infty} x^{s/(s+1)}/k_1^{(h)} = 0$. For a proof, we refer to Lemma 2.2 in [7]. Therefore, the following lemma holds.

Lemma 2.3. *Suppose that $\tau > qk_2^{(h)}/((p-1)k_1^{(a)})$ and (1.13) is satisfied. Then the initial value problem (2.6) has a unique bounded solution $\bar{v}(t)$ for all $t > 0$.*

The maximum principle implies that $H(x, t) \leq \bar{v}(t)$ for $x \in \bar{\Omega}$, $t > 0$.

Step 3: Proof of Theorem 1.1

It is easy to see from (2.5) that $A(x, t) \rightarrow 0$ uniformly on $\bar{\Omega}$ as $t \rightarrow +\infty$. Hence, it remains to consider the behavior of $H(x, t)$ as $t \rightarrow +\infty$.

Let $\Sigma_{h,D}(x)$ be the unique solution of (1.12). Put $W(x, t) = H(x, t) - \Sigma_{h,D}(x)$. Then it is a solution of the following initial-boundary value problem:

$$\tau \frac{\partial W}{\partial t} = D A_h W - \mu_h(x) W + \rho_h(A, H, x) \frac{A^r}{H^s} \quad \text{for } x \in \Omega, t > 0,$$

$$B_h W = 0 \quad \text{for } x \in \partial\Omega, t > 0,$$

$$W(x, 0) = H_0(x) - \Sigma_{h,D}(x) \quad \text{for } x \in \Omega.$$

We note that by (2.1) and (2.5)

$$\rho_h(A, H, x) \frac{A^r}{H^s} \leq C_h \frac{C^r e^{-rk_1^{(a)}t}}{[e^{-k_2^{(h)}t/\tau} \min_{x \in \bar{\Omega}} H_0(x)]^s} = C' e^{-[rk_1^{(a)} - sk_2^{(h)}/\tau]t}.$$

Put $\omega_0 = \max_{x \in \bar{\Omega}} (H_0(x) - \Sigma_{h,D}(x))$ and let $\omega(t)$ be the solution of the initial value problem

$$\tau \frac{d\omega}{dt} = -k_1^{(h)} \omega + C' e^{-[rk_1^{(a)} - sk_2^{(h)}/\tau]t} \quad \text{for } t > 0, \quad \text{and} \quad \omega(0) = \omega_0.$$

Then by the maximum principle, we see that $W(x, t) \leq \omega(t)$ for all $x \in \bar{\Omega}$ and $t > 0$. Since

$$\omega(t) = \left\{ \omega_0 + C' \frac{1 - e^{-[rk_1^{(a)} - (sk_2^{(h)} + k_1^{(h)})/\tau]t}}{\tau rk_1^{(a)} - sk_2^{(h)} - k_1^{(h)}} \right\} e^{-k_1^{(h)}t/\tau},$$

and since

$$\tau rk_1^{(a)} - (sk_2^{(h)} + k_1^{(h)}) > k_2^{(h)}[qr/(p-1) - (s+1)] + k_2^{(h)} - k_1^{(h)} > 0$$

by virtue of $\tau > qk_2^{(h)}/[(p-1)k_1^{(a)}]$ and (1.9), we have the desired estimate.

2.2. Proof of Theorem 1.2

First, we derive a lower bound of $H(x, t)$. By the maximum principle, $H(x, t) \geq H_1(x, t) + H_2(x, t)$ for all $x \in \bar{\Omega}$, $t > 0$, where H_1 and H_2 are unique solutions of

$$(2.9) \quad \begin{cases} \tau \frac{\partial H_1}{\partial t} = D A_h H_1 - \mu_h(x) H_1 & \text{for } x \in \Omega, t > 0, \\ B_h H_1 = 0 & \text{for } x \in \partial\Omega, t > 0, \\ H_1(x, 0) = H_0(x) & \text{for } x \in \Omega \end{cases}$$

and

$$(2.10) \quad \begin{cases} \tau \frac{\partial H_2}{\partial t} = D A_h H_2 - \mu_h(x) H_2 + \sigma_h(x) & \text{for } x \in \Omega, t > 0, \\ B_h H_2 = 0 & \text{for } x \in \partial\Omega, t > 0, \\ H_2(x, 0) = 0 & \text{for } x \in \Omega. \end{cases}$$

By the maximum principle again, we see easily

$$H_1(x, t) \geq e^{-k_2^{(h)}t/\tau} \min_{x \in \bar{\Omega}} H_0(x), \quad H_2(x, t) \geq \max\{0, \delta_h - \Gamma_h e^{-k_1^{(h)}t/\tau}\},$$

where δ_h and Γ_h were defined in (1.14). Put $h_m = \min_{x \in \bar{\Omega}} H_0(x)$. Since $\delta_h - \Gamma_h e^{-k_1^{(h)} t/\tau} < 0$ if $t < t_0 = (\tau \log(\Gamma_h/\delta_h))/k_1^{(h)}$ and $\delta_h - \Gamma_h e^{-k_1^{(h)} t/\tau} = 0$ when $t = t_0$, we see that

$$(2.11) \quad H(x, t) \geq \begin{cases} h_m e^{-k_2^{(h)} t_0/\tau} & \text{for } 0 < t < t_0, \\ h_m e^{-k_2^{(h)} t/\tau} + \delta_h - \Gamma_h e^{-k_1^{(h)} t/\tau} & \text{for } t \geq t_0 \end{cases}$$

for all $x \in \bar{\Omega}$. We consider the case where $t \geq t_0$. Letting

$$f(\zeta) = h_m e^{-k_2^{(h)}(t_0+\zeta)/\tau} + \delta_h - \Gamma_h e^{-k_1^{(h)}(t_0+\zeta)/\tau} \quad \text{for } \zeta \geq 0,$$

we obtain the following: if $h_m(\delta_h/\Gamma_h)^{k_2^{(h)}/k_1^{(h)}} \leq \delta_h k_1^{(h)}/k_2^{(h)}$, then $f(\zeta)$ increases monotonically for $\zeta > 0$; if $h_m(\delta_h/\Gamma_h)^{k_2^{(h)}/k_1^{(h)}} > \delta_h k_1^{(h)}/k_2^{(h)}$, then $f(\zeta)$ has the global minimum at

$$\zeta_0 = \frac{\tau}{k_2^{(h)} - k_1^{(h)}} \log \left[\left(\frac{\delta_h}{\Gamma_h} \right)^{(k_2^{(h)} - k_1^{(h)})/k_1^{(h)}} \frac{h_m k_2^{(h)}}{\Gamma_h k_1^{(h)}} \right] > 0.$$

Therefore, we get a lower bound of $H(x, t)$ after some calculations, that is,

$$(2.12) \quad H(x, t) \geq \min \left\{ h_m \left(\frac{\delta_h}{\Gamma_h} \right)^{k_2^{(h)}/k_1^{(h)}}, \delta_h \frac{k_1^{(h)}}{k_2^{(h)}} \right\}.$$

Having obtained a lower bound, the rest of the proof of Theorem 1.2 is exactly the same as that of Theorem 1.1.

2.3. Proof of Proposition 1.3

To verify Proposition 1.3, we follow the proof of Lemma 3.9 in [8] where the ODE case was treated. We use the assumption $H^q > \rho_a A^{p-1}/\mu_a$ and (1.9) to show that $\tau H_t \leq D A_h H - H(k_1^{(h)} - c H^\alpha)$ for some positive constants c and α . Since $H \rightarrow 0$ as $t \rightarrow +\infty$, for any $\delta_0 > 0$ there is a $T > 0$ such that $0 < c H^\alpha \leq \delta_0$ for $t \geq T$, so that $\tau H_t \leq D A_h H - (k_1^{(h)} - \delta_0) H$. Thus, we obtain the decay estimate: $0 < H(x, t) \leq C e^{-(k_1^{(h)} - \delta_0)t/\tau}$ for $t > T$. From this and the assumption of the proposition, we have $A(x, t) \leq \text{const. } H^{q/(p-1)} \leq \text{const. } e^{-(k_1^{(a)} - \delta_0)qt/[\tau(p-1)]}$. On the other hand, Lemma 2.1 yields a lower bound: $A(x, t) \geq \text{const. } e^{-k_2^{(a)} t}$. Comparing these bounds, we conclude that $-k_2^{(a)} + q(k_1^{(h)} - \delta_0)/[\tau(p-1)] \leq 0$. Since δ_0 is arbitrary, we obtain the assertion.

3. Almost decoupled patterns

3.1. Existence

In this section we prove Theorem 1.5. We start with preparing some symbols which are used throughout this section. Letting $\Sigma_{a,e}(x)$ and $\Sigma_{h,D}(x)$

be solutions of (1.11) and (1.12), we also use the notation of (1.14). Since $\max_{x \in \bar{\Omega}} \sigma_a(x) > 0$ and $\max_{x \in \bar{\Omega}} \sigma_h(x) > 0$, it is clear by the maximum principle that

$$(3.1) \quad \begin{cases} \Sigma_{a,\varepsilon}(x) > 0 & \text{and} \quad \|\Sigma_{a,\varepsilon}\|_{L^\infty} \leq \max_{x \in \bar{\Omega}} \sigma_a(x)/k_1^{(a)}, \\ \Sigma_{h,D}(x) \geq \delta_h > 0. \end{cases}$$

Here δ_h depends only on D , A_h , μ_h and σ_h .

