On the Existence and the Asymptotic Behavior of Nonoscillatory Solutions of Second Order Quasilinear Difference Equations

By

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(Dedicated to Professor Kusano Takaši on the occasion of his 80th birthday)

Abstract. For the second-order quasilinear difference equation $\Delta(p_n|\Delta x_n|^{\alpha-1}\Delta x_n) + q_n|x_{n+1}|^{\beta-1}x_{n+1} = 0$, $\alpha > \beta$, we establish a necessary and sufficient condition for the existence of slowly growing and slowly decaying positive solution. In particular, when $p_n = 1$, the precise asymptotic forms of its slowly growing positive solutions are obtained.

Key Words and Phrases. Emden-Fowler difference equations, Asymptotic behavior of solutions, Positive solutions, Slowly growing solutions, Slowly decaying solutions, Regularly varying functions, Slowly varying functions.

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1. Introduction

The aim of this paper is to investigate nonoscillatory solutions of the second-order quasilinear difference equation

$$(\Omega) \quad \Delta(p_n|\Delta x_n|^{\alpha-1}\Delta x_n) + q_n|x_{n+1}|^{\beta-1}x_{n+1} = 0,$$

where α and β are positive constants such that $\alpha < \beta$, $p = \{p_n\}$, $q = \{q_n\}$ are positive real sequence and the operator Δ is the usual forward difference operator defined as $\Delta x_n = x_{n+1} - x_n$. Equation (Ω) is said to be *half-linear* if $\alpha = \beta$, while in the case $\alpha \neq \beta$ is called Emden-Fowler equation, whereby it is *super-half-linear* or *sub-half-linear* according as $\alpha < \beta$ or $\alpha > \beta$.

By a solution of (Ω) we mean a real sequence $x = \{x_n\}$ satisfying (Ω) and is not trivial. A solution x of equation (Ω) is called oscillatory if for every $M \in \mathbb{N}$ there exist $m, n \in \mathbb{N}$, $M \leq m < n$ such that $x_m x_n < 0$, otherwise, it is nonoscillatory. In other words, a solution x is called nonoscillatory if it is eventually positive or eventually negative. If $x = \{x_n\}$ is a solution of (Ω) , then clearly $-x = \{-x_n\}$ is also a solution. Thus, in studying the nonoscillation of (Ω) , for the sake of simplicity we restrict ourself to solution which are

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eventually positive. Any such solution $\{x_n\}$ is eventually strongly monotone and belongs to one of the two classes listed below (see [6, Lemma 1]):

$$\mathbf{M}^+ = \{x \text{ solution of } (\Omega) : x_n > 0, \Delta x_n > 0 \text{ eventually}\},$$

$$\mathbf{M}^- = \{x \text{ solution of } (\Omega) : x_n > 0, \Delta x_n < 0 \text{ eventually}\}.$$

For any solution x of (Ω) denote by $x^{[1]} = \{x_n^{[1]}\}$ its quasi-difference $x_n^{[1]} = p_n |\Delta x_n|^{\alpha-1} \Delta x_n$. Thus, the classes $\mathbf{M}^+$ and $\mathbf{M}^-$ can be a-priori divided into the following subclasses:

$$\mathbf{M}_{\infty, l}^+ = \{x \in \mathbf{M}^+ : \lim_n x_n = \infty, \lim_n x_n^{[1]} = l, 0 < l < \infty\},$$

$$\mathbf{M}_{\infty, 0}^+ = \{x \in \mathbf{M}^+ : \lim_n x_n = \infty, \lim_n x_n^{[1]} = 0\},$$

$$\mathbf{M}_{k, l}^+ = \{x \in \mathbf{M}^+ : \lim_n x_n = k, 0 < k < \infty, \lim_n x_n^{[1]} = l, 0 < l < \infty\},$$

$$\mathbf{M}_{k, 0}^+ = \{x \in \mathbf{M}^+ : \lim_n x_n = k, 0 < k < \infty, \lim_n x_n^{[1]} = 0\},$$

$$\mathbf{M}_{k, \infty}^- = \{x \in \mathbf{M}^- : \lim_n x_n = k, 0 < k < \infty, \lim_n x_n^{[1]} = -\infty\},$$

$$\mathbf{M}_{0, \infty}^- = \{x \in \mathbf{M}^- : \lim_n x_n = 0, \lim_n x_n^{[1]} = \infty\},$$

$$\mathbf{M}_{k, l}^- = \{x \in \mathbf{M}^- : \lim_n x_n = k, 0 < k < \infty, \lim_n x_n^{[1]} = -l, 0 < l < \infty\},$$

$$\mathbf{M}_{0, l}^- = \{x \in \mathbf{M}^- : \lim_n x_n = 0, \lim_n x_n^{[1]} = -l, 0 < l < \infty\}.$$

There has been much interest on the study of oscillatory and nonoscillatory behavior of solutions of (Ω) , which is mostly described by means of the convergence or divergence of the next two series:

$$S_p = \sum_{k=1}^{\infty} \frac{1}{p_k^{1/\alpha}}, \quad \text{and} \quad S_q = \sum_{k=1}^{\infty} q_k,$$

considering cases

$$(H_1): S_p = \infty, S_q = \infty,$$

$$(H_2): S_p = \infty, S_q < \infty,$$

$$(H_3): S_p < \infty, S_q = \infty,$$

$$(H_4): S_p < \infty, S_q < \infty.$$

If (H_1) holds, then (Ω) is oscillatory (see [1, Theorem 5.1.1]), so the existence of solutions belonging to classes $\mathbf{M}^+$, $\mathbf{M}^-$ have been considered in the remaining

three cases. Depending on the case, some of the above listed classes is proved to be empty (see [7, Proposition 1.1, Theorem 4.1], [6, Theorem 5]):

- Theorem 1.1.** (i) If (H_2) holds, then $M_{k,l}^+ = \emptyset$ and $M^- = \emptyset$.
(ii) If (H_3) holds, then $M_{k,l}^- = \emptyset$ and $M^+ = \emptyset$.
(iii) If (H_4) holds, then $M_{\infty,l}^+ = M_{\infty,0}^+ = M_{k,\infty}^- = M_{0,\infty}^- = \emptyset$.

Thus, if (H_2) holds, any positive solution belongs to the set

$$M^+ = M_{\infty,l}^+ \cup M_{\infty,0}^+ \cup M_{k,0}^+.$$

The existence of solutions belonging to classes $M_{\infty,l}^+$, $M_{k,0}^+$ can be completely characterized (see [6, Theorems 2 and 3]):

Theorem 1.2. Assume (H_2) .

- (i) $M_{\infty,l}^+ \neq \emptyset$ if and only if

$$T_\beta = \sum_{n=1}^{\infty} q_n \left(\sum_{k=1}^n \frac{1}{p_k^{1/\alpha}} \right)^\beta < \infty.$$

- (ii) $M_{k,0}^+ \neq \emptyset$ if and only if

$$R_\alpha = \sum_{n=1}^{\infty} \left(\frac{1}{p_n} \sum_{k=n}^{\infty} q_k \right)^{1/\alpha} < \infty.$$

On the other hand, if (H_3) holds, any positive solution belongs to the set

$$M^- = M_{0,l}^- \cup M_{0,\infty}^- \cup M_{k,\infty}^-,$$

and solutions belonging to classes $M_{0,l}^-$, $M_{k,\infty}^-$ have been completely characterized in [7, Theorem 3.1]:

Theorem 1.3. Assume (H_3) .

- (i) $M_{0,l}^- \neq \emptyset$ if and only if

$$W_\beta = \sum_{n=1}^{\infty} q_n \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k^{1/\alpha}} \right)^\beta < \infty.$$

- (ii) $M_{k,\infty}^- \neq \emptyset$ if and only if

$$V_\alpha = \sum_{n=1}^{\infty} \left(\frac{1}{p_{n+1}} \sum_{k=1}^n q_k \right)^{1/\alpha} < \infty.$$

Thus we are led to the next two questions:

- (a) under what conditions does equation (Ω) possess solutions from the class $M_{\infty,0}^+$ if (H_2) is satisfied and solutions from the class $M_{0,\infty}^-$ if (H_3) holds?
- (b) what can be said about the asymptotic behavior of such solutions as $n \rightarrow \infty$?

In the existing literature, only partial solutions to the first problem has been given and to our knowledge there is no answer to the second question. Therefore, the purpose of this paper is two-fold. One is to completely characterize, for of the sub-half linear equation (Ω) , the existence of solutions from the classes $M_{\infty,0}^+$, $M_{0,\infty}^-$, called in what follows, respectively, *slowly growing positive solutions* and *slowly decaying positive solutions*. The other is to establish exact asymptotic forms of such solutions. Second problem, however, is very difficult in general, and so we propose to downsize the problem slightly by limiting ourselves to slowly growing positive solutions of equation (Ω) with $p_n \equiv 1$ and coefficient $q = \{q_n\}$ which is regularly varying sequence. Then, using properties of regular varying sequence, one can determine the asymptotic behavior of regularly varying slowly growing solutions explicitly and accurately, thereby acquiring complete information about the structure of positive solutions of equation

$$(\Omega_1) \quad A(|\Delta x_n|^{\alpha-1} \Delta x_n) + q_n |x_{n+1}|^{\beta-1} x_{n+1} = 0,$$

in the framework of regular variation.

2. Sharp conditions for the existence of slowly growing and slowly decaying solutions of (Ω)

The problem of the existence of slowly growing and slowly decaying positive solutions has been completely solved in the half-linear case (see [8, Theorem 2.1] and [7, Theorem 3.2]).

Theorem 2.1. (i) Assume (H_2) . If equation (Ω) with $\beta = \alpha$ is nonoscillatory, then it does not have solutions in the class $M_{\infty,0}^+$ if and only if $R_\alpha < \infty$ and $T_\alpha < \infty$.

(ii) Assume (H_3) . If equation (Ω) with $\beta = \alpha$ is nonoscillatory, then it does not have solutions in the class $M_{0,\infty}^-$ if and only if $W_\alpha < \infty$ and $V_\alpha < \infty$.

In the case $\alpha > \beta$, it was shown in [9, Theorem 2]:

Proposition 2.1. If (H_2) holds and $R_\alpha = \infty$, $T_\beta < \infty$, then equation (Ω) has solutions in the class $M_{\infty,0}^+$.

