Maximal Regularity of Hölder Type for Abstract Parabolic Evolution Equations*

By

Atsushi YAGI
(Osaka University, Japan)

Abstract. This paper shows Hölder type maximal regularity for the Cauchy problem of linear abstract parabolic evolution equations in Banach spaces. In author’s monograph [20, Chapter 3], such regularity has already been established in the case when the domains of the linear operators appearing in the equations are temperately varying with respect to the time variable. This paper is devoted to handling the more general case when the domains are completely varying. Maximal regularity is known to be an essential property of linear parabolic differential equations.

Key Words and Phrases. Abstract parabolic equation, Maximal regularity.

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1. Introduction

We study the maximal regularity of solutions to the Cauchy problem for a linear abstract parabolic evolution equation

\[
\begin{aligned}
\frac{dU}{dt} + A(t)U &= F(t), \quad 0 < t \leq T, \\
U(0) &= U_0,
\end{aligned}
\]

in a Banach space \(X\). Here, \(A(t), 0 \leq t \leq T\), is a family of densely defined, closed linear operators acting in \(X\) and each \(-A(t)\) is assumed to be the generator of an analytic semigroup on \(X\), more precisely to satisfy the conditions (2.1) and (2.2).

Let the initial value \(U_0\) be in \(\mathcal{D}(A(0)^{\beta})\) with \(0 < \beta \leq 1\) and let the external force function \(F\) belong to the space of weighted Hölder continuous functions \(\mathbb{F}^\beta,\sigma((0, T]; X)\) (whose precise definition will be given in Section 2). It is then possible to show that both the derivative \(dU/dt\) and \(A(t)U\) belong to the same function space. Such a regularity property of solutions is called maximal regularity of Hölder type and is one of essential properties of solutions of parabolic evolution equations. We note that the space \(\mathbb{F}^\beta,\sigma((0, T]; X)\) which

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may seem to be somewhat artificial is used in view of the fact that, for any operator $A$ satisfying (2.1) and (2.2), $\frac{d(e^{-tA}U_0)}{dt} = -Ae^{-tA}U_0$ always belongs to $\mathcal{F}^{\beta,\alpha}((0,T];X)$ provided that $U_0$ is in $\mathcal{D}(A^\beta)$ (see Theorem 2.1). In other words, the weighted Hölder function space $\mathcal{F}^{\beta,\alpha}((0,T];X)$ is a natural function space in which to consider maximal regularity for (1.1). It was first introduced by Osaki-Yagi [11] for treating semilinear abstract parabolic evolution equations.

Maximal regularity of Hölder type was first studied in the papers of Da Prato-Grisvard [1, 2]. They use the methods of operational equations, that is, they rewrite the evolution equation $\frac{dU}{dt} + A(t)U = F(t)$ into an operational equation $\mathcal{B}U + \mathcal{A}U = F$ in the space of Hölder continuous functions, where $\mathcal{B}U = \frac{dU}{dt}$ and $\mathcal{A}U = A(t)U$, and devise a solution formula $U = (\mathcal{B} + \mathcal{A})^{-1}F$. These results were afterward developed and applied to partial differential equations by Lunardi [10]. Favini-Yagi [5] then handled degenerate abstract evolution equations by extending the methods; similarly, Guidotti [8] handled singular abstract parabolic evolutions. In the meantime, Favini-Yagi [7, Section 3.4] employ semigroup methods to show similar maximal regularity results, that is, they utilize the evolution operator $U(t,s)$ which gives the solution of (1.1) via Duhamel’s formula $U(t) = U(t,0)U_0 + \int_0^t U(t,s)F(s)ds$. These methods were then developed in [20, Chapter 3] for farther reaching results. Although semigroup methods are more technical than those of operational equations in making use of refined properties of $U(t,s)$, they can yield more general results. In fact, the initial values $U_0$ are allowed to be in $\mathcal{D}(A(0)^\beta)$ with any exponent $0 < \beta \leq 1$ and the external force functions $F(t)$ are allowed to have a singular behavior like $t^{\beta-1}$ at $t = 0$. This provides much needed flexibility in the study of nonlinear parabolic problems.

We say that linear operators $A(t)$ have temperately varying domains if there exists an exponent $0 < \nu \leq 1$ such that for any $0 \leq s, t \leq T$, it holds that

\begin{equation}
\mathcal{D}(A(t)) \subset \mathcal{D}(A(s)^\nu).
\end{equation}

(If (1.2) is satisfied with $\nu = 1$, this means that $A(t)$ have constant domains.) When (1.2) fails for any $0 < \nu \leq 1$, we say that $A(t)$ have completely varying domains. As mentioned before, the author has already established the desired maximal regularity [20, Chapter 3] in the former case. So, in this paper we are concerned with the general case of completely varying domains. We will make the assumption that $A(t)^{-1}$ is strongly, continuously differentiable with respect to $t$ together with the range condition (2.3) and estimate (2.4). Such assumptions were first introduced by Tanabe [12] for studying (1.1); indeed, according to [12, 17] (cf. also [18]), (2.3) and (2.4) allow us to construct the evolution operator $U(t,s)$ which gives the mentioned representation of the solution $U(t)$ of (1.1). The maximal regularity to be shown, i.e., $U_0 \in \mathcal{D}(A(0)^\beta)$ and
$F \in F^{\beta,\alpha}((0,T];X)$ imply $dU/dt$, $A(t)U \in F^{\beta,\alpha}((0,T];X)$, is not necessarily an automatic consequence of such a construction of $U(t,s)$. We need some more refined properties on $U(t,s)$ than those obtained in [12, 17] which must be derived from the conditions (2.3) and (2.4) alone.

Unfortunately, when $A(t)$ have completely varying domains, we do not have any convenient sufficient conditions which imply (2.3) and (2.4). This is a quite different situation from the case of temperately varying domains. For example, as Problem 5.1 in Section 5 shows, one can no longer expect existence of any fixed intermediate space $Y$ such that $\mathcal{R}(d(A(t)^{-1})/dt) \subset \mathcal{D}(A(t)^{\nu})$ for all $0 \leq t \leq T$. Verification of (2.3) and (2.4) is therefore not so easy. We shall present in Section 5, however, some application to singular diffusion equations.

As well known in the large literature (see for instance [7, 9, 10, 13, 14, 20]), the theory of abstract parabolic evolution equations can in fact cover various concrete parabolic differential equations. As in [18, 19], we have to treat evolution equations in which the coefficient operators $-A(t)$ are not necessarily the generator of an analytic semigroup but merely the generator of an infinitely differentiable semigroup. Or as in [6, 7], $A(t)$ may be a multivalued linear operator of $X$. Analogous maximal regularity for these generalized equations has been studied by Favini-Yagi [5] and more recently by Favaron [3]. Nevertheless establishing such a regularity in a satisfactory manner for these extended classes of evolution equations seems to remain as a hard problem.

2. Structural assumptions and preliminary

Let $X$ be a Banach space with norm $\| \cdot \|$. Consider a family of densely defined, closed linear operators $A(t)$, $0 \leq t \leq T$, acting in $X$. We assume that each $A(t)$ has its spectrum $\sigma(A(t))$ in a fixed sectorial open domain

\[ \sigma(A(t)) \subset \Sigma = \{ \lambda \in \mathbb{C}; |\arg \lambda| < \omega \}, \]

where $0 < \omega < \pi/2$, and its resolvent $(\lambda - A(t))^{-1}$ satisfies

\[ \| (\lambda - A(t))^{-1} \|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}, \quad \lambda \notin \Sigma, \ 0 \leq t \leq T, \]

with some constant $M > 0$. As $\sigma(A(t))$ is a closed set, (2.1) implicitly means that $0 \notin \sigma(A(t))$, namely, $A(t)^{-1}$ is a bounded operator on $X$. We further assume that $A(t)^{-1}$ is strongly, continuously differentiable on $X$ for $0 \leq t \leq T$ and that its derivative satisfies the range condition

\[ \mathcal{R} \left( \frac{dA(t)^{-1}}{dt} \right) \subset \mathcal{D}(A(t)^{\nu}), \quad 0 \leq t \leq T, \]
as well as the estimate

\[(2.4) \quad \left\| A(t)^{y} \frac{dA(t)^{-1}}{dt} \right\|_{\mathcal{L}(X)} \leq N, \quad 0 \leq t \leq T, \]

for some exponent \(0 < y \leq 1\) and some constant \(N > 0\).

Let us next list several immediate consequences from the structural assumptions (2.1), (2.2), (2.3) and (2.4) on \(A(t)\).