Fix $\varepsilon > 0$ and $D > 0$. For simplicity, we put $m_a = \max_{x \in \bar{\Omega}} \sigma_a(x)/k_1^{(a)}$. Let $(A(x), H(x))$ be a stationary solution of (1.1)–(1.3). Putting $A = \Sigma_{a,\varepsilon} + \phi$, $H = \Sigma_{h,D} + \psi$, we seek a solution $(\phi(x), \psi(x))$ of the following equations:

$$(3.2) \quad \begin{cases} \varepsilon^2 A_a \phi - \mu_a(x) \phi + F_A^0 \phi + F_H^0 \psi = -F^0 - R_1(\phi, \psi), \\ D A_h \psi - \mu_h(x) \psi + G_A^0 \phi + G_H^0 \psi = -G^0 - R_2(\phi, \psi) \end{cases}$$

with boundary conditions $B_a \phi = B_h \psi = 0$. Here we have set

$$(3.3) \quad F^0 = \rho_a(\Sigma_{a,\varepsilon}, \Sigma_{h,D}, x) \frac{\Sigma_{a,\varepsilon}^p}{\Sigma_{h,D}^q},$$

$$(3.4) \quad F_A^0 = \frac{\partial \rho_a}{\partial A}(\Sigma_{a,\varepsilon}, \Sigma_{h,D}, x) \frac{\Sigma_{a,\varepsilon}^p}{\Sigma_{h,D}^q} + \rho_a(\Sigma_{a,\varepsilon}, \Sigma_{h,D}, x) p \frac{\Sigma_{a,\varepsilon}^{p-1}}{\Sigma_{h,D}^q},$$

$$(3.5) \quad F_H^0 = \frac{\partial \rho_a}{\partial H}(\Sigma_{a,\varepsilon}, \Sigma_{h,D}, x) \frac{\Sigma_{a,\varepsilon}^p}{\Sigma_{h,D}^q} - \rho_a(\Sigma_{a,\varepsilon}, \Sigma_{h,D}, x) q \frac{\Sigma_{a,\varepsilon}^p}{\Sigma_{h,D}^{q+1}}$$

and G^0 , G_A^0 and G_H^0 are defined by the same formulas as above but ρ_a , p and q are replaced by ρ_h , r and s , respectively. Moreover, $R_1(\phi, \psi)$ and $R_2(\phi, \psi)$ are defined by

$$(3.6) \quad R_1(\phi, \psi) = \rho_a(\Sigma_{a,\varepsilon} + \phi, \Sigma_{h,D} + \psi, x) \frac{(\Sigma_{a,\varepsilon} + \phi)^p}{(\Sigma_{h,D} + \psi)^q} - (F^0 + F_A^0 \phi + F_H^0 \psi),$$

$$(3.7) \quad R_2(\phi, \psi) = \rho_h(\Sigma_{a,\varepsilon} + \phi, \Sigma_{h,D} + \psi, x) \frac{(\Sigma_{a,\varepsilon} + \phi)^r}{(\Sigma_{h,D} + \psi)^s} - (G^0 + G_A^0 \phi + G_H^0 \psi).$$

Define operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}_D$ by

$$\mathcal{L}_\varepsilon = L_\varepsilon + F_A^0 \quad \text{and} \quad \mathcal{L}_D = L_D + G_H^0,$$

where $L_\varepsilon = \varepsilon^2 A_a - \mu_a(x)$ and $L_D = D A_h - \mu_h(x)$. Since L_ε and L_D are invertible, we can prove by the maximum principle that

$$(3.8) \quad \|L_\varepsilon^{-1}\|_{B(C^0(\bar{\Omega}))} \leq 1/k_1^{(a)}, \quad \|L_D^{-1}\|_{B(C^0(\bar{\Omega}))} \leq 1/k_1^{(h)}.$$

Using these estimates, we will show the following lemma.

Lemma 3.1. *The operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}_D$ are invertible if m_a is sufficiently small. Moreover, we have*

$$(3.9) \quad \|\mathcal{L}_\varepsilon^{-1}\|_{B(C^0(\bar{\Omega}))} \leq 2/k_1^{(a)}$$

if $m_a < 1$ and $m_a^{p-1} < \delta_h^q k_1^{(a)} / (2pC_a)$. On the other hand,

$$(3.10) \quad \|\mathcal{L}_D^{-1}\|_{B(C^0(\bar{\Omega}))} \leq 2/k_1^{(h)},$$

provided $m_a^r < \delta_h^{s+1} k_1^{(h)} / (2C_h(\delta_h + s))$.

Proof. We verify the invertibility of $\mathcal{L}_\varepsilon$ and the estimate (3.9). As to $\mathcal{L}_D$, the same argument leads to (3.10).

Since we have

$$\mathcal{L}_\varepsilon \phi = L_\varepsilon(\phi + L_\varepsilon^{-1}[F_A^0 \phi])$$

for $\phi \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $\mathcal{L}_\varepsilon$ is invertible if $\|L_\varepsilon^{-1}[F_A^0 \cdot]\|_{B(C^0(\bar{\Omega}))} < 1$. We obtain from (1.5), (3.1), (3.4) and (3.8) that

$$\|L_\varepsilon^{-1}[F_A^0 \phi]\|_{L^\infty} \leq \frac{C_a m_a^{p-1}}{k_1^{(a)} \delta_h^q} (m_a + p) \|\phi\|_{L^\infty}.$$

Letting $m_a < 1$, we see that $m_a < p$ because of $p > 1$. Therefore, we have

$$\|L_\varepsilon^{-1}[F_A^0 \phi]\|_{L^\infty} \leq 2p \frac{C_a m_a^{p-1}}{k_1^{(a)} \delta_h^q} \|\phi\|_{L^\infty},$$

which implies that the assertion holds if $m_a^{p-1} < \delta_h^q k_1^{(a)} / (2pC_a)$. $\square$

In the following, we assume that m_a is sufficiently small so that both (3.9) and (3.10) hold. We introduce complete metric spaces X and Y for m_a sufficiently small:

$$(3.11) \quad X = \{\phi \in C^0(\bar{\Omega}) \mid \|\phi\|_X \leq \kappa\}, \quad Y = \{\psi \in C^0(\bar{\Omega}) \mid \|\psi\|_Y \leq \kappa\},$$

where $\|\phi\|_X = \|\phi\|_{L^\infty} / m_a^p$ and $\|\psi\|_Y = \|\psi\|_{L^\infty} / m_a^r$. Moreover, κ is a positive constant to be fixed later, and m_a is taken so small that $\kappa m_a^r < \delta_h / 3$. Hence, if $\psi \in Y$, then

$$(3.12) \quad \Sigma_{h,D}(x) + \psi(x) > \frac{2\delta_h}{3} \quad \text{for all } x \in \bar{\Omega}.$$

Defining a mapping $\mathcal{F}(\phi, \psi) = (\mathcal{F}_1(\phi, \psi), \mathcal{F}_2(\phi, \psi))$ from $X \times Y$ into $C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ by

$$\mathcal{F}_1(\phi, \psi) = \mathcal{L}_\varepsilon^{-1}[-F^0 - R_1(\phi, \psi) - F_H^0 \psi],$$

$$\mathcal{F}_2(\phi, \psi) = \mathcal{L}_D^{-1}[-G^0 - R_2(\phi, \psi) - G_A^0 \phi],$$

we see that if (ϕ_*, ψ_*) is a fixed point of $\mathcal{F}(\phi, \psi)$, then it is a solution of (3.2). Indeed, since $\mathcal{L}_\varepsilon^{-1}$ maps $L^P(\Omega)$ into $W_N^{2,P}(\Omega) = \{\phi \in W^{2,P}(\Omega) \mid \mathbf{B}_a \phi|_{\partial\Omega} = 0\}$ for any $1 < P < \infty$, we see that $\phi_* \in C^{1+\alpha}(\bar{\Omega})$ by the Sobolev embedding theorem by taking $P > N$, where $0 < \alpha < 1$. Similarly, $\psi_* \in C^{1+\alpha}(\bar{\Omega})$. Hence the functions $-F^0 - R_1(\phi, \psi) - F_H^0 \psi$ and $-G^0 - R_2(\phi, \psi) - G_A^0 \phi$ are both Hölder continuous on $\bar{\Omega}$. Therefore, the standard elliptic regularity theory implies that (ϕ_*, ψ_*) is a classical solution of (3.2).

We have the following estimates:

Lemma 3.2. *Assume $r \geq 1$. Let $\phi, \hat{\phi} \in X$ and $\psi, \hat{\psi} \in Y$. There is a positive constant $C(\kappa, \rho)$ depending on κ , C_a and C_h besides (p, q, r, s) such that*

$$(3.13) \quad \|R_1(\phi, \psi)\|_{L^\infty} \leq C(\kappa, \rho) C(\delta_h) c(m_a) m_a^p,$$

$$(3.14) \quad \|R_2(\phi, \psi)\|_{L^\infty} \leq C(\kappa, \rho) C(\delta_h) c(m_a) m_a^r,$$

$$(3.15) \quad \|R_1(\phi, \psi) - R_1(\hat{\phi}, \hat{\psi})\|_{L^\infty} \leq C(\kappa, \rho) C(\delta_h) (m_a^{p-1} \|\phi - \hat{\phi}\|_{L^\infty} + m_a^p \|\psi - \hat{\psi}\|_{L^\infty}),$$

$$(3.16) \quad \|R_2(\phi, \psi) - R_2(\hat{\phi}, \hat{\psi})\|_{L^\infty} \leq C(\kappa, \rho) C(\delta_h) (m_a^{r-1} \|\phi - \hat{\phi}\|_{L^\infty} + m_a^r \|\psi - \hat{\psi}\|_{L^\infty}).$$

Here, $C(\delta_h)$ and $c(m_a)$ are positive constants depending only on δ_h and on m_a , respectively, in addition to (p, q, r, s) . Moreover, $c(m_a)$ tends to 0 as $m_a \downarrow 0$.

Lemma 3.2 can be proved by using (3.1), (3.6), (3.7), (3.11) and the following lemma.

Lemma 3.3. *Let $\alpha > 0$, $\beta > 0$, $\phi \in X$ and $\psi \in Y$. Then we have*

$$\left\| \frac{(\Sigma_{a,\varepsilon} + \phi)^\alpha}{(\Sigma_{h,D} + \psi)^\beta} - \frac{\Sigma_{a,\varepsilon}^\alpha}{\Sigma_{h,D}^\beta} \right\|_{L^\infty} \leq C(\delta_h) C(\kappa) (m_a^{(p-1)\alpha} + m_a^{\beta_* r}) m_a^\alpha,$$

where $\beta_* = \min\{\beta, 1\}$, and $C(\kappa)$ is a positive constant depending only on α , β and κ .

This lemma follows easily from the following elementary estimates:

$$(3.17) \quad |a_+^\lambda - b_+^\lambda| \leq \begin{cases} \lambda \max\{a_+^{\lambda-1}, b_+^{\lambda-1}\} |a_+ - b_+| & \text{if } \lambda \geq 1, \\ |a_+ - b_+|^\lambda & \text{if } 0 < \lambda < 1 \end{cases}$$

in which $a_+ = \max\{0, a\}$.

Now we start to verify that $\mathcal{F}$ is a contraction mapping. For $\phi, \hat{\phi} \in X$ and $\psi, \hat{\psi} \in Y$, we define $\|(\phi, \psi) - (\hat{\phi}, \hat{\psi})\|_{X \times Y} = \|\phi - \hat{\phi}\|_X + \|\psi - \hat{\psi}\|_Y$.