Also, some sufficient conditions for nonexistence of solutions in the class $M_{\infty,0}^+$ and $M_{0,\infty}^-$ have been proved by Cecchi, Došlá, Marini in [7, Theorems

2.4 and 3.3]. However, until now, no effective sharp conditions for the existence of these type of solutions has been established. Thus, we will here solve this problem by proving the conjecture that the converse of Proposition 2.1 would hold.

Theorem 2.2. *Assume $S_p = \infty$. Sub-half linear equation (Ω) has slowly growing solutions if and only if $R_\alpha = \infty$ and $T_\beta < \infty$.*

Theorem 2.3. *Assume $S_p < \infty$. Sub-half linear equation (Ω) has slowly decaying solutions if and only if $W_\alpha < \infty$ and $V_\beta = \infty$.*

We first give the proof of the “only if” part of Theorem 2.2, and consequently, with the assertion of Proposition 2.1 we provide sharp conditions for the existence of slowly growing positive solutions of (Ω) .

Proof of the “only if” part of Theorem 2.2. Suppose that (Ω) has a solution $x \in M_{\infty,0}^+$, satisfying $x_n > 0$ and $x_n^{[1]} > 0$ for $n \geq N \geq 1$. Since $x_n^{[1]} \rightarrow 0$, $n \rightarrow \infty$ and x is increasing for $n \geq N$, summing (Ω) we get for $n \geq N$

$$(2.1) \quad x_n^{[1]} = \sum_{k=n}^{\infty} q_k x_{k+1}^\beta \geq x_{n+1}^\beta \sum_{k=n}^{\infty} q_k.$$

Denote $Q_n = \sum_{k=n}^{\infty} q_k$, and note that from (2.1) we have $Q_n \rightarrow 0$, $n \rightarrow \infty$. The sequence $\{p_n(\Delta x_n)^\alpha / Q_n\}$ is increasing, because

$$\Delta \left(\frac{p_n(\Delta x_n)^\alpha}{Q_n} \right) = \frac{q_n[p_n(\Delta x_n)^\alpha - x_{n+1}^\beta Q_n]}{Q_n Q_{n+1}} \geq 0, \quad n \geq N.$$

We claim that

$$(2.2) \quad \liminf_{n \rightarrow \infty} \frac{p_n(\Delta x_n)^\alpha}{Q_n x_n^\alpha} = 0.$$

In fact, assume on the contrary that (2.2) does not hold. Then, there exists constants $c > 0$ and $N_1 \geq N$ such that

$$(2.3) \quad x_n^{[1]} = p_n(\Delta x_n)^\alpha \geq c Q_n x_n \quad \text{for all } n \geq N_1.$$

Without loss of generality we may assume that

$$(2.4) \quad x_n \geq 1, \quad x_n^{[1]} \leq 1 \quad \text{and} \quad Q_n \leq 1 \quad \text{for all } n \geq N_1.$$

From (2.3) we have

$$(x_n^{[1]})^{\beta/\alpha} \geq c^{\beta/\alpha} Q_n^{\beta/\alpha} x_n^{\beta/\alpha}, \quad n \geq N_1$$

so multiplying by $q_n x_{n+1}^\beta = -\Delta x_n^{[1]} > 0$, $n \geq N$ and using (2.4), we get

$$(2.5) \quad (x_n^{[1]})^{\beta/\alpha} (-\Delta x_n^{[1]}) \geq c^{\beta/\alpha} q_n Q_n^{\beta/\alpha} = c^{\beta/\alpha} (-\Delta Q_n) Q_n^{\beta/\alpha}, \quad n \geq N_1.$$

Since

$$(x_n^{[1]})^{\beta/\alpha} (-\Delta x_n^{[1]}) \leq (x_n^{[1]})^{-\beta/\alpha} (-\Delta x_n^{[1]}) \leq \int_{x_{n+1}^{[1]}}^{x_n^{[1]}} s^{-\beta/\alpha} ds, \quad \text{and}$$

$$Q_n^{\beta/\alpha} (-\Delta Q_n) \geq \int_{Q_{n+1}}^{Q_n} s^{\beta/\alpha} ds,$$

we obtain from (2.5)

$$c^{\beta/\alpha} \int_{Q_{n+1}}^{Q_n} s^{\beta/\alpha} ds \leq \int_{x_{n+1}^{[1]}}^{x_n^{[1]}} s^{-\beta/\alpha} ds,$$

which summing from n to ∞ and using that $Q_n \rightarrow 0$ and $x_n^{[1]} \rightarrow 0$ as $n \rightarrow \infty$, yields

$$-c^{\beta/\alpha} \frac{\alpha}{\alpha + \beta} Q_{n+1}^{1+\beta/\alpha} \leq -\frac{\alpha}{\alpha - \beta} (x_{n+1}^{[1]})^{1-\beta/\alpha}, \quad n \geq N_1.$$

Thus,

$$(2.6) \quad x_{n+1}^{[1]} \leq c^{\beta/(\alpha-\beta)} \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^{\alpha/(\alpha-\beta)} Q_{n+1}^{(\alpha+\beta)/(\alpha-\beta)}, \quad n \geq N_1,$$

which combined with (2.3) gives

$$x_n \leq \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^{\alpha/(\alpha-\beta)} c^{\beta/(\alpha-\beta)-1} Q_n^{2\beta/(\alpha-\beta)} \leq \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^{\alpha/(\alpha-\beta)} c^{\beta/(\alpha-\beta)-1}, \quad n \geq N_1,$$

contradicting assumption that $x_n \rightarrow \infty$, $n \rightarrow \infty$, and verifying the truth of (2.2).

Then, there is a sufficiently large $N_2 \geq N$ such that

$$\begin{aligned} \frac{x_n}{2} \leq x_n - x_{N_2} &= \sum_{k=N_2}^{n-1} \Delta x_k = \sum_{k=N_2}^{n-1} \left(\frac{Q_k}{p_k} \right)^{1/\alpha} \left(\frac{p_k (\Delta x_k)^\alpha}{Q_k} \right)^{1/\alpha} \\ &\leq \left(\frac{p_n (\Delta x_n)^\alpha}{Q_n} \right)^{1/\alpha} \sum_{k=N_2}^{n-1} \left(\frac{Q_k}{p_k} \right)^{1/\alpha}, \quad n \geq N_2 \end{aligned}$$

or

$$\sum_{k=N_2}^{n-1} \left(\frac{Q_k}{p_k} \right)^{1/\alpha} \geq \frac{1}{2} \left(\frac{p_n (\Delta x_n)^\alpha}{Q_n x_n^\alpha} \right)^{-1/\alpha},$$

proving, due to (2.2), that $R_\alpha = \infty$.

To prove the second condition, we start from the equality $\Delta x_n = p_n^{-1/\alpha} (x_n^{[1]})^{1/\alpha}$ and using that $\{x_n^{[1]}\}$ is a decreasing, we have

$$(2.7) \quad x_{n+1} = x_N + \sum_{k=1}^n \left(\frac{1}{p_k} \right)^{1/\alpha} (x_k^{[1]})^{1/\alpha} \geq (x_n^{[1]})^{1/\alpha} \sum_{k=1}^n \left(\frac{1}{p_k} \right)^{1/\alpha}, \quad n \geq N.$$

Then, because

$$\frac{-\Delta x_n^{[1]}}{(x_n^{[1]})^{\beta/\alpha}} \leq \int_{x_{n+1}^{[1]}}^{x_n^{[1]}} s^{-\beta/\alpha} ds,$$

we get from (2.7)

$$q_n \left(\sum_{k=1}^n \left(\frac{1}{p_k} \right)^{1/\alpha} \right)^\beta \leq \frac{q_n x_{n+1}^\beta}{(x_{n+1}^{[1]})^{\beta/\alpha}} \leq \frac{-\Delta x_n^{[1]}}{(x_n^{[1]})^{\beta/\alpha}} \leq \int_{x_{n+1}^{[1]}}^{x_n^{[1]}} s^{-\beta/\alpha} ds, \quad n \geq N.$$

Summing the last inequality from N to ∞ , we come to the desired conclusion that

$$T_\beta = \sum_{n=N}^{\infty} q_n \left(\sum_{k=1}^n \left(\frac{1}{p_k} \right)^{1/\alpha} \right)^\beta \leq \frac{\alpha}{\alpha - \beta} (x_{N+1}^{[1]})^{1-\beta/\alpha}. \quad \square$$

Now, Theorem 2.3 can be easily proved by using Theorem 2.2 and the following simple property called *the reciprocal principle* (see [7]), which links solutions of (Ω) with the difference equation

$$(\Omega_r) \quad \Delta(q_n^{-1/\beta} |\Delta y_n|^{1/\beta-1} \Delta y_n) + p_{n+1}^{-1/\alpha} |y_{n+1}|^{1/\alpha-1} y_{n+1} = 0,$$

so-called *the reciprocal equation*. Namely, if x is a solution of (Ω) , then, $y = \{y_n\}$ where $y_n = x_n^{[1]}$ is a solution of (Ω_r) . If y is a solution of (Ω_r) , then its quasi-difference

$$y_n^{[1]} = q_n^{-1/\beta} |\Delta y_n|^{1/\beta-1} \Delta y_n$$

satisfies $y_n^{[1]} = -x_{n+1}$, so that $z = \{z_n\}$, where $z_n = y_n^{[1]}$, is a solution of

$$\Delta(p_{n+1} |Az_n|^{\alpha-1} \Delta z_n) + q_{n+1} |z_{n+1}|^{\beta-1} z_{n+1} = 0.$$

For solution x of (Ω) and y for (Ω_r) , with $y_n = x_n^{[1]}$, it holds that

$$\begin{aligned} x \in \mathbf{M}_{k,\infty}^- &\Leftrightarrow y \in \mathbf{M}_{\infty,l}^+ \\ x \in \mathbf{M}_{0,\infty}^- &\Leftrightarrow y \in \mathbf{M}_{\infty,0}^+ \\ x \in \mathbf{M}_{0,l}^- &\Leftrightarrow y \in \mathbf{M}_{k,0}^+ \end{aligned}$$

Since $1/\alpha < 1/\beta$ equation (Ω_r) is also sub-half linear. Clearly, (H_3) holds for (Ω) if and only if (H_2) holds for (Ω_r) and the series W_α , V_β for (Ω) plays the same role as the series T_β , R_α for (Ω_r) and vice versa. Therefore, the assertion of Theorem 2.3 follows by applying Theorem 2.2 to the reciprocal equation (Ω_r) .