Since \(A(t)^{-1}\) is continuously differentiable, we have

\[\sup_{0 \leq t \leq T} \|d(A(t)^{-1})/dt\|_{\mathcal{L}(X)} < \infty.\]

As \(A(t)^{-1} = A(0)^{-1} + \int_{0}^{t} d(A(s)^{-1})/dsds\), this then implies that

\[\|A(t)^{-1}\|_{\mathcal{L}(X)} \leq D, \quad 0 \leq t \leq T,\]

with some constant \(D > 0\). Consequently, there exists some constant \(\delta > 0\) independent of \(t\) such that, if \(|\lambda| \leq \delta\), then \(\lambda \in \rho(A(t))\). Therefore,

\[\sigma(A(t)) \subset \Sigma_{\delta} = \{\lambda \in C; |\arg \lambda| < \omega \text{ and } |\lambda| > \delta\}, \quad 0 \leq t \leq T.\]

As may be well known, (2.2) together with (2.1) yields that each \(-A(t)\) generates an analytic semigroup \(e^{-\tau A(t)}\), \(0 \leq \tau < \infty\), on \(X\). And for \(\tau > 0\), it is given by the Dunford integral

\[(2.5) \quad e^{-\tau A(t)} = \frac{1}{2\pi i} \int_{\Gamma} e^{-\tau \lambda} (\lambda - A(t))^{-1} d\lambda, \quad 0 \leq \tau \leq T,\]

in the space \(\mathcal{L}(X)\), where \(\Gamma\) is a suitable integral contour lying in \(C - \Sigma_{\delta}\). Furthermore, for \(0 < \theta < \infty\), let \(A(t)^{\theta}\) denote the fractional power of \(A(t)\) with exponent \(\theta\). Then, for \(\tau > 0\), \(A(t)^{\theta} e^{-\tau A(t)}\) is given by

\[(2.6) \quad A(t)^{\theta} e^{-\tau A(t)} = \frac{1}{2\pi i} \int_{\Gamma'} \lambda^{\theta} e^{-\tau \lambda} (\lambda - A(t))^{-1} d\lambda, \quad 0 \leq \tau \leq T, 0 \leq \theta < \infty,\]

where \(\Gamma'\) is a suitable integral contour lying \(C - \Sigma_{\delta} - (-\infty, 0]\). For example, we can set \(\Gamma' = \Gamma^1 \cup \Gamma^2\) in such a way that \(\Gamma^1 : \lambda = \delta e^{i\theta}, \, |\theta| \leq \omega\), and \(\Gamma^2 : \lambda = re^{i\theta}, \, \delta \leq r < \infty\). It is known that, for \(0 \leq \theta \leq 2\),

\[(2.7) \quad \|A(t)^{\theta} e^{-\tau A(t)}\|_{\mathcal{L}(X)} \leq C\tau^{-\theta}, \quad 0 \leq \tau \leq T, \quad \tau > 0.\]

The differentiability of \(A(t)^{-1}\) implies that the resolvent \((\lambda - A(t))^{-1}\), \(\lambda \notin \Sigma_{\delta}\), too, is strongly differentiable for \(0 \leq t \leq T\) and its derivative is given as

\[(2.8) \quad \frac{d}{dt} (\lambda - A(t))^{-1} = -A(t)(\lambda - A(t))^{-1} \frac{dA(t)^{-1}}{dt} A(t)(\lambda - A(t))^{-1}.\]
Then, (2.2) and (2.4) yield that
\[
\frac{\partial}{\partial t} (\lambda - A(t))^{-1} \leq C|\lambda|^{-\nu}, \quad 0 \leq t \leq T, \lambda \notin \Sigma.
\]
Moreover, this estimate can be generalized to the following form. For any \(0 \leq \theta \leq \nu\),
\[
\left\| A(t)^{\theta} \frac{\partial}{\partial t} (\lambda - A(t))^{-1} \right\|_{\mathscr{L}(X)} \leq C|\lambda|^{-\theta - \nu}, \quad 0 \leq t \leq T, \lambda \notin \Sigma.
\]
Utilizing this we obtain that, for \(0 \leq \theta \leq 2\),
\[
\| A(t)^{\theta} e^{-\tau A(t)} - A(s)^{\theta} e^{-\tau A(s)} \|_{\mathscr{L}(X)} \leq C\tau^{\theta - 1}|t - s|, \quad 0 \leq s, t \leq T, \tau > 0.
\]
For \(n = 1, 2, 3, \ldots\), let \(A_n(t)\) be the Yosida approximation of \(A(t)\) which is indeed defined by
\[
A_n(t) = nA(t)(n + A(t))^{-1}, \quad 0 \leq t \leq T.
\]
By definition, \(A_n(t)\) are all a bounded operator on \(X\). We can easily observe that each \(A_n(t)\) satisfies the same condition as (2.1) (with the same \(\omega\)) and satisfies the estimate
\[
\|(\lambda - A_n(t))^{-1}\|_{\mathscr{L}(X)} \leq \frac{\tilde{M}}{|\lambda|}, \quad 0 \leq t \leq T, \lambda \notin \Sigma,
\]
with some constant \(\tilde{M}\) independent of \(n\). As the domain of \(A_n(t)\) is the whole space \(X\), it is trivial that the same condition as (2.3) is satisfied by \(A_n(t)\). But, more strongly, \(A_n(t)\) does satisfy the estimate
\[
\left\| A_n(t)^{\nu} \frac{dA_n(t)}{dt}^{-1} \right\|_{\mathscr{L}(X)} \leq \tilde{N}, \quad 0 \leq t \leq T,
\]
with some uniform constant \(\tilde{N}\) for \(n\). Proof of (2.13) is however not so easy. It is actually accomplished by using the following fact whose proof was given by [20, Propositions 3.3 and 3.4].

**Proposition 2.1.** For \(n = 1, 2, 3, \ldots\), it holds that \(A_n(t)^{\nu} A(t)^{-\nu} = (1 + n^{-1} A(t))^{-\nu}, \ 0 \leq t \leq T\). As a consequence, \(\|A_n(t)^{\nu} A(t)^{-\nu}\|_{\mathscr{L}(X)} \leq C\) and, as \(n \to \infty\), \(A_n(t)^{\nu} A(t)^{-\nu} \to I\) strongly on \(X\) for \(0 \leq t \leq T\).

The family of Yosida approximations \(A_n(t)\) is thus verified to satisfy the similar conditions as (2.1), (2.2) and (2.4) with the same \(\omega\) and \(\nu\) as \(A(t)\) and with suitable constants \(\tilde{M}\) and \(\tilde{N}\) uniform for \(n\). Naturally, the similar estimate as (2.7) and (2.11) hold true uniformly for \(A_n(t)\).
In addition, it is also known that, as \( n \to \infty \),
\[
(\lambda - A_n(t))^{-1} \to (\lambda - A(t))^{-1}
\]
strongly on \( X \) for \( \lambda \notin \Sigma \).

Throughout the paper, we shall denote by \( C \) a universal constant which is determined only by \( \omega, \nu, M, N, D \) and \( \delta \). So it may change from occurrence to occurrence.

**Remark 2.1.** In \([20, \text{Chapter 3}]\), \( A(t) \) was assumed to satisfy in addition to (1.2) a Hölder condition of the form
\[
\| A(t)^{-1} [A(t)^{-1} - A(s)^{-1}] \|_{\mathcal{X}(X)} \leq N|t - s|^\mu, \quad 0 \leq s, t \leq T,
\]
with some exponent \( 0 < \mu \leq 1 \) such that \( \mu + \nu > 1 \). We may then regard (2.4) as an extreme case of this where, roughly speaking, \( \nu = 0 \) and \( \mu > 1 \).

**Remark 2.2.** In \([17]\), \( d(A(t)^{-1})/dt \) was assumed to satisfy the estimate
\[
\left\| A(t)(\lambda - A(t))^{-1} \frac{dA(t)^{-1}}{dt} \right\|_{\mathcal{X}(X)} \leq \frac{N}{|\lambda|}, \quad \lambda \notin \Sigma, 0 \leq t \leq T,
\]
for some exponent \( 0 < \nu \leq 1 \). Actually, this condition is equivalent to (2.4). By the moment inequality of the fractional powers of operators, (2.4) readily implies (2.15); meanwhile, according to Watanabe \([15]\), (2.15) implies (2.4) in a restrictive sense that the \( \nu \) in (2.4) must be replaced by an arbitrarily smaller exponent than that of (2.15).

Let us here introduce a space of external force functions \( F(t) \).

**Definition.** For two exponents \( 0 < \sigma < \beta \leq 1 \), \( \mathcal{X}^{\beta, \sigma}((0, T]; X) \) denotes the space of \( X \)-valued continuous functions defined on \((0, T]\) (resp. \([0, T]\)) if \( 0 < \beta < 1 \) (resp. if \( \beta = 1 \)) such that:

1. When \( \beta < 1 \), \( t^{1-\beta}F(t) \) has a limit as \( t \to 0 \). (By definition, even if \( \beta = 1 \), this assertion remains true.)
2. \( F \) is Hölder continuous with exponent \( \sigma \) and with weight \( s^{1-\beta+\sigma} \), i.e.,
\[
\sup_{0 \leq s < t \leq T} \frac{s^{1-\beta+\sigma}\|F(t) - F(s)\|}{(t - s)^\beta} = \sup_{0 \leq t \leq T} \sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma}\|F(t) - F(s)\|}{(t - s)^\beta} < \infty.
\]
3. As \( t \to 0 \),
\[
\omega_F(t) = \sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma}\|F(t) - F(s)\|}{(t - s)^\beta} \to 0.
\]

It is easy to see that \( \mathcal{X}^{\beta, \sigma}((0, T]; X) \) is a Banach space with norm
\[
\| F \|_{\mathcal{X}^{\beta, \sigma}} = \sup_{0 \leq t \leq T} t^{1-\beta}\|F(t)\| + \sup_{0 \leq s < t \leq T} \frac{s^{1-\beta+\sigma}\|F(t) - F(s)\|}{(t - s)^\beta}.
\]
The following result gives us a motivation to introduce such a function space.