By (3.13), we see that

$$\begin{aligned} \|\mathcal{F}_1(\phi, \psi)\|_{L^\infty} &\leq \frac{2}{k_1^{(a)}} (\|F_H^0\|_{L^\infty} \|\psi\|_{L^\infty} + \|F^0\|_{L^\infty} + \|R_1(\phi, \psi)\|_{L^\infty}) \\ &\leq \frac{2}{k_1^{(a)}} \left[\frac{C_a}{\delta_h^q} \left\{ \left(1 + \frac{q}{\delta_h}\right) \kappa m_a^r + 1 \right\} m_a^p + C(\kappa, \rho) C(\delta_h) c(m_a) m_a^p \right]. \end{aligned}$$

Hence

$$(3.18) \quad \frac{\|\mathcal{F}_1(\phi, \psi)\|_{L^\infty}}{m_a^p} \leq \frac{\kappa}{2} \left[\left(1 + \frac{q}{\delta_h}\right) \kappa m_a^r + 1 \right] + \frac{2}{k_1^{(a)}} C(\kappa, \rho) C(\delta_h) c(m_a),$$

where $\kappa = 4C_a/(k_1^{(a)}\delta_h^q)$. Therefore, if we choose m_a so small that

$$\left(1 + \frac{q}{\delta_h}\right) \kappa m_a^r \leq \frac{1}{2} \quad \text{and} \quad \frac{2}{k_1^{(a)}} C(\kappa, \rho) C(\delta_h) c(m_a) \leq \frac{\kappa}{4},$$

then $\mathcal{F}_1(\phi, \psi) \in X$. Similarly, from (3.14), we have

$$\|\mathcal{F}_2(\phi, \psi)\|_{L^\infty} \leq \frac{2}{k_1^{(h)}} \left[\frac{C_h}{\delta_h^s} \{ (m_a^p + r m_a^{p-1}) \kappa + 1 \} m_a^r + C(\kappa, \rho) C(\delta_h) c(m_a) m_a^r \right].$$

We choose $\kappa = \max\{4C_a/(k_1^{(a)}\delta_h^q), 4C_h/(k_1^{(h)}\delta_h^s)\}$ to obtain

$$(3.19) \quad \frac{\|\mathcal{F}_2(\phi, \psi)\|_{L^\infty}}{m_a^r} \leq \frac{\kappa}{2} [(m_a^p + r m_a^{p-1}) \kappa + 1] + \frac{2}{k_1^{(h)}} C(\kappa, \rho) C(\delta_h) c(m_a).$$

Hence, $\mathcal{F}_2(\phi, \psi) \in Y$ if m_a satisfies

$$(m_a^p + r m_a^{p-1}) \kappa \leq \frac{1}{2} \quad \text{and} \quad \frac{2}{k_1^{(h)}} C(\kappa, \rho) C(\delta_h) c(m_a) \leq \frac{\kappa}{4}.$$

We thus have verified that $\mathcal{F}$ maps $X \times Y$ into itself.

Next, it follows from (3.15) that

$$\begin{aligned} \|\mathcal{F}_1(\phi, \psi) - \mathcal{F}_1(\hat{\phi}, \hat{\psi})\|_{L^\infty} &\leq \|\mathcal{L}_\varepsilon^{-1}[F_H^0(\psi - \hat{\psi})]\|_{L^\infty} + \|\mathcal{L}_\varepsilon^{-1}[R_1(\phi, \psi) - R_1(\hat{\phi}, \hat{\psi})]\|_{L^\infty} \\ &\leq \frac{2}{k_1^{(a)}} \left(C_a \frac{m_a^p}{\delta_h^q} + C_a q \frac{m_a^p}{\delta_h^{q+1}} \right) \|\psi - \hat{\psi}\|_{L^\infty} \\ &\quad + \frac{2}{k_1^{(a)}} C(\kappa, \rho) C(\delta_h) (m_a^{p-1} \|\phi - \hat{\phi}\|_{L^\infty} + m_a^p \|\psi - \hat{\psi}\|_{L^\infty}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{\|\mathcal{F}_1(\phi, \psi) - \mathcal{F}_1(\hat{\phi}, \hat{\psi})\|_{L^\infty}}{m_a^p} \\ & \leq \frac{2}{k_1^{(a)}} \left(C_a \frac{1}{\delta_h^q} + C_a q \frac{1}{\delta_h^{q+1}} \right) \frac{\|\psi - \hat{\psi}\|_{L^\infty}}{m_a^r} m_a^r \\ & \quad + \frac{2}{k_1^{(a)}} C(\kappa, \rho) C(\delta_h) \left(m_a^{p-1} \frac{\|\phi - \hat{\phi}\|_{L^\infty}}{m_a^p} + m_a^r \frac{\|\psi - \hat{\psi}\|_{L^\infty}}{m_a^r} \right), \end{aligned}$$

which implies

$$\|\mathcal{F}_1(\phi, \psi) - \mathcal{F}_1(\hat{\phi}, \hat{\psi})\|_X \leq \frac{1}{2} (\|\phi - \hat{\phi}\|_X + \|\psi - \hat{\psi}\|_Y)$$

provided m_a is sufficiently small. Similarly, we obtain, by (3.16), that

$$\begin{aligned} & \|\mathcal{F}_2(\phi, \psi) - \mathcal{F}_2(\hat{\phi}, \hat{\psi})\|_Y \\ & \leq (\|\mathcal{L}_D^{-1}[G_A^0(\phi - \hat{\phi})]\|_{L^\infty} + \|\mathcal{L}_D^{-1}[R_2(\phi, \psi) - R_2(\hat{\phi}, \hat{\psi})]\|_{L^\infty})/m_a^r \\ & \leq C(m_a^{p-1}\|\phi - \hat{\phi}\|_X + m_a^r\|\psi - \hat{\psi}\|_Y) \\ & \leq \frac{1}{2} (\|\phi - \hat{\phi}\|_X + \|\psi - \hat{\psi}\|_Y) \end{aligned}$$

for m_a sufficiently small, where C is a positive constant independent of m_a . Therefore, we see that $\mathcal{F}$ is a contraction mapping in $X \times Y$. Consequently, there exists a unique fixed point (ϕ_*, ψ_*) of $\mathcal{F}$ and $(A_*, H_*) = (\Sigma_{a,e} + \phi_*, \Sigma_{h,D} + \psi_*)$ gives a stationary solution of (1.1)–(1.3) with $\|\phi_*\|_{L^\infty} \leq \kappa m_a^p$ and $\|\psi_*\|_{L^\infty} \leq \kappa m_a^r$.

Next we turn to the case $0 < r < 1$. We assume that $\min_{x \in \bar{\Omega}} \sigma_a(x) > 3\kappa k_2^{(a)} m_a^p$. By the maximum principle, $\Sigma_{a,e}(x) > 3\kappa m_a^p$ for all $x \in \bar{\Omega}$. Note that $m_a > m_a^p$ since $m_a < 1$ and $p > 1$. Therefore, for ϕ and ψ which satisfy $\|\phi\|_{L^\infty} \leq \kappa m_a^p$ and $\|\psi\|_{L^\infty} \leq \kappa m_a^r$, we see that

$$(3.20) \quad \Sigma_{a,e}(x) + t\phi(x) \geq 2\kappa m_a^p,$$

$$\Sigma_{h,D}(x) + t\psi(x) \geq \delta_h - \kappa m_a^r \geq \frac{2\delta_h}{3} \quad (x \in \bar{\Omega}, 0 \leq t \leq 1).$$

Under these assumptions, the same line of arguments as in the case $r \geq 1$ can be applied. Therefore, we see that $\mathcal{F}$ becomes a contraction mapping in $X \times Y$ by using (3.20) and the following lemma:

Lemma 3.4. *Let $0 < \alpha < 1$, $\beta > 0$, $\phi \in X$ and $\psi \in Y$. Then we have*

$$\left\| \frac{1}{(\Sigma_{a,\varepsilon} + \phi)^\alpha (\Sigma_{h,D} + \psi)^\beta} - \frac{1}{\Sigma_{a,\varepsilon}^\alpha \Sigma_{h,D}^\beta} \right\|_{L^\infty} \leq C(\delta_h) C(\kappa) \frac{1 + m_a^{\beta_* r}}{m_a^{2p}},$$

where $\beta_* = \min\{\beta, 1\}$, $C(\kappa)$ is a positive constant depending on κ .

The estimate of $R_2(\phi, \psi) - R_2(\hat{\phi}, \hat{\psi})$ is delicate because this contains the factor $(A_* + \phi)^{r-1}$. To avoid this difficulty we use the weighted norm $\|\cdot\|_Y$ instead of the L^∞ -norm in (3.19), which we handle as follows:

$$\begin{aligned} (3.21) \quad & |R_2(\phi, \psi) - R_2(\hat{\phi}, \hat{\psi})| \\ & \leq \left\| \left(\rho_h(\Sigma_{a,\varepsilon} + \phi, \Sigma_{h,D} + \psi, x) \right. \right. \\ & \quad - \rho_h(\Sigma_{a,\varepsilon} + \hat{\phi}, \Sigma_{h,D} + \hat{\psi}, x) \frac{(\Sigma_{a,\varepsilon} + \phi)^r}{(\Sigma_{h,D} + \psi)^s} \\ & \quad + \rho_h(\Sigma_{a,\varepsilon} + \hat{\phi}, \Sigma_{h,D} + \hat{\psi}, x) \left(\frac{(\Sigma_{a,\varepsilon} + \phi)^r}{(\Sigma_{h,D} + \psi)^s} - \frac{(\Sigma_{a,\varepsilon} + \hat{\phi})^r}{(\Sigma_{h,D} + \hat{\psi})^s} \right) \\ & \quad - \frac{\partial \rho_h}{\partial A}(\Sigma_{a,\varepsilon}, \Sigma_{h,D}, x) \frac{\Sigma_{a,\varepsilon}^r}{\Sigma_{h,D}^s} (\phi - \hat{\phi}) \\ & \quad \left. - r \rho_h(\Sigma_{a,\varepsilon}, \Sigma_{h,D}, x) \frac{\Sigma_{a,\varepsilon}^{r-1}}{\Sigma_{h,D}^s} (\phi - \hat{\phi}) - G_H^0(\psi - \hat{\psi}) \right\|_{L^\infty}. \end{aligned}$$

