Nonlinear Neumann Problems with Constraints

By

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Abstract. We consider a C^1 -functional ψ defined on the “Neumann” Sobolev space $W_n^{1,p}(\Omega)$. If M is a C^1 -submanifold, then for $\psi|_M$ we show that any local $C_n^1(\overline{\Omega})$ -minimizer is also a local $W_n^{1,p}(\Omega)$ -minimizer. Then we use this general result on local minimizers to show that a nonlinear parametric Neumann problem driven by the p -Laplace differential operator and restricted on a sphere, has at least three distinct smooth solutions.

Key Words and Phrases. $W^{1,p}$ and C^1 local minimizers, C^1 -manifold, Lagrange multiplier, Duality map, Positive solutions, p -Laplacian.

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1. Introduction

Brézis-Nirenberg [3] proved that, if $\Omega \subseteq \mathbf{R}^N$ is a bounded domain with a smooth boundary, $f : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is a Carathéodory function with subcritical growth in the $\zeta \in \mathbf{R}$ variable, $F(z, \zeta) = \int_0^\zeta f(z, s)ds$ and $\varphi : H_0^1(\Omega) \rightarrow \mathbf{R}$ is the C^1 -functional, defined by

$$\varphi(u) = \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega F(z, u(z))dz, \quad \forall u \in H_0^1(\Omega),$$

then if $u_0 \in H_0^1(\Omega)$ is a local $C_0^1(\overline{\Omega})$ -minimizer of φ , then $u_0 \in C_0^1(\overline{\Omega})$ and it is also a local $H_0^1(\Omega)$ -minimizer of φ . This result turned out to be a very useful tool in proving existence and multiplicity results for semilinear equations. The result was extended to the “Dirichlet” Sobolev space $W_0^{1,p}(\Omega)$ ($1 < p < +\infty$) by García Azorero-Manfredi-Peral-Alonso [6] (see also Guo-Zhang [14] where $p \geq 2$). The extension to the “Neumann” Sobolev space $W_n^{1,p}(\Omega) = \overline{C_n^1(\overline{\Omega})}^{\|\cdot\|}$,

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where

$$C_n^1(\bar{\Omega}) = \left\{ u \in C^1(\bar{\Omega}) : \frac{\partial u}{\partial n} \Big|_{\partial\Omega} = 0 \right\}$$

and $\|\cdot\|$ is the usual norm of the Sobolev space $W^{1,p}(\Omega)$ ($1 < p < +\infty$), was proved by Motreanu-Motreanu-Papageorgiou [21] (see also Iannizzotto-Papageorgiou [16] for an extension to nonsmooth locally Lipschitz functionals). Recently Motreanu-Papageorgiou [23] extended the result to more general nonlinear nonhomogeneous differential operators and presented an alternative simpler proof, which avoids the complicated estimations of the previous ones. Finally, we mention the related works of Gasiński-Papageorgiou [8, 9, 11, 12] (for Neumann problems with p -Laplacian), Fan [5] and Gasiński-Papageorgiou [10] (for anisotropic Sobolev spaces), Khan-Motreanu [18] (for general Banach spaces), Winkert [26] (for unconstrained Neumann problems) and Winkert [27] (for nonsmooth functionals with nonlinear Neumann boundary condition).

The aim of this paper is to extend this result to local minimizers of functionals defined on submanifolds of the space $W_n^{1,p}(\Omega)$ (with $1 < p < +\infty$). Throughout this paper $\Omega \subseteq \mathbf{R}^N$ is a bounded domain with a C^2 -boundary $\partial\Omega$. More precisely, let $g : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ and $f : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ be two Carathéodory functions with subcritical growth, i.e.,

$$|g(z, \zeta)| \leq a(z) + c|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega,$$

$$|f(z, \zeta)| \leq \hat{a}(z) + \hat{c}|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega,$$

with $a, \hat{a} \in L^\infty(\Omega)$, $c, \hat{c} > 0$ and $1 < r < p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

We set

$$G(z, \zeta) = \int_0^\zeta g(z, \zeta) dz \quad \text{and} \quad F(z, \zeta) = \int_0^\zeta f(z, \zeta) dz$$

and consider the C^1 -functionals $\psi, \varphi : W_n^{1,p}(\Omega) \rightarrow \mathbf{R}$, defined by

$$\psi(u) = \int_\Omega G(z, u(z)) dz, \quad \forall u \in W_n^{1,p}(\Omega),$$

$$\varphi(u) = \frac{1}{p} \|\nabla u\|_p^p - \int_\Omega F(z, u(z)) dz, \quad \forall u \in W_n^{1,p}(\Omega).$$