**Theorem 2.1.** Let $A$ be a densely defined, closed linear operator satisfying (2.1) and (2.2), and let $U_0 \in \mathcal{D}(A^\beta)$ with $0 < \beta \leq 1$. Then, it holds true that $Ae^{-tA}U_0 \in \mathcal{D}^{\beta,\sigma}((0, T]; X)$ for any $\sigma$ such that $0 < \sigma < \beta$.

For the proof, see [20, Theorem 2.27].

We conclude this section by noticing some important convergence properties of $e^{-tA}$ as $t \to 0$.

**Theorem 2.2.** Let $A$ be a densely defined, closed linear operator satisfying (2.1) and (2.2). Then, for $0 < \theta < 1$, $t^\theta A^\theta e^{-tA}$ converges as $t \to 0$ to 0 strongly on $X$. For $0 \leq \theta < 1$, $t^{-\theta}[e^{-tA} - 1]A^{-\theta}$ converges as $t \to 0$ to 0 strongly on $X$.

For the proof, see [20, (2.130) and (2.131)].

3. Basic properties of evolution operator

Let $A(t)$, $0 \leq t \leq T$, satisfy the structural assumptions (2.1)–(2.4). This section is devoted to reviewing construction of an evolution operator for $A(t)$. We will argue along the similar way as in [17, Theorem 1].

3.1. Evolution operator for $A_n(t)$

For $n = 1, 2, 3, \ldots$, let $A_n(t)$ be the Yosida approximation of $A(t)$. The evolution operator $U_n(t, s)$, $0 \leq s \leq t \leq T$, for $A_n(t)$ is defined as a solution to the initial value problem for a differential equation

$$
\begin{align*}
\frac{dU_n}{dt} + A_n(t)U_n &= 0, \quad s \leq t \leq T, \\
U_n(s) &= I,
\end{align*}
$$

in $\mathcal{L}(X)$ with initial time $0 \leq s < T$, $I$ being the identity operator of $X$. Therefore, by definition,

$$
\frac{\partial}{\partial t} U_n(t, s) = -A_n(t)U_n(t, s), \quad 0 \leq s \leq t \leq T,
$$

and $U_n(s, s) = I$. On the other hand, $U_n(t, s)$ can be observed to be differentiable for $s$, too, and to satisfy the equation

$$
\frac{\partial}{\partial s} U_n(t, s) = U_n(t, s)A_n(s), \quad 0 \leq s \leq t \leq T.
$$
It is then immediate to verify from (3.1) and (3.2) that $U_n(t, s)$ satisfies the property

$$U_n(t, r) U_n(r, s) = U_n(t, s), \quad 0 \leq r \leq t \leq T.$$  

Indeed, (3.1) and (3.2) imply that $U_n(t, r) U_n(r, s)$ is independent of $r$.

### 3.2. Integral equations for $U_n(t, s)$

Let us introduce two integral equations satisfied by $U_n(t, s)$.

On account of (3.1), we see that

$$U_n(t, s) = \int_s^t \frac{\partial}{\partial \tau} \left[ e^{-(t-\tau)A_n(\tau)} U_n(\tau, s) \right] d\tau$$

$$= \int_s^t \left[ \frac{\partial}{\partial \tau} e^{-(t-\tau)A_n(\tau)} - e^{-(t-\tau)A_n(\tau)} A_n(\tau) \right] U_n(\tau, s) d\tau.$$

Hence, it is obtained that

$$U_n(t, s) = e^{-(t-s)A_n(s)} + \int_s^t P_n(t, \tau) U_n(\tau, s) d\tau,$$

where

$$P_n(t, s) = \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) e^{-(t-s)A_n(s)}, \quad 0 \leq s \leq t \leq T.$$

Similarly, on account of (3.2),

$$U_n(t, s) = e^{-(t-s)A_n(t)} = -\int_s^t U_n(t, \tau) \left[ A_n(\tau) e^{-(t-s)A_n(\tau)} + \frac{\partial}{\partial \tau} e^{-(t-s)A_n(\tau)} \right] d\tau$$

$$= -\int_s^t U_n(t, \tau) \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial s} \right) e^{-(t-s)A_n(\tau)} d\tau,$$

where

$$Q_n(t, s) = \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) e^{-(t-s)A_n(t)}, \quad 0 \leq s \leq t \leq T.$$

Furthermore, we operate $A_n(t)$ to this equality and put $W_n(t, s) = A_n(t) U_n(t, s) - A_n(t) e^{-(t-s)A_n(t)}, \quad 0 \leq s \leq t \leq T$. Then,

$$W_n(t, s) = R_n(t, s) - \int_s^t W_n(t, \tau) Q_n(\tau, s) d\tau,$$

where $R_n(t, s)$ is given by

$$R_n(t, s) = e^{-(t-s)A_n(t)} Q_n(\tau, s) d\tau.$$
Remark 3.1. We have here introduced newly the integral equation (3.7) for $A_n(t) U_n(t,s) - A_n(t) e^{-(t-s)A_n(t)}$, although in the paper [17], we handled an integral equation satisfied by $A_n(t) U_n(t,s) - A_n(s) e^{-(t-s)A_n(s)}$. This modification is, in fact, essential for establishing the maximal regularity. For instance, it is impossible to derive directly the refined properties of the evolution operator like (4.1) and (4.2) from the mentioned integral equation, namely, [17, (1.10)].

3.3. Convergence of $U_n(t,s)$

We now prove strong convergence of $U_n(t,s)$ and $W_n(t,s)$ as $n \to \infty$ on the basis of (2.14) by employing the dominate convergence theorems on the Volterra type integral equations, see [20, Theorems 1.31 and 1.32].

Consider first the equation (3.4). From (2.5) and (3.5) it follows that

$$P_n(t,s) = \frac{1}{2\pi i} \int_R e^{-(t-s)\lambda} \frac{\partial}{\partial s} (\lambda - A_n(s))^{-1} d\lambda.$$ 

Therefore, the same estimate as (2.9) obtained by replacing $A(t)$ with $A_n(t)$ yields that

$$\|P_n(t,s)\|_{\mathcal{D}(X)} \leq C(t-s)^{\nu-1}, \quad 0 \leq s < t \leq T.$$

In addition, by (2.12) and (2.14), $P_n(t,s)$ converges as $n \to \infty$ to the operator

$$P(t,s) = \frac{1}{2\pi i} \int_R e^{-(t-s)\lambda} \frac{\partial}{\partial s} (\lambda - A(s))^{-1} d\lambda, \quad 0 \leq s < t \leq T,$$

strongly on $X$. Then by virtue of [20, Theorem 1.31], it is deduced that $U_n(t,s)$ satisfies the uniform estimate

$$\|U_n(t,s)\|_{\mathcal{D}(X)} \leq C, \quad 0 \leq s \leq t \leq T,$$

and as well converges to a bounded operator $U(t,s)$ strongly on $X$ for each $0 \leq s \leq t \leq T$. Indeed, the limit $U(t,s)$ can be characterized as a solution to an integral equation

$$U(t,s) = e^{-(t-s)A(s)} + \int_s^t P(t,\tau) U(\tau,s) d\tau.$$

Consider next the equation (3.7). From (3.6) it is seen that

$$Q_n(t,s) = \frac{1}{2\pi i} \int_R e^{-(t-s)\lambda} \frac{\partial}{\partial t} (\lambda - A_n(t))^{-1} d\lambda.$$ 

Then, by the same reason as for $P_n(t,s)$, we obtain that $Q_n(t,s)$ satisfies the uniform estimates

$$\|Q_n(t,s)\|_{\mathcal{D}(X)} \leq C(t-s)^{\nu-1}, \quad 0 \leq s < t \leq T.$$
and as well converges as \( n \to \infty \) to the operator

\[
Q(t,s) = \frac{1}{2\pi i} \int e^{-(t-s)\lambda} \frac{\partial}{\partial t} (\lambda - A(t))^{-1} d\lambda, \quad 0 \leq s < t \leq T,
\]

strongly on \( X \).

Verification of the similar uniform estimate and the strong convergence on \( R_n(t,s) \) is, however, more delicate.

