Radial and Nonradial Solutions for a Semilinear Elliptic System of Schrödinger Type

By

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Abstract. In this article we consider the system of equations $\Delta u_i = p_i(x)f_i(u_1, \dots, u_d)$ for $i = 1, \dots, d$ on $\mathbf{R}^N$, $N \geq 3$ and $d \in \{1, 2, 3, 4, \dots\}$. We prove that the considered system has a bounded positive entire solution under some conditions on p_i and f_i . Also, we give a necessary condition as well as a sufficient condition for a positive radial solution to be large. The method of proving theorems is essentially based on a successive approximation. Furthermore, a non-radially symmetric solution is obtained by using a lower and upper solution method.

Key Words and Phrases. Entire solution, Large solution, Elliptic system.

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1. Introduction

In this paper we study the existence of solutions for the semilinear elliptic system

$$(1) \quad \begin{cases} \Delta u_1 = p_1(x)f_1(u_1, \dots, u_d), \\ \dots \\ \Delta u_d = p_d(x)f_d(u_1, \dots, u_d), \end{cases} \quad \text{in } \mathbf{R}^N,$$

where $N \geq 3$, $d \geq 1$, and the functions p_j , f_j ($j = 1, \dots, d$) are supposed to satisfy the following hypothesis:

(P1) $p_j : \mathbf{R}^N \rightarrow [0, \infty)$ are locally Hölder continuous functions of exponent $\alpha \in (0, 1)$;

(C1) $f_j : [0, \infty)^d \rightarrow [0, \infty)$ are continuously differentiable in each variable, $f_j(0, \dots, 0) = 0$, and $f_j(s_1, \dots, s_d) > 0$ if $s_i > 0$ for some $i = 1, \dots, d$;

(C2) f_j are non-decreasing in each variable;

(C3) $\int_1^\infty [F(s)]^{-1/2} ds = \infty$ ($F(s) = \int_0^s \sum_{i=1}^d f_i(t, \dots, t) dt$).

Similar problems to those we are analyzing here are related to steady-state reaction-diffusion, subsonic fluid flows, electric potentials of some bodies and control theory.

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As an example, the first motivation for studying the above problem stems from the article [8] where the reader observe that such problems arise from the description of the basic stochastic control theory. The controls are to be designed so that the state of the system is constrained to some region. Finding optimal controls is then shown to be equivalent to finding large solutions for a second order semilinear elliptic partial differential equation. In terms of the dynamic programming approach, an explosive solution of (1) corresponds to a value function (or Bellman function) associated to an infinite exit cost (see [8]).

Another motivation comes from the work of [3] where the parabolic problem corresponding to system (1) are models of steady state of non-linear heat conduction through a 2-components mixture.

The numerous applications that lead to favorable answers in order to establish new and significant results for problem (1) as well as the recent results in the field motivate the study of more generally class of problems (1).

The main results of this paper are the following two theorems. They substantially solve the open problem proposed in [7] and complete the results of [2] where only sufficient conditions are obtained.

Theorem 1. *Suppose that (P1), (C1)–(C3) are satisfied. If there exists a positive number ε such that*

$$(2) \quad \int_0^\infty t^{1+\varepsilon} \sum_{j=1}^d \varphi_j(t) dt < \infty \quad \text{where } \varphi_j(t) = \max_{|x|=t} p_j(x),$$

and $r^{2N-2} \sum_{j=1}^d \varphi_j(r)$ is nondecreasing for large r , then system (1) has a nonnegative nontrivial bounded solution on $\mathbf{R}^N$. If, on the other hand, p_j satisfy

$$(3) \quad \int_0^\infty t \sum_{j=1}^d \psi_j(t) dt = \infty \quad \text{where } \psi_j(t) = \min_{|x|=t} p_j(x),$$

and $r^{2N-2} \sum_{j=1}^d \psi_j(r)$ is nondecreasing for large r , then system (1) has no nonnegative nontrivial entire bounded radial solution on $\mathbf{R}^N$.