Suppose that $c \in \mathbf{R}$ is not a critical value of the functional φ and consider

$$M = \{u \in W_n^{1,p}(\Omega) : \varphi(u) = c\}.$$

This constraint set is very general and natural. It incorporates as special cases spherical constraints (see Section 3), affine constraints and other constraints described by a C^1 -manifold. Such constraints arise in applications. From the Ljusternik theorem (see e.g., Papageorgiou-Kyritsi [24, pp. 74–75] and Zeidler [30, p. 290]) we know that M is a C^1 -Banach manifold in $W_n^{1,p}(\Omega)$ of codimension 1. Our goal in this paper is to relate local $C_0^1(\bar{\Omega})$ and $W_n^{1,p}(\Omega)$ minimizers of $\psi|_M$. Such a result for the Hilbert space $H_0^1(\Omega)$ was proved by Tehrani [25]. To the best of our knowledge no such result exists for the space $W_n^{1,p}(\Omega)$ ($1 < p < +\infty$). In the next section we prove a theorem of this type and in Section 3 we use it to obtain a multiplicity theorem for a constrained parametric Neumann problem driven by the p -Laplacian.

2. Local minimizers in $C_n^1(\bar{\Omega})$ and $W_n^{1,p}(\Omega)$

Keeping the notation established in Section 1, we prove a theorem on local minimizers. In what follows, $\mathcal{F} : W_n^{1,p}(\Omega) \rightarrow W_n^{1,p}(\Omega)^*$ is the normalized duality map for the Sobolev space $W_n^{1,p}(\Omega)$. We know that $\mathcal{F}$ is homeomorphism (see e.g., Gasiński-Papageorgiou [7, p. 316] and Zeidler [29, p. 861]). Also, by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W_n^{1,p}(\Omega)^*, W_n^{1,p}(\Omega))$.

Theorem 2.1. *If $u_0 \in M$ is a local $C_0^1(\bar{\Omega})$ -minimizer of $\psi|_M$, i.e., there exists $\delta_0 > 0$, such that*

$$\psi(u_0) \leq \psi(u_0 + h), \quad \forall h \in C_n^1(\bar{\Omega}), \quad \|h\|_{C_n^1(\bar{\Omega})} \leq \delta_0, \quad u_0 + h \in M$$

and

$$(2.1) \quad \langle \psi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle < 0,$$

then $u_0 \in M \cap C_0^1(\bar{\Omega})$ and it is also a local $W_n^{1,p}(\Omega)$ -minimizer of $\psi|_M$, i.e., there exists $\delta_1 > 0$, such that

$$\psi(u_0) \leq \psi(u_0 + h), \quad \forall h \in W_n^{1,p}(\Omega), \quad \|h\| \leq \delta_1, \quad u_0 + h \in M.$$

Proof. Since $c \in \mathbf{R}$ is a regular value of φ and u_0 is a local $C_n^1(\bar{\Omega})$ -minimizer of $\psi|_M$, by the Lagrange multiplier rule (see e.g., Papageorgiou-Kyritsi [24, p. 76]), we can find $\lambda \neq 0$ (the Lagrange multiplier), such that

$$\langle \psi'(u_0), h \rangle = \lambda \langle \varphi'(u_0), h \rangle, \quad \forall h \in C_n^1(\bar{\Omega}),$$

so

$$\langle \psi'(u_0), h \rangle = \lambda \langle \varphi'(u_0), h \rangle, \quad \forall h \in W_n^{1,p}(\Omega)$$

and thus

$$A(u_0) = \frac{1}{\lambda} N_g(u_0) + N_f(u_0),$$

where $A : W_n^{1,p}(\Omega) \rightarrow W_n^{1,p}(\Omega)^*$ is the nonlinear map, defined by

$$\langle A(u), y \rangle = \int_{\Omega} \|\nabla u\|^{p-2} (\nabla u, \nabla y)_{\mathbf{R}^N} dz, \quad \forall u, y \in W_n^{1,p}(\Omega)$$

and

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)), \quad N_f(u)(\cdot) = f(\cdot, u(\cdot)), \quad \forall u \in W_n^{1,p}(\Omega).$$

Then, from Motreanu-Papageorgiou [22], we have

$$\begin{cases} -\Delta_p u_0(z) = \frac{1}{\lambda} g(z, u_0(z)) + f(z, u_0(z)) & \text{a.e. in } \Omega, \\ \frac{\partial u_0}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

We know that $u_0 \in L^\infty(\Omega)$ (see e.g. Hu-Papageorgiou [15]). Then the nonlinear regularity theory implies that $u_0 \in C_n^1(\bar{\Omega})$ (see Lieberman [19]).