**Proposition 3.1.** \( R_n(t,s) \) satisfies the estimates

\[
\|R_n(t,s)\|_{\mathcal{L}(X)} \leq C(t-s)^{v-1}, \quad 0 \leq s < t \leq T,
\]

and converges as \( n \to \infty \) to a bounded operator \( R(t,s) \), \( 0 \leq s < t \leq T \), strongly on \( X \).

**Proof.** We decompose \( R_n(t,s) \) into

\[
R_n(t,s) = \int_s^t \left[ A_n(t) e^{-(t-\tau)A_n(\tau)} - A_n(t) e^{-(t-\tau)} A_n(\tau) \right] Q_n(\tau,s)d\tau \\
\quad - \int_s^t A_n(t) e^{-(t-\tau)} A_n(\tau) e^{-\tau} A_n(\tau) R_n(\tau,s)d\tau,
\]

where \( \rho \) is any fixed exponent such that \( 0 < \rho < v \). Then, by the same reason as for (3.13), it is seen that, for \( 0 \leq \theta \leq v \),

\[
\|A_n(t)\|_{\mathcal{L}(X)} \leq C(t-s)^{v-\theta-1}, \quad 0 \leq s < t \leq T.
\]

Therefore, this together with (2.7) (in which \( A(t) \) is replaced with \( A_n(t) \)) implies

\[
\|A_n(t) e^{-(t-\tau)} A_n(\tau) e^{-\tau} A_n(\tau)\|_{\mathcal{L}(X)} \leq C(t-s)^{v-1} (t-s)^{v-\rho-1}.
\]

In the meantime, the estimate (2.11) obtained by replacing \( A(t) \) with \( A_n(t) \) yields that

\[
\|A_n(t) e^{-(t-\tau)} A_n(\tau) - A_n(t) e^{-(t-\tau)} A_n(\tau)\|_{\mathcal{L}(X)} \leq C(t-s)^{v-1}.
\]

The desired estimate on \( R_n(t,s) \) is now immediate.

Strong convergence is also obvious from that of \( Q_n(t,s) \). Indeed, \( R_n(t,s) \) is observed to converge to the operator

\[
R(t,s) = \int_s^t \left[ A(\tau) e^{-(t-\tau)} A(\tau) - A(t) e^{-(t-\tau)} A(\tau) \right] Q(\tau,s)d\tau \\
\quad - \int_s^t A(\tau) e^{-(t-\tau)} A(\tau) e^{-\tau} A(\tau) R(\tau,s)d\tau
\]

strongly on \( X \). Of course, both (3.16) and (3.17) remain true even if we replace \( A_n(t) \) with \( A(t) \).
It is ready to apply [20, Theorem 1.32] to $W_n(t,s)$. As a result, we conclude that $W_n(t,s)$ satisfies the uniform estimate

\[
\|W_n(t,s)\|_{\mathcal{F}(X)} \leq C(t - s)^{-1}, \quad 0 \leq s < t \leq T,
\]

and as well converges as $n \to \infty$ to a bounded operator $W(t,s)$ strongly on $X$. As before, $W(t,s)$ is characterized as a solution to the integral equation

\[
W(t,s) = R(t,s) - \int_s^t W(t,\tau)Q(\tau,s)d\tau.
\]

The last result then provides strong convergence of $A_n(t)U_n(t,s)$. As $A_n(t)U_n(t,s) = A_n(t)e^{-(t-s)A_n(t)} + W_n(t,s)$, we verify that $A_n(t)U_n(t,s)$ satisfies the uniform estimate

\[
\|A_n(t)U_n(t,s)\|_{\mathcal{F}(X)} \leq C(t - s)^{-1}, \quad 0 \leq s < t \leq T,
\]

and converges as $n \to \infty$ to the bounded operator $A(t)e^{-(t-s)A(t)} + W(t,s)$ strongly on $X$ for each $0 \leq s < t \leq T$. It is quite easy to see that $A(t)U(t,s) = A(t)e^{-(t-s)A(t)} + W(t,s)$.

In this way, we have arrive at the following theorem (more detailed arguments, see [17]).

**Theorem 3.1.** Under (2.1), (2.2), (2.3) and (2.4), there exists a unique family of bounded operators $U(t,s)$ on $X$ defined for $0 \leq s \leq t \leq T$ with the following properties: 1) $U(t,s)$ has the property (3.3) together with $U(s,s) = I$; 2) $U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$ with the estimate $\|U(t,s)\|_{\mathcal{F}(X)} \leq C$; 3) $A(t)U(t,s)$ is strongly continuous for $0 \leq s < t \leq T$ with the estimate $\|A(t)U(t,s)\|_{\mathcal{F}(X)} \leq C(t - s)^{-1}$; and 4) $U(t,s)$ is strongly differentiable for $t > s$ and its derivative is given by $(\partial U/\partial t)(t,s) = -A(t)U(t,s)$.

### 3.4. Cauchy problem

In this subsection we want to consider the Cauchy problem (1.1) and to construct a unique solution by using the evolution operator $U(t,s)$. Let $0 < \beta \leq 1$, let $F$ be such that

\[
F \in \mathcal{F}^{\beta,\sigma}((0,T];X), \quad 0 < \sigma < \beta,
\]

and let $U_0$ be taken in $X$.

**Theorem 3.2.** Under (2.1), (2.2), (2.3) and (2.4), let $F$ satisfy (3.22) and let $U_0$ be in $X$. Then, (1.1) possesses a unique solution $U$ in the function space:

\[
U \in C([0,T];X) \cap C^1((0,T];X), \quad A(t)U \in C((0,T];X).
\]
Moreover, $U$ is necessarily given by

$$U(t) = U(t,0)U_0 + \int_0^t U(t,\tau)F(\tau)d\tau, \quad 0 \leq t \leq T.$$  \hspace{1cm} (3.24)

**Proof.** Although the space (3.22) was not specifically used in the study of Cauchy problem in [17], the arguments for proving Property 5) of [17, Theorem 1] is still available in the present case. So, we will here sketch only the outline of proof.

For $n = 1, 2, 3, \ldots$, put

$$U_n(t) = U_n(t,0)U_0 + \int_0^t U_n(t,\tau)F(\tau)d\tau, \quad 0 \leq t \leq T.$$  

Clearly, $U_n(t)$ converges to the function $U(t)$ in (3.24) pointwise. In the meantime, by (3.1) it is directly verified that

$$\frac{dU_n}{dt}(t) = -A_n(t)U_n(t) + F(t), \quad 0 < t < T.$$  

Therefore, for any $0 < \varepsilon < T$, we have

$$U_n(t) = U_n(\varepsilon) + \int_\varepsilon^t [F(\tau) - A_n(\tau)U_n(\tau)]d\tau, \quad \varepsilon \leq t \leq T.$$  \hspace{1cm} (3.25)

On the other hand, we write $A_n(t)U_n(t)$ in the form

$$A_n(t)U_n(t) = A_n(t)U_n(t,0)U_0 + \int_0^t A_n(t)U_n(t,\tau)[F(\tau) - F(t)]d\tau$$

$$+ \int_0^t W_n(t,\tau)d\tau F(t) + [1 - e^{-tA_n(t)}]F(t).$$

Then, by (3.19) and (3.21), $A_n(t)U_n(t)$ is estimated by

$$\|A_n(t)U_n(t)\| \leq Ct^{-1}, \quad 0 < t \leq T,$$  \hspace{1cm} (3.26)

and converges as $n \to \infty$ to a function

$$W(t) = A(t)U(t,0)U_0 + \int_0^t A(t)U(t,\tau)[F(\tau) - F(t)]d\tau$$

$$+ \int_0^t W(t,\tau)d\tau F(t) + [1 - e^{-tA(t)}]F(t)$$

pointwise. Here, $W(t)$ is seen to be an $X$-valued continuous function for $0 < t \leq T$ and to satisfy the relation $A(t)U(t) = W(t)$ for $0 < t \leq T$. In view of (3.26), it follows from (3.25) that

$$U(t) = U(\varepsilon) + \int_\varepsilon^t [F(\tau) - A(\tau)U(\tau)]d\tau, \quad \varepsilon \leq t \leq T,$$

which means that $U(t)$ is a $C^1$ solution of (1.1).
The uniqueness of solution in (3.23) is verified by using the differentiability of \( U_n(t,s) \) for \( s \) verified in (3.2).

4. Maximal regularity

Before showing the maximal regularity of solution of (1.1), we have to investigate more refined properties of \( U(t,s) \) than those announced in Theorem 3.1.

Let \( A(t) \), \( 0 \leq t \leq T \), be a family of densely defined, closed linear operators in \( X \) satisfying the assumptions (2.1)–(2.4), and let \( U(t,s) \) be the evolution operator for \( A(t) \).

4.1. Refined properties of \( U(t,s) \)

Under (2.1), (2.2), (2.3) and (2.4), we can show the following properties of \( U(t,s) \).