Theorem 2. *Assume that $p_j : \mathbf{R}^N \rightarrow [0, \infty)$ ($j = 1, \dots, d$) are spherically symmetric continuous functions (i.e. $p_j(x) = p_j(|x|)$). If f_j ($j = 1, \dots, d$) satisfy (C1)–(C3), then the problem (1) has a nonnegative nontrivial entire radial solution. Suppose furthermore that $r^{2N-2} \sum_{j=1}^d p_j(r)$ is nondecreasing for large r . If p_j satisfies*

$$(4) \quad \int_0^\infty \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_j(s) ds dt = \infty \quad \text{for all } j = 1, \dots, d,$$

then any nonnegative nontrivial solution $(u_1, \dots, u_d)$ of (1) is large. Conversely, if (1) has a nonnegative entire large solution, then p_j satisfy

$$(5) \quad \int_0^\infty r^{1+\varepsilon} \sum_{j=1}^d p_j(r) dr = \infty,$$

for every $\varepsilon > 0$.

2. Preliminary result

We recall the following definition of lower and upper solution which are our main tool in the proof of the solvability of problem (1).

Definition 1. A function $(w_1, \dots, w_d) \in [C_{loc}^{2,\alpha}(\mathbf{R}^N)]^d$ ($\alpha \in (0, 1)$) is called a lower solution of the problem (1) if

$$\Delta w_i \geq p_i(x) f_i(w_1, \dots, w_d) \quad \text{in } \mathbf{R}^N \quad \text{for all } i = 1, \dots, d.$$

Definition 2. We say that $(v_1, \dots, v_d) \in [C_{loc}^{2,\alpha}(\mathbf{R}^N)]^d$ ($\alpha \in (0, 1)$) is an upper solution of the problem (1) if

$$\Delta v_i \leq p_i(x) f_i(v_1, \dots, v_d) \quad \text{in } \mathbf{R}^N \quad \text{for all } i = 1, \dots, d.$$

We need the following lemma which can be found in [4, Theorem 5.1, pp. 146]:

Lemma 3. Make the same assumptions on p_j and f_j ($j = 1, \dots, d$) as in Theorem 1. If the problem (1) has a pair of upper and lower bounded solutions $(v_1, \dots, v_d)$ and $(w_1, \dots, w_d)$ fulfilling $w_i(x) \leq v_i(x)$, $i = 1, \dots, d$, $\forall x \in \mathbf{R}^N$ then there exists a bounded function $(u_1, \dots, u_d)$ belonging to $[C_{loc}^{2,\alpha}(\mathbf{R}^N)]^d$ ($\alpha \in (0, 1)$) with

$$w_i(x) \leq u_i(x) \leq v_i(x), \quad i = 1, \dots, d, \quad \forall x \in \mathbf{R}^N$$

and satisfying (1).

3. Proofs of main results

In this section we give the proofs of Theorems 1 and 2.

Proof of the Theorem 1. Assume that (2) holds. In this section, we use the method of upper and lower solutions to show the existence of positive bounded solutions of the problem (1). Our aim is to construct an upper

solution $(v_1, \dots, v_d)$ and a lower solution $(w_1, \dots, w_d)$ for the problem (1). On this purpose, we first prove the existence of $(w_1, \dots, w_d)$ to the system

$$(6) \quad \begin{cases} \Delta w_1(r) = \varphi_1(r)f_1(w_1, \dots, w_d) & \text{for } r := |x|, \\ \dots & \\ \Delta w_d(r) = \varphi_d(r)f_d(w_1, \dots, w_d) & \text{for } r := |x|, \end{cases} \quad \text{in } \mathbf{R}^N.$$

Observe that we can rewrite (6) as follows:

$$\begin{cases} (r^{N-1}w_1(r))' = r^{N-1}\varphi_1(r)f_1(w_1, \dots, w_d), \\ \dots \\ (r^{N-1}w_d(r))' = r^{N-1}\varphi_d(r)f_d(w_1, \dots, w_d). \end{cases}$$