To show that $u_0 \in M \cap C_n^1(\bar{\Omega})$ is also a local $W_n^{1,p}(\Omega)$ -minimizer of $\psi|_M$, we argue indirectly. So, suppose that u_0 is not a local $W_n^{1,p}(\Omega)$ -minimizer of $\psi|_M$. Exploiting the compactness of the embedding of $W_n^{1,p}(\Omega)$ into $L^r(\Omega)$ (Sobolev embedding theorem), we can easily check that ψ is sequentially weakly continuous on $W_n^{1,p}(\Omega)$ and φ is sequentially weakly lower semicontinuous. We claim that we can find $h_\varepsilon \in W_n^{1,p}(\Omega)$, such that

$$(2.2) \quad \begin{aligned} \psi(u_0 + h_\varepsilon) &= \inf \{ \psi(u_0 + h) : \|h\|^p \leq \varepsilon, \varphi(u_0 + h) \leq c \} \\ &= m_\varepsilon < \psi(u_0). \end{aligned}$$

To this end, let $\{h_n\}_{n \geq 1} \subseteq W_n^{1,p}(\Omega)$ be a minimizing sequence. From the growth hypothesis on $f(z, \cdot)$, we have

$$|F(z, \zeta)| \leq c_1(1 + |\zeta|^r) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbf{R},$$

with $c_1 > 0$. Evidently the sequence $\{h_n\}_{n \geq 1} \subseteq W_n^{1,p}(\Omega)$ is bounded. So, we may assume that

$$(2.3) \quad h_n \rightharpoonup h_\varepsilon \quad \text{weakly in } W_n^{1,p}(\Omega),$$

$$(2.4) \quad h_n \rightarrow h_\varepsilon \quad \text{in } L^p(\Omega).$$

From (2.3) and the weak continuity of ψ , we have

$$(2.5) \quad \psi(u_0 + h_n) \rightarrow \psi(u_0 + h_\varepsilon).$$

Also from (2.3) and the sequential weak lower semicontinuity of φ , we have

$$(2.6) \quad \varphi(u_0 + h_\varepsilon) \leq \liminf_{n \rightarrow +\infty} \varphi(u_0 + h_n).$$

From (2.3), (2.5) and (2.6), we have

$$\psi(u_0 + h_\varepsilon) = m_\varepsilon, \quad \|h_\varepsilon\|^p \leq \varepsilon \quad \text{and} \quad \varphi(u_0 + h_\varepsilon) \leq c,$$

which proves the claim. For every $\varepsilon > 0$, we introduce the set

$$D_\varepsilon = \{h \in W_n^{1,p}(\Omega) : \|h\|^p \leq \varepsilon, \varphi(u_0 + h) \leq c\}.$$

Let $\varepsilon_n \searrow 0$. We have just seen that for every $n \geq 1$, we can find $h_n \in D_{\varepsilon_n}$, such that

$$(2.7) \quad \psi(u_0 + h_n) = m_{\varepsilon_n} = \inf_{h \in D_{\varepsilon_n}} \psi(u_0 + h), \quad \forall n \geq 1.$$

Claim 1. $\varphi(u_0 + h_n) = c$ for all $n \geq n_0$.

We proceed by contradiction. So, suppose that by passing to a subsequence if necessary, we can find $h_n \in D_{\varepsilon_n}$ for $n \geq 1$, such that one of the following two cases is true:

Case (a): $\varphi(u_0 + h_n) < c$ and $\|h_n\|^p < \varepsilon_n$ for all $n \geq 1$.

In this case, we have

$$\psi'(u_0 + h_n) = 0$$

(see (2.7)), so

$$\langle \psi'(u_0 + h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle = 0, \quad \forall n \geq 1.$$

But $h_n \rightarrow 0$ in $L^r(\Omega)$ and ψ' is continuous on $L^r(\Omega)$. So, passing to the limit as $n \rightarrow +\infty$, we obtain

$$\langle \psi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle = 0,$$

which contradicts (2.1).

Case (b): $\varphi(u_0 + h_n) < c$ and $\|h_n\|^p = \varepsilon_n$ for all $n \geq 1$.