For \( 0 \leq \theta < 1 + \nu \), \( U(t,s) \) satisfies the range condition

\[
\mathcal{R}(U(t,s)) \subseteq \mathcal{R}(A(t)^0), \quad 0 \leq s < t \leq T,
\]

with the estimates

\[
\|A(t)^0 U(t,s)\|_{\mathcal{L}(X)} \leq C_\theta (t-s)^{-\theta}, \quad 0 \leq s < t \leq T.
\]

The last estimate can be generalized as follows. For \( 0 \leq \varphi \leq 1 \) and \( 0 \leq \varphi \leq \theta < 1 + \nu \),

\[
\|A(t)^0 U(t,s) A(s)^{-\varphi}\|_{\mathcal{L}(X)} \leq C_\varphi (t-s)^{-\theta}, \quad 0 \leq s < t \leq T.
\]

For \( 0 \leq \theta < 1 \), \( U(t,s) A(s)^0 \) admits a bounded extension on \( X \) and its extension (which will be denoted again by \( U(t,s) A(s)^0 \)) satisfies the estimate

\[
\|U(t,s) A(s)^0\|_{\mathcal{L}(X)} \leq C_\theta (t-s)^{-\theta}, \quad 0 \leq s < t \leq T.
\]

Consider the difference of \( U(t,s) \) and the semigroup \( e^{-(t-s)A(t)} \). For \( 0 \leq \theta < 1 + \nu \) and \( 0 \leq \varphi \leq 1 \), we have

\[
\|A(t)^0 [U(t,s) - e^{-(t-s)A(t)}] A(s)^{-\varphi}\|_{\mathcal{L}(X)} \leq C_\varphi (t-s)^{\theta + \nu - \theta}, \quad 0 \leq s < t \leq T.
\]

Meanwhile, the difference of \( A(t)^0 U(t,s) A(s)^{-\theta} \) and \( e^{-(t-s)A(s)} \) is estimated as follows. For \( k = 0 \) and 1, it holds true that

\[
\|A(t)^k U(t,s) A(s)^{-k} - e^{-(t-s)A(s)}\|_{\mathcal{L}(X)} \leq C(t-s)^{\nu}, \quad 0 \leq s < t \leq T.
\]
On the other hand, for $0 < \theta < 1$,

\begin{equation}
(4.7) \quad \|A(t)^{\theta} U(t,s) A(s)^{-\theta} - e^{-(t-s)A(s)} \|_{\mathcal{F}(X)} \leq \begin{cases} CT(1-v)(t-s)^v, & 0 \leq s \leq t \leq T, \text{ if } v < 1, \\ C \log((t-s)^{1} + 1)(t-s), & 0 \leq s \leq t \leq T, \text{ if } v = 1. \end{cases}
\end{equation}

Let us now describe the proof for these properties.

For $0 \leq \theta < 1 + \nu$, operate $A(t)^{\theta-1}$ to (3.20) from the left hand side to obtain that

\begin{equation}
(4.8) \quad A(t)^{\theta-1} W(t,s) = A(t)^{\theta-1} R(t,s) - \int_s^t A(t)^{\theta-1} W(t,\tau) Q(\tau,s) d\tau.
\end{equation}

Here, $Q(\tau,s)$ satisfies $\|Q(\tau,s)\|_{\mathcal{F}(X)} \leq C(t-s)^{\nu-1}$ due to (3.13). In the mean time, by the same techniques as in the proof of Proposition 3.1, we can derive for $A(t)^{\theta-1} R(t,s)$ the estimate

\[ \|A(t)^{\theta-1} R(t,s)\|_{\mathcal{F}(X)} \leq C_0(t-s)^{-\theta}, \quad 0 \leq s < t \leq T. \]

As $\nu - \theta > -1$, these mean that we can treat (4.8) as an integral equation satisfied by $A(t)^{\theta-1} W(t,s)$. Hence, (4.1) and (4.2), as well as (4.5) for $\varphi = 0$, are verified.

If $0 \leq \theta < 1$, then it follows from (3.11) that

\begin{equation}
(4.9) \quad U(t,s) A(s)^{\theta} = A(s)^{\theta} e^{-(t-s)A(s)} + \int_s^t P(t,\tau) U(\tau,s) A(s)^{\theta} d\tau.
\end{equation}

Here, $P(t,\tau)$ satisfies $\|P(t,\tau)\|_{\mathcal{F}(X)} \leq C(t-\tau)^{\nu-1}$ due to (3.9), and (2.7) yields that $\|A(s)^{\theta} e^{-(t-s)A(s)}\|_{\mathcal{F}(X)} \leq C_0(t-s)^{-\theta}$. As $-\theta > -1$, these mean that (4.9) can be treated as an integral equation satisfied by $U(t,s) A(s)^{\theta}$. Hence, $U(t,s) A(s)^{\theta}$ is actually a bounded operator on $X$ and as well satisfies (4.4).

Let $0 \leq \varphi \leq 1$ and $0 \leq \varphi \leq \theta < 1 + \nu$. Operating $A(s)^{-\varphi}$ to (4.8) from the right hand side, we have

\[ A(t)^{\theta-1} W(t,s) A(s)^{-\varphi} = A(t)^{\theta-1} R(t,s) A(s)^{-\varphi} - \int_s^t A(t)^{\theta-1} W(t,\tau) Q(\tau,s) A(s)^{-\varphi} d\tau. \]

Here, $Q(\tau,s) A(s)^{-\varphi}$ is estimated as follows. Write

\[ Q(t,s) A(s)^{-\varphi} = Q(t,s) |A(s)^{-\varphi} - A(t)^{-\varphi}| + Q(t,s) A(t)^{-\varphi}. \]

Then, by (4.13) in Lemma 4.1 below, we have

\begin{equation}
(4.10) \quad \|Q(t,s) [A(s)^{-\varphi} - A(t)^{-\varphi}]\|_{\mathcal{F}(X)} \leq C(t-s)^{\varphi+\nu-1}.
\end{equation}
In the meantime, from (3.14) it follows that
\[
\|Q(t, s)A(t)^{-\theta}\|_{\mathcal{F}(X)} \leq C \int_R e^{-(t-s)\Re \hat{\lambda}} \left| \frac{\partial}{\partial \hat{\lambda}} (\hat{\lambda} - A(t))^{-1} \cdot A(t)^{-\theta} \right|_{\mathcal{F}(X)} |d\hat{\lambda}|
\]
\[
\leq C \int_R |\hat{\lambda}|^{-\nu} e^{-(t-s)\Re \hat{\lambda}} |d\hat{\lambda}|
\]
\[
\leq C(t-s)^{\theta \nu - 1}.
\]

In this calculation, when \( \nu + \varphi < 1 \), we shifted \( \Gamma \) to \( \Gamma_0 : \hat{\lambda} = re^{\pm i\alpha}, 0 < r < \infty \); on the other hand, when \( \nu + \varphi \geq 1 \), we shifted \( \Gamma \) to \( \Gamma_{(t-s)^{-1}} \). Here and in what follows, \( \Gamma_R, 0 < R < \infty \), stands for an integral contour such that \( \Gamma_R = \Gamma_{R1} \cup \Gamma_{R2} \) with \( \Gamma_{R1} : \hat{\lambda} = Re^{\theta}, -\omega < |\omega| < \pi, \) and \( \Gamma_{R2} : \hat{\lambda} = re^{\pm i\alpha}, R < r < \infty \). More generally, by the same technique it is possible to verify that, for any \( \rho \) such that \( 0 \leq \rho \leq \nu \),
\[
(4.11) \quad \|A(t)^{\rho} Q(t, s)A(t)^{-\theta}\|_{\mathcal{F}(X)} \leq C(t-s)^{\theta \nu - \rho - 1}
\]
(cf. (3.15)). Anyway, we thus obtained that
\[
(4.12) \quad \|Q(t, s)A(s)^{-\theta}\|_{\mathcal{F}(X)} \leq C(t-s)^{\theta \nu - 1}.
\]

Next, \( A(t)^{-1} R(t, s)A(s)^{-\theta} \) is estimated as follows. As \( R(t, s) \) is given by (3.18), we can write
\[
A(t)^{-1} R(t, s)A(s)^{-\theta} = \int_s^t [A(\tau)^{\theta} e^{-(t-\tau)A(\tau)} - A(\tau)^{\theta} e^{-(t-\tau)A(\tau)}] Q(\tau, s)A(s)^{-\theta} d\tau
\]
\[
+ \int_s^t A(\tau)^{\theta-\rho} e^{-(t-\tau)A(\tau)} A(\tau)^{\theta} Q(\tau, s)A(v)^{-\theta} - A(s)^{-\theta}] d\tau
\]
\[
- \int_s^t A(\tau)^{\theta-\rho} e^{-(t-\tau)A(\tau)} A(\tau)^{\theta} Q(\tau, s)A(v)^{-\theta} d\tau
\]
\[
= Q_1 + Q_2 + Q_3,
\]
using an exponent \( \rho \). By (2.11) and (4.12), we observe that \( \|Q_i\|_{\mathcal{F}(X)} \leq C(t-s)^{\theta + 2v - \theta} \). As for \( \rho \) in \( Q_i \) (i = 2, 3), we will take \( \rho = 0 \) if \( 0 \leq \theta < 1 \) and \( \theta - 1 < \rho < \theta \) if \( 1 \leq \theta < 1 + v \). Then, by (2.7), (4.10), (4.11) and (4.13), we obtain that \( \|Q_2\|_{\mathcal{F}(X)} \leq C(t-s)^{\theta + \nu - \theta} \). And, by (2.7) and (4.11), we obtain that \( \|Q_3\|_{\mathcal{F}(X)} \leq C(t-s)^{\theta + \nu - \theta} \). Therefore,
\[
\|A(t)^{-1} R(t, s)A(s)^{-\theta}\|_{\mathcal{F}(X)} \leq C(t-s)^{\theta + \nu - \theta}.
\]