3. Normalized regularly varying sequences

Once the existence of solutions of (Ω) has been established, the next task is to acquire detailed information about qualitative properties of its solutions. This is a very difficult problem to solve in general, but the recent development of asymptotic analysis of ordinary differential equations (see [10, 11, 12, 13, 14, 15, 16, 17, 18]) by means of regular variation functions, suggests investigating the discrete problem in the framework of regularly varying sequences. Therefore, we first state here definitions and some basic properties of regularly varying sequences and then, we prove some other properties which will be essential in establishing our main results on the asymptotic behavior of non-oscillatory solutions stated and proved in the next section (Theorems 4.1, 4.2 and 4.3).

Two main approaches are known in the basic theory of regularly varying sequences: the approach due to Karamata [5], based on a definition that can be understood as a direct discrete counterpart of simple and elegant continuous definition, and the approach due to Galambos and Seneta, based on purely sequential definition.

Definition 3.1 (Karamata [5]). A positive sequence $y = \{y_n\}$, $n \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{n \rightarrow \infty} \frac{y_{[\lambda n]}}{y_n} = \lambda^\rho \quad \text{for } \forall \lambda > 0,$$

where $[u]$ denotes the integer part of u .

Definition 3.2 (Galambos and Seneta [4]). A positive sequence $y = \{y_n\}$, $n \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if there exists a positive sequence $\{\alpha_n\}$ satisfying

$$\lim_{n \rightarrow \infty} \frac{y_n}{\alpha_n} = C, \quad 0 < C < \infty \quad \lim_{n \rightarrow \infty} n \frac{\Delta \alpha_{n-1}}{\alpha_n} = \rho.$$

If $\{y_n\}$ is regularly varying of index $\rho = 0$ it is said to be *slowly varying*. The totality of regularly varying sequences of index ρ and slowly varying sequences is denoted, respectively, by $\mathcal{RV}(\rho)$ and $\mathcal{SV}$.

There exist various necessary and sufficient conditions for a sequence of positive numbers to be regularly varying (see [3, 4, 19, 20]) and consequently each one of them could be used to define regularly varying sequence. The one that is the most important is the following Representation theorem (see [3, Theorem 3]), while some other representation formula for regularly varying sequences were established in [20, Lemma 1].

Proposition 3.1 (Representation theorem). *A positive sequence $\{y_k\}$, $k \in \mathbf{N}$ is regularly varying of index $\rho \in \mathbf{R}$ if and only if there exists sequences $\{c_k\}$ and $\{\delta_k\}$ such that*

$$\lim_{k \rightarrow \infty} c_k = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{k \rightarrow \infty} \delta_k = 0,$$

and

$$(3.1) \quad y_k = c_k k^\rho \exp \left(\sum_{i=1}^k \frac{\delta_i}{i} \right).$$

In particular, the following property of regularly varying sequences follows from Proposition 3.1.

Proposition 3.2. *Let $y \in \mathcal{SV}$. Then, for every $\varepsilon > 0$*

$$\lim_{n \rightarrow \infty} n^\varepsilon y_n = \infty, \quad \lim_{n \rightarrow \infty} n^{-\varepsilon} y_n = 0.$$

That the class of regularly varying sequences is well suited for the study of the second order linear difference equations and the half-linear difference equations has been shown by Matucci and Řehák in [19, 20, 21]. They introduced the concept of normalized regularly varying sequences which play a relevant role in the study of the asymptotic behavior of nonoscillatory solutions of linear and half-linear difference equations.

Definition 3.3. A positive sequence $y = \{y_n\}$ is said to be *normalized regularly varying of index ρ* if it satisfies

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{n \Delta y_n}{y_n} = \rho.$$

If $\rho = 0$, then y is called a *normalized slowly varying sequence*.

In what follows symbols $\mathcal{NSV}$ and $\mathcal{NRV}(\rho)$ is used to denote the set of all normalized slowly varying and normalized regularly varying sequences of index ρ . Typical examples of regularly varying sequences are:

$$\{\log k\} \in \mathcal{NSV}, \quad \{k^\rho \log k\} \in \mathcal{NRV}(\rho), \quad \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}.$$

In fact, Matucci and Řehák established necessary and sufficient conditions for solutions of second order linear difference equation

$$(3.3) \quad \Delta^2 x_n + f_n x_{n+1} = 0,$$

to be regularly varying, where $f = \{f_n\}$ is a negative sequence of real numbers (in [19, Theorem 3.1, 3.2]) or an arbitrary sequence (in [21]).

Theorem 3.1 ([21, Theorem 3.1]). *Let $A \in (-\infty, 1/4)$ and $\rho_1 < \rho_2$ be the real roots of the quadratic equation $\rho^2 - \rho - A = 0$. Equation (3.3) has a fundamental set of solutions*

$$u_n = n^{\rho_1} L_n \in \mathcal{NRV}(\rho_1) \quad \text{and} \quad v_n = n^{\rho_2} \tilde{L}_n \in \mathcal{NRV}(\rho_2)$$

if and only if

$$(3.4) \quad \lim_{n \rightarrow \infty} n \sum_{k=n}^{\infty} f_k = A,$$

where $L, \tilde{L} \in \mathcal{NSV}$ with

$$\tilde{L}_n \sim \frac{1}{(1 - 2\rho_1)L_n}, \quad n \rightarrow \infty.$$

Moreover, every eventually positive solution x of (3.3) is normalized regularly varying, with $x \in \mathcal{NRV}(\rho_1) \cup \mathcal{NRV}(\rho_2)$.

Same authors obtained in [20, 21] analog results for the second order half-linear difference equation.

The feature of the present work is to demonstrate that the class of normalized regularly varying sequences, which has so far been confined to the existence of positive solutions of difference equations, can also be effectively applied to determine the asymptotic forms of positive solutions explicitly and accurately. For that purpose, we need some basic properties of normalized regularly varying sequences listed below and proved in [3, 19, 20].

Proposition 3.3 ([20, Lemma 2]). *$y \in \mathcal{NRV}(\rho)$ if and only if $y_n = n^\rho L_n$, where $L = \{L_n\} \in \mathcal{NSV}$.*

Proposition 3.4 ([19, Lemma 2.3]). *Let $u \in \mathcal{NRV}(\rho_1)$ and $v \in \mathcal{NRV}(\rho_2)$. Then, $uv \in \mathcal{NRV}(\rho_1 + \rho_2)$ and $1/u \in \mathcal{NRV}(-\rho_1)$.*

Proposition 3.5 ([3, Lemma 1]). *If $y \in \mathcal{NRV}(\rho)$, then $\lim_{n \rightarrow \infty} y_{n+1}/y_n = 1$.*

From Proposition 3.4 it follows that if $l \in \mathcal{NSV}$, so does $l^m \in \mathcal{NSV}$ for every $m \in \mathbb{N}$. Moreover, if $y \in \mathcal{NRV}(\rho)$, then in the representation formula

(3.1), we have $c_k \equiv C = \text{const}$. Therefore, by Proposition 3.1 it is clear that $l^\gamma \in \mathcal{NSV}$ for every $\gamma \in \mathbf{R}$.

In view of Proposition 3.3, if for $y \in \mathcal{NRV}(\rho)$

$$\lim_{n \rightarrow \infty} \frac{y_n}{n^\rho} = \lim_{t \rightarrow \infty} L_n = \text{const} > 0,$$

then $y = \{y_n\}$ is said to be a *trivial normalized regularly varying sequence of index ρ* and is denoted by $y \in \text{tr} - \mathcal{NRV}(\rho)$. Otherwise y is said to be a *nontrivial normalized regularly varying sequence of index ρ* , denoted by $y \in \text{ntr} - \mathcal{NRV}(\rho)$. According to these terminologies a solution $x \in \mathbf{M}_{k,0}^+ \cup \mathbf{M}_{k,\infty}^-$ is a trivial $\mathcal{NSV}$ -solution and a solution $x \in \mathbf{M}_{\infty,l}^+ \cup \mathbf{M}_{0,l}^-$ is a trivial $\mathcal{NRV}(1)$ -solution of (Ω) .

Next, we prove some properties of normalized regularly varying sequence, useful for the development of asymptotic behavior of solutions of (Ω_1) in the subsequent section. We start by recalling two variants of the Stolz-Cezaro theorem.

Lemma 3.1. *If $f = \{f_n\}$ is a strictly increasing sequence of positive real numbers, such that $\lim_{n \rightarrow \infty} f_n = \infty$, then for any sequence $g = \{g_n\}$ of positive real numbers one has the inequalities:*

$$\liminf_{n \rightarrow \infty} \frac{\Delta f_n}{\Delta g_n} \leq \liminf_{n \rightarrow \infty} \frac{f_n}{g_n} \leq \limsup_{n \rightarrow \infty} \frac{f_n}{g_n} \leq \limsup_{n \rightarrow \infty} \frac{\Delta f_n}{\Delta g_n}.$$

In particular, if the sequence $\{\Delta f_n / \Delta g_n\}$ has a limit then

$$(3.5) \quad \lim_{n \rightarrow \infty} \frac{f_n}{g_n} = \lim_{n \rightarrow \infty} \frac{\Delta f_n}{\Delta g_n}.$$

Lemma 3.2. *Let $f = \{f_n\}$, $g = \{g_n\}$ be sequences of positive real numbers, such that*

- (i) $\lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} g_n = 0$;
- (ii) *the sequence g is strictly monotone;*
- (iii) *the sequence $\{\Delta f_n / \Delta g_n\}$ has a limit.*

Then, a sequence $\{f_n / g_n\}$ is convergent and (3.5) hold.

The following results concerns monotonicity of normalized regularly varying sequences.