By virtue of the Lagrange multiplier rule, we can find $\vartheta_n \in \mathbf{R}$, such that

$$(2.8) \quad \langle \psi'(u_0 + h_n), y \rangle = \vartheta_n p \|h_n\|^{p-2} \langle \mathcal{F}(h_n), y \rangle, \quad \forall y \in W_n^{1,p}(\Omega).$$

In (2.8) we choose $y = \mathcal{F}^{-1}(\varphi'(u_0))$. Then

$$\langle \psi'(u_0 + h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle = \vartheta_n p \|h_n\|^{p-2} \langle \mathcal{F}(h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle, \quad \forall n \geq 1.$$

Passing to the limit as $n \rightarrow +\infty$ and using the continuity of ψ' on $W_n^{1,p}(\Omega)^*$ and the hypothesis (see (2.1)), we have

$$(2.9) \quad \begin{aligned} 0 &> \langle \psi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle \\ &= \lim_{n \rightarrow +\infty} (\vartheta_n p \|h_n\|^{p-2} \langle \mathcal{F}(h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle). \end{aligned}$$

From (2.9), it follows that we can find $\xi_0 > 0$, such that

$$(2.10) \quad |\mathfrak{I}_n| p \|h_n\|^{p-1} \|\mathcal{F}^{-1}(\varphi'(u_0))\|_* \geq \xi_0, \quad \forall n \geq n_0$$

(recall that $\|\mathcal{F}(h_n)\|_* = \|h_n\|$ for all $n \geq 1$). Let

$$v_n = \frac{h_n}{\|h_n\|}, \quad \forall n \geq 1.$$

Then $\|v_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$(2.11) \quad v_n \rightarrow v \quad \text{weakly in } W_n^{1,p}(\Omega),$$

$$(2.12) \quad v_n \rightarrow v \quad \text{in } L^r(\Omega).$$

In (2.8) we choose $y = v_n$ and we have

$$(2.13) \quad \langle \psi'(u_0 + h_n), v_n \rangle = \mathfrak{I}_n p \|h_n\|^{p-2} \langle \mathcal{F}(h_n), v_n \rangle, \quad \forall n \geq 1.$$

Since

$$\varphi(u_0 + h_n) < c,$$

we have that

$$\varphi(u_0 + (1-t)h_n) < c, \quad \forall t \in (0, 1) \text{ small,}$$

and so

$$\langle \psi'(u_0 + h_n), h_n \rangle \leq 0,$$

thus

$$\langle \psi'(u_0 + h_n), v_n \rangle \leq 0$$

and so we have

$$(2.14) \quad \langle \psi'(u_0), v \rangle = \lim_{n \rightarrow +\infty} \mathfrak{I}_n p \|h_n\|^{p-1} \leq 0$$

(see (2.11) and (2.13)).

From (2.10), we have

$$|\mathfrak{I}_n| p \|h_n\|^{p-1} \geq \frac{\xi_0}{\|\mathcal{F}^{-1}(\varphi'(u_0))\|} > 0, \quad \forall n \geq n_0$$

(from the hypothesis of the theorem we know that $\mathcal{F}^{-1}(\varphi'(u_0)) \neq 0$).

Returning to (2.14), we see that

$$(2.15) \quad \langle \psi'(u_0), v \rangle < 0.$$

Recall that

$$(2.16) \quad \psi'(u_0) = \lambda \varphi'(u_0),$$

so

$$\langle \psi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle = \lambda \langle \varphi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle = \lambda \|\varphi'(u_0)\|_*^2$$

(since $\mathcal{F}^{-1}$ is the normalized duality map of $W_n^{1,p}(\Omega)^*$; see Gasiński-Papageorgiou [7, p. 315] or Zeidler [30, 861]) and thus

$$\lambda < 0$$

(see (2.1)).

By hypothesis, we have that

$$\varphi(u_0 + h_n) < c, \quad \forall n \geq 1,$$

hence

$$\varphi(u_0 + h_n) < \varphi(u_0), \quad \forall n \geq 1,$$

so

$$\varphi(u_0 + h_n) - \varphi(u_0) < 0, \quad \forall n \geq 1.$$

By virtue of the mean value theorem, we can find $t_n \in (0, 1)$, such that

$$\langle \varphi'(u_0 + t_n h_n), h_n \rangle < 0, \quad \forall n \geq 1,$$

so

$$\langle \varphi'(u_0 + t_n h_n), v_n \rangle < 0, \quad \forall n \geq 1.$$

Passing to the limit as $n \rightarrow +\infty$, we obtain

$$\langle \varphi'(u_0), v \rangle \leq 0$$

(see (2.11) and recall that $h_n \rightarrow 0$ in $W_n^{1,p}(\Omega)$), so

$$\lambda \langle \varphi'(u_0), v \rangle \geq 0$$

(since $\lambda < 0$) and thus

$$(2.17) \quad \langle \psi'(u_0), v \rangle \geq 0$$

(see (2.16)).