Since our problem is to estimate the terms with the exponent $r - 1$, we consider the second and fourth terms, I_2 and I_4 , of the right hand side of (3.21). By the mean value theorem we obtain

$$\begin{aligned} \|I_2 + I_4\|_{L^\infty} & \leq r C_h \int_0^1 \left\| \frac{(\Sigma_{a,\varepsilon} + \hat{\phi} + t(\phi - \hat{\phi}))^{r-1}}{(\Sigma_{h,D} + \hat{\psi} + t(\psi - \hat{\psi}))^s} - \frac{\Sigma_{a,\varepsilon}^{r-1}}{\Sigma_{h,D}^s} \right\|_{L^\infty} dt \|\phi - \hat{\phi}\|_{L^\infty} \\ & \quad + r \int_0^1 \left\| \frac{\partial \rho_h}{\partial A}(\Sigma_{a,\varepsilon} + t\hat{\phi}, \Sigma_{h,D} + t\hat{\psi}, x) \right\|_{L^\infty} dt \|\hat{\phi}\|_{L^\infty} \frac{\Sigma_{a,\varepsilon}^{r-1}}{\Sigma_{h,D}^s} \|\phi - \hat{\phi}\|_{L^\infty} \\ & \quad + r \int_0^1 \left\| \frac{\partial \rho_h}{\partial H}(\Sigma_{a,\varepsilon} + t\hat{\phi}, \Sigma_{h,D} + t\hat{\psi}, x) \right\|_{L^\infty} dt \|\hat{\psi}\|_{L^\infty} \frac{\Sigma_{a,\varepsilon}^{r-1}}{\Sigma_{h,D}^s} \|\phi - \hat{\phi}\|_{L^\infty}. \end{aligned}$$

Noting that $r - 1 < 0$, we see from Lemma 3.4 that

$$\begin{aligned}
\|I_2 + I_4\|_{L^\infty} &\leq C(\delta_h)C(\kappa, \rho) \frac{1 + m_a^{s,r}}{m_a^{p(1-r)}} \|\phi - \hat{\phi}\|_{L^\infty} + C(\delta_h)C(\kappa, \rho) \frac{m_a^p}{m_a^{p(1-r)}} \|\phi - \hat{\phi}\|_{L^\infty} \\
&\quad + C(\delta_h)C(\kappa, \rho) \frac{m_a^r}{m_a^{p(1-r)}} \|\phi - \hat{\phi}\|_{L^\infty} \\
&\leq C(\delta_h)C(\kappa, \rho) m_a^r m_a^{r(p-1)} [(1 + m_a^{s,r}) + m_a^p + m_a^r] \|\phi - \hat{\phi}\|_X.
\end{aligned}$$

This implies $\|I_2 + I_4\|_Y \leq C(\delta_h)C(\kappa, \rho) m_a^{r(p-1)} \|\phi - \hat{\phi}\|_X$. Therefore, combining other estimates, we obtain

$$\|R_2(\phi, \psi) - R_2(\hat{\phi}, \hat{\psi})\|_Y \leq C(\delta_h)C(\kappa, \rho)c(m_a)(\|\phi - \hat{\phi}\|_X + \|\psi - \hat{\psi}\|_Y).$$

3.2. Stability

We here recall that $A_*(x) = \Sigma_{a,e}(x) + \phi_*(x)$ and $H_*(x) = \Sigma_{h,D}(x) + \psi_*(x)$; and that $\max_{x \in \bar{\Omega}} \Sigma_{a,e}(x) \leq m_a$, $\min_{x \in \bar{\Omega}} \Sigma_{h,D}(x) = \delta_h$, $\|\phi_*\|_{L^\infty(\Omega)} \leq \kappa m_a^p$ and $\|\psi_*\|_{L^\infty(\Omega)} \leq \kappa m_a^r$. Note also that $\min_{x \in \bar{\Omega}} \Sigma_{a,e}(x) \geq \min_{x \in \bar{\Omega}} \sigma_a(x)/k_2^{(a)}$ and $\min_{x \in \bar{\Omega}} \psi_*(x) \geq 0$. Therefore, we may assume that (i) $\max_{x \in \bar{\Omega}} A_*(x) \leq 2m_a$, (ii) $\min_{x \in \bar{\Omega}} H_*(x) \geq \delta_h$, and (iii) in the case $0 < r < 1$, there exists a positive constant c_{A_*} such that $\min_{x \in \bar{\Omega}} A_*(x) \geq c_{A_*} m_a^p$, whenever $\max_{x \in \bar{\Omega}} \sigma_a(x) \leq m_0$.

The goal in this subsection is to prove the following proposition.

Proposition 3.5. *There exists positive constants m_l and K such that the following statements hold for $0 < m_a \leq m_l$. Let C_ϕ and C_ψ be positive constants satisfying*

$$(3.22) \quad 3K(C_\phi + C_\psi) \leq \begin{cases} \min\{2, \delta_h\} & \text{if } r \geq 1, \\ \min\{c_{A_*}, \delta_h\} & \text{if } 0 < r < 1. \end{cases}$$

If the solution $(A(x, t), H(x, t))$ of the initial-boundary value problem (1.1)–(1.3) satisfies

$$\begin{aligned}
(3.23) \quad \|A(\cdot, 0) - A_*\|_{L^\infty} &\leq \begin{cases} C_\phi m_a^p & \text{if } 0 < r < 1 \\ C_\phi m_a & \text{if } r \geq 1 \end{cases} \quad \text{and} \\
\|H(\cdot, 0) - H_*\|_{L^\infty} &\leq C_\psi,
\end{aligned}$$

then

$$\begin{aligned}
(3.24) \quad \|A(\cdot, t) - A_*\|_{L^\infty} &\leq \begin{cases} 2K(C_\phi + C_\psi) m_a^p e^{-\lambda t} & \text{if } 0 < r < 1 \\ 2K(C_\phi + C_\psi) m_a e^{-\lambda t} & \text{if } r \geq 1, \end{cases} \\
\|H(\cdot, t) - H_*\|_{L^\infty} &\leq 2K(C_\phi + C_\psi) e^{-\lambda t}
\end{aligned}$$

for all $t > 0$. Here, λ is any number satisfying

$$(3.25) \quad 0 < 8\lambda < \min\{k_1^{(a)}, k_1^{(h)}/\tau\}.$$

To begin with, we put

$$A(x, t) = A_*(x) + \phi(x, t) \quad \text{and} \quad H(x, t) = H_*(x) + \psi(x, t)$$

and write down the equations satisfied by (ϕ, ψ) :

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \varepsilon^2 A_a \phi - \mu_a \phi + \rho_a(A_* + \phi, H_* + \psi, x) \frac{(A_* + \phi)^p}{(H_* + \psi)^q} - \rho_a(*) \frac{A_*^p}{H_*^q}, \\ \tau \frac{\partial \psi}{\partial t} &= D A_h \psi - \mu_h \psi + \rho_h(A_* + \phi, H_* + \psi, x) \frac{(A_* + \psi)^r}{(H_* + \psi)^s} - \rho_h(*) \frac{A_*^r}{H_*^s}, \end{aligned}$$

where we have defined

$$\rho_a(*) = \rho_a(A_*(x), H_*(x), x) \quad \text{and} \quad \rho_h(*) = \rho_h(A_*(x)H_*(x), x).$$

By the mean value theorem we have

$$\begin{aligned} &\rho_a(A_* + \phi, H_* + \psi, x) \frac{(A_* + \phi)^p}{(H_* + \psi)^q} - \rho_a(*) \frac{A_*^p}{H_*^q} \\ &= [\rho_a(A_* + \phi, H_* + \psi, x) - \rho_a(A_*, H_*, x)] \frac{(A_* + \phi)^p}{(H_* + \psi)^q} + \rho_a(*) \left[\frac{(A_* + \phi)^p}{(H_* + \psi)^q} - \frac{A_*^p}{H_*^q} \right] \\ &= \frac{(A_* + \phi)^p}{(H_* + \psi)^q} \left(\frac{\partial \rho_a}{\partial A}(A_* + \theta_1 \phi, H_* + \theta_1 \psi, x) \phi + \frac{\partial \rho_h}{\partial H}(A_* + \theta_1 \phi, H_* + \theta_1 \psi, x) \psi \right) \\ &\quad + \rho_a(A_* + \phi, H_* + \psi, x) \left[p \frac{(A_* + \theta_2 \phi)^{p-1}}{(H_* + \theta_2 \psi)^q} \phi - q \frac{(A_* + \theta_2 \phi)^p}{(H_* + \theta_2 \psi)^{q+1}} \psi \right] \end{aligned}$$

with appropriate constants $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ and also a similar expression for $\rho_h(A_* + \phi, H_* + \psi, x)(A_* + \phi)^r/(H_* + \psi)^s - \rho_h(*)A_*^r/H_*^s$. Hence, we define

$$(3.26) \quad \begin{aligned} \gamma_{11}(x, t) &= \frac{(A_* + \phi)^p}{(H_* + \psi)^q} \frac{\partial \rho_a}{\partial A}(A_* + \theta_1 \phi, H_* + \theta_1 \psi, x) \\ &\quad + p \rho_a(A_* + \phi, H_* + \psi, x) \frac{(A_* + \theta_2 \phi)^{p-1}}{(H_* + \theta_2 \psi)^q}, \end{aligned}$$

$$(3.27) \quad \begin{aligned} \gamma_{12}(x, t) &= \frac{(A_* + \phi)^p}{(H_* + \psi)^q} \frac{\partial \rho_a}{\partial H}(A_* + \theta_1 \phi, H_* + \theta_1 \psi, x) \\ &\quad - q \rho_a(A_* + \phi, H_* + \psi, x) \frac{(A_* + \theta_2 \phi)^p}{(H_* + \theta_2 \psi)^{q+1}}, \end{aligned}$$

$$(3.28) \quad \gamma_{21}(x, t) = \frac{(A_* + \phi)^r}{(H_* + \psi)^s} \frac{\partial \rho_h}{\partial A} (A_* + \theta_3 \phi, H_* + \theta_3 \psi, x) \\ + r \rho_h(A_* + \phi, H_* + \psi, x) \frac{(A_* + \theta_4 \phi)^{r-1}}{(H_* + \theta_4 \psi)^s},$$

$$(3.29) \quad \gamma_{22}(x, t) = \frac{(A_* + \phi)^r}{(H_* + \psi)^q} \frac{\partial \rho_h}{\partial H} (A_* + \theta_3 \phi, H_* + \theta_3 \psi, x) \\ - s \rho_h(A_* + \phi, H_* + \psi, x) \frac{(A_* + \theta_4 \phi)^r}{(H_* + \theta_4 \psi)^{s+1}}$$

with the understanding that the θ_j 's are constants in the interval $(0, 1)$ determined by the mean value theorem.