Then radial solutions of (6) are any solution $(w_1, \dots, w_d)$ of the integral equations

$$\begin{cases} w_1(r) = \frac{1}{d} + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} \varphi_1(s) f_1(w_1(s), \dots, w_d(s)) ds dt, \\ \dots \\ w_d(r) = \frac{1}{d} + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} \varphi_d(s) f_d(w_1(s), \dots, w_d(s)) ds dt. \end{cases}$$

To establish a solution to this system, we use successive approximation. Define sequences $\{w_j^k\}_{j=1, \dots, d}^{k \geq 1}$ on $[0, \infty)$ by

$$\begin{cases} w_1^0 = \dots = w_d^0 = \frac{1}{d}, & r \geq 0, \\ w_i^k(r) = \frac{1}{d} + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} \varphi_i(s) f_i(w_1^{k-1}(s), \dots, w_d^{k-1}(s)) ds dt, & i = 1, \dots, d. \end{cases}$$

We remark that, for all $r \geq 0$, $j = 1, \dots, d$ and $k \in \mathbf{N}$

$$w_j^k(r) \geq \frac{1}{d}.$$

Moreover, proceeding by induction we conclude $\{w_j^k\}_{j=1, \dots, d}^{k \geq 1}$ are non-decreasing sequence on $[0, \infty)$. We note that $\{w_j^k\}_{j=1, \dots, d}^{k \geq 1}$ satisfy

$$\begin{cases} [r^{N-1}(w_1^k)]' = r^{N-1}\varphi_1(r)f_1(w_1^{k-1}, \dots, w_d^{k-1}), \\ \dots \\ [r^{N-1}(w_d^k)]' = r^{N-1}\varphi_d(r)f_d(w_1^{k-1}, \dots, w_d^{k-1}). \end{cases}$$

By the monotonicity of $\{w_j^k\}_{j=1,\dots,d}^{k \geq 1}$ we have the inequalities

$$(7) \quad \begin{cases} [r^{N-1}(w_1^k)']' = r^{N-1}\varphi_1(r)f_1(w_1^{k-1}, \dots, w_d^{k-1}) \\ \leq r^{N-1}\varphi_1(r) \sum_{i=1}^d f_i\left(\sum_{j=1}^d w_j^k(r), \dots, \sum_{j=1}^d w_j^k(r)\right), \\ \dots \\ [r^{N-1}(w_d^k)']' = r^{N-1}\varphi_d(r)f_d(w_1^{k-1}, \dots, w_d^{k-1}) \\ \leq r^{N-1}\varphi_d(r) \sum_{i=1}^d f_i\left(\sum_{j=1}^d w_j^k(r), \dots, \sum_{j=1}^d w_j^k(r)\right). \end{cases}$$

Thus, summing up inequalities, we have

$$(8) \quad \left[r^{N-1} \sum_{i=1}^d (w_i^k)' \right]' \leq r^{N-1} \sum_{i=1}^d \varphi_i(r) \sum_{i=1}^d f_i\left(\sum_{j=1}^d w_j^k(r), \dots, \sum_{j=1}^d w_j^k(r) \right).$$

Choose $R > 0$ so that $r^{2N-2} \sum_{j=1}^d \varphi_j(r)$ are non-decreasing for $r \geq R$. We are now ready to show that $w_j^k(R)$ and $(w_j^k(R))'$, both of which are nonnegative, are bounded above independent of k . To do this, let

$$\phi_j^R = \max\{\varphi_j(r) : 0 \leq r \leq R\}, \quad j = 1, \dots, d.$$

Using this and the fact that $(w_j^k)' \geq 0$, we note that (7) yields

$$\begin{cases} [(w_1^k)']' \leq \phi_1^R \sum_{i=1}^d f_i\left(\sum_{i=1}^d w_i^k, \dots, \sum_{i=1}^d w_i^k\right), \\ \dots \\ [(w_d^k)']' \leq \phi_d^R \sum_{i=1}^d f_i\left(\sum_{i=1}^d w_i^k, \dots, \sum_{i=1}^d w_i^k\right). \end{cases}$$