Theorem 3.2. *$y \in \mathcal{NRV}(\rho)$ if and only if there exists a finite number ρ such that $\{n^{-\mu} y_n\}$ is eventually increasing for each $\mu < \rho$ and $\{n^{-\tau} y_n\}$ is eventually decreasing for each $\tau > \rho$.*

Proof. First, notice that from (3.2), using Proposition 3.5, we have

$$\lim_{n \rightarrow \infty} \frac{(n+1)\Delta y_n}{y_{n+1}} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{n\Delta y_n}{y_n} \cdot \frac{y_n}{y_{n+1}} = \rho,$$

implying

$$(3.6) \quad \frac{y_n}{y_{n+1}} = 1 - \frac{\rho + o(1)}{n+1}, \quad n \rightarrow \infty.$$

If we assume that $y \in \mathcal{NRV}(\rho)$, we have

$$\begin{aligned} & (n+1)^{-\mu} y_{n+1} - n^{-\mu} y_n \\ &= (n+1)^{-\mu} y_{n+1} \left(1 - \frac{n^{-\mu}}{(n+1)^{-\mu}} \left(1 - \frac{\rho + o(1)}{n+1} \right) \right) \\ &= (n+1)^{-\mu-1} y_{n+1} \left[\frac{(n+1)^{1-\mu} - n^{-\mu}(n+1)}{(n+1)^{-\mu}} + \frac{n^{-\mu}}{(n+1)^{-\mu}} (\rho + o(1)) \right] \\ &= (n+1)^{-\mu-1} y_{n+1} (-\mu + \rho + o(1)), \end{aligned}$$

and since $\mu < \rho$, we conclude that $\{n^{-\mu} y_n\}$ is eventually increasing. Similarly, for $\tau > \rho$, we can show that $\{n^{-\tau} y_n\}$ is eventually decreasing. On the contrary, assuming that for each μ, τ , $\mu < \rho < \tau$ sequences $\{n^{-\mu} y_n\}$ and $\{n^{-\tau} y_n\}$ are, respectively, eventually increasing and eventually decreasing, for n sufficiently large we have

$$\left(\frac{n}{n+1} \right)^\tau \leq \frac{y_n}{y_{n+1}} \leq \left(\frac{n}{n+1} \right)^\mu,$$

so that

$$(n+1) \left(1 - \left(1 - \frac{1}{n+1} \right)^\mu \right) \leq (n+1) \left(1 - \frac{y_n}{y_{n+1}} \right) \leq (n+1) \left(1 - \left(1 - \frac{1}{n+1} \right)^\tau \right),$$

which yields

$$\mu \leq \liminf_{n \rightarrow \infty} \frac{(n+1)\Delta y_n}{y_{n+1}} \leq \limsup_{n \rightarrow \infty} \frac{(n+1)\Delta y_n}{y_{n+1}} \leq \tau.$$

Now, choosing μ and τ arbitrarily close to ρ , we get $y \in \mathcal{NRV}(\rho)$. $\square$

Theorem 3.3. *If $f = \{f_n\} \in \mathcal{NSV}$ is a strictly decreasing sequence, such that $\lim_{n \rightarrow \infty} f_n = 0$, then for each $\gamma \in \mathbf{R}$*

$$(3.7) \quad \lim_{n \rightarrow \infty} f_n^{-\gamma} \sum_{k=n}^{\infty} f_k^{\gamma-1} (-\Delta f_k) = \frac{1}{\gamma}.$$

If $g = \{g_n\} \in \mathcal{NPSV}$ is a strictly increasing sequence such that $\lim_{n \rightarrow \infty} g_n = \infty$, then

$$(3.8) \quad \lim_{n \rightarrow \infty} g_n^{-\gamma} \sum_{k=1}^{n-1} g_k^{\gamma-1} \Delta g_k = \frac{1}{\gamma}.$$

Proof. The statement clearly hold for $\gamma = 1$, so let $\gamma \neq 1$. For the strictly decreasing sequence $\{f_n\} \in \mathcal{NPSV}$, using the mean value theorem we have for all $n \in N$,

$$\mathcal{V}f_{n+1}^{\gamma-1}(-\Delta f_n) \leq f_n^\gamma - f_{n+1}^\gamma \leq \mathcal{V}f_n^{\gamma-1}(-\Delta f_n), \quad \text{if } \gamma > 1,$$

$$\mathcal{V}f_n^{\gamma-1}(-\Delta f_n) \leq f_n^\gamma - f_{n+1}^\gamma \leq \mathcal{V}f_{n+1}^{\gamma-1}(-\Delta f_n), \quad \text{if } \gamma < 1,$$

implying

$$\frac{1}{\gamma} \cdot \frac{f_n^{\gamma-1}(-\Delta f_n)}{f_n^{\gamma-1}(-\Delta f_n)} \leq \frac{f_n^{\gamma-1}(-\Delta f_n)}{f_n^\gamma - f_{n+1}^\gamma} \leq \frac{1}{\gamma} \cdot \frac{f_n^{\gamma-1}(-\Delta f_n)}{f_{n+1}^{\gamma-1}(-\Delta f_n)}, \quad \text{if } \gamma > 1,$$

$$\frac{1}{\gamma} \cdot \frac{f_n^{\gamma-1}(-\Delta f_n)}{f_{n+1}^{\gamma-1}(-\Delta f_n)} \leq \frac{f_n^{\gamma-1}(-\Delta f_n)}{f_n^\gamma - f_{n+1}^\gamma} \leq \frac{1}{\gamma} \cdot \frac{f_n^{\gamma-1}(-\Delta f_n)}{f_n^{\gamma-1}(-\Delta f_n)}, \quad \text{if } \gamma < 1,$$

which using Proposition 3.5 gives

$$(3.9) \quad \lim_{n \rightarrow \infty} \frac{f_n^{\gamma-1}(-\Delta f_n)}{f_n^\gamma - f_{n+1}^\gamma} = \frac{1}{\gamma}.$$

Now, using (3.9), by Stolz-Cesaro's theorem we obtain

$$\lim_{n \rightarrow \infty} f_n^{-\gamma} \sum_{k=n}^{\infty} f_k^{\gamma-1}(-\Delta f_k) = \lim_{n \rightarrow \infty} \frac{f_n^{\gamma-1}(-\Delta f_n)}{-\Delta f_n^\gamma} = \frac{1}{\gamma}.$$

Assertion for strictly increasing normalized slowly varying sequence $\{g_n\}$ can be verified in a similar manner. $\square$

The following theorem can be concerned as *the discrete analog of the Karamata's integration theorem* and plays a central role in the proof of our main results in the Section 4.

Theorem 3.4. Let $l = \{l_n\} \in \mathcal{NPSV}_n$.

- (i) If $\alpha > -1$, then $\lim_{n \rightarrow \infty} (n^{\alpha+1} l_n)^{-1} \sum_{k=1}^n k^\alpha l_k = 1/(1+\alpha)$;
- (ii) If $\alpha < -1$, then $\lim_{n \rightarrow \infty} (n^{\alpha+1} l_n)^{-1} \sum_{k=n}^{\infty} k^\alpha l_k = -1/(1+\alpha)$;
- (iii) If $\sum_{k=1}^{\infty} l_k/k < \infty$, then $\sum_{k=n}^{\infty} l_k/k \in \mathcal{NPSV}$ and $\lim_{n \rightarrow \infty} l_n^{-1} \sum_{k=n}^{\infty} l_k/k = \infty$;
- (iv) If $\sum_{k=1}^{\infty} l_k/k = \infty$, then $\sum_{k=1}^n l_k/k \in \mathcal{NPSV}$ and $\lim_{n \rightarrow \infty} y_n^{-1} \sum_{k=1}^n y_k/k = \infty$.

Proof. (i) Let $\alpha > -1$. Proposition 3.2 and Theorem 3.2 ensures that $\{n^{\alpha+1}l_n\}$ is eventually increasing sequence such that $n^{\alpha+1}l_n \rightarrow \infty$, $n \rightarrow \infty$. By application of Lemma 3.1, using (3.2) and Proposition 3.5 we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^{\alpha+1}l_n}{\sum_{k=1}^n k^\alpha l_k} &= \lim_{n \rightarrow \infty} \frac{(n+1)^{\alpha+1} \Delta l_n + l_n \Delta n^{\alpha+1}}{(n+1)^\alpha l_{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{n \Delta l_n}{l_n} \cdot \frac{l_n}{l_{n+1}} + \lim_{n \rightarrow \infty} \frac{l_n}{l_{n+1}} \cdot \frac{(n+1)^{\alpha+1} - n^{\alpha+1}}{(n+1)^\alpha} \\ &= 1 + \alpha. \end{aligned}$$

(ii) Because $\alpha < -1$, the sequence $\{n^{\alpha+1}l_n\}$ is eventually decreasing satisfying that $n^{\alpha+1}l_n \rightarrow 0$, $n \rightarrow \infty$. Moreover, since $n^{(\alpha-1)/2}l_n \rightarrow 0$, $n \rightarrow \infty$,

$$n^\alpha l_n = \frac{n^{(\alpha-1)/2}l_n}{n^{-(\alpha+1)/2}} \leq \frac{1}{n^{-(\alpha+1)/2}}, \quad \text{for large } n,$$

so because $-(\alpha+1)/2 > 1$, the series $\sum_{n=1}^\infty n^\alpha l_n$ is convergent. Thus, $\sum_{k=n}^\infty k^\alpha l_k \rightarrow 0$, as $n \rightarrow \infty$ and application of Lemma 3.2 gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^{\alpha+1}l_n}{\sum_{k=n}^\infty k^\alpha l_k} &= \frac{(n+1)^{\alpha+1} \Delta l_n + l_n \Delta n^{\alpha+1}}{-n^\alpha l_n} \\ &= - \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{\alpha+1} \frac{n \Delta l_n}{l_n} - \lim_{n \rightarrow \infty} \frac{(n+1)^{\alpha+1} - n^{\alpha+1}}{n^\alpha} \\ &= -(1 + \alpha). \end{aligned}$$

(iii) If $\sum_{k=1}^\infty l_k/k < \infty$, denote $L_n = \sum_{k=n}^\infty l_k/k$, so that $\{L_n\}$ is decreasing and $L_n \rightarrow 0$, $n \rightarrow \infty$. By application of Lemma 3.2 we get

$$\lim_{n \rightarrow \infty} n \frac{\Delta L_n}{L_n} = \lim_{n \rightarrow \infty} n \frac{n^{-1}l_n}{L_n} = - \lim_{n \rightarrow \infty} \frac{l_n}{L_n} = - \lim_{n \rightarrow \infty} \frac{\Delta l_n}{\Delta L_n} = \lim_{n \rightarrow \infty} \frac{\Delta l_n}{n^{-1}l_n} = 0,$$

implying that $L = \{L_n\} \in \mathcal{NSV}$ and $\lim_{n \rightarrow \infty} L_n/l_n = \infty$.

