Classical Solutions to a Linear Schrödinger Evolution Equation Involving a Coulomb Potential with a Moving Center of Mass

By

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Abstract. This paper is concerned with Cauchy problems for the linear Schrödinger evolution equation $(i(\partial/\partial t) + \Delta + |x - a(t)|^{-1} + V_1(x, t))u(x, t) = f(x, t)$ in $\mathbf{R}^N \times [0, T]$, subject to initial condition: $u(\cdot, 0) \in H^2(\mathbf{R}^N) \cap H_2(\mathbf{R}^N)$, where $i := \sqrt{-1}$, $N \geq 3$, $T > 0$ and $a : [0, T] \rightarrow \mathbf{R}^N$ expresses the center of the Coulomb potential, V_1 and f are another real-valued potential and an inhomogeneous term, respectively, while $H_2(\mathbf{R}^N) := \{v \in L^2(\mathbf{R}^N); |x|^2 v \in L^2(\mathbf{R}^N)\}$. We show that under some conditions on V_1 and f the equation has a classical solution $u(\cdot) \in C^1([0, T]; L^2(\mathbf{R}^N)) \cap C([0, T]; H^2(\mathbf{R}^N) \cap H_2(\mathbf{R}^N))$.

Key Words and Phrases. Schrödinger equation, Coulomb potential with a moving center of mass, Potentials singular at infinity, Existence and uniqueness of classical solutions, Energy estimates.

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1. Introduction

In this paper we consider Cauchy problems for the Schrödinger equation:

$$(SE) \quad \begin{cases} i \frac{\partial u}{\partial t}(x, t) + \Delta u(x, t) + \frac{u(x, t)}{|x - a(t)|} \\ \quad + V_1(x, t)u(x, t) = f(x, t), & (x, t) \in \mathbf{R}^N \times [0, T], \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^N \end{cases}$$

in $L^2(\mathbf{R}^N)$, $N \geq 3$, under the assumption (which is the same as in Baudouin, Kavian and Puel [1]) that the function $a : [0, T] \rightarrow \mathbf{R}^N$ and the real-valued potential $V_1 : \mathbf{R}^N \times [0, T] \rightarrow \mathbf{R}$ satisfy

- (a) $a \in W^{2,1}(0, T)^N = W^{2,1}(0, T; \mathbf{R}^N)$,
- (V1) $\langle x \rangle^{-2} V_1 \in W^{1,1}(0, T; L^\infty(\mathbf{R}^N))$,
- (V2) $\langle x \rangle^{-2} \nabla V_1 \in L^1(0, T; L^\infty(\mathbf{R}^N))^N$.

Here we employ the usual notations of function spaces. Namely, we denote the Lebesgue and L^2 -type Sobolev spaces by

$$L^p = L^p(\mathbf{R}^N), \quad p \in [1, \infty], \quad H^s = H^s(\mathbf{R}^N), \quad s = 1, 2,$$

with norm $\|\cdot\|_{L^p}$ and

$$\|v\|_{H^1} := (\|v\|^2 + \|\nabla v\|^2)^{1/2} = \|(1 - \Delta)^{1/2}v\|, \quad \|v\|_{H^2} := \|(1 - \Delta)v\|.$$

Also we use the abbreviation for L^2 -norm and inner product:

$$\|\cdot\| := \|\cdot\|_{L^2}, \quad (\cdot, \cdot) := (\cdot, \cdot)_{L^2}.$$

Setting $\langle x \rangle := (1 + |x|^2)^{1/2}$, we can define as

$$H_s = H_s(\mathbf{R}^N) := \{v \in L^2(\mathbf{R}^N); \langle x \rangle^s v \in L^2(\mathbf{R}^N)\}, \quad s > 0.$$