Then (ϕ, ψ) solves the following initial-boundary value problem:

$$(3.30) \quad \frac{\partial \phi}{\partial t} = \varepsilon^2 A_a \phi - (\mu_a - \gamma_{11})\phi + \gamma_{12}\psi \quad \text{for } x \in \Omega, t > 0,$$

$$(3.31) \quad \tau \frac{\partial \psi}{\partial t} = D A_h \psi - (\mu_h - \gamma_{22})\psi + \gamma_{21}\phi \quad \text{for } x \in \Omega, t > 0,$$

$$(3.32) \quad \mathbf{B}_a \phi = \mathbf{B}_h \psi = 0 \quad \text{for } x \in \partial\Omega, t > 0,$$

$$(3.33) \quad \phi(x, 0) = \phi_0(x), \quad \psi(x, 0) = \psi_0(x) \quad \text{for } x \in \Omega.$$

First we derive estimates of the coefficients γ_{ij} ($i, j = 1, 2$).

Lemma 3.6. *Let T be an arbitrary positive number, and $\phi(x, t)$, $\psi(x, t)$ be any continuous functions on the closed domain $\bar{\Omega} \times [0, T]$ satisfying*

$$(3.34) \quad \|\phi\|_{L^\infty(\Omega \times (0, T))} \leq \begin{cases} 2m_a & \text{if } r \geq 1 \\ c_{A_*} m_a^p / 2 & \text{if } 0 < r < 1, \end{cases} \quad \text{and}$$

$$\|\psi\|_{L^\infty(\Omega \times (0, T))} \leq \delta_h / 2.$$

Then there exist positive constants C_{ij} , with $i, j = 1, 2$, independent of m_a in the interval $0 < m_a \leq m_0$ such that

$$(i) \quad |\gamma_{11}(x, t)| \leq C_{11} m_a^{p-1};$$

$$(ii) \quad |\gamma_{12}(x, t)| \leq C_{12} m_a^p;$$

$$(iii) \quad |\gamma_{21}(x, t)| \leq \begin{cases} C_{21} m_a^{r-1} & \text{if } r \geq 1 \\ C_{21} m_a^{p(r-1)} & \text{if } 0 < r < 1; \end{cases}$$

$$(iv) \quad |\gamma_{22}(x, t)| \leq C_{22} m_a^r \\ \text{for all } x \in \bar{\Omega} \text{ and } 0 \leq t \leq T.$$

Proof. By (3.34), we see that $\|A_* + \phi\|_{L^\infty(\Omega)} \leq \max_{x \in \bar{\Omega}} A_*(x) + 2m_a \leq 4m_a$ and $\min_{x \in \bar{\Omega}} (H_*(x) + \psi(x)) \geq \min_{x \in \bar{\Omega}} H_*(x) - \delta_h/2 \geq \delta_h/2$. In addition, $\min_{x \in \bar{\Omega}} (A_*(x) + \phi(x)) \geq \min_{x \in \bar{\Omega}} A_*(x) - c_{A_*} m_a^p/2 \geq c_{A_*} m_a^p/2$ if $0 < r < 1$. Let $0 \leq \vartheta \leq 1$. If $\alpha \geq 0$ and $\beta > 0$ then

$$\frac{(A_* + \vartheta\phi)^\alpha}{(H_* + \vartheta\psi)^\beta} \leq \frac{(4m_a)^\alpha}{(\delta_h/2)^\beta} \leq 2^{2\alpha+\beta} (\delta_h)^{-\beta} m_a^\alpha.$$

If $\alpha < 0$ and $\beta > 0$, then

$$\frac{(A_* + \vartheta\phi)^\alpha}{(H_* + \vartheta\psi)^\beta} \leq \frac{(c_{A_*} m_a^p/2)^\alpha}{(\delta_h/2)^\beta} = c_{A_*}^\alpha \delta_h^{-\beta} m_a^{\alpha p}.$$

Combining these observations and assumptions (1.5) and (1.6), we obtain the assertions of the lemma. $\square$

First we choose $m_l > 0$ so small that

$$(3.35) \quad C_{11} m_a^{p-1} \leq \frac{k_1^{(a)}}{2} \quad \text{and} \quad C_{22} m_a^r \leq \frac{k_1^{(h)}}{2}$$

if $0 < m_a \leq m_l$. Then

$$(3.36) \quad \mu_a(x) - \gamma_{11}(x, t) \geq \frac{k_1^{(a)}}{2} \quad \text{and} \quad \mu_h(x) - \gamma_{22}(x, t) \geq \frac{k_1^{(h)}}{2}$$

for all $x \in \bar{\Omega}$ and $0 \leq t \leq T$ as long as ϕ and ψ satisfy (3.34).

Lemma 3.7. Assume that m_l is chosen so that (3.35) holds for $0 < m_a \leq m_l$. Let (ϕ, ψ) be a solution of the initial-boundary value problem (3.30)–(3.33). For λ satisfying (3.25) put

$$(3.37) \quad U(x, t) = e^{\lambda t} \phi(x, t) \quad \text{and} \quad V(x, t) = e^{\lambda t} \psi(x, t).$$

If (ϕ, ψ) satisfies (3.34) in the domain $\Omega \times (0, T)$ and $0 < m_a \leq m_l$, then

$$(3.38) \quad |||U|||_T \leq \|\phi_0\|_{L^\infty(\Omega)} + \frac{4C_{12}m_a^p}{k_1^{(a)}} |||V|||_T,$$

$$(3.39) \quad |||V|||_T \leq \|\psi_0\|_{L^\infty(\Omega)} + \frac{4C_{21}m_a^{p(r-1)}}{k_1^{(h)}} |||U|||_T \quad \text{if } 0 < r < 1,$$

$$(3.40) \quad |||V|||_T \leq \|\psi_0\|_{L^\infty(\Omega)} + \frac{4C_{21}m_a^{r-1}}{k_1^{(h)}} |||U|||_T \quad \text{if } r \geq 1,$$

where $|||f|||_T = \|f\|_{L^\infty(\Omega \times (0, T))}$.

Proof. Multiply both sides of (3.30) by $|\phi|^{\ell-2}\phi$, with $\ell \geq 2$, and then integrate the resulting equation over Ω . After integration by parts one obtains by virtue of the boundary conditions (3.32) that

$$\begin{aligned} \frac{1}{\ell} \frac{d}{dt} \int_{\Omega} |\phi|^{\ell} dx &= -\varepsilon^2(\ell-1) \int_{\Omega} |\phi|^{\ell-2} \sum_{i,j=1}^N d_{ij}^{(a)} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx \\ &\quad - \int_{\Omega} (\mu_a - \gamma_{11}) |\phi|^{\ell} dx + \int_{\Omega} \gamma_{12} \psi |\phi|^{\ell-2} \phi dx \\ &\leq -\varepsilon^2(\ell-1) \int_{\Omega} |\phi|^{\ell-2} |\nabla \phi|^2 dx - \frac{k_1^{(a)}}{2} \int_{\Omega} |\phi|^{\ell} dx + \int_{\Omega} |\gamma_{12} \psi| |\phi|^{\ell-1} dx, \end{aligned}$$

where we have used (3.34) and (3.36). By making use of Hölder's inequality and then Young's inequality, we obtain

$$\begin{aligned} (3.41) \quad \frac{1}{\ell} \frac{d}{dt} \int_{\Omega} |\phi|^{\ell} dx + \varepsilon^2(\ell-1) \int_{\Omega} |\phi|^{\ell-2} |\nabla \phi|^2 dx + \frac{k_1^{(a)}}{4} \int_{\Omega} |\phi|^{\ell} dx \\ \leq K_a(\ell)^{\ell-1} \int_{\Omega} |\gamma_{12} \psi|^{\ell} dx, \quad \text{where } K_a(\ell) = \frac{4(\ell-1)}{k_1^{(a)} \ell}. \end{aligned}$$

Therefore,

$$\begin{aligned} (3.42) \quad \int_{\Omega} |\phi(x, t)|^{\ell} dx &\leq \int_{\Omega} |\phi_0(x)|^{\ell} dx e^{k_1^{(a)} \ell t/4} \\ &\quad + \ell K_a(\ell)^{\ell-1} \int_0^t e^{-k_1^{(a)} \ell(t-s)/4} \int_{\Omega} |\gamma_{12}(x, t)|^{\ell} |\psi(x, t)|^{\ell} dx dt. \end{aligned}$$

In exactly the same way, from (3.31) we obtain

$$\begin{aligned} (3.43) \quad \frac{\tau}{\ell} \frac{d}{dt} \int_{\Omega} |\psi|^{\ell} dx + D(\ell-1) \int_{\Omega} |\psi|^{\ell-2} |\nabla \psi|^{\ell} dx + \frac{k_1^{(h)}}{4} \int_{\Omega} |\psi|^{\ell} dx \\ \leq K_h(\ell)^{\ell-1} \int_{\Omega} |\gamma_{21} \phi|^{\ell} dx, \quad \text{where } K_h(\ell) = \frac{4(\ell-1)}{k_1^{(h)} \ell} \end{aligned}$$

and

$$\begin{aligned} (3.44) \quad \int_{\Omega} |\psi(x, t)|^{\ell} dx &\leq \int_{\Omega} |\psi_0(x)|^{\ell} dx e^{-k_1^{(h)} \ell t/(4\tau)} \\ &\quad + \frac{\ell}{\tau} K_h(\ell)^{\ell-1} \int_0^t e^{-k_1^{(h)} \ell(t-s)/(4\tau)} \int_{\Omega} |\gamma_{21}(x, t)|^{\ell} |\phi(x, t)|^{\ell} dx dt. \end{aligned}$$

Set

$$U_\ell(t) = \int_{\Omega} |e^{\lambda t} \phi(x, t)|^\ell dx \quad \text{and} \quad V_\ell(t) = \int_{\Omega} |e^{\lambda t} \psi(x, t)|^\ell dx.$$

Then, since $-2\lambda l > -k_1^{(a)}\ell/4$ and $-2\lambda l > -k_1^{(h)}\ell/(4\tau)$, in view of Lemma 3.6 we have from (3.42) and (3.44) that

$$(3.45) \quad U_\ell(t) \leq U_\ell(0)e^{-\lambda \ell t} + \ell K_a(\ell)^{\ell-1} (C_{12}m_a^p)^\ell \int_0^t e^{-\lambda \ell(t-s)} V_\ell(s) ds,$$

$$(3.46) \quad V_\ell(t) \leq V_\ell(0)e^{-\lambda \ell t} + \frac{\ell}{\tau} K_h(\ell)^{\ell-1} (C_{21}m_a^{p(r-1)})^\ell \int_0^t e^{-\lambda \ell(t-s)} U_\ell(s) ds.$$