This implies that

$$\sum_{i=1}^d [(w_i^k)']' \leq \left(\sum_{i=1}^d \phi_i^R \right) \sum_{i=1}^d f_i\left(\sum_{i=1}^d w_i^k, \dots, \sum_{i=1}^d w_i^k \right).$$

On the other hand, multiplying this equation by $(\sum_{i=1}^d w_i^k)'$ and integrating we also have

$$(9) \quad \left[\left(\sum_{i=1}^d w_i^k(r) \right)' \right]^2 \leq 2 \left(\sum_{i=1}^d \phi_i^R \right) \int_1^{\sum_{i=1}^d w_i^k(r)} \sum_{i=1}^d f_i(s, \dots, s) ds, \quad 0 \leq r \leq R.$$

Integrating the above inequality, we see that

$$\int_1^{\sum_{i=1}^d w_i^k(R)} \left[\int_1^t \sum_{i=1}^d f_i(s, \dots, s) ds \right]^{-1/2} dt \leq \sqrt{2 \left(\sum_{i=1}^d \phi_i^R \right)} R.$$

It follows from the above relation and by the assumption C3) that $\sum_{i=1}^d w_i^k(R)$ is bounded above independent of k . Using this fact in (9) shows that the same is true of $(\sum_{i=1}^d w_i^k(R))'$. Thus, the sequences $w_i^k(R)$ and $(w_i^k(R))'$ are bounded above independent of k . Finally, we show that the non-decreasing sequences w_i^k is bounded for all $r \geq 0$ and all k . Multiplying (8) by $r^{N-1}(\sum_{i=1}^d w_i^k)'$ and integrating gives

$$\begin{aligned} & \left[r^{N-1} \left(\sum_{i=1}^d w_i^k(r) \right)' \right]^2 \\ & \leq \left[R^{N-1} \left(\sum_{i=1}^d w_i^k(R) \right)' \right]^2 \\ & \quad + 2 \int_R^r s^{2N-2} \sum_{i=1}^d \varphi_i(s) \sum_{i=1}^d f_i \left(\sum_{i=1}^d w_i^k, \dots, \sum_{i=1}^d w_i^k \right) \left(\sum_{i=1}^d w_i^k \right)' ds, \end{aligned}$$

for $r \geq R$. Noting that, by the monotonicity of $s^{2N-2} \sum_{i=1}^d \varphi_i(s)$ for $s \geq R$, we get

$$\left[r^{N-1} \left(\sum_{i=1}^d w_i^k(r) \right)' \right]^2 \leq C + 2r^{2N-2} \sum_{i=1}^d \varphi_i(r) F \left(\sum_{i=1}^d w_i^k(r) \right),$$

where $C = [R^{N-1}(\sum_{i=1}^d w_i^k(R))']^2$, which yields

$$(10) \quad \left(\sum_{i=1}^d w_i^k \right)' \leq \sqrt{C} r^{1-N} + \sqrt{2 \sum_{i=1}^d \varphi_i(r)} \left[F \left(\sum_{i=1}^d w_i^k(r) \right) \right]^{1/2},$$

and hence

$$\frac{d}{dr} \int_{\sum_{i=1}^d w_i^k(R)}^{\sum_{i=1}^d w_i^k(r)} [F(t)]^{-1/2} dt \leq \sqrt[3]{C} r^{1-N} \left[F \left(\sum_{i=1}^d w_i^k(r) \right) \right]^{-1/2} + \left(2 \sum_{i=1}^d \varphi_i(r) \right)^{1/2}.$$