It is easy to see that H_s is the image of H^s under the Fourier transform, with norm $\|v\|_{H_s} := \|\langle x \rangle^s v\|$. In this connection, it is useful to introduce

$$\|v\|_{H^m \cap H_s} := (\|v\|_{H^m}^2 + \|v\|_{H_s}^2)^{1/2} \quad (m, s = 1, 2).$$

Let X be a Banach space with the norm $\|\cdot\|_X$ and $p \in [1, \infty]$. Then $L^p(I; X)$ is the Banach space of X -valued L^p functions over I :

$$L^p(I; X) := \left\{ u; \|u\|_{L^p(I; X)} := \left(\int_I \|u(t)\|_X^p dt \right)^{1/p} < \infty \right\},$$

while $W^{m,p}(I; X)$ is a class of u such that $(\partial/\partial t)^j u \in L^p(I; X)$ for every $0 \leq j \leq m$. Similarly, $C^k(I; X)$ (resp. $C_w(I; X)$) is the space of all X -valued C^k -functions (resp. weakly continuous functions).

Before stating our result we review the main theorem in [4]. In [4] we have rewritten the original proof in [1] correctly and established [1, Theorems 1 and 2] with an inhomogeneous term.

Theorem 1.1 ([4, Theorem 1.2]). *In addition to conditions (a), (V1) and (V2) assume that f satisfies*

$$(1.1) \quad f \in W^{1,1}(0, T; L^2(\mathbf{R}^N)) \cap L^1(0, T; H^1(\mathbf{R}^N) \cap H_2(\mathbf{R}^N)).$$

Then (SE) with initial value $u_0 \in H^2(\mathbf{R}^N) \cap H_2(\mathbf{R}^N)$ has a unique solution satisfying

$$(1.2) \quad u \in W^{1,\infty}(0, T; L^2(\mathbf{R}^N)) \cap C_w([0, T]; H^2(\mathbf{R}^N) \cap H_2(\mathbf{R}^N)),$$

$$(1.3) \quad u \in L^\infty(0, T; H^2(\mathbf{R}^N) \cap H_2(\mathbf{R}^N)) \cap C([0, T]; H^1(\mathbf{R}^N) \cap H_1(\mathbf{R}^N))$$

and the energy estimate:

$$(1.4) \quad \left\| \frac{\partial u}{\partial t}(t) \right\| + \|u(t)\|_{H^2 \cap H_2} \leq C_0(\|u_0\|_{H^2 \cap H_2} + \|f\|_F),$$

where $C_0 > 0$ is a constant depending on a , V_1 and T , while $\|f\|_F$ is given as follows:

$$(1.5) \quad \|f\|_F := \|f\|_{L^\infty(0,T;L^2)} + \int_0^T \left(\left\| \frac{\partial f}{\partial t}(t) \right\| + \alpha_1 \|f(t)\|_{H^1} + \|f(t)\|_{H_2} \right) dt,$$

where $\alpha_1 := \|(d/dt)a\|_{L^\infty(0,T)}$.

In this context the purpose of this paper is to prove an assertion stronger than Theorem 1.1:

Theorem 1.2. *In addition to conditions (a), (V1) and (V2) assume that f satisfies (1.1). Then (SE) with initial value $u_0 \in H^2(\mathbf{R}^N) \cap H_2(\mathbf{R}^N)$ has a unique classical solution*

$$(1.6) \quad u \in C^1([0, T]; L^2(\mathbf{R}^N)) \cap C([0, T]; H^2(\mathbf{R}^N) \cap H_2(\mathbf{R}^N))$$

satisfying the energy estimate (1.4).

This type of result has already been obtained by Wüller [5] and Yajima [6, 7] (for details, see [4]).

Remark 1. If a , V_1 and f are defined on $[-T, T]$, then we can obtain a unique solution satisfying

$$u \in C^1([-T, T]; L^2(\mathbf{R}^N)) \cap C([-T, T]; H^2(\mathbf{R}^N) \cap H_2(\mathbf{R}^N))$$

(see, e.g., Okazawa [3, Remark 1.3]).