(iv) If $\sum_{k=1}^\infty l_k/k = \infty$, denoting $\lambda_n = \sum_{k=1}^n l_k/k$, because $\{\lambda_n\}$ is increasing sequence such that $\lambda_n \rightarrow \infty$, $n \rightarrow \infty$, applying Lemma 3.1 we get

$$\begin{aligned} \lim_{n \rightarrow \infty} n \frac{\Delta \lambda_n}{\lambda_n} &= \lim_{n \rightarrow \infty} n \frac{(n+1)^{-1}l_{n+1}}{\lambda_n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \lim_{n \rightarrow \infty} \frac{l_{n+1}}{\lambda_n} \\ &= \lim_{n \rightarrow \infty} \frac{\Delta l_{n+1}}{\Delta \lambda_n} = \lim_{n \rightarrow \infty} (n+1) \frac{\Delta l_{n+1}}{l_{n+1}} = 0, \end{aligned}$$

implying that $\lambda = \{\lambda_n\} \in \mathcal{NPSV}$ and

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{l_n} = \lim_{n \rightarrow \infty} \frac{\lambda_n}{l_{n+1}} \cdot \frac{l_{n+1}}{l_n} = \infty. \quad \square$$

4. The asymptotic behavior of slowly growing solutions of (Ω_1)

This section is concerned with the problem of the existence of slowly growing positive solutions for equation (Ω_1) with coefficient $q = \{q_n\}$ which is normalized regularly varying sequence of index σ expressed as

$$(4.1) \quad q_n = n^\sigma \omega_n, \quad \omega = \{\omega_n\} \in \mathcal{NPSV}.$$

If $\{y_n\}$ is an eventually positive solution of (Ω) , then there are positive constants c_1 and c_2 such that

$$(4.2) \quad c_1 \leq y_n \leq c_2 n \quad \text{for all large } n.$$

Therefore, the regularity index ρ of regularly varying solutions x of (Ω_1) satisfies $0 \leq \rho \leq 1$. If x is a slowly growing normalized regularly varying solution of index ρ , it may be expressed as $x_n = n^\rho \xi_n$, $\xi = \{\xi_n\} \in \mathcal{NPSV}$. There, if $\rho = 1$, then slowly varying part of x satisfies $\xi_n = x_n/n \rightarrow \infty$, $n \rightarrow \infty$ and if $\rho = 0$, then $\xi_n = x_n \rightarrow 0$, $n \rightarrow \infty$. This means that the class of slowly growing regularly varying solutions of (Ω_1) if non-empty, is divided into three types of subclasses composed of regularly varying solutions belonging to

$$ntr - \mathcal{NPSV} \quad \text{or} \quad \mathcal{NRV}(\rho) \quad \text{for some } \rho \in (0, 1) \quad \text{or} \quad ntr - \mathcal{NRV}(1),$$

each of which can be thoroughly understood by means of regular variation as shown by our main results (Theorems 4.1, 4.2, 4.3).

4.1. Preparatory results

To prove our main results, we use, beside properties of normalized regularly varying sequence, the next two results: the discrete analog of the Lebesgue dominated convergence theorem and the fixed point theorem, which was proved in [22] and which can be considered as a discrete analog of Schauder's fixed point theorem.

Lemma 4.1. *Let $\{a_{m,k}\}$ be a double real sequence, $a_{m,k} \geq 0$ for $m, k \in \mathbb{N}$ such that $\lim_{m \rightarrow \infty} a_{m,k} = A_k$ for every $k \in \mathbb{N}$. Assume that the series $\sum_{k=1}^{\infty} a_{m,k}$ is totally convergent, that is, there exists a sequence $\{\alpha_k\}$ such that $a_{m,k} \leq \alpha_k$ with $\sum_{k=1}^{\infty} \alpha_k < \infty$. Then, the series $\sum_{k=1}^{\infty} A_k$ converges and*

$$\lim_{m \rightarrow \infty} \sum_{k=1}^{\infty} a_{m,k} = \sum_{k=1}^{\infty} A_k.$$

Lemma 4.2. *Suppose X is a Banach space and K is closed, bounded and convex subset of X . If $\mathcal{F} : K \rightarrow X$ is a continuous mapping such that $\mathcal{F}(K) \subset K$ and $\mathcal{F}(K)$ is uniformly Cauchy, then $\mathcal{F}$ has a fixed point in K .*

To simplify the “only if” part of the proof of our main results we prove the next result.

Lemma 4.3. *If $q \in \mathcal{NRV}(\sigma)$ is expressed with (4.1) and $x = \{x_n\}$ with $x_n = n^\rho \xi_n$, $\xi \in \mathcal{NSV}$, $\rho \in [0, 1]$ is a solution of (Ω_1) belonging to the class $\mathbf{M}_{\infty,0}^+$, then only one of the following three statements holds:*

(i) $\sigma + \rho\beta = -1$ and

$$(4.3) \quad x_n \sim n \left(\sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \omega_k \xi_k^\beta \right)^{1/\alpha} \in \mathcal{NRV}(1), \quad n \rightarrow \infty;$$

(ii) $-1 < (\sigma + \rho\beta + 1)/\alpha < 0$ and

$$(4.4) \quad x_n \sim \frac{n^{(\sigma+\rho\beta+1)/\alpha+1} \omega_n^{1/\alpha} \xi_n^{\beta/\alpha}}{[-(\sigma + \rho\beta + 1)]^{1/\alpha} ((\sigma + \rho\beta + 1)/\alpha + 1)} \\ \in \mathcal{NRV}\left(\frac{\sigma + \rho\beta + 1}{\alpha} + 1\right), \quad n \rightarrow \infty;$$

(iii) $(\sigma + \rho\beta + 1)/\alpha = -1$ and

$$(4.5) \quad x_n \sim \alpha^{-1/\alpha} \sum_{n=n_0}^{n-1} k^{-1} \omega_k^{1/\alpha} \xi_k^{\beta/\alpha} \in \mathcal{NSV}.$$

Proof. Suppose that (Ω_1) has a solution $x \in \mathbf{M}_{\infty,0}^+ \cap \mathcal{NRV}(\rho)$ with $\rho \in [0, 1]$, satisfying $x_n > 0$ and $\Delta x_n > 0$ for $n \geq n_0 \geq 1$. Such a solution is expressed as $x_n = n^\rho \xi_n$, $\xi \in \mathcal{NSV}$. Using (4.1) we have

$$(4.6) \quad (\Delta x_n)^\alpha = \sum_{k=n}^{\infty} q_k x_{k+1}^\beta \sim \sum_{k=n}^{\infty} q_k x_k^\beta = \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \omega_k \xi_k^\beta, \quad n \geq n_0.$$

The fact that $\Delta x_n \rightarrow 0$ $n \rightarrow \infty$ implies the convergence of the last series in (4.6), so it must be $\sigma + \rho\beta \leq -1$. We first consider the case $\sigma + \rho\beta = -1$. Then,

$$(4.7) \quad \Delta x_n = \left(\sum_{k=n}^{\infty} q_k x_{k+1}^\beta \right)^{1/\alpha} \sim \left(\sum_{k=n}^{\infty} k^{-1} \omega_k \xi_k^\beta \right)^{1/\alpha} \in \mathcal{NSV}$$

(see Theorem 3.4-(iii)) and summing (4.7) from n_0 to $n-1$, with application of Theorem 3.4-(i) we get (4.3).

Next we consider the case $\sigma + \rho\beta < -1$. An application of Theorem 3.4-(ii) in (4.6) implies

$$(4.8) \quad \Delta x_n = \left(\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta} \right)^{1/\alpha} \sim \frac{n^{(\sigma+\rho\beta+1)/\alpha} \omega_n^{1/\alpha} \xi_n^{\beta/\alpha}}{[-(\sigma + \rho\beta + 1)]^{1/\alpha}}, \quad n \rightarrow \infty.$$

Because $\sum_{n=1}^{\infty} \Delta x_n$ is divergent series, we conclude from (4.8) that it must be $(\sigma + \rho\beta + 1)/\alpha \geq -1$, so we distinguish the two possibilities:

$$(a) \quad \frac{\sigma + \rho\beta + 1}{\alpha} > -1, \quad (b) \quad \frac{\sigma + \rho\beta + 1}{\alpha} = -1.$$

If (a) holds, summing (4.8) from n_0 to n , by Theorem 3.4-(i), we get

$$x_{n+1} \sim x_n \sim \frac{n^{(\sigma+\rho\beta+1)/\alpha+1} \omega_n^{1/\alpha} \xi_n^{\beta/\alpha}}{[-(\sigma + \rho\beta + 1)]^{1/\alpha} ((\sigma + \rho\beta + 1)/\alpha + 1)}, \quad n \rightarrow \infty,$$

which shows that

$$x \in \mathcal{NRV} \left(\frac{\sigma + \rho\beta + 1}{\alpha} + 1 \right) \quad \text{and} \quad 0 < \frac{\sigma + \rho\beta + 1}{\alpha} + 1 < 1.$$

On the other hand if (b) holds, (4.8) becomes

$$\Delta x_n \sim \alpha^{-1/\alpha} n^{-1} \omega_n^{1/\alpha} \xi_n^{\beta/\alpha} \quad n \rightarrow \infty,$$

and by summing from n_0 to $n-1$ we obtain (4.5). $\square$

4.2. Existence and asymptotic behavior of normalized regularly varying solutions of index $\rho \in (0, 1)$

First, we verify the necessary and sufficient condition for the existence of normalized regularly varying solutions of index $\rho \in (0, 1)$ with the precise asymptotic behavior at infinity and that the regularity index ρ of such solution is uniquely determined by α , β and the regularity index σ of q .