Comparing (2.15) and (2.17) we reach a contradiction. Thus, we have finished the proof of Claim 1.

Using Claim 1 and the Lagrange multiplier rule for mixed problems (see Ioffe-Tichomirov [17, p. 74]), we can find $\mu_n \in \mathbf{R}$ and $\vartheta_n \leq 0$, such that

$$(2.18) \quad \psi'(u_0 + h_n) = \mu_n \varphi'(u_0 + h_n) + \vartheta_n p \|h_n\|^{p-2} \mathcal{F}(h_n) \quad \forall n \geq n_0.$$

First suppose that the sequence $\{\vartheta_n\}_{n \geq 1}$ is bounded, i.e., there exists c_2 , such that

$$|\vartheta_n| \leq c_2 \quad \forall n \geq n_0.$$

We have

$$(2.19) \quad \begin{aligned} \langle \psi'(u_0 + h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle &= \mu_n \langle \varphi'(u_0 + h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle \\ &\quad + \vartheta_n p \|h_n\|^{p-2} \langle \mathcal{F}(h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle. \end{aligned}$$

Suppose that, at least for a subsequence, we have $\mu_n \rightarrow 0$. Since the sequence $\{h_n\}_{n \geq 1} \subseteq W_n^{1,p}(\Omega)$ is bounded, the sequence $\{\varphi'(u_0 + h_n)\}_{n \geq 1} \subseteq W_n^{1,p}(\Omega)^*$ is bounded too. Therefore, passing to the limit as $n \rightarrow +\infty$ in (2.19), we obtain

$$\langle \psi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle = 0,$$

which contradicts (2.1).

So, by passing to a subsequence if necessary, we may assume that

$$|\mu_n| \geq \eta_0 > 0, \quad \forall n \geq n_0.$$

Passing to the limit as $n \rightarrow +\infty$ in (2.19) and recalling that $h_n \rightarrow 0$ in $W_n^{1,p}(\Omega)$, we have

$$\begin{aligned} 0 &> \langle \psi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle \\ &= \left(\lim_{n \rightarrow +\infty} \mu_n \right) \langle \varphi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle \\ &= \left(\lim_{n \rightarrow +\infty} \mu_n \right) \|\varphi'(u_0)\|_*^2 \end{aligned}$$

(see (2.19)), so

$$(2.20) \quad \mu_n \leq -\eta_0 < 0, \quad \forall n \geq n_1 \geq n_0.$$

Now suppose that the sequence $\{\vartheta_n\}_{n \geq 1}$ is unbounded. Passing to a subsequence if necessary, we may assume that $|\vartheta_n| \rightarrow +\infty$ as $n \rightarrow +\infty$.

Claim 2. There exists $\hat{\eta}_1 > 0$, such that

$$|\mu_n| \geq \hat{\eta}_1, \quad \forall n \geq n_2.$$

We argue by contradiction. So, suppose that at least for a subsequence we have $\mu_n \rightarrow 0$. Let

$$v_n = \frac{h_n}{\|h_n\|}, \quad \forall n \geq 1.$$

Then $\|v_n\| = 1$ for all $n \geq 1$ and so, passing to a subsequence if necessary, we may assume that

$$(2.21) \quad v_n \rightarrow v \quad \text{weakly in } W_n^{1,p}(\Omega),$$

$$(2.22) \quad v_n \rightarrow v \quad \text{in } L^r(\Omega).$$

From (2.18), we have

$$(2.23) \quad \langle \psi'(u_0 + h_n), v_n \rangle = \mu_n \langle \varphi'(u_0 + h_n), v_n \rangle + \mathfrak{I}_n p \|h_n\|^{p-2} \langle \mathcal{F}(h_n), v_n \rangle.$$

Since the sequence $\{\varphi'(u_0 + h_n)\}_{n \geq 1} \subseteq W_n^{1,p}(\Omega)^*$ is bounded and $\|v_n\| = 1$, we infer that

$$(2.24) \quad \mu_n \langle \varphi'(u_0 + h_n), v_n \rangle \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Hence, if in (2.23) we pass to the limit as $n \rightarrow +\infty$ and use the fact that $h_n \rightarrow 0$ in $W_n^{1,p}(\Omega)$ as well as (2.24), we obtain

$$(2.25) \quad \langle \psi'(u_0), v \rangle = \lim_{n \rightarrow +\infty} (\mathfrak{I}_n p \|h_n\|^{p-1}).$$