In Section 2 we then prepare two lemmas. In Section 3 we will show Theorem 1.2. In [4] we have already showed the existence and uniqueness of a strong solution u to (SE) satisfying (1.2) and (1.3). Thus it suffices to show that u satisfies (1.6). (1.6) will follow from the trick as in Okazawa [3, Proof of Lemma 3.10 (a)].

2. Preliminaries

In this section we prepare two lemmas. They are related to the Gronwall type lemma (see Brézis [2, Lemma A.5]) and the Lebesgue convergence theorem, respectively.

Lemma 2.1. *Let $m(\cdot) \in L^1(0, T)$ be a nonnegative function and $\phi(\cdot) \in L^\infty(0, T)$. Assume that*

$$||\phi(t)|^2 - |\phi(s)|^2| \leq 2 \left| \int_s^t m(r) |\phi(r)| dr \right| \quad \forall t, s \in [0, T].$$

Then $|\phi|(\cdot)$ is absolutely continuous on $[0, T]$:

$$||\phi(t)| - |\phi(s)|| \leq \left| \int_s^t m(r) dr \right| \quad \forall t, s \in [0, T].$$

Lemma 2.2. Let $\varphi \in L^1(0, T)$ and $\psi \in L^\infty(0, T)$. Then

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_s^t \left(\int_r^{r+h} \varphi(\tau) d\tau \right) \psi(r) dr = \int_s^t \varphi(r) \psi(r) dr \quad \forall t, s \in [0, T].$$

At the endpoints, the left-hand side is regarded as a one-sided limit.

Proof. Let $s \leq t$ be fixed and $h > 0$ satisfy $t + h \leq T$. Then we have

$$\frac{1}{h} \int_s^t \left(\int_r^{r+h} \varphi(\tau) d\tau \right) \psi(r) dr = \int_s^{t+h} \varphi(\tau) \frac{1}{h} \left(\int_{\tau-h}^\tau \tilde{\psi}(r) dr \right) d\tau,$$

where

$$\tilde{\psi}(r) = \begin{cases} \psi(r), & r \in (s, t), \\ 0, & \text{otherwise.} \end{cases}$$

Put

$$\Phi_h(\tau) := \varphi(\tau) \frac{1}{h} \int_{\tau-h}^\tau \tilde{\psi}(r) dr.$$

Then for a.a. $\tau \in (s, t)$, we have

$$\lim_{h \downarrow 0} \Phi_h(\tau) = \varphi(\tau) \psi(\tau),$$

$$|\Phi_h(\tau)| \leq |\varphi(\tau)| \cdot \|\psi\|_{L^\infty(0, T)}.$$

Thus the assertion follows from the Lebesgue convergence theorem. In the same way as above, we can show the case of $h < 0$ satisfying $0 \leq t + h$. $\square$

3. Classical solutions of the Schrödinger equation

In this section we show that u satisfies (1.6). First, let us show that a unique solution of (SE) belongs to $C([0, T]; H_2(\mathbf{R}^N))$.

Lemma 3.1. Let u be a unique solution to (SE). Then

$$u(\cdot) \in C([0, T]; H_2(\mathbf{R}^N)).$$

Proof. Since $u(\cdot) \in C_w([0, T]; H_2(\mathbf{R}^N))$, it suffices to show that

$$(3.1) \quad \|u(\cdot)\|_{H_2} \in C[0, T].$$

Put $B_n(x) := \langle x \rangle^2 (1 + n^{-1} \langle x \rangle^2)^{-1}$. Then $B_n(x) \leq n$ and

$$\begin{aligned} \frac{1}{2} \frac{d}{dr} \|B_n u(r)\|^2 &= \operatorname{Re} \left(\frac{\partial u}{\partial r}(r), B_n^2 u(r) \right) \\ &= \operatorname{Im}(-\Delta u(r) + f(r), B_n^2 u(r)) \\ &= \operatorname{Im}(2(\nabla B_n) \cdot \nabla u(r) + B_n f(r), B_n u(r)). \end{aligned}$$