From (3.45) it follows that

$$U_\ell(t) \leq U_\ell(0)e^{-\lambda \ell t} + \lambda^{-1} K_a(\ell)^{\ell-1} (C_{12}m_a^p)^\ell \sup_{0 < s < T} V_\ell(s)$$

for all $0 \leq t \leq T$. This implies that

$$\sup_{0 < t < T} U_\ell(t)^{1/\ell} \leq U_\ell(0)^{1/\ell} + \lambda^{-1/\ell} K_a(\ell)^{(\ell-1)/\ell} C_{12}m_a^p \sup_{0 < s < T} V_\ell(s)^{1/\ell}.$$

Letting $\ell \rightarrow \infty$, we have

$$(3.47) \quad \sup_{x \in \Omega, 0 < s < T} |U(x, s)| \leq \sup_{x \in \Omega} |U(x, 0)| + [4/k_1^{(a)}] (C_{12}m_a^p) \sup_{x \in \Omega, 0 < s < T} |V(x, s)|.$$

In a similar way we see from (3.44) that

$$(3.48) \quad \sup_{x \in \Omega, 0 < s < T} |V(x, s)| \leq \sup_{x \in \Omega} |V(x, 0)| \\ + [4/k_1^{(h)}] (C_{21}m_a^{p(r-1)}) \sup_{x \in \Omega, 0 < s < T} |U(x, s)|.$$

This completes the proof for the case $0 < r < 1$. The proof of the case $r \geq 1$ is the same and hence is omitted. $\square$

Proof of Proposition 3.5. We treat the case $0 < r < 1$ first. Choose $m_l > 0$ so small that

$$(3.49) \quad \frac{16C_{12}C_{21}}{k_1^{(a)}k_1^{(h)}} m_a^{pr} \leq \frac{1}{2} \quad \text{and} \quad \frac{4C_{21}}{k_1^{(a)}} m_a^{pr} \leq \frac{1}{2} \quad \text{for } 0 < m_a \leq m_l.$$

Put

$$(3.50) \quad K = \max \left\{ \frac{4C_{12}}{k_1^{(h)}}, 2 \right\}$$

and assume that positive constants C_ϕ and C_ψ satisfy

$$(3.51) \quad 3K(C_\phi + C_\psi) \leq \min\{c_{A_*}, \delta_h\}.$$

We claim that if

$$(3.52) \quad \|\phi_0\|_{L^\infty} \leq C_\phi m_a^p \quad \text{and} \quad \|\psi_0\|_{L^\infty} \leq C_\psi,$$

then

$$(3.53) \quad \sup_{x \in \Omega, 0 < t < +\infty} |e^{\lambda t} \phi(x, t)| \leq 2K(C_\phi + C_\psi) m_a^p \quad \text{and} \\ \sup_{x \in \Omega, 0 < t < +\infty} |e^{\lambda t} \psi(x, t)| \leq 2K(C_\phi + C_\psi).$$

Since $K \geq 2$, we see that $U(x, t)$ satisfies $\sup_{x \in \Omega} |U(x, t)| < KC_\phi m_a^p$ and $\sup_{x \in \Omega} |V(x, t)| < KC_\psi$ for $t > 0$ sufficiently small. Hence we assume that there exists a positive number T such that

$$(3.54) \quad \sup_{x \in \Omega} |U(x, t)| < 2K(C_\phi + C_\psi) m_a^p, \\ \sup_{x \in \Omega} |V(x, t)| < 2K(C_\phi + C_\psi) \quad \text{for } 0 \leq t < T$$

and

$$(3.55) \quad \sup_{x \in \Omega} |U(x, T)| = 2K(C_\phi + C_\psi) m_a^p \quad \text{or} \quad \sup_{x \in \Omega} |V(x, T)| = 2K(C_\phi + C_\psi).$$

Bound $\|U\|_T$ on the right-hand side of (3.39) by (3.38) and then use (3.49). We thus obtain

$$(3.56) \quad \|V\|_T \leq 2C_\psi + C_\phi.$$

From (3.56) and (3.38) follows that

$$(3.57) \quad \|U\|_T \leq ((1 + K)C_\phi + 2KC_\psi) m_a^p.$$

Since $K > 1$, we see that $\|U\|_T < 2K(C_\phi + C_\psi) m_a^p$ and $\|V\|_T < 2K(C_\phi + C_\psi)$. This is a contradiction. Therefore, we conclude that (3.53) holds true.

In the case where $r \geq 1$ we argue along the same line as above. Inequality (3.39) is replaced by (3.40), and by making use of these estimates, we can relax the condition on the smallness of the initial data ϕ_0 considerably. In place of $A_* + \phi$ in (3.26)–(3.29) we use $(A_* + \phi)_+ = \max\{A_* + \phi, 0\}$. Obviously, Lemma 3.6 still holds in the case $r \geq 1$ even after this modification. Let C_ϕ, C_ψ be positive numbers satisfying

$$3(C_\phi + C_\psi) \leq \min\{2, \delta_h\}.$$

We choose m_l so small that

$$(3.58) \quad \frac{16C_{12}C_{21}m_a^{p+r-1}}{k_1^{(a)}k_1^{(h)}} \leq \frac{1}{2}, \quad \frac{4C_{12}m_a^{p-1}}{k_1^{(a)}} \leq \frac{1}{2},$$

$$\frac{4C_{21}m_a^r}{k_1^{(h)}} \leq \frac{1}{2} \quad \text{for } 0 < m_a < m_l.$$

We assume that

$$\|\phi_0\|_{L^\infty(\Omega)} \leq C_\phi m_a \quad \text{and} \quad \|\psi_0\|_{L^\infty(\Omega)} \leq C_\psi.$$

We repeat the argument above with $K = 1$ and obtain from (3.40) and (3.38) the estimate

$$\|V\|_T \leq 2C_\psi + C_\phi$$

and then, by making use of (3.58) this time,

$$\|U\|_T \leq \left(\frac{3}{2}C_\phi + C_\psi\right)m_a.$$

Again we reach a contradiction to (3.55). Thus the proof of the proposition is now complete. $\square$

To prove the collapse of patterns in the case where $0 < r < 1$, one needs to relax the condition on the size of the admissible initial data $\phi_0(x)$.

In what follows we assume that $A_0(x) \geq \Sigma_{a,\varepsilon}(x)$ and $H_0(x) \geq \Sigma_{h,D}(x)$. Then by the maximum principle we see that $A(x, t) \geq \Sigma_{a,\varepsilon}(x)$ and $H(x, t) \geq \Sigma_{h,D}(x)$ for all $x \in \bar{\Omega}$ and $t \geq 0$. Therefore, when estimating the coefficients $\gamma_{i,j}$, we may use the following estimates

$$(3.59) \quad \frac{(A_* + \mathfrak{I}\phi)^\alpha}{(H_* + \mathfrak{I}\psi)^\beta} \leq \begin{cases} \delta_h^{-\beta}(A_* + |\phi|)^\alpha & \text{if } \alpha > 0, \\ \delta_h^{-\beta}(\min_{x \in \bar{\Omega}} \Sigma_{a,\varepsilon}(x))^\alpha & \text{if } \alpha < 0 \end{cases}$$

as long as $\phi \geq -\phi_*$ and $\psi \geq -\psi_*$ and $0 \leq \mathfrak{I} \leq 1$.

Let θ_* be any constant such that

$$(3.60) \quad 1 > \theta_* > \max\{1/p, 1-r\}.$$

Choose m_l so that

$$(3.61) \quad \max \left\{ \frac{16C_{12}C_{21}m_a^{p(r-1+\theta_*)}}{k_1^{(a)}k_1^{(h)}}, \frac{4C_{12}m_a^{p(1-\theta_*)}}{k_1^{(a)}}, \frac{4C_{21}m_a^r}{k_1^{(h)}} \right\} \leq \frac{1}{2}$$

if $0 < m_a \leq m_l$.

Let C_ϕ and C_ψ be positive numbers satisfying

$$3(C_\phi + C_\psi)m_a^{\theta_*p} \leq m_a \quad \text{and} \quad 3(C_\phi + C_\psi) \leq \delta_h/2.$$

Under these assumptions, we can easily prove that if

$$(3.62) \quad \begin{aligned} -\phi_*(x) &\leq \phi_0(x) \leq C_\phi m_a^{\theta_*p} \quad \text{and} \\ -\psi_*(x) &\leq \psi_0(x) \leq C_\psi \quad \text{for } x \in \bar{\Omega}, \end{aligned}$$

then

$$(3.63) \quad \begin{aligned} \sup_{x \in \Omega, 0 < t < +\infty} |e^{\lambda t} \phi(x, t)| &\leq 2(C_\phi + C_\psi)m_a^{\theta_*p} \quad \text{and} \\ \sup_{x \in \Omega, 0 < t < +\infty} |e^{\lambda t} \psi(x, t)| &\leq 2(C_\phi + C_\psi). \end{aligned}$$

This will be used in the next subsection.

3.3. Collapse to almost decoupled patterns

We here give a proof of Theorem 1.7, that is, we demonstrate the existence of solutions $(A(x, t), H(x, t))$ of (1.1)–(1.3) which come to the neighborhood of $(A_*(x), H_*(x))$ specified by Proposition 3.5.

If $H_0(x) \geq \max_{x \in \bar{\Omega}} \Sigma_{h,D}(x)$, then $\Psi(x, t) = H(x, t) - \Sigma_{h,D}(x)$ satisfies

$$\begin{cases} \tau \frac{\partial \Psi}{\partial t} = D A_h \Psi - \mu_h(x) \Psi + \rho_h(A, H, x) \frac{A^r}{H^s} & \text{for } x \in \Omega, t > 0, \\ B_h \Psi = 0 & \text{for } x \in \partial\Omega, t > 0, \\ \Psi(x, 0) = H_0(x) - \Sigma_{h,D}(x) & \text{for } x \in \Omega. \end{cases}$$

Hence, the maximum principle implies $H(x, t) \geq \Sigma_{h,D}(x)$ for all $x \in \bar{\Omega}$, $t \geq 0$.

In the following we assume $H(x, t) \geq \delta_h$.