From Claim 1, we know that

$$\varphi(u_0 + h_n) = \varphi(u_0) = c, \quad \forall n \geq n_0.$$

The mean value theorem implies that we can find $t_n \in (0, 1)$, such that

$$0 = \varphi(u_0 + h_n) - \varphi(u_0) = \langle \varphi'(u_0 + t_n h_n), h_n \rangle, \quad \forall n \geq n_0,$$

so

$$0 = \langle \varphi'(u_0 + t_n h_n), v_n \rangle, \quad \forall n \geq n_0.$$

Note that

$$t_n h_n \rightarrow 0 \quad \text{in } W_n^{1,p}(\Omega)$$

and so

$$\varphi'(u_0 + t_n h_n) \rightarrow \varphi'(u_0) \quad \text{in } W_n^{1,p}(\Omega)^*.$$

Therefore, in the limit as $n \rightarrow +\infty$, we have

$$(2.26) \quad 0 = \langle \varphi'(u_0), v \rangle.$$

By virtue of the Ljusternik theorem, we know that the tangent space to the C^1 -manifold M at u_0 is given by

$$T_M(u_0) = \{y \in W_n^{1,p}(\Omega) : \langle \varphi'(u_0), y \rangle = 0\}.$$

From (2.26), we have that $v \in T_M(u_0)$. By hypothesis u_0 is a local $C_0^1(\bar{\Omega})$ -minimizer of $\psi|_M$. Hence

$$\langle \psi'(u_0), y \rangle = 0, \quad \forall y \in T_M(u_0) \cap C_0^1(\bar{\Omega}),$$

so

$$\langle \psi'(u_0), y \rangle = 0, \quad \forall y \in T_M(u_0),$$

thus

$$\langle \psi'(u_0), v \rangle = 0$$

(since $v \in T_M(u_0)$) and finally

$$(2.27) \quad \lim_{n \rightarrow +\infty} (\mathfrak{I}_n p \|h_n\|^{p-1}) = 0$$

(see (2.25)).

In (2.19) we pass to the limit as $n \rightarrow +\infty$. Using (2.1), we have

$$\begin{aligned} 0 &> \langle \psi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle \\ &\geq \left(\liminf_{n \rightarrow +\infty} \mu_n \right) \|\varphi'(u_0)\|_*^2 + \lim_{n \rightarrow +\infty} (\mathfrak{I}_n p \|h_n\|^{p-1}) \|\mathcal{F}^{-1}(\varphi'(u_0))\|^2 \end{aligned}$$

(since $\mathfrak{I}_n \leq 0$ and $\|\mathcal{F}(h_n)\|_* = \|h_n\|$), so

$$(2.28) \quad 0 > \left(\liminf_{n \rightarrow +\infty} \mu_n \right) \|\varphi'(u_0)\|_*^2,$$

which contradicts the fact that $\mu_n \rightarrow 0$. This proves Claim 2.

From Claim 2, we have

$$\mu_n < 0 \quad \forall n \geq n_2$$

(see also (2.28)). From (2.19) and since by hypothesis $|\mathfrak{I}_n| \rightarrow +\infty$, we have

$$\begin{aligned} &\frac{1}{\mathfrak{I}_n} \langle \psi'(u_0 + h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle \\ &= \frac{\mu_n}{\mathfrak{I}_n} \langle \varphi'(u_0 + h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle + p \|h_n\|^{p-2} \langle \mathcal{F}(h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle. \end{aligned}$$

Passing to the limit as $n \rightarrow +\infty$ and recalling that $h_n \rightarrow 0$ in $W_n^{1,p}(\Omega)$, $\mathfrak{g}_n \rightarrow -\infty$, we obtain

$$\lim_{n \rightarrow +\infty} \frac{\mu_n}{\mathfrak{g}_n} \langle \varphi'(u_0 + h_n), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle = 0,$$

so

$$\left(\lim_{n \rightarrow +\infty} \frac{\mu_n}{\mathfrak{g}_n} \right) \|\varphi'(u_0)\|_* = 0$$

and thus

$$(2.29) \quad \lim_{n \rightarrow +\infty} \frac{\mu_n}{\mathfrak{g}_n} = 0$$

(since $c = \varphi(u_0)$ is a regular value of φ).