As (4.5) is already known in the case \( \varphi = 0 \), this joined with (4.12) yields directly the desired estimate (4.5) for the general \( 0 < \varphi \leq 1 \).
Lemma 4.1. For $0 \leq \varphi \leq 1$,

\[ \| A(t)^{-\varphi} - A(s)^{-\varphi} \|_{\mathcal{L}(X)} \leq C|t-s|^\varphi, \quad 0 \leq t, s \leq T. \]

**Proof.** It suffices to consider the case when $0 < \varphi < 1$. Dividing 1 as $1 = (1 - \varphi) + \varphi$, we obtain by (2.2) and (2.9) that

\[ \| (\lambda - A(t))^{-1} - (\lambda - A(s))^{-1} \|_{\mathcal{L}(X)} \leq C|\lambda|^{-\varphi - 1}(|\lambda|^{-\varphi}|t-s|)^\varphi, \quad \lambda \notin \Sigma. \]

Then,

\[ \| A(t)^{-\varphi} - A(s)^{-\varphi} \|_{\mathcal{L}(X)} \leq \frac{1}{2\pi} \int_{\Gamma} |\lambda|^{-\varphi}\| (\lambda - A(t))^{-1} - (\lambda - A(s))^{-1} \|_{\mathcal{L}(X)} \]

\[ \leq C \int_{\Gamma} |\lambda|^{-\varphi - 1}|d\lambda||t-s|^\varphi \leq C|t-s|^\varphi. \]

In order to show other properties, we have to generalize (2.11) as follows.

Lemma 4.2. For $0 \leq \varphi \leq 1$ and $1 \leq \theta \leq 2$,

\[ \|[A(t)^{\theta}e^{-\tau A(t)} - A(s)^{\theta}e^{-\tau A(s)}]A(s)^{-\varphi}\|_{\mathcal{L}(X)} \leq C\tau^{-\theta - 1}|t-s|^{\varphi}|t-s|^\varphi. \]

For $0 \leq \varphi \leq \theta < 1$,

\[ \|[A(t)^{\theta}e^{-\tau A(t)} - A(s)^{\theta}e^{-\tau A(s)}]A(s)^{-\varphi}\|_{\mathcal{L}(X)} \]

\[ \leq \begin{cases} C\Gamma(\theta - \varphi - \nu + 1)\tau^{\nu - \theta - 1}|t-s|^\nu(t^\varphi + |t-s|^\varphi), & \text{if } \varphi < \theta \text{ or } \nu < 1, \\ C|t-s|[\log(\tau^{-1} + 1) + \tau^{-\theta}|t-s|^\varphi + 1], & \text{if } \varphi = \theta \text{ and } \nu = 1. \end{cases} \]

For $0 \leq \varphi \leq 1$,

\[ \|[e^{-\tau A(t)} - e^{-\tau A(s)}]A(s)^{-\varphi}\|_{\mathcal{L}(X)} \leq C\tau^{-1}|t-s|^\varphi(t^\varphi + |t-s|^\varphi). \]

**Proof.** Let us first prove (4.14). For $0 \leq \varphi \leq 1$ and $1 \leq \theta \leq 2$, we write

\[ [A(t)^{\theta}e^{-\tau A(t)} - A(s)^{\theta}e^{-\tau A(s)}]A(s)^{-\varphi} = \int_{s}^{t} \frac{\partial}{\partial r}A(r)^{\theta}e^{-\tau A(r)} \cdot [A(s)^{-\varphi} - A(r)^{-\varphi}]dr \]

\[ + \int_{s}^{t} \frac{\partial}{\partial r}A(r)^{\theta}e^{-\tau A(r)} \cdot A(r)^{-\varphi}dr = E_1 + E_2. \]

Here, it is observed from (2.6) that

\[ \left\| \frac{\partial}{\partial r}A(r)^{\theta}e^{-\tau A(r)} \right\|_{\mathcal{L}(X)} \]

\[ \leq \frac{1}{2\pi} \int_{\Gamma_0} |\lambda|^{\theta-\varphi} \Re \lambda \left\| \frac{\partial}{\partial r}(\lambda - A(r))^{-1} \right\|_{\mathcal{L}(X)} |d\lambda| \]

\[ \leq C \int_{\Gamma_0} |\lambda|^{\theta-\varphi} \Re \lambda |d\lambda| \leq C\tau^{\nu-\theta-1}. \]
In this calculation, we shifted the integral contour $\Gamma'$ to $\Gamma_0$. Then, this together with (4.13) yields that \( \|E_1\|_{X(\mathcal{X})} \leq C \tau^{v-\theta-1}(t-s)^{\phi+1} \).

Similarly, it is observed that
\[
\left\| \frac{\partial}{\partial r} A(r)^\theta e^{-rA(s)} \cdot A(r)^{-\phi} \right\|_{X(\mathcal{X})} \leq C \int_{\Gamma'} |\lambda|^{\theta-\phi} e^{-r \text{Re} \lambda} |d\lambda| \leq C \tau^{v+\phi-\theta-1}.
\]

In this calculation, when $\theta - v - \phi = -1$, i.e., $\phi = \theta = v = 1$, we shifted $\Gamma'$ to $\Gamma_{t-1}$; in the other case, we shifted $\Gamma'$ to $\Gamma_0$. Therefore, we obtain that \( \|E_2\|_{X(\mathcal{X})} \leq C \tau^{\theta+v-\theta-1}|t-s| \). Thus, (4.14) has been verified.

The proof of (4.15) is quite similar to that of (4.14). Only difference is that we can no longer use the shift of $\Gamma'$ to $\Gamma_{t-1}$ when $\theta - v - \phi = -1$, i.e., $\phi = \theta < 1$ and $v = 1$, because the function $\lambda^\theta$ which is not holomorphic on $(-\infty, 0]$ remains in the integral.

The proof of (4.16) is also carried out in a similar way as in the favorable case $\theta = 1$ of (4.14). We can again utilize the shift of $\Gamma'$ to $\Gamma_{t-1}$ as above.

Let $0 \leq \phi \leq 1$ and $0 \leq \phi \leq \theta < 1 + v$. Then by Lemma 4.2, we observe that
\[
\| [A(t)^\theta e^{-(t-s)A(t)} - A(s)^\theta e^{-(t-s)A(s)}] A(s)^{-\phi} \|_{X(\mathcal{X})} \leq C \log((t-s)^{-1} + 1)(t-s)^{v+\phi-\theta}.
\]

The known estimate (4.5) jointed with this then yields (4.3).

Set $\theta = \phi = 0$ (resp. 1) in (4.5). Then, (4.6) for $k = 0$ (resp. (4.14)). It is similar for (4.7).

We have thus accomplished the proof of all the properties (4.1)–(4.7).

We want to conclude this subsection by remarking some other property of $U(t, s)$ which is an immediate consequence of (4.5) and Lemma 4.2 but is needed in the subsequent subsection. For $0 \leq \phi \leq 1$ and $0 \leq \phi \leq \theta < 1 + v$,

\[
(4.17) \quad \| A(t)^\theta U(t, s) A(s)^{-\phi} - A(s)^\theta e^{-(t-s)A(s)} \|_{X(\mathcal{X})} \leq C \log((t-s)^{-1} + 1)(t-s)^{v+\phi-\theta}, \quad 0 \leq s < t \leq T.
\]

Indeed, due to (4.5), (4.14) and (4.15),
\[
\| A(t)^\theta U(t, s) A(s)^{-\phi} - A(s)^\theta e^{-(t-s)A(s)} \|_{X(\mathcal{X})} \leq \| A(t)^\theta [U(t, s) - e^{-(t-s)A(t)}] A(s)^{-\phi} \|_{X(\mathcal{X})}
\]
\[
+ \| [A(t)^\theta e^{-(t-s)A(t)} - A(s)^\theta e^{-(t-s)A(s)}] A(s)^{-\phi} \|_{X(\mathcal{X})}
\]
\[
\leq C \log((t-s)^{-1} + 1)(t-s)^{v+\phi-\theta}.
\]
4.2. Proof of maximal regularity

We are now in a position to show the maximal regularity of solutions to (1.1).