Let $M(t)$ be a solution of the following initial value problem:

$$(3.64) \quad \begin{cases} \frac{dM}{dt} = -k_1^{(a)} M + \frac{C_a}{\delta_h^q} M^p + \max_{x \in \bar{\Omega}} \sigma_a(x), \\ M(0) = \max_{x \in \bar{\Omega}} A_0(x). \end{cases}$$

Note that the algebraic equation (1.18) has two positive roots κ_a and K_a satisfying (1.19) and (1.20) when m_a is sufficiently small. Therefore, if $0 < M(0) < K_a$, then the solution $M(t)$ of (3.64) is bounded for all $t > 0$ and it approaches κ_a as $t \rightarrow +\infty$. We observe from (1.19) that

$$(3.65) \quad \kappa_a = m_a + O(m_a^p) \quad \text{as } m_a \downarrow 0.$$

Now, assuming $\max_{x \in \bar{\Omega}} A_0(x) < K_a$, we consider the following initial-boundary value problem:

$$(3.66) \quad \begin{cases} \frac{\partial \bar{A}}{\partial t} = \varepsilon^2 \mathcal{A}_a \bar{A} - \mu_a(x) \bar{A} + \frac{C_a}{\delta_h^q} \bar{A}^p + \sigma_a(x) & \text{for } x \in \Omega, t > 0, \\ \mathcal{B}_a \bar{A} = 0 & \text{for } x \in \partial\Omega, t > 0, \\ \bar{A}(x, 0) = A_0(x) & \text{for } x \in \Omega. \end{cases}$$

By the maximum principle, we see $M(t) \geq \bar{A}(x, t)$ for all $x \in \bar{\Omega}$, $t > 0$. Since $M(t)$ converges to κ_a as $t \rightarrow +\infty$, we obtain

$$(3.67) \quad \limsup_{t \rightarrow +\infty} \max_{x \in \bar{\Omega}} \bar{A}(x, t) \leq \kappa_a.$$

It is clear, by the maximum principle again, that the solution $A(x, t)$ of (1.1)–(1.3) satisfies

$$(3.68) \quad A(x, t) \leq \bar{A}(x, t) \quad \text{for all } x \in \bar{\Omega}, t > 0.$$

Therefore, it follows from (3.67) and (3.68) that $\max_{x \in \bar{\Omega}} A(x, t) \leq 2\kappa_a$ whenever t is sufficiently large, say $t \geq T$, and m_a is sufficiently small.

To consider the asymptotic behavior of $(A(x, t), H(x, t))$, we need more precise estimate of $A(x, t)$ for t sufficiently large. We see that $\bar{A}(x, t)$ converges to $\bar{A}_*(x)$ as $t \rightarrow +\infty$, where $\bar{A}_*(x)$ is a solution of

$$(3.69) \quad \begin{cases} \varepsilon^2 \mathcal{A}_a \bar{A}_* - \mu_a(x) \bar{A}_* + \frac{C_a}{\delta_h^q} \bar{A}_*^p + \sigma_a(x) = 0 & \text{for } x \in \Omega, \\ \mathcal{B}_a \bar{A}_* = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Moreover, the convergence is uniform in $x \in \bar{\Omega}$. Indeed, it follows from (3.67) that $\max_{x \in \bar{\Omega}} \bar{A}(x, t) \leq 2\kappa_a$ and $\max_{x \in \bar{\Omega}} \bar{A}_*(x) \leq 2\kappa_a$ for $t > T$. Letting $\Phi(x, t) = \bar{A}(x, t + T) - \bar{A}_*(x)$, we see that Φ satisfies

$$(3.70) \quad \begin{cases} \Phi_t = \varepsilon^2 \mathcal{A}_a \Phi - \mu_a(x) \Phi + \frac{C_a}{\delta_h^q} p(\bar{A}_* + \theta \Phi)^{p-1} \Phi & \text{for } x \in \Omega, t > 0, \\ \mathcal{B}_a \Phi = 0 & \text{for } x \in \partial\Omega, t > 0, \\ \Phi(x, 0) = A_0(x) - \bar{A}_*(x) & \text{for } x \in \Omega. \end{cases}$$

Since we can take m_a so small that $-\mu_a(x) + pC_a(\bar{A}_* + \theta \Phi)^{p-1}/\delta_h^q < -2\lambda_0 < 0$ where λ_0 is a positive constant, we obtain by the maximum principle that

$$|\Phi(x, t)| \leq e^{-\lambda_0 t} \max_{x \in \bar{\Omega}} |A_0(x) - \bar{A}_*(x)|,$$

which implies $\lim_{t \rightarrow +\infty} \|\Phi(t)\|_{L^\infty} = 0$.

It follows from (3.68) and the arguments above that there exists a $T_* > 0$ such that

$$(3.71) \quad A(x, t) \leq \bar{A}_*(x) + m_a^p \quad \text{for all } x \in \bar{\Omega}, t > T_*.$$

Since $\bar{A}_*(x) = O(m_a)$ as $m_a \downarrow 0$, we can apply the same argument as that used to prove the existence of the stationary solution $(A_*(x), H_*(x))$, we obtain

$$0 \leq \bar{A}_*(x) - \Sigma_{a,\varepsilon}(x) \leq \bar{\kappa} m_a^p \quad \text{for } x \in \bar{\Omega}$$

whenever m_a is sufficiently small, in which $\bar{\kappa}$ is an appropriate positive constant. Recall that $A_*(x) = \Sigma_{a,\varepsilon}(x) + \phi_*(x)$ with $\|\phi_*\|_{L^\infty} \leq \kappa m_a^p$. Hence, if we set $A(x, t + T_*) = A_*(x) + \phi_a(x, t)$, then ϕ_a must satisfy $\|\phi_a(t)\|_\infty \leq (\bar{\kappa} + 1)m_a^p$. Therefore, the solution $A(x, t)$ satisfies

$$(3.72) \quad |A(x, t) - A_*(x)| \leq (\kappa + \bar{\kappa} + 1)m_a^p$$

for t sufficiently large.

Similarly, we can show

$$(3.73) \quad |H(x, t) - H_*(x)| \leq (\kappa + \bar{\kappa}_h + 1)m_a^r,$$

where $\bar{\kappa}_h$ is an appropriate positive constant. Indeed, by using the estimate $\max_{x \in \bar{\Omega}} A(x, t) \leq 2m_a$ for $t > T_*$, we see

$$(3.74) \quad H(x, t) \leq \bar{H}(x, t) \quad \text{for all } x \in \bar{\Omega}, t > T_*,$$

where $\bar{H}(x, t)$ is a solution of the following initial-boundary value problem:

$$(3.75) \quad \begin{cases} \frac{\partial \bar{H}}{\partial t} = D A_h \bar{H} - \mu_h(x) \bar{H} + C_h \frac{(2m_a)^r}{\delta_h^s} + \sigma_h(x) & \text{for } x \in \Omega, t > 0, \\ B_h \bar{H} = 0 & \text{for } x \in \partial\Omega, t > 0, \\ \bar{H}(x, 0) = H(x, T_*) & \text{for } x \in \Omega. \end{cases}$$

We obtain that $\bar{H}(x, t)$ converges to $\bar{H}_*(x)$ as $t \rightarrow +\infty$ uniformly in $x \in \bar{\Omega}$. Here $\bar{H}_*(x)$ is a solution of

$$(3.76) \quad \begin{cases} D A_h \bar{H}_* - \mu_h(x) \bar{H}_* + C_h \frac{(2m_a)^r}{\delta_h^s} + \sigma_h(x) = 0 & \text{for } x \in \Omega, \\ B_h \bar{H}_* = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

It is easily seen that $\bar{H}_* = \Sigma_{h,D}(x) + O(m_a^r)$ as $m_a \downarrow 0$.

Consequently, it follows from (3.72) and (3.73) that there are positive constants K_ϕ and K_ψ such that

$$\|A(\cdot, t) - A_*\|_{L^\infty} \leq K_\phi m_a^p, \quad \|H(\cdot, t) - H_*\|_{L^\infty} \leq K_\psi m_a^r.$$

Hence we can apply Proposition 3.5 to finish the proof of Theorem 1.7.

4. Concluding remarks

In order to explain the meaning of the various conditions on the initial data leading to the collapse of patterns, it is convenient to introduce the following semi-bounded strips in the first quadrant of the AH -plane. First we define two functions $\Phi(A)$ and $\Psi(A)$:

$$\Phi(A) = \left[\frac{C_a(p-1)}{k_1^{(a)}(p-1) - qk_2^{(h)}/\tau} A^{p-1} \right]^{1/q}, \quad \Psi(A) = \left[(\Gamma_h/\delta_h)^{k_2^{(h)}/k_1^{(h)}} \frac{C_a}{k_1^{(a)}} A^{p-1} \right]^{1/q}.$$

In defining $\Phi(A)$ we assume that $\tau > qk_2^{(h)}/[k_1^{(a)}(p-1)]$. For a positive number A_1 , define semi-bounded strips $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ by

$$\begin{aligned} \mathcal{S}_1(A_1) &= \{(A, H) \mid 0 < A < A_1, H > \Phi(A_1)\}, \\ \mathcal{S}_2(A_1) &= \{(A, H) \mid 0 < A < \min\{A_1, 1\}, H > \Psi(A_1)\}, \\ \mathcal{S}_3 &= \left\{ (A, H) \mid 0 < A < K_a, H \geq \max_{x \in \bar{\Omega}} \Sigma_{h,D}(x) \right\}. \end{aligned}$$

Let $\mathcal{E}_0 = \{(A_0(x), H_0(x)) \mid x \in \bar{\Omega}\}$. Then Condition (1.13) is equivalent to the condition $\mathcal{E}_0 \subset \mathcal{S}_1(\max_{x \in \bar{\Omega}} A_0(x))$, Condition (1.15) is equivalent to the condition $\mathcal{E}_0 \subset \mathcal{S}_2(\max_{x \in \bar{\Omega}} A_0(x))$ and Condition (1.20) is equivalent to the condition $\mathcal{E}_0 \subset \mathcal{S}_3$.

For a solution $(A(x, t), H(x, t))$ of the initial-boundary value problem (1.1)–(1.3), set

$$\mathcal{E}(t) = \{(A(x, t), H(x, t)) \mid x \in \bar{\Omega}\} \quad \text{for } t \geq 0.$$

We may summarize our results as follows: If $\mathcal{E}(t_0)$ happens to be contained in one of the *inhibitor-dominant strips* $\mathcal{S}_1(A_1)$, $\mathcal{S}_2(A_1)$, and $\mathcal{S}_3$ for some A_1 at certain time t_0 , then the solution $(A(x, t), H(x, t))$ converges uniformly to $(A_*(x), H_*(x))$ as $t \rightarrow +\infty$. Here we have defined $A_*(x) \equiv 0$ if $\sigma_a(x) \equiv 0$ and $H_*(x) \equiv 0$ if $\sigma_h(x) \equiv 0$. (Whenever $\sigma_a \equiv 0$, the almost decoupled pattern is in fact decoupled.)