From (2.18), we have

$$(2.30) \quad \begin{aligned} & \mu_n A(u_0 + h_n) + \mathfrak{g}_n p A(h_n) \\ &= N_g(u_0 + h_n) + \mu_n N_f(u_0 + h_n) - \mathfrak{g}_n p |h_n|^{p-2} h_n. \end{aligned}$$

There are two possibilities.

Case (i): There exists $M_1 > 0$, such that $|\mathfrak{g}_n| \leq M_1$ for all $n \geq 1$.

From Claim 2, we know that

$$\mu_n < -\hat{\eta}_1, \quad \forall n \geq n_2.$$

So, from (2.30), we obtain

$$(2.31) \quad A(u_0 + h_n) + \frac{\mathfrak{g}_n}{\mu_n} A(h_n) = \frac{1}{\mu_n} N_g(u_0 + h_n) + N_f(u_0 + h_n) - \frac{\mathfrak{g}_n p}{\mu_n} |h_n|^{p-2} h_n.$$

Recall that since u_0 is a local $C_n^1(\overline{\Omega})$ -minimizer of $\psi|_M$, then from the Lagrange multiplier rule and (2.1), we can find $\eta < 0$, such that

$$(2.32) \quad A(u_0) = N_f(u_0) + \frac{1}{\eta} N_g(u_0).$$

So, from (2.31) and (2.32), we have

$$\begin{aligned} & A(u_0 + h_n) - A(u_0) + \frac{\mathfrak{g}_n}{\mu_n} A(h_n) \\ &= \frac{1}{\mu_n} N_g(u_0 + h_n) + N_f(u_0 + h_n) - \frac{\mathfrak{g}_n p}{\mu_n} |h_n|^{p-2} h_n - N_f(u_0) - \frac{1}{\eta} N_g(u_0), \end{aligned}$$

so

$$\begin{aligned}
 (2.33) \quad & -\operatorname{div} \left(\|\nabla(u_0 + h_n)\|^{p-2} \nabla(u_0 + h_n) - \|\nabla u_0\|^{p-2} \nabla u_0 + \frac{\vartheta_n}{\mu_n} \|\nabla h_n\|^{p-2} \nabla h_n \right) \\
 &= \frac{1}{\mu_n} g(z, u_0 + h_n) + f(z, u_0 + h_n) \\
 &\quad - \frac{\vartheta_n p}{\mu_n} |h_n|^{p-2} h_n - f(z, u_0) - \frac{1}{\eta} g(z, u_0) \quad \text{in } \Omega
 \end{aligned}$$

and

$$(2.34) \quad \frac{\partial h_n}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

We set

$$\begin{aligned}
 a_1(z, \xi) &= \|\nabla(u_0(z) + \xi)\|^{p-2} (\nabla u_0(z) + \xi) \\
 &\quad - \|\nabla u_0(z)\|^{p-2} \nabla u_0(z) + \frac{\vartheta_n}{\mu_n} \|\xi\|^{p-2} \xi
 \end{aligned}$$

and then the left hand side of (2.33) becomes $-\operatorname{div} a_1(z, \nabla h_n(z))$. Then

$$h_n \in L^\infty(\Omega) \quad \text{and} \quad \|h_n\|_\infty \leq \hat{M}, \quad \forall n \geq 1,$$

for some $\hat{M} > 0$ (see Hu-Papageorgiou [15]). Invoking Theorem 2 of Lieberman [19] (see also García Azorero-Manfredi-Peral Alonso [6, Lemma 4.1] and Motreanu-Motreanu-Papageorgiou [21, Proposition 2.5]), we can find $\beta \in (0, 1)$ and $M_2 > 0$, such that

$$(2.35) \quad h_n \in C_n^{1,\beta}(\bar{\Omega}) \quad \text{and} \quad \|h_n\|_{C_n^{1,\beta}(\bar{\Omega})} \leq M_2, \quad \forall n \geq 1.$$

Recall that $C_n^{1,\beta}(\bar{\Omega})$ is embedded compactly in $C_n^1(\bar{\Omega})$ and $h_n \rightarrow 0$ in $W_n^{1,p}(\Omega)$. So, from (2.35), it follows that

$$(2.36) \quad h_n \rightarrow 0 \quad \text{in } C_n^1(\bar{\Omega}).$$

Case (ii): The sequence $\{\vartheta_n\}_{n \geq 1}$ is unbounded.