**Theorem 4.1.** Under (2.1), (2.2), (2.3) and (2.4), let $0 < \beta \leq 1$. Let

$$F \in \mathcal{F}^{\beta,\sigma}((0, T]; X), \quad 0 \leq \sigma < \min\{v, \beta\},$$

and

$$U_0 \in \mathcal{D}(A(0)^\beta).$$

Then, the solution $U$ to (1.1) given by (3.24) possesses the regularity:

$$A(t)^\beta U \in \mathcal{C}([0, T]; X),$$

$$\frac{dU}{dt}, \quad A(t)U \in \mathcal{F}^{\beta,\sigma}((0, T]; X),$$

with the estimates

$$\|A(t)^\beta U\| \leq C\|A(0)^\beta U_0\| + \|F\|_{\mathcal{F}^{\beta,\sigma}},$$

$$\left\|\frac{dU}{dt}\right\|_{\mathcal{F}^{\beta,\sigma}} + \|A(t)U\|_{\mathcal{F}^{\beta,\sigma}} \leq C\|A(0)^\beta U_0\| + \|F\|_{\mathcal{F}^{\beta,\sigma}}.$$  

**Proof.** Let us first verify (4.20). For $0 < t \leq T$, we have $A(t)^\beta U(t) = A(t)^{\beta-1}A(t)U(t)$. Since $A(\cdot)^{\beta-1} \in \mathcal{C}^{1-\beta}([0, T]; \mathcal{L}(X))$ due to (4.13), (3.23) implies that $A(t)^\beta U$ is continuous for $t > 0$. So, it suffices to prove that $A(t)^\beta U$ is continuous at $t = 0$. We write

$$A(t)^\beta U(t) = A(0)^\beta U_0 + [A(t)^\beta U(t, 0)A(0)^{-\beta} - 1]A(0)^\beta U_0$$

$$+ \int_0^t A(t)^\beta U(t, \tau)[F(\tau) - F(t)]d\tau$$

$$+ \int_0^t A(t)^\beta[U(t, \tau) - e^{-(t-\tau)A(t)}]d\tau F(t)$$

$$+ t^{\beta-1}[1 - e^{-tA(t)}]A(t)^{\beta-1}t^{1-\beta}F(t).$$

Then, by the properties of $U(t, s)$ and $F(t)$, it is possible to observe that, as $t \to 0$, every term in the right hand side except $A(0)^\beta U_0$ converges to 0, that is, $A(t)^\beta U(t)$ converges to $A(0)^\beta U_0$. Hence, (4.20) is verified. At the same time, (4.22) is also verified directly from this expression.
Let us next prove (4.21). We decompose $A(t)U(t)$ into the form

$$A(t)U(t) = A(0)e^{-tA(0)}U_0 + [A(t)U(t,0)A(0)^{-\beta} - A(0)^{1-\beta}e^{-tA(0)}]A(0)^\beta U_0$$

$$+ \int_0^t A(t)U(t,\tau)[F(\tau) - F(t)]d\tau + \int_0^t W(t,\tau)d\tau F(t)$$

$$+ [1 - e^{-tA(t)}]F(t) = W_1(t) + W_2(t) + W_3(t) + W_4(t) + W_5(t),$$

where $W(t,s) = A(t)[U(t,s) - e^{-(t-s)A(t)}].$ By Theorem 2.1, we already know that $W_1$ is in $\mathcal{F}^{\beta,\sigma}((0,T];X).$ So, it suffices to prove that $W_i \in \mathcal{F}^{\beta,\sigma}((0,T];X)$ for other $2 \leq i \leq 5.$

**Proof for $W_2.$** We use (4.17) to conclude that, as $t \to 0,$

$$t^{1-\beta}\|W_2(t)\| \leq Ct^v \log(t^{-1} + 1) \to 0.$$

Meanwhile, we have

$$W_2(t) - W_2(s) = [A(t)U(t,s)A(s)^{-1} - e^{-(t-s)A(s)}]A(s)U(s,0)A(0)^{-\beta}A(0)^\beta U_0$$

$$+ [e^{-(t-s)A(s)} - 1]A(s)^{-\sigma}A(s)^{1+\sigma}U(s,0)A(0)^{-\beta}A(0)^\beta U_0$$

$$- [e^{-(t-s)A(0)} - 1]A(0)^{-\sigma}A(0)^{1-\beta+\sigma}e^{-sA(0)}A(0)^\beta U_0.$$  

Then, by (4.3) and (4.5), it is observed that

$$\|W_2(t) - W_2(s)\| \leq C[\|s^{\beta-1}(t-s)^{\gamma} + s^{\beta-\sigma-1}(t-s)^{\sigma}\|\|A(0)^\beta U_0\|].$$

Hence, as $\sigma < v,$ $W_2$ satisfies (2.16). It is also verified by Theorem 2.2 and (4.17) that $W_2$ satisfies (2.17), too.

**Proof for $W_3.$** It is easy to see that $\lim_{t \to 0} t^{1-\beta}W_3(t) = 0.$ In addition, $W_3(t) - W_3(s)$ is decomposed as

$$W_3(t) - W_3(s) = \int_s^t A(t)U(t,\tau)[F(\tau) - F(t)]d\tau$$

$$+ \left\{ \int_0^s W(t,\tau)d\tau + [e^{-(t-s)A(t)} - e^{-tA(t)}] \right\}[F(s) - F(t)]$$

$$+ [A(t)U(t,s)A(s)^{-1} - e^{-(t-s)A(s)}] \int_0^s A(s)U(s,\tau)[F(\tau) - F(s)]d\tau$$

$$+ [e^{-(t-s)A(s)} - 1]A(s)^{-\sigma} \int_0^s A(s)^\sigma W(s,\tau)[F(\tau) - F(s)]d\tau$$

$$+ [e^{-(t-s)A(s)} - 1] \int_0^s A(s)e^{-(t-\tau)A(s)}[F(\tau) - F(s)]d\tau.$$
For the last term, we apply the known result [20, (3.21)], and for other terms we apply (4.2), (4.5) and (4.6). Then, we have
\[ \| W_3(t) - \tilde{W}_3(s) \| \leq C[s^{\beta-1}(t-s)^\nu + s^{\beta-\sigma-1}(t-s)^\sigma]\| F \|_{\mathcal{F}^{\beta,\sigma}}. \]
Hence, \( W_3 \) satisfies (2.16).

As \( F(t) \) satisfies (2.17), this condition implies that every term in the right hand side also satisfies the condition (2.17).

**Proof for \( W_4 \).** It is clear that \( \lim_{t \to 0} t^{1-\beta} W_4(t) = 0 \). We decompose \( W_4(t) - W_4(s) \) into
\[
W_4(t) - W_4(s) = \int_s^t W(t, \tau) d\tau F(t) + \int_0^s W(t, \tau) d\tau [F(t) - F(s)] \\
+ [A(t) U(t, s) A(s)^{-1} - e^{-(t-s)A(s)}] \int_s^t W(s, \tau) d\tau F(s) \\
+ [e^{-(t-s)A(s)} - 1]A(s)^{-\sigma} \int_0^s A(s)^\sigma W(s, \tau) d\tau F(s) \\
+ \int_0^s [A(s)e^{-(t-\tau)A(s)} - A(t)e^{-(t-\tau)A(t)}] d\tau F(s) \\
+ [A(t) U(t, s) A(s)^{-1} - e^{-(t-s)A(s)}][1 - e^{-sA(s)}] F(s).
\]
Then, by (4.5), (4.6) and (4.14),
\[ \| W_4(t) - W_4(s) \| \leq C[s^{\beta-1}(t-s)^\nu + t^{\nu}s^{\beta-\sigma-1}(t-s)^\sigma]\| F \|_{\mathcal{F}^{\beta,\sigma}}. \]
Hence, \( W_4 \) satisfies (2.16) and (2.17).

**Proof for \( W_5 \).** It is easy to see that \( \lim_{t \to 0} t^{\beta-1} W_5(t) = 0 \). In addition, \( W_5(t) - W_5(s) \) is decomposed as
\[
W_5(t) - W_5(s) = [1 - e^{-tA(t)}] [F(t) - F(s)] - [e^{-tA(t)} - e^{-tA(s)}] F(s) \\
- [e^{-(t-s)A(s)} - 1]A(s)^{-\sigma} A(s)^\sigma e^{-sA(s)} F(s).
\]
Then, by (4.16),
\[ \| W_5(t) - W_5(s) \| \leq C s^{\beta-\sigma-1}(t-s)^\sigma\| F \|_{\mathcal{F}^{\beta,\sigma}}. \]
It is also verified that \( W_5 \) satisfies (2.17).

We have thus shown that \( A(t) U \) belongs to \( \mathcal{F}^{\beta,\sigma}([0, T]; X) \) and satisfies the estimate (4.23). As \( dU/dt = -A(t) U + F \), it is the same for \( dU/dt \). □
5. Applications

Let us apply our abstract results to concrete differential equations.