Theorem 4.1. *Suppose that $q \in \mathcal{NRV}(\sigma)$. Equation (Ω_1) possesses a normalized regularly varying solution x of index $\rho \in (0, 1)$ if and only if*

$$(4.9) \quad \sigma \in (-\alpha - 1, -\beta - 1),$$

in which case ρ is given by

$$(4.10) \quad \rho = \frac{\sigma + \alpha + 1}{\alpha - \beta},$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$(4.11) \quad x_n \sim \left[\frac{n^{\alpha+1} q_n}{\alpha \rho^\alpha (1-\rho)} \right]^{1/(\alpha-\beta)} \quad \text{as } n \rightarrow \infty.$$

Proof. **The “only if” part:** Suppose that $x \in \mathcal{NRV}(\rho)$ with $\rho \in (0, 1)$. Then, $x_n = n^\rho \zeta_n$, $\zeta \in \mathcal{NSV}$ and it is clear that only case (ii) of Lemma 4.3 is admissible for x . Thus it must be

$$\rho = \frac{\sigma + \rho\beta + 1}{\alpha} + 1,$$

implying that the regularity index of x is given by (4.10), which combined with the hypothesis $\rho \in (0, 1)$, determines the range of σ as $-\alpha - 1 < \sigma < -\beta - 1$. Using (4.10) we obtain

$$[-(\sigma + \rho\beta + 1)]^{1/\alpha} \left(\frac{\sigma + \rho\beta + 1}{\alpha} + 1 \right) = (\alpha(1-\rho)\rho^\alpha)^{1/\alpha},$$

and

$$n^{(\sigma+\rho\beta+1)/\alpha+1} \omega_n^{1/\alpha} \zeta_n^{\beta/\alpha} = (n^{\alpha+1} q_n)^{1/\alpha} x_n^{\beta/\alpha}.$$

then, from (4.4) we obtain that the asymptotic form of normalized regularly varying solution x of index ρ is given by (4.11). Thus the “only if” part of the Theorem 4.1 has been proved.

The “if” part: Denote

$$X_{n+1} = \left[\frac{n^{\alpha+1} q_n}{\alpha \rho^\alpha (1-\rho)} \right]^{1/(\alpha-\beta)}, \quad n \geq 1,$$

and $\lambda = \alpha \rho^\alpha (1-\rho)$, where ρ is given by (4.10). Clearly, $X = \{X_n\} \in \mathcal{NRV}(\rho)$. Notice that (4.9) implies that $\rho \in (0, 1)$, so that by Proposition 3.2, $X_n \rightarrow \infty$ and $X_n/n \rightarrow 0$ as $n \rightarrow \infty$. We begin, by proving that sequence X satisfy the asymptotic relation

$$(4.12) \quad \sum_{k=1}^n \left(\sum_{j=k}^{\infty} q_j X_{j+1}^\beta \right)^{1/\alpha} \sim X_{n+1} \quad n \rightarrow \infty.$$

Using (4.1) and that $X_n \in \mathcal{NRV}(\rho)$ is expressed in the form

$$X_{n+1} = \lambda^{-1/(\alpha-\beta)} n^\rho \omega_n^{1/(\alpha-\beta)},$$

by application of Theorem 3.4-(ii) we get

$$\begin{aligned}
(4.13) \quad \sum_{k=n}^{\infty} q_k X_{k+1}^{\beta} &= \lambda^{-\beta/(\alpha-\beta)} \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \omega_k^{\alpha/(\alpha-\beta)} = \lambda^{-\beta/(\alpha-\beta)} \sum_{k=n}^{\infty} k^{\alpha(\rho-1)-1} \omega_k^{\alpha/(\alpha-\beta)} \\
&\sim \lambda^{-\beta/(\alpha-\beta)} \frac{n^{\alpha(\rho-1)} \omega_n^{\alpha/(\alpha-\beta)}}{\alpha(1-\rho)}, \quad n \rightarrow \infty.
\end{aligned}$$

From (4.13), applying Theorem 3.4-(i), we obtain the desired asymptotic relation for X :

$$\begin{aligned}
\sum_{k=1}^n \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} &\sim \lambda^{-\beta/\{\alpha(\alpha-\beta)\}} (\alpha(1-\rho))^{-1/\alpha} \sum_{k=1}^n k^{\rho-1} \omega_k^{1/(\alpha-\beta)} \\
&\sim \lambda^{-\beta/\{\alpha(\alpha-\beta)\}} (\alpha(1-\rho))^{-1/\alpha} \frac{n^{\rho} \omega_n^{1/(\alpha-\beta)}}{\rho} \\
&= \lambda^{-\beta/\{\alpha(\alpha-\beta)\}} \cdot \lambda^{-1/\alpha} n^{\rho} \omega_n^{1/(\alpha-\beta)} = X_{n+1}, \quad n \rightarrow \infty.
\end{aligned}$$

Thus, there exists $n_0 > 1$ such that

$$(4.14) \quad \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \leq 2X_n, \quad n \geq n_0.$$

Let such n_0 be fixed. We may assume that X_n is increasing for $n \geq n_0$. Since from (4.12) we have

$$\sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \sim X_n, \quad n \rightarrow \infty,$$

there exists $n_1 > n_0 + 1$ such that

$$(4.15) \quad \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \geq \frac{X_n}{2}, \quad n \geq n_1.$$

Let such n_1 be fixed. We choose constants $\kappa \in (0, 1)$ and $K > 1$ such that

$$(4.16) \quad \kappa^{1-\beta/\alpha} \leq \frac{1}{2}, \quad K^{1-\beta/\alpha} \geq 4 \quad \text{and} \quad K \geq 2\kappa \frac{X_{n_1}}{X_{n_0}}.$$

Consider the Banach space Υ_{n_0} of all real sequences $x = \{x_n\}_{n=n_0}^{\infty}$ with norm

$$\|x\| = \sup_{n \geq n_0} \frac{|x_n|}{n},$$

and define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ to be the set of real sequences $x \in \Upsilon_{n_0}$ satisfying

$$(4.17) \quad \begin{cases} \kappa X_{n_1} \leq x_n \leq KX_n, & \text{for } n_0 \leq n \leq n_1, \\ \kappa X_n \leq x_n \leq KX_n, & \text{for } n \geq n_1, \end{cases}$$

which is clearly bounded, closed and convex subset of Υ_{n_0} . We will define the operator $\mathcal{F} : \mathcal{X} \rightarrow \Upsilon_{n_0}$ by

$$(4.18) \quad (\mathcal{F}x)_n = \kappa X_{n_1} + \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j x_{j+1}^{\beta} \right)^{1/\alpha}, \quad n \geq n_0,$$

and show that $\mathcal{F}$ has a fixed point by using Lemma 4.2. Namely, the operator $\mathcal{F}$ has the following properties:

(i) *Operator $\mathcal{F}$ maps $\mathcal{X}$ into itself:* Let $x \in \mathcal{X}$. Using (4.14), (4.16), (4.17) and (4.18), we get

$$\begin{aligned} (\mathcal{F}x)_n &\leq \kappa X_{n_1} + K^{\beta/\alpha} \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \\ &\leq \frac{K}{2} X_{n_0} + 2K^{\beta/\alpha} X_n \leq \frac{K}{2} X_n + \frac{K}{2} X_n = KX_n, \quad n \geq n_0. \end{aligned}$$

On the other hand, using (4.15), (4.16), (4.17) and (4.18), we have

$$\mathcal{F}x_n \geq \kappa X_{n_1} \quad \text{for } n_0 \leq n \leq n_1,$$

and

$$\mathcal{F}x_n \geq \kappa^{\beta/\alpha} \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \geq \kappa^{\beta/\alpha} \frac{X_n}{2} \geq \kappa X_n, \quad n \geq n_1.$$

This shows that $(\mathcal{F}x)_n \in \mathcal{X}$, for all $n \geq n_0$, that is, $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$.

(ii) *Operator $\mathcal{F}$ is continuous on $\mathcal{X}$:* Let $\varepsilon > 0$ and $\{x^{(m)}\}$ be a sequence in $\mathcal{X}$, such that $\lim_{m \rightarrow \infty} \|x^{(m)} - x\| = 0$. Since $\mathcal{X}$ is closed, $x \in \mathcal{X}$. Let $x^{(m)} = \{x_n^{(m)}\}$, $x = \{x_n\}$. For every $n \geq n_0$ we have

$$|(\mathcal{F}x^{(m)})_n - (\mathcal{F}x)_n| \leq \sum_{k=n_0}^{n-1} \left| \left(\sum_{j=k}^{\infty} q_j (x_{j+1}^{(m)})^{\beta} \right)^{1/\alpha} - \left(\sum_{j=k}^{\infty} q_j x_{j+1}^{\beta} \right)^{1/\alpha} \right|.$$

From (4.13) we obtain

$$(4.19) \quad \left(\sum_{k=n}^{\infty} q_k X_{k+1}^{\beta} \right)^{1/\alpha} \sim \lambda^{-\beta/\{\alpha(\alpha-\beta)\}} \frac{n^{\rho-1} \omega_n^{1/(\alpha-\beta)}}{[\alpha(1-\rho)]^{1/\alpha}}, \quad n \rightarrow \infty,$$

and because the normalized regularly varying sequence on the right-hand side of the previous relation tends to 0 as $n \rightarrow \infty$, $\sum_{j=n_0}^{\infty} q_j X_{j+1}^{\beta}$ is convergent. Thus, the series $\sum_{j=n_0}^{\infty} q_j (x_{j+1}^{(m)})^{\beta}$ is totally convergent, because $q_j (x_{j+1}^{(m)})^{\beta} \leq K^{\beta} q_j X_{j+1}^{\beta}$ for every $j \geq n_0$, $m \in \mathbb{N}$. Then, by Lemma 4.1 it holds for every $k \geq n_0$

$$\lim_{m \rightarrow \infty} \left| \left(\sum_{j=k}^{\infty} q_j (x_{j+1}^{(m)})^{\beta} \right)^{1/\alpha} - \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \right| = 0,$$

which shows that for all large $m \in \mathbb{N}$, we have

$$\frac{|(\mathcal{F}x^{(m)})_n - (\mathcal{F}x)_n|}{n} \leq \varepsilon.$$

Therefore, $\|\mathcal{F}x^{(m)} - \mathcal{F}x\| \rightarrow 0$ as $m \rightarrow \infty$, i.e. $\mathcal{F}$ is continuous on $\mathcal{X}$.