Integrating this equality from s to t , it follows from the Cauchy-Schwarz inequality that

$$| \|B_n u(t)\|^2 - \|B_n u(s)\|^2 | \leq 2 \left| \int_s^t (2 \| \nabla B_n |\nabla u(r)| \| + \|B_n f(r)\|) \|B_n u(r)\| dr \right|.$$

We see from Lemma 2.1, $B_n(x) \leq \langle x \rangle^2$ and $|\nabla B_n(x)| \leq 2|x|$ that

$$| \|B_n u(t)\| - \|B_n u(s)\| | \leq \left| \int_s^t (4 \| |x| \nabla u(r) \| + \| \langle x \rangle^2 f(r) \|) dr \right|.$$

For $\varphi \in H^2 \cap H_2$, we can see $\| |x| \nabla \varphi \| \leq \|\varphi\|_{H^2 \cap H_2}$ (see [4, Proof of Lemma 3.1]). Noting that $B_n(x) \rightarrow \langle x \rangle^2$ ($n \rightarrow \infty$), we have

$$| \|u(t)\|_{H_2} - \|u(s)\|_{H_2} | \leq 4C_0 |t - s| (\|u_0\|_{H^2 \cap H_2} + \|f\|_F) + \left| \int_s^t \|f(r)\|_{H_2} dr \right|.$$

Thus we obtain (3.1). $\square$

Next, we shall show that $u \in C^1([0, T]; L^2(\mathbf{R}^N))$. We use the new unknown function

$$(3.2) \quad v(y, t) := u(x, t) = u(y + a(t), t).$$

Then v satisfies

$$(3.3) \quad \begin{cases} i \frac{\partial v}{\partial t} + \Delta v - i \left(\frac{da}{dt}(t) \cdot \nabla \right) v + \frac{v}{|y|} \\ \quad + V_1(y + a(t), t) v = f(y + a(t), t), & (y, t) \in \mathbf{R}^N \times [0, T], \\ v(y, 0) = u_0(y + a(0)), & y \in \mathbf{R}^N. \end{cases}$$

Roughly speaking, u and v belong to the same class.

Let us show that $u \in C^1([0, T]; L^2(\mathbf{R}^N))$ step by step.

Lemma 3.2. *Let v and u be a unique solution to (3.3) and (SE), respectively. Then*

- (a) $\partial v / \partial t \in C_w([0, T]; L^2(\mathbf{R}^N))$.
- (b) $\|(\partial v / \partial t)(\cdot)\| \in C[0, T]$.
- (c) $\partial v / \partial t \in C([0, T]; L^2(\mathbf{R}^N))$.
- (d) $\partial u / \partial t \in C([0, T]; L^2(\mathbf{R}^N))$.

Proof. Note that the solution v also belongs to both classes (1.2) and (1.3).

(a) We see from (3.3) that

$$i \frac{\partial v}{\partial t} = -\Delta v + i \left(\frac{da}{dt}(t) \cdot \nabla \right) v - \frac{v}{|y|} - V_1(y + a(t), t)v + f(y + a(t), t).$$

(1.2) guarantees the weak continuity of the first term on the right-hand side. On the other hand, it follows from Lemma 3.1, (1.3) and condition (1.1) that the other terms on the right-hand side are strongly continuous in $L^2(\mathbf{R}^N)$. Thus $\partial v / \partial t$ is weakly continuous in $L^2(\mathbf{R}^N)$.