The meaning of the conditions (1.13), (1.15) and (1.20) on initial data may be best illustrated in FIGURE 1 for the most frequently used case $(p, q, r, s) = (2, 1, 2, 0)$:

$$f(A, H) = -A + \frac{A^2}{H} + \sigma_a, \quad g(A, H) = -H + A^2 + \sigma_h,$$

where σ_a and σ_h are nonnegative constants. In this case, $\Phi(A) = [1/(1 - 1/\tau)]A$ and $\Psi(A) = A$.

Fig. 1. The shaded rectangle in each figure shows an “inhibitor-dominant” strip. Fig. a) depicts Case I (Theorem 1.1), Fig. c1) and Fig. c2) display Case II (Theorem 1.2) and Fig. b) depicts Case III (Theorem 1.7). If $\sigma_a > 0$ is sufficiently small compared with σ_h , then $f(A, H) = g(A, H) = 0$ has three nonnegative solutions as in Fig. b), c1), c2). The smallest (in regard to A) is the almost decoupled pattern (A_*, H_*) . If $\sigma_a = \sigma_h = 0$, then we set $(A_*, H_*) = (0, 0)$. Once the set $\mathcal{E}(t)$ is contained in one of those strips at some moment t , the solution converges uniformly to (A_*, H_*) as $t \rightarrow +\infty$. In the figures the filled circle stands for the point (A_*, H_*) ; the filled square stands for a stationary solution.

There are reaction-diffusion systems other than (GM) that admit the collapse of patterns. For example, the activator-inhibitor system with saturation, also proposed by Gierer and Meinhardt,

$$(SGM) \quad \begin{cases} a_t = D_a \Delta a - \mu a + \rho \left(c \frac{a^p}{h^q(1 + \kappa a^p)} + \rho_0 \right), \\ h_t = D_h \Delta h - \nu h + c' \rho' \frac{a^r}{h^s}, \end{cases}$$

has the form (1.1) with $\rho_a = c\rho/(1 + \kappa a^p)$. Here, κ is a positive constant and (1.5) is satisfied uniformly on each compact subinterval of $0 \leq A < +\infty$, which is sufficient to prove all the statements. The following model proposed by MacWilliams [6] also has the same property:

$$(M) \quad \begin{cases} \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - u + \frac{u^p}{u^p + v^q} + \sigma_a, \\ \tau \frac{\partial v}{\partial t} = D \Delta v - v + \frac{\alpha u^r}{u^r + \beta} + \sigma_h, \end{cases}$$

where $\varepsilon, D, \alpha, \beta, \tau$ are positive constants, and σ_a, σ_h are nonnegative functions. The exponents (p, q, r) satisfy (1.9) with $s = 0$. MacWilliams used this system to simulate hydra transplantation experiments.

In view of Theorem 1.5, we can regard the collapse of patterns as a transient phenomenon en route to the almost decoupled pattern (A_*, H_*) whenever the basic production term σ_a for the activator is sufficiently small. In the case $\sigma_a \equiv \sigma_h \equiv 0$, due to the factors v^q and v^r in the denominator, $(u, v) = (0, 0)$ is not an equilibrium of (GM). Rather, it is a sort of singular rest point of saddle-point type, by which we mean that it is approachable only within a sector-like (or cusp-like) domain. The case $\sigma_a \equiv \sigma_h \equiv 0$ can be viewed as a limit of $\sigma_a(x) \rightarrow 0$ and $\sigma_h(x) \rightarrow 0$ subject to a certain condition on the size of $\|\sigma_a\|_\infty$ relative to $\|\sigma_h\|_\infty$. On the other hand, most reaction-diffusion systems do not have singularities in their nonlinearity. For example, the reaction terms of the FitzHugh-Nagumo system are polynomials; for a certain range of parameters it has a constant steady-state solution which is globally stable as an equilibrium of the ordinary differential equations, i.e., it models an excitable medium. In such a case patterns do collapse for a wide range of initial data. In particular, any solution is attracted to the constant steady-state solution once it enters a certain neighborhood. The almost decoupled pattern (A_*, H_*) of the Gierer-Meinhardt system with $\sigma_h(x) \not\equiv 0$ shares the same property. In general, the presence of multiple stable equilibria or closed orbits makes the dynamics very complicated. It appears that some solutions of

the initial-boundary value problem visit the neighborhood of several unstable equilibria or closed orbits one after another before completing a long journey to the final rest state.

It is equally important to know how the basic production terms affect the shape of stationary solutions. In this direction we obtained the following results in a separate paper [17], which were somewhat surprising to us. In the case of a homogeneous medium with $\sigma_h(x) \equiv 0$, there are several results on the existence of stationary solutions of (1.1)–(1.3) with boundary spikes (see, e.g., [18, 5, 10, 9, 11, 20, 12]). We consider the case where $\sigma_a(x)$ is a function of x and see the influence of $\sigma_a(x)$ upon the shape of stationary solutions. In what follows we restrict ourselves to the simplest situation, that is to say, we consider the limiting system, called the *shadow system*, obtained by letting $D \rightarrow +\infty$ in spatial dimension one:

$$(4.1) \quad \frac{\partial A}{\partial t} = \varepsilon^2 \frac{\partial^2 A}{\partial x^2} - A + \frac{A^p}{\xi^q} + \sigma_a(x) \quad \text{for } 0 < x < l, t > 0,$$

$$(4.2) \quad \tau \frac{d\xi}{dt} = -\xi + \frac{1}{l\xi^s} \int_0^l A^r dx \quad \text{for } t > 0,$$

$$(4.3) \quad \frac{\partial A}{\partial x}(0, t) = \frac{\partial A}{\partial x}(l, t) = 0 \quad \text{for } t > 0.$$

We state our results after introducing a function which is crucial to describe the asymptotic form of the activator. Let $w(y)$ be the solution of the following boundary value problem:

$$(4.4) \quad \begin{cases} w'' - w + w^p = 0, \text{ and } w > 0 & \text{for } 0 < y < +\infty, \\ w'(0) = 0, \quad \lim_{y \rightarrow +\infty} w(y) = 0. \end{cases}$$

It is well-known that the solution w is unique and decays exponentially as $y \rightarrow +\infty$: $\sup_{0 < y < \infty} e^y w(y) < +\infty$. Let $\Phi(y)$ be a solution of

$$(4.5) \quad \begin{cases} \Phi'' - \Phi + p w^{p-1} \Phi + p w^{p-1} = 0, & \text{for } 0 < y < +\infty, \\ \Phi'(0) = 0, \quad \lim_{y \rightarrow +\infty} \Phi(y) = 0. \end{cases}$$

This problem is known to have unique solutions (see, e.g., pp. 330–331 of [11]). The following theorem tells us how the term $\sigma_a(x)$ affects the shape of stationary solutions.

Theorem 4.1. *Assume $\max_{0 \leq x \leq l} \sigma_a(x) > 0$. Moreover, assume $\min_{0 \leq x \leq l} \sigma_a(x) > 0$ if $0 < r < 1$. There exists an $\varepsilon_0 > 0$ such that for each*

$\varepsilon \in (0, \varepsilon_0)$ the shadow system has a pair of stationary solutions $(A_{1,\varepsilon}(x), \xi_{1,\varepsilon})$ and $(A_{2,\varepsilon}(x), \xi_{2,\varepsilon})$ satisfying

$$(4.6) \quad A_{1,\varepsilon}(x) = \xi_{1,\varepsilon}^{q/(p-1)} \left\{ w\left(\frac{x}{\varepsilon}\right) + o(1) \right\} + \sigma_a(x) + \sigma_a(0)\Phi\left(\frac{x}{\varepsilon}\right) + o(1),$$

$$(4.7) \quad \xi_{1,\varepsilon} = \left\{ \varepsilon \left(\frac{1}{l} \int_0^\infty w(z)^r dz + o(1) \right) \right\}^{-(p-1)/[qr-(p-1)(s+1)]},$$

$$(4.8) \quad A_{2,\varepsilon}(x) = \xi_{2,\varepsilon}^{q/(p-1)} \left\{ w\left(\frac{l-x}{\varepsilon}\right) + o(1) \right\} + \sigma_a(x) + \sigma_a(l)\Phi\left(\frac{l-x}{\varepsilon}\right) + o(1),$$

$$(4.9) \quad \xi_{2,\varepsilon} = \left\{ \varepsilon \left(\frac{1}{l} \int_0^\infty w(z)^r dz + o(1) \right) \right\}^{-(p-1)/[qr-(p-1)(s+1)]},$$

as $\varepsilon \downarrow 0$. Here, the terms $o(1)$ in (4.6) and (4.8) are uniform in $x \in [0, l]$.

Rigorously speaking, we should notice that $\sigma_a(x)$ is not smooth in general, and hence the term $\sigma_a(x)$ in (4.6) and (4.8) should be replaced with $\Sigma_\varepsilon(x)$ which is the solution of (1.11). As to the stability of these stationary solutions, for example, we have

Theorem 4.2. *Let $r = 2$ and $1 < p < 5$. For each $\alpha \in (0, \alpha_0)$ where α_0 is a sufficiently small number, one can choose an $\varepsilon_1 > 0$ so that for $0 < \varepsilon < \varepsilon_1$, there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that*

- (i) *if $0 < \tau < \tau_1$, then $(A_{1,\varepsilon}(x), \xi_{1,\varepsilon})$ is asymptotically stable; and if $0 < \tau < \tau_2$, then $(A_{2,\varepsilon}(x), \xi_{1,\varepsilon})$ is asymptotically stable;*
- (ii) *$(A_{1,\varepsilon}(x), \xi_{1,\varepsilon})$ is unstable if $\tau > \tau_1$ and $(A_{2,\varepsilon}(x), \xi_{1,\varepsilon})$ is unstable if $\tau > \tau_2$.*

To sum up, under certain conditions on r there exist stable steady-state solutions which have a very large spike at one of the end points of the interval, superimposed onto the distribution of the basic production term $\sigma_a(x)$. In these solutions, therefore, the boundary spike is the major part and the contribution of $\sigma_a(x)$ is relatively small as far as resulting patterns are concerned. This, however, does not rule out the possibility of the existence of a stable stationary solution with an interior spike.

The principal role of σ_a seems to stabilize the system (i.e., to guarantee the boundedness of solutions) and avoid the collapse of patterns, without affecting very much the resulting patterns. This is an important observation from a biological view point. However, the situation changes drastically when we include the inhomogeneity in the cross-reaction coefficient $\rho_a(A, H, x)$. For example, Ren [13] discussed this problem.

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