Integrating this and using the fact that

$$\begin{aligned} \left(2 \sum_{i=1}^d \varphi_i(s) \right)^{1/2} &= \left(2s^{1+\varepsilon} \sum_{i=1}^d \varphi_i(s) s^{-1-\varepsilon} \right)^{1/2} \\ &\leq s^{1+\varepsilon} \left(\sum_{i=1}^d \varphi_i(s) \right) + s^{-1-\varepsilon} \end{aligned}$$

for each $\varepsilon > 0$, we have

$$\begin{aligned}
 (11) \quad & \int_{\sum_{i=1}^d w_i^k(R)}^{\sum_{i=1}^d w_i^k(r)} [F(t)]^{-1/2} dt \leq \sqrt[2]{C} \int_R^r t^{1-N} \left[F \left(\sum_{i=1}^d w_i^k(t) \right) \right]^{-1/2} dt \\
 & + \left[\int_R^r t^{1+\varepsilon} \sum_{i=1}^d \varphi_i(t) dt + \int_R^r t^{-1-\varepsilon} dt \right] \\
 & \leq \sqrt[2]{C} \left[F \left(\sum_{i=1}^d w_i^k(R) \right) \right]^{-1/2} \int_R^r t^{1-N} dt \\
 & + \left[\int_R^r t^{1+\varepsilon} \left(\sum_{i=1}^d \varphi_i(t) \right) dt + \frac{1}{\varepsilon R^\varepsilon} \right].
 \end{aligned}$$

The above relation is needed in proving the bounded of the function $\{w_j^k\}_{j=1,\dots,d}^{k \geq 1}$ in the following. Indeed, since for each $\varepsilon > 0$ the right side of this inequality is bounded independent of k (note that $w_i^k(t) \geq 1/d$ for all $i = 1, \dots, d$), so is the left side and hence, in light of C3), the sequence $\{\sum_{i=1}^d w_i^k\}_{k \geq 1}$ is a bounded sequence and so $\{w_j^k\}_{j=1,\dots,d}^{k \geq 1}$ are bounded sequence. Thus $\{w_j^k\}_{j=1,\dots,d}^{k \geq 1} \rightarrow \{w_j\}_{j=1,\dots,d}$ as $k \rightarrow \infty$ and the limit functions $\{w_j\}_{j=1,\dots,d}$ are positive entire solutions of system (6). It is obvious that every solution of (6) is a lower solution to (1).

We let M be the least upper bound of $\sum_{i=1}^d w_i$ and note that

$$M = \lim_{r \rightarrow \infty} \sum_{i=1}^d w_i(r).$$

Now let $\psi_i(t) = \min_{|x|=t} p_i(x)$ ($i = 1, \dots, d$) and $\{v_j\}_{j=1,\dots,d}$ be the positive increasing bounded solutions of

$$\begin{cases} v_1(r) = M + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} \psi_1(s) f_1(v_1(s), \dots, v_d(s)) ds dt, \\ \dots \\ v_d(r) = M + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} \psi_d(s) f_d(v_1(s), \dots, v_d(s)) ds dt, \end{cases}$$

which, of course, satisfies (6) with w_j ($j = 1, \dots, d$) replaced with v_j ($j = 1, \dots, d$) and φ_j ($j = 1, \dots, d$) replaced with ψ_j ($j = 1, \dots, d$). It is also clear that $v_j \geq M$. If we modify the proof of the lower solution $(w_1, \dots, w_d)$ we obtain the existence of an upper solution $(v_1, \dots, v_d)$. Hence the standard upper-lower solution principle implies that (1) has a solution $(u_1, \dots, u_d)$ such that

$$w_i(x) \leq u_i(x) \leq v_i(x), \quad i = 1, \dots, d,$$

(see Lemma 3).

Now assume that (3) holds. Assume to the contrary that there exist a nonnegative nontrivial entire bounded radial solution $(u_1, \dots, u_d)$ on $\mathbf{R}^N$ for the system (1). Assuming $M_i = \sup_{x \in \mathbf{R}^N} u_i(x)$ ($i = 1, \dots, d$) and knowing that $u'_i \geq 0$, we get $\lim_{r \rightarrow \infty} u_i = M_i$. Thus there exists $R > 0$ such that $u_i \geq M_i/2$ ($i = 1, \dots, d$) for $r \geq R$. From conditions of f_i , it follows that

$$f_i(u_1, \dots, u_d) \geq f_i(M_1/2, \dots, M_d/2) := c_0^i, \quad \text{for } r \geq R \text{ and } i = 1, \dots, d.$$