(b) Let $s, t \in [0, T]$. Noting that (1.4) and condition (1.1), it suffices to show that

$$(3.4) \quad \left\| \left\| \frac{\partial v}{\partial t}(t) \right\| - \left\| \frac{\partial v}{\partial s}(s) \right\| \right\| \leq \left\| \int_s^t \left(\gamma_1(r) \|u(r)\|_{H^2 \cap H_2} + \left\| \frac{\partial f}{\partial r}(r) \right\| + \alpha_1 \|f(r)\|_{H^1} \right) dr \right\|,$$

where $\gamma_1 \in L^1(0, T)$ is given by

$$\gamma_1(t) := \left\| \frac{d^2 a}{dt^2}(t) \right\| + \left\| \frac{1}{\langle x \rangle^2} \frac{\partial V_1}{\partial t}(t) \right\|_{L^\infty} + \alpha_1 \left\| \frac{\nabla V_1(t)}{\langle x \rangle^2} \right\|_{L^\infty}.$$

For the proof difference quotients are useful:

$$(D_h \varphi)(y, t) := \frac{1}{h} [\varphi(y, t+h) - \varphi(y, t)] = \frac{1}{h} \int_t^{t+h} \frac{\partial \varphi}{\partial s}(y, s) ds.$$

In the same way as in [4, Section 3.2] we can show that

$$(3.5) \quad \left| \|(D_h v)(t)\| - \|(D_h v)(s)\| \right| \leq \left| \int_s^t I_h(r) dr \right|,$$

where

$$(3.6) \quad \begin{aligned} I_h(t) := & \left\| \left(D_h \frac{da}{dt} \right)(t) \cdot \nabla v(t) \right\| + \|D_h(V_1(\cdot + a(t), t))v(t)\| \\ & + \|D_h(f(\cdot + a(t), t))\|. \end{aligned}$$

In fact, let $r \in [s, t]$. Then we see from (3.3) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dr} \|(D_h v)(r)\|^2 &= \operatorname{Re} \int_{\mathbf{R}^N} \left(D_h \frac{da}{dr} \right)(r) \cdot \nabla v(y, r) \overline{(D_h v)(y, r)} dy \\ &\quad - \operatorname{Im} \int_{\mathbf{R}^N} D_h(V_1(y + a(r), r)) v(y, r) \overline{(D_h v)(y, r)} dy \\ &\quad + \operatorname{Im} \int_{\mathbf{R}^N} D_h(f(y + a(r), r)) \overline{(D_h v)(y, r)} dy. \end{aligned}$$

Integrating this inequality from s to t , we have by the Cauchy-Schwarz inequality that

$$| \|(D_h v)(t)\|^2 - \|(D_h v)(s)\|^2 | \leq 2 \left| \int_s^t I_h(r) \|(D_h v)(r)\| dr \right|.$$

Thus (3.5) is a consequence of Lemma 2.1. First we estimate the second term on the right-hand side of (3.6). Since $\|\langle x + a(t) \rangle^2 v(t)\| = \|u(t)\|_{H_2}$, we have

$$\|D_h(V_1(\cdot + a(t), t))v(t)\| \leq \left\| \frac{1}{\langle \cdot + a(t) \rangle^2} \frac{1}{h} \int_t^{t+h} \frac{\partial}{\partial s} V_1(\cdot + a(s), s) ds \right\|_{L^\infty} \|u(t)\|_{H_2}.$$

Noting that $\langle x + a \rangle \leq (1 + |a - b|)\langle x + b \rangle$, we obtain

$$\begin{aligned} &\left\| \frac{1}{\langle \cdot + a(t) \rangle^2} \frac{1}{h} \int_t^{t+h} \frac{\partial}{\partial s} V_1(\cdot + a(s), s) ds \right\|_{L^\infty} \\ &\leq \left\| \frac{1}{h} \int_t^{t+h} (1 + |a(t) - a(s)|)^2 \frac{1}{\langle \cdot + a(s) \rangle^2} \frac{\partial}{\partial s} V_1(\cdot + a(s), s) ds \right\|_{L^\infty} \\ &\leq (1 + \alpha_1 h)^2 \frac{1}{h} \int_t^{t+h} \left\| \frac{1}{\langle \cdot + a(s) \rangle^2} \frac{\partial}{\partial s} V_1(\cdot + a(s), s) \right\|_{L^\infty} ds. \end{aligned}$$

The first and third terms on the right-hand side of (3.6) are estimated in the same way. Letting $h \rightarrow 0$, we obtain (3.4) from (3.5) by virtue of Lemma 2.2.