(iii) $\mathcal{F}(\mathcal{X})$ is uniformly Cauchy: For $x \in \mathcal{X}$ and $m > n > n_0$ we have

$$\begin{aligned} & \left| \frac{(\mathcal{F}x)_m}{m} - \frac{(\mathcal{F}x)_n}{n} \right| \\ & \leq \left| \frac{\kappa X_{n_1}}{m} - \frac{\kappa X_{n_1}}{n} \right| + \left| \frac{1}{m} \sum_{k=n_0}^{m-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} - \frac{1}{n} \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \right| \\ & \leq \frac{2\kappa X_{n_1}}{n} + \frac{1}{m} \left| \sum_{k=n_0}^{m-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} - \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \right| \\ & \quad + \left| \frac{1}{m} - \frac{1}{n} \right| \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \\ & \leq \frac{2\kappa X_{n_1}}{n} + \frac{1}{m} \sum_{k=n}^{m-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} + \frac{2}{n} \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} \\ & \leq \frac{2\kappa X_{n_1}}{n} + K^{\beta/\alpha} \left(\sum_{j=n}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} + \frac{2K^{\beta/\alpha}}{n} \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha}. \end{aligned}$$

Since, by Stolz-Cesaro theorem and (4.19)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} = \lim_{n \rightarrow \infty} \left(\sum_{j=n}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} = 0,$$

for any $\varepsilon > 0$, there exists an integer $N \geq n_0$, such that for $n \geq N$

$$\frac{2\kappa X_{n_1}}{n} < \frac{\varepsilon}{3}, \quad K^{\beta/\alpha} \left(\sum_{j=n}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} < \frac{\varepsilon}{3}, \quad \frac{2K^{\beta/\alpha}}{n} \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{1/\alpha} < \frac{\varepsilon}{3}.$$

Thus,

$$\left| \frac{(\mathcal{F}x)_m}{m} - \frac{(\mathcal{F}x)_n}{n} \right| < \varepsilon$$

for $m > n \geq N$, which means that $\mathcal{F}(\mathcal{X})$ is uniformly Cauchy.

Thus all the hypotheses of Lemma 4.2 are fulfilled implying the existence of a fixed point $x \in \mathcal{X}$ of $\mathcal{F}$, which satisfies

$$(4.20) \quad x_n = \kappa X_{n_1} + \sum_{k=n_0}^{n-1} \left(\sum_{j=k}^{\infty} q_j x_{j+1}^{\beta} \right)^{1/\alpha}, \quad n \geq n_0.$$

It is clear that x is a positive solution of (Ω_1) which belongs to the class $M_{\infty,0}^+$. Moreover, from (4.17) x satisfies

$$(4.21) \quad 0 < \liminf_{n \rightarrow \infty} \frac{x_n}{X_n} \leq \limsup_{n \rightarrow \infty} \frac{x_n}{X_n} < \infty.$$

Application of Lemma 3.1, using (4.12) and (4.20), yields

$$\begin{aligned} L &= \limsup_{n \rightarrow \infty} \frac{x_n}{X_n} \leq \limsup_{n \rightarrow \infty} \frac{\Delta x_n}{\Delta X_n} = \limsup_{n \rightarrow \infty} \frac{(\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta})^{1/\alpha}}{(\sum_{k=n}^{\infty} q_k X_{k+1}^{\beta})^{1/\alpha}} \\ &\leq \left(\limsup_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta}}{\sum_{k=n}^{\infty} q_k X_{k+1}^{\beta}} \right)^{1/\alpha} \leq \left(\limsup_{n \rightarrow \infty} \frac{-q_n x_{n+1}^{\beta}}{-q_n X_{n+1}^{\beta}} \right)^{1/\alpha} \\ &= \left(\limsup_{n \rightarrow \infty} \frac{x_{n+1}}{X_{n+1}} \right)^{\beta/\alpha} = L^{\beta/\alpha}. \end{aligned}$$

Since $\beta < \alpha$, from above we conclude that

$$(4.22) \quad 0 < L \leq 1.$$

Similarly, we can see that

$$l = \liminf_{n \rightarrow \infty} \frac{x_n}{X_n}$$

satisfies

$$(4.23) \quad 1 \leq l < \infty.$$

From (4.22) and (4.23) we obtain that $l = L = 1$, which means that $x_n \sim X_n$, $n \rightarrow \infty$ and ensures that x is a regularly varying solution of (Ω_1) with requested regularity index and the asymptotic formula given by (4.11). $\square$

4.3. Existence and asymptotic behavior of normalized regularly varying solutions of index $\rho \in \{0, 1\}$

By Theorem 4.1 equation (Ω_1) possess $\mathcal{NRV}(\rho)$ -solutions, $\rho \in (0, 1)$, if the regularity index σ of q satisfies $\sigma \in (-\alpha - 1, -\beta - 1)$. Thus, we now proceed further studying two border cases in which the regularity index of q satisfies $\sigma = -\alpha - 1$ and $\sigma = -\beta - 1$. We establish necessary and sufficient condition for the existence of slowly growing solutions of equation (Ω_1) belonging to the classes $ntr - \mathcal{NSV}$ and $ntr - \mathcal{NRV}(1)$ and determine the exact asymptotic forms of such solutions.

Theorem 4.2. *Suppose that $q \in \mathcal{NRV}(\sigma)$. Equation (Ω_1) possesses non-trivial normalized slowly varying solutions if and only if*

$$(4.24) \quad \sigma = -\alpha - 1 \quad \text{and} \quad \sum_{n=1}^{\infty} (nq_n)^{1/\alpha} = \infty,$$

and all such solutions of (Ω_1) enjoy the precise asymptotic property

$$(4.25) \quad x_n \sim \left[\frac{\alpha - \beta}{\alpha^{1+1/\alpha}} \sum_{k=1}^{n-1} (kq_k)^{1/\alpha} \right]^{\alpha/(\alpha-\beta)}, \quad n \rightarrow \infty.$$

Proof. **The “only if” part:** Suppose that $x \in \mathcal{NSV}$. Then, clearly only the statement (iii) of Lemma 4.3 could hold. Therefore, $\rho = 0$ and so $\sigma = -\alpha - 1$. Denoting

$$y_n = \alpha^{-1/\alpha} \sum_{k=n_0}^{n-1} k^{-1} \omega_k^{1/\alpha} \zeta_k^{\beta/\alpha},$$

since $x_n = \xi_n \sim y_n$, $n \rightarrow \infty$, we transform (4.5) into differential asymptotic relation for $y = \{y_n\}$:

$$(4.26) \quad y_n^{-\beta/\alpha} \Delta y_n \sim \alpha^{-1/\alpha} n^{-1} \omega_n^{1/\alpha} = \alpha^{-1/\alpha} (nq_n)^{1/\alpha}, \quad n \rightarrow \infty.$$

Summing (4.26) from n_0 to $n - 1$ and using Theorem 3.3 we obtain

$$(4.27) \quad \frac{\alpha}{\alpha - \beta} y_n^{1-\beta/\alpha} \sim \alpha^{-1/\alpha} \sum_{k=n_0}^{n-1} (kq_k)^{1/\alpha}, \quad n \rightarrow \infty.$$

Because $1 - \beta/\alpha > 0$, $y_n^{1-\beta/\alpha} \rightarrow \infty$, $n \rightarrow \infty$, so that (4.27) yields that the second condition in (4.24) is satisfied as well as that the asymptotic expression for x is

$$x_n \sim y_n \sim \left[\frac{\alpha - \beta}{\alpha^{1+1/\alpha}} \sum_{k=1}^{n-1} (kq_k)^{1/\alpha} \right]^{\alpha/(\alpha-\beta)}, \quad n \rightarrow \infty.$$

This completes the “only if” part of the proof of Theorem 4.2.

The “if” part: Since $q \in \mathcal{N}\mathcal{B}\mathcal{V}(-\alpha-1)$, due to (4.1), it is expressed as $q_n = n^{-\alpha-1}\omega_n$, $\omega \in \mathcal{N}\mathcal{S}\mathcal{V}$. Define the sequence $Y = \{Y_n\}$ with

$$Y_n = \left[\frac{\alpha - \beta}{\alpha^{1+1/\alpha}} \sum_{k=1}^{n-1} (kq_k)^{1/\alpha} \right]^{\alpha/(\alpha-\beta)}, \quad n \geq 1.$$

In view of the assumption (4.24), $Y \in ntr - \mathcal{N}\mathcal{S}\mathcal{V}$ by Theorem 3.4-(iv). Thus, by Theorem 3.4-(ii) we have

$$\begin{aligned} (4.28) \quad \sum_{k=n}^{\infty} q_k Y_{k+1}^{\beta} &= \sum_{k=n}^{\infty} k^{-\alpha-1} \omega_k Y_{k+1}^{\beta} \sim \frac{n^{-\alpha} \omega_n Y_{n+1}^{\beta}}{\alpha} \\ &= \frac{nq_n Y_{n+1}^{\beta}}{\alpha} \sim \frac{nq_n Y_n^{\beta}}{\alpha}, \quad n \rightarrow \infty. \end{aligned}$$

Summing (4.28) from 1 to $n-1$, with the help of Theorem 3.4-(iii) and Theorem 3.3, we obtain

$$\begin{aligned} &\sum_{k=1}^{n-1} \left(\sum_{j=k}^{\infty} q_j Y_{j+1}^{\beta} \right)^{1/\alpha} \\ &\sim \alpha^{-1/\alpha} \sum_{k=1}^{n-1} (kq_k)^{1/\alpha} \left[\frac{\alpha - \beta}{\alpha^{1+1/\alpha}} \sum_{j=1}^{k-1} (jq_j)^{1/\alpha} \right]^{\beta/(\alpha-\beta)} \\ &= \alpha^{-1/\alpha} \left(\frac{\alpha - \beta}{\alpha^{1+1/\alpha}} \right)^{\beta/(\alpha-\beta)} \sum_{k=1}^{n-1} \Delta \left(\sum_{j=1}^{k-1} (jq_j)^{1/\alpha} \right) \left(\sum_{j=1}^{k-1} (jq_j)^{1/\alpha} \right)^{\beta/(\alpha-\beta)} \\ &\sim \left(\frac{\alpha - \beta}{\alpha^{1+1/\alpha}} \right)^{\beta/(\alpha-\beta)} \cdot \frac{\alpha - \beta}{\alpha^{1+1/\alpha}} \left(\sum_{j=1}^{n-1} (jq_j)^{1/\alpha} \right)^{\alpha/(\alpha-\beta)} = Y_n, \quad n \rightarrow \infty. \end{aligned}$$