Using this we get the following

$$(12) \quad \begin{aligned} u_1 &\geq u_1(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} \psi_1(s) f_1(u_1(s), \dots, u_d(s)) ds dt \\ &\dots \\ u_d &\geq u_d(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} \psi_d(s) f_d(u_1(s), \dots, u_d(s)) ds dt. \end{aligned}$$

Rearranging the terms, and by using these conditions in (12) follows

$$\sum_{i=1}^d u_i(r) \geq dm_1 + m_2 \frac{1}{N} \int_R^r t \sum_{i=1}^d \psi_i(t) dt \rightarrow \infty \quad \text{as } r \rightarrow \infty,$$

where $m_1 = \min\{u_1(0), \dots, u_d(0)\}$ and $m_2 := \min\{c_0^1, \dots, c_0^d\}$. A contradiction to the boundedness of $\sum_{i=1}^d u_i(r)$. This proves Theorem 1.

Proof of the Theorem 2. It is known from [6, 12, Theorem 2] that the problem

$$(13) \quad \Delta z(r) = \sum_{i=1}^d p_i(r) \sum_{i=1}^d f_i(z(r), \dots, z(r)) \quad \text{for } r := |x|, \quad x \in \mathbf{R}^N$$

has a non-negative non-trivial entire solution. Moreover, for each $R > 0$, there exists $c_R > 0$ such that $z(R) \leq c_R$. Due to the fact that z is radial, we have

$$z(r) = z(0) + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} \sum_{i=1}^d p_i(s) \sum_{i=1}^d f_i(z(s), \dots, z(s)) ds dt \quad \text{for all } r \geq 0.$$

We choose $\beta_1 \in (0, z(0)]$. Define the sequences $\{u_j^k\}_{j=1, \dots, d}^{k \geq 1}$ on $[0, \infty)$ by

$$\begin{cases} u_1^0 = \dots = u_d^0 = \beta_1 & \text{for all } r \geq 0 \\ u_1^k(r) = \beta_1 + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_1(s) f_1(u_1^{k-1}(s), \dots, u_d^{k-1}(s)) ds dt, \\ \dots \\ u_d^k(r) = \beta_1 + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_d(s) f_d(u_1^{k-1}(s), \dots, u_d^{k-1}(s)) ds dt, \end{cases} \quad d \geq 1.$$

With the same arguments as in the proof of Theorem 1 we obtain that $\{u_j^k\}_{j=1,\dots,d}^{k \geq 1}$ are non-decreasing sequence on $[0, \infty)$. Because $z'(r) \geq 0$ follows $0 < \beta_1 \leq z(0) \leq z(r)$ for all $r \geq 0$ and so

$$\begin{aligned} u_1^1(r) &= \beta_1 + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_1(s) f_1(u_1^0(s), \dots, u_d^0(s)) ds dt \\ &\leq z(0) + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} \sum_{i=1}^d p_i(s) \sum_{i=1}^d f_i(z(s), \dots, z(s)) ds dt = z(r). \end{aligned}$$

Thus $u_1^1(r) \leq z(r)$. Similar arguments show that

$$u_j^k \leq z(r) \quad (j = 1, \dots, d) \quad \text{for all } r \in [0, \infty) \text{ and } k \geq 1.$$

Thus, for every $x \in \mathbf{R}^N$ the sequence $u_j^k(|x|)$ ($j = 1, \dots, d$) have subsequences, denoted again by $u_j^k(|x|)$ ($j = 1, \dots, d$), converging and

$$(u_1(|x|), \dots, u_d(|x|)) := \left(\lim_{k \rightarrow \infty} u_1^k(|x|), \dots, \lim_{k \rightarrow \infty} u_d^k(|x|) \right),$$

is an entire radial solution of system (1).