(c) is a direct consequence of (a) and (b).

(d) Note that

$$(3.7) \quad \frac{\partial v}{\partial t} - \frac{da}{dt} \cdot \nabla v \in C([0, T]; L^2(\mathbf{R}^N)).$$

Let $\varphi \in C_0^\infty(\mathbf{R}^N)$. Then, by virtue of (3.2), we have

$$\begin{aligned} \left(\frac{\partial u}{\partial t}(t), \varphi \right) &= \left(\frac{\partial v}{\partial t}(t, \cdot - a(t)) - \left(\frac{da}{dt}(t) \cdot \nabla \right) v(t, \cdot - a(t)), \varphi \right) \\ &= \left(\frac{\partial v}{\partial t}(t, \cdot) - \left(\frac{da}{dt}(t) \cdot \nabla \right) v(t, \cdot), \varphi(\cdot + a(t)) \right). \end{aligned}$$

It follows from condition (a) that $\varphi(\cdot + a(\cdot)) \in C([0, T]; L^2(\mathbf{R}^N))$. Thus we obtain

$$\frac{\partial u}{\partial t} \in C_w([0, T]; L^2(\mathbf{R}^N)).$$

On the other hand, we see from (3.2) and (3.7) that

$$\left\| \frac{\partial u}{\partial t}(\cdot) \right\| = \left\| \frac{\partial v}{\partial t}(\cdot) - \left(\frac{da}{dt}(\cdot) \cdot \nabla \right) v(\cdot) \right\| \in C[0, T].$$

Therefore the assertion follows. $\square$

Remark 2. At the last part of proof of Lemma 3.2(b) we use Lemma 2.2. However, we do not need Lemma 2.2 if we use the approximations of a , V_1 and f (See [4, Proof of Lemma 3.4]).

Now we are in a position to prove that $u \in C([0, T]; H^2(\mathbf{R}^N))$.

Lemma 3.3. *Let u be a unique solution of (SE). Then*

$$\Delta u \in C([0, T]; L^2(\mathbf{R}^N)).$$

Proof. In the same way as in Lemma 3.1 it suffices to show that

$$\|\Delta u(\cdot)\| \in C[0, T].$$

First we show that $\Delta v \in C([0, T]; L^2(\mathbf{R}^N))$, where v is a unique solution to (3.3). We see from (3.3) that

$$\Delta v = -i \frac{\partial v}{\partial t} + i \left(\frac{da}{dt}(t) \cdot \nabla \right) v - \frac{v}{|y|} - V_1(y + a(t), t)v + f(y + a(t), t).$$

It follows from Lemmas 3.1, 3.2(c), (1.3) and condition (1.1) that all terms on the right-hand side are strongly continuous in $L^2(\mathbf{R}^N)$. Thus we have $\Delta v \in C([0, T]; L^2(\mathbf{R}^N))$. This yields that $\|\Delta u(\cdot)\| = \|\Delta v(\cdot)\| \in C[0, T]$. $\square$

Proof of Theorem 1.2. Combining Lemmas 3.1–3.3, we obtain (1.6). $\square$

Remark 3. We can obtain the same conclusion of Theorem 1.2, even if we replace $V_0 = |x - a(t)|^{-1}$ with $|x - a(t)|^{-\alpha}$ in (SE), where α is a positive

constant such that $\alpha \leq 2$ and $\alpha < N/2$. In this case we use not only Hardy's but also Rellich's inequalities.

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