5.1. Model equation

We begin with considering a model problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} + |x-t|^{-\alpha} u &= f(x, t), \quad 0 < x < 1, \quad 0 < t \leq 1, \\
\lambda(t) &= u(x, 0) = u_0(x), \quad 0 < x < 1,
\end{aligned}
\]

in the spatial interval \((0,1)\), where \(\alpha\) is a positive exponent such that \(\alpha > 1\). The underlying space \(X\) is set as \(X = L_p(0, 1)\), where \(1 \leq p < \infty\). For \(0 \leq t \leq 1\), let \(A(t)\) be a multiplicative operator in \(X\) by the function \(|x-t|^{-\alpha}\). We easily see that each \(A(t)\) is a sectorial operator of \(X\) with angle 0, and its domain is given by \(D(A(t)) = \{u \in X; \ |x-t|^{-\alpha} u \in X\} = |x-t|^{-\alpha} L_p(0, 1)\). Furthermore, its fractional power \(A(t)^{\frac{1}{2}}\) is a multiplicative operator by the function \(|x-t|^{-\alpha}\) with domain \(D(A(t)^{\frac{1}{2}}) = |x-t|^{-\alpha} L_p(0, 1)\). Therefore, \(A(t)\) never satisfies (1.2). On the other hand, \(A(t)^{-1}\) is a multiplicative operator by the function \(|x-t|^{-\alpha}\). So, as \(\alpha > 1\), \(A(t)^{-1}\) is strongly differentiable on \(X\) with the derivative \([d(A(t)^{-1})/dt] f = |x-t|^{-\alpha} \text{sign}(t-x)] f\) for \(f \in X\). Therefore, \(A(t)\) fulfills the structural assumptions (2.3) and (2.4) with \(v = (\alpha - 1)/\alpha\).

Let \(0 < \beta \leq 1\). Then, by Theorem 4.1, for any \(u_0 \in D(A(0)^{\beta})\) and any \(f \in \mathcal{F}^{\beta,\sigma}((0,1]; X), 0 < \sigma < \min\{v, \beta\}\), the solution \(u\) to (5.1) enjoys the maximal regularity \(du/dt \in \mathcal{F}^{\beta,\sigma}((0,1]; X)\).

5.2. Some singular parabolic equations

Let us next consider the Cauchy problem for two singular equations

\[
\begin{aligned}
\frac{\partial u}{\partial t} + A[m(x, t)^{-1}] u &= f(x, t), \quad (x, t) \in \Omega \times (0, T], \\
m(x, t)^{-1} u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T], \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{\partial u}{\partial t} + m(x, t)^{-1} \Delta u &= f(x, t), \quad (x, t) \in \Omega \times (0, T], \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T], \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{aligned}
\]
respectively, in a \( \mathcal{C}^2 \) bounded domain \( \Omega \subset \mathbb{R}^n \). Here, \( m(x, t) \geq 0 \) is a given nonnegative function in \( \Omega \) such that
\[
(5.4) \quad \text{the set } \{ x \in \Omega; m(x, t) = 0 \} \text{ is a null subset of } \Omega \text{ for } 0 \leq t \leq T.
\]

We assume also that \( m \) satisfies the conditions:
\[
(5.5) \quad \begin{cases}
m \in \mathcal{C}([0, T]; \mathcal{C}^1(\overline{\Omega})) \cap \mathcal{C}^1([0, T]; L^1(\Omega)), \\
|\frac{\partial m}{\partial t}(x, t)| \leq Cm(x, t)^\beta, \quad \text{a.e. } x \in \Omega, 0 \leq t \leq T,
\end{cases}
\]
with some exponent \( 0 < \beta \leq 1 \). Then, according to [6, Example 6.6] (cf. also [7, Examples 4.5 and 4.6]), Problems (5.2) and (5.3) can be formulated as an abstract problem of the form (1.1) in the spaces \( H^{-1}(\Omega) \) and \( H^0_0(\Omega) \), respectively.

**Problem (5.2).** Set \( X = H^{-1}(\Omega) \) and set \( A(t) = Lm(t)^{-1} \), \( 0 \leq t \leq T \), where \( L \) is the realization of the Laplace operator \(-\Delta\) in the space \( H^{-1}(\Omega) \) under the Dirichlet boundary conditions on \( \partial \Omega \). The domain of \( L \) is given by \( \mathcal{D}(L) = H^1_0(\Omega) \). Meanwhile, \( m(t) \) is a multiplicative operator by the function \( m(t, x) \) from \( H^1_0(\Omega) \) into \( H^{-1}(\Omega) \), and \( m(t)^{-1} \) is its inverse. As seen by Lemma 5.1 below, \( \mathcal{D}(m(t)^{-1}) = \mathcal{R}(m(t)^{-1}) \) is dense in \( H^{-1}(\Omega) \). Therefore, \( A(t) \) is densely defined in \( X \) with \( \mathcal{D}(A(t)) = \mathcal{R}(m(t)) \). As shown in [6, Example 6.6], under (5.5), \( A(t) \) satisfies (2.1) and (2.2) as well (2.3) and (2.4) with \( v = \alpha/2 \).

The following assertion is therefore valid. Let \( 0 < \beta \leq 1 \). Then, for any \( u_0 \in \mathcal{D}(A(0)^{\beta}) \) and any \( f \in \mathcal{F}^{\beta, \alpha}((0, T]; X) \), \( 0 < \sigma < \min\{v, \beta\} \), (5.2) possesses a unique solution with the maximal regularity \( du/\partial t \in \mathcal{F}^{\beta, \sigma}((0, T]; X) \).

**Problem (5.3).** It is known that Problems (5.2) and (5.3) are conjugate. So it is natural to set \( X = H^1_0(\Omega) \) and to handle (5.3) in this space. We now set \( A(t) = m(t)^{-1}L \), \( 0 \leq t \leq T \). The domain of \( A(t) \) is given by \( \mathcal{D}(A(t)) = L^{-1}(\mathcal{R}(m(t))) \). Since \( L^{-1} \) is an isomorphism from \( H^{-1}(\Omega) \) onto \( H^1_0(\Omega) \), Lemma 5.1 provides that \( L^{-1}(\mathcal{R}(m(t))) \) is dense in \( H^1_0(\Omega) \). Furthermore, as shown in [6, Example 6.6], under (5.5), \( A(t) \) satisfies (2.1), (2.2), (2.3), and (2.4) with \( v = \alpha/2 \). Hence, Problem (5.3) also enjoys the similar maximal regularity as (5.2).

**Lemma 5.1.** The range \( \mathcal{R}(m(t)) \) of \( m(t) \) is dense in \( H^{-1}(\Omega) \).

**Proof.** Let \( u \in H^1_0(\Omega) \). For \( n = 1, 2, 3, \ldots \), put \( u_n = n|m(t)u|/(n + m(t)) \). Then, \( nu/(n + m(t)) \) is in \( H^1(\Omega) \) and \( [nu/(n + m(t))]_{\partial \Omega} = 0 \), i.e., \( nu/(n + m(t)) \in H^1_0(\Omega) \). Consequently, \( u_n \in \mathcal{R}(m(t)) \). In the meantime, as \( n \to \infty \), \( nm(t)/(n + m(t)) \to 1 \) for a.e. \( x \in \Omega \). This means that \( u_n \) converges to \( u \) in \( L^2(\Omega) \), a fortiori in \( H^{-1}(\Omega) \). We have thus verified that \( H^1_0(\Omega) \) is contained in \( \mathcal{R}(m(t)) \).
(closure in $H^{-1}(\Omega)$). Since $H^1_0(\Omega)$ is dense in $H^{-1}(\Omega)$, we obtain the desired result.

We will conclude this section by remarking several known results on the problems (5.2) and (5.3). For constructing a unique solution for (5.2) or (5.3), (5.4) is not necessary; $m(x,t) \geq 0$ can vanish on some open subset of $\Omega$ (see Favini-Yagi [6, 7]). But in such a case, $Lm(t)^{-1}$ or $m(t)^{-1}L$ is not a single valued operator but a multivalued linear operator. One can treat (5.2) in the space $L_p(\Omega), 1 < p < \infty$, too. But, as shown by Favini-Lorenzi-Tanabe-Yagi [4], one can prove only that $\| (\lambda - Lm(t)^{-1})^{-1} \|_{L_p} \leq C/|\lambda|^\rho, \lambda \notin \Sigma$, with some exponent $0 < \rho < 1$, that is, (2.2) is no longer verified. In these general cases, some strong regularity for the solutions to (5.2) and (5.3) was studied by the papers Favini-Yagi [5] and Favaron [3]. The maximal regularity of $L_p$ type was studied by Weber [16] by different techniques.

References


nuna adreso:
Department of Applied Physics
Osaka University
Suita, Osaka 565-0871
Japan

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