Let $(u_1, \dots, u_d)$ be any non-negative non-trivial entire radial solution of (1) and suppose that p_j ($j = 1, \dots, d$) satisfies (4). Since u_i ($i = 1, \dots, d$) is non-trivial and non-negative, there exists $R > 0$ so that $u_i(R) > 0$. Since $u_i' \geq 0$, we get $u_i(r) \geq u_i(R)$ for $r \geq R$ and thus from

$$\begin{cases} u_1(r) = u_1(0) + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_1(s) f_1(u_1(s), \dots, u_d(s)) ds dt, \\ \dots \\ u_d(r) = u_d(0) + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_d(s) f_d(u_1(s), \dots, u_d(s)) ds dt, \end{cases}$$

we get

$$\begin{cases} u_i(r) = u_i(0) + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_i(s) f_i(u_1(s), \dots, u_d(s)) ds dt, & i = 1, \dots, d \\ \geq u_i(R) + f_i(u_1(R), \dots, u_d(R)) \int_R^r \frac{1}{t^{N-1}} \int_R^t s^{N-1} p_i(s) ds dt \rightarrow \infty & \text{as } r \rightarrow \infty. \end{cases}$$

Conversely, if f_i ($i = 1, \dots, d$) satisfy (C1)–(C3) and $(w_1, \dots, w_d)$ is a non-negative entire large solution of (1), then w_i satisfy

$$\begin{cases} [r^{N-1}(w_1)']' = p_1(x) f_1(w_1, \dots, w_d), \\ \dots \\ [r^{N-1}(w_d)']' = p_d(x) f_d(w_1, \dots, w_d). \end{cases}$$

Then, using the monotonicity of $r^{2N-2} \sum_{j=1}^d p_j(r)$ we can apply similar arguments used in obtaining Theorem 1 to get

$$\left(\sum_{i=1}^d w_i^k(r) \right)' \leq \sqrt{C} r^{1-N} + \sqrt{2 \sum_{i=1}^d \varphi_i(r)} \sqrt{F \left(\sum_{i=1}^d w_i^k(r) \right)}.$$

This conclusion and relation (11) imply that

$$\begin{aligned} (14) \quad \int_{\sum_{i=1}^d w_i(R)}^{\sum_{i=1}^d w_i(r)} [F(t)]^{-1/2} dt &\leq \frac{\sqrt[2]{C}}{[F(\sum_{i=1}^d w_i(R))]^{1/2}} \int_R^r t^{1-N} dt \\ &\quad + \left(\int_R^r t^{1+\varepsilon} \sum_{i=1}^d p_i(t) dt + \frac{1}{\varepsilon R^\varepsilon} \right) \\ &\leq C_R + \int_R^r t^{1+\varepsilon} p_i(t) dt, \end{aligned}$$

where

$$C_R = \sqrt[2]{C} \left[F \left(\sum_{i=1}^d w_i(R) \right) \right]^{-1/2} \frac{R^{N-2}}{N-2} + \frac{1}{\varepsilon R^\varepsilon}.$$

By taking $r \rightarrow \infty$ in (14) we obtain that p_j ($j = 1, \dots, d$) satisfies (5). This completes the proof of the theorem.

We conclude this paper with some remarks:

Remark 1. The time-independent Schrödinger equation in quantum mechanics is

$$(h^2/2m)\Delta u = (V - E)u$$

where $h = 6.625 \cdot 10^{-27}$ erg sec is the Planck constant, m is the mass of a particle moving under the action of a force field described by the potential $V(x, y, z)$ whose wave function is $u(x, y, z, t)$ and the quantity E is the total energy of the particle, problems which falls into the class of equations discussed here.

Remark 2. If (C1)–(C3) are satisfied then

$$\int_1^\infty \left(\int_0^t f_i(s, \dots, s) \right)^{-1/2} ds dt = \infty \quad \text{for all } i = 1, \dots, d.$$

Remark 3 (see [1]). If C1)–C2) and

$$\int_1^\infty \left(\sum_{i=1}^d f_i(s, \dots, s) \right)^{-1} ds = \infty,$$

are satisfied, then

$$\int_1^\infty \left(\int_0^t \sum_{i=1}^d f_i(s, \dots, s) \right)^{-1/2} ds dt = \infty.$$

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