Taking that $Y = \{Y_n\}$ satisfies the same asymptotic relation (4.12), proceeding exactly as in the proof of Theorem 4.1, replacing X_n with Y_n , $\mathcal{N}\mathcal{S}\mathcal{V}$ -solution x of equation (Ω_1) such that

$$\kappa Y_n \leq x_n \leq KY_n, \quad \text{for large } n,$$

is obtained by the application of the discrete analog of Schauder's fixed point theorem, while application of Lemma 3.1 proves that the solution x enjoys the precise asymptotic behavior (4.25). $\square$

Remark 4.1. It is easy to see that for a sequence $q \in \mathcal{NRV}(-\alpha - 1)$

$$\sum_{n=1}^{\infty} (nq_n)^{1/\alpha} = \infty \quad \Leftrightarrow \quad R_\alpha = \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} q_k \right)^{1/\alpha} = \infty.$$

Theorem 4.3. Suppose that $q \in \mathcal{NRV}(\sigma)$. Equation (Ω_1) possesses non-trivial $\mathcal{NRV}(1)$ -solutions if and only if

$$(4.29) \quad \sigma = -\beta - 1 \quad \text{and} \quad \sum_{n=1}^{\infty} n^\beta q_n < \infty,$$

and all such solutions of (Ω_1) has one and the same asymptotic behavior

$$(4.30) \quad x_n \sim n \left[\frac{\alpha - \beta}{\alpha} \sum_{k=n}^{\infty} k^\beta q_k \right]^{1/(\alpha - \beta)}, \quad n \rightarrow \infty.$$

Proof. **The “only if” part:** Suppose $x \in ntr - \mathcal{NRV}(1)$. For such x only case (i) in Lemma 4.3 is possible and x must satisfy the asymptotic relation (4.3). Therefore, $\rho = 1$ i.e. $\sigma = -\beta - 1$. Since $x_n = n\xi_n$, $\xi \in \mathcal{NSV}$, from (4.3) we get

$$(4.31) \quad \xi_n \sim \left(\sum_{k=n}^{\infty} k^{-1} \omega_k \xi_k^\beta \right)^{1/\alpha}, \quad n \rightarrow \infty.$$

Letting

$$\eta_n = \sum_{k=n}^{\infty} k^{-1} \omega_k \xi_k^\beta,$$

we transform (4.31) into

$$(4.32) \quad -\eta_n^{-\beta/\alpha} \Delta \eta_n \sim n^{-1} \omega_n = n^\beta q_n, \quad n \rightarrow \infty.$$

Since $\alpha > \beta$ and $\eta_n \sim \xi_n \rightarrow 0$ as $n \rightarrow \infty$, summing (4.32) from n to ∞ , Theorem 3.3 yields the relation

$$(4.33) \quad \eta_n \sim \left[\frac{\alpha - \beta}{\alpha} \sum_{k=n}^{\infty} k^\beta q_k \right]^{\alpha/(\alpha - \beta)}, \quad n \rightarrow \infty,$$

and assuring the truth of the second condition in (4.29). Combining (4.3), (4.31) and (4.33), we conclude that

$$x_n \sim n \zeta_n \sim n \eta_n^{1/\alpha} \sim n \left[\frac{\alpha - \beta}{\alpha} \sum_{k=n}^{\infty} k^{\beta} q_k \right]^{1/(\alpha - \beta)}, \quad n \rightarrow \infty.$$

The “if” part: Defining the sequence $z = \{z_n\}$ and $Z = \{Z_n\}$ with

$$z_n = \frac{\alpha - \beta}{\alpha} \sum_{k=n}^{\infty} k^{\beta} q_k, \quad Z_n = n z_n^{1/(\alpha - \beta)}, \quad n \geq 1.$$

By Theorem 3.4-(iii), $z \in \mathcal{NSV}$, so that $Z \in \mathcal{NRV}(1)$. Since $q \in \mathcal{NRV}(-\beta - 1)$ could be expressed as $q_n = n^{-\beta - 1} \omega_n$, $\omega \in \mathcal{NSV}$, using Proposition 3.5 we have

$$\begin{aligned} \sum_{k=n}^{\infty} q_k Z_{k+1}^{\beta} &\sim \sum_{k=n}^{\infty} q_k Z_k^{\beta} = \sum_{k=n}^{\infty} k^{\beta} q_k z_k^{\beta/(\alpha - \beta)} \\ &= \frac{\alpha}{\alpha - \beta} \sum_{k=n}^{\infty} z_k^{\beta/(\alpha - \beta)} (-\Delta z_k) \sim z_n^{\alpha/(\alpha - \beta)} \in \mathcal{NSV}, \quad n \rightarrow \infty, \end{aligned}$$

which by Theorem 3.4-(ii) implies

$$(4.34) \quad \sum_{k=1}^{n-1} \left(\sum_{j=k}^{\infty} q_j Z_{j+1}^{\beta} \right)^{1/\alpha} \sim n z_n^{1/(\alpha - \beta)} = Z_n, \quad n \rightarrow \infty.$$

Then, starting from the asymptotic relation (4.34) instead of (4.12), as in the proof of Theorem 4.1, solution $x = \{x_n\}$ such that

$$\kappa Z_n \leq x_n \leq K Z_n, \quad \text{for large } n,$$

is obtained by the application of the discrete analog of Schauder’s fixed point theorem, while application of Lemma 3.2 proves that such solution x satisfies $x_n \sim Z_n$, $n \rightarrow \infty$. $\square$

We conclude with a remark that our main results (Theorems 4.1, 4.2, 4.3) combined with Theorem 2.2 provide thorough and complete information about the existence and the asymptotic behavior of normalized regularly varying solutions of equation (Ω) . Namely, all solutions of equation (Ω_1) are oscillatory if and only if

$$T_{\beta} = \sum_{n=1}^{\infty} n^{\beta} q_n = \infty$$

(see [7, Theorem 2.2]). Since in view of (4.1),

$$T_\beta = \sum_{n=1}^{\infty} n^{\beta+\sigma} \omega_n$$

it is easy to see that

$$T_\beta < \infty \quad \Leftrightarrow \quad \begin{array}{ll} \text{(i)} & \sigma < -\beta - 1, \text{ or} \\ \text{(ii)} & \sigma = -\beta - 1 \text{ and } \sum_{n=1}^{\infty} l_n/n = \sum_{n=1}^{\infty} n^\beta q_n < \infty. \end{array}$$

Therefore, our analysis of positive solutions of (Ω_1) should be made on the premise that the regularity index σ of q satisfies $\sigma \leq -\beta - 1$. Then, in view of Theorem 1.2 the differential equation (Ω_1) always possesses solutions in $M_{\infty, l}^+ \neq \emptyset$, that is, trivial normalized regularly varying solutions of index 1.

We denote by $\mathcal{R}$ the totality of normalized regularly varying solutions of (Ω_1) . Using notation

$$\begin{aligned} M_{\mathcal{P}\mathcal{V}}^+ &= M^+ \cap \mathcal{N}\mathcal{P}\mathcal{V} & \text{and} & & M_{\mathcal{R}\mathcal{V}}^+(\rho) &= M^+ \cap \mathcal{N}\mathcal{R}\mathcal{V}(\rho), \\ T_{\mathcal{P}\mathcal{V}}^+ &= M^+ \cap tr - \mathcal{N}\mathcal{P}\mathcal{V} & \text{and} & & T_{\mathcal{R}\mathcal{V}}^+(\rho) &= M^+ \cap tr - \mathcal{N}\mathcal{R}\mathcal{V}(\rho), \\ N_{\mathcal{P}\mathcal{V}}^+ &= M^+ \cap ntr - \mathcal{N}\mathcal{P}\mathcal{V} & \text{and} & & N_{\mathcal{R}\mathcal{V}}^+(\rho) &= M^+ \cap ntr - \mathcal{N}\mathcal{R}\mathcal{V}(\rho), \end{aligned}$$

we have the following result, which provide a structure of positive solutions of (Ω_1) in the framework of regular variation.

Theorem 4.4. *Let $q \in \mathcal{N}\mathcal{R}\mathcal{V}(\sigma)$.*

- (i) *If $\sigma < -\alpha - 1$, then $\mathcal{R} = T_{\mathcal{R}\mathcal{V}}^+(1) \cup T_{\mathcal{P}\mathcal{V}}^+$;*
- (ii) *If $\sigma = -\alpha - 1$ and $R_\alpha < \infty$, then $\mathcal{R} = T_{\mathcal{R}\mathcal{V}}^+(1) \cup T_{\mathcal{P}\mathcal{V}}^+$;*
- (iii) *If $\sigma = -\alpha - 1$ and $R_\alpha = \infty$, then $\mathcal{R} = T_{\mathcal{R}\mathcal{V}}^+(1) \cup N_{\mathcal{P}\mathcal{V}}^+$;*
- (iv) *If $\sigma \in (-\alpha - 1, -\beta - 1)$, then $\mathcal{R} = T_{\mathcal{R}\mathcal{V}}^+(1) \cup M_{\mathcal{R}\mathcal{V}}^+((\sigma + \alpha + 1)/(\alpha - \beta))$;*
- (v) *If $\sigma = -\beta - 1$ and $T_\beta < \infty$, then $\mathcal{R} = T_{\mathcal{R}\mathcal{V}}^+(1) \cup N_{\mathcal{R}\mathcal{V}}^+(1)$;*
- (vi) *If $\sigma = -\beta - 1$ and $T_\beta = \infty$, then $\mathcal{R} = \emptyset$.*

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