Passing to a subsequence if necessary, we may assume that $|\vartheta_n| \rightarrow +\infty$. Recall that $\vartheta_n < 0$ and because of (2.29), we may assume that

$$\frac{\mu_n}{\vartheta_n} \in (0, 1], \quad \forall n \geq 1.$$

From (2.31), we have

$$(2.37) \quad \begin{aligned} & \frac{\mu_n}{g_n} A(u_0 + h_n) + A(h_n) \\ &= \frac{1}{g_n} N_g(u_0 + h_n) + \frac{\mu_n}{g_n} N_f(u_0 + h_n) - p|h_n|^{p-2}h_n, \quad \forall n \geq 1. \end{aligned}$$

Of course (2.32) remains valid. Then from (2.37) and (2.32), it follows that

$$\begin{aligned} & \frac{\mu_n}{g_n} (A(u_0 + h_n) - A(u_0)) + A(h_n) \\ &= \frac{1}{g_n} N_g(u_0 + h_n) + \frac{\mu_n}{g_n} N_f(u_0 + h_n) \\ & \quad - p|h_n|^{p-2}h_n - \frac{\mu_n}{g_n} N_f(u_0) - \frac{\mu_n}{g_n \eta} N_g(u_0) \end{aligned}$$

so

$$(2.38) \quad \begin{aligned} & -\operatorname{div} \left(\frac{\mu_n}{g_n} \|\nabla(u_0 + h_n)\|^{p-2} \nabla(u_0 + h_n) - \frac{\mu_n}{g_n} \|\nabla u_0\|^{p-2} \nabla u_0 + \|\nabla h_n\|^{p-2} \nabla h_n \right) \\ &= \frac{1}{g_n} g(z, u_0 + h_n) + \frac{\mu_n}{g_n} f(z, u_0 + h_n) - p|h_n|^{p-2}h_n \\ & \quad - \frac{\mu_n}{g_n} f(z, u_0) - \frac{\mu_n}{g_n \eta} g(z, u_0) \quad \text{in } \Omega \end{aligned}$$

and

$$(2.39) \quad \frac{\partial h_n}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

We set

$$\begin{aligned} a_2(z, \xi) &= \frac{\mu_n}{g_n} \|\nabla u_0(z) + \xi\|^{p-2} (\nabla u_0(z) + \xi) \\ & \quad - \frac{\mu_n}{g_n} \|\nabla u_0(z)\|^{p-2} \nabla u_0(z) + \|\xi\|^{p-2} \xi. \end{aligned}$$

Then the left hand side of (2.38) becomes $-\operatorname{div} a_2(z, \nabla h_n(z))$ and as before via Theorem 2 of Lieberman [19], we conclude that (2.36) remains true.

Since u_0 is a local $C_n^1(\bar{\Omega})$ -minimizer of $\psi|_M$ and

$$u_0 + h_n \in M \cap C_n^1(\bar{\Omega}), \quad \forall n \geq n_0$$

(see the first part of the proof and Claim 1), we have

$$\psi(u_0) \leq \psi(u_0 + h_n), \quad \forall n \geq n_3$$

(see (2.36)).

But this contradicts the choice of h_n (see (2.2)). This proves that u_0 is also a local $W_n^{1,p}(\Omega)$ -minimizer of $\psi|_M$. $\square$

3. Multiple solutions for constrained parametric equations

In this section, we consider the following nonlinear parametric Neumann problem constrained on a sphere:

$$(P)_\lambda \quad \begin{cases} -\Delta_p u(z) + |u(z)|^{p-2} u(z) = \lambda g(z, u(z)) & \text{a.e. in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \|u\| = \eta > 0, \lambda > 0. \end{cases}$$

Here Δ_p denotes the p -Laplace differential operator, defined by

$$\Delta_p u(z) = \operatorname{div}(\|\nabla u(z)\|^{p-2} \nabla u(z)), \quad \forall u \in W^{1,p}(\Omega),$$

with $2 \leq p < +\infty$.

The hypotheses on the reaction term are the following:

H: $g : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is a Carathéodory function, such that $g(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) there exist $a \in L^\infty(\Omega)_+$, $c > 0$ and $r \in [p, p^*)$, such that

$$|g(z, \zeta)| \leq a(z) + c|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega \text{ and all } \zeta \in \mathbf{R};$$

(ii) we have

$$g(z, \zeta)\zeta > 0 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \neq 0.$$

Remark 3.1. Hypothesis H on g are minimal. We only require subcritical growth (see $H(i)$) and a strict sign condition (see $H(ii)$).

We set

$$G(z, \zeta) = \int_0^\zeta g(z, s) ds$$

and consider the C^1 -functionals $\psi, \varphi : W_n^{1,p}(\Omega) \rightarrow \mathbf{R}$, defined by

$$\psi(u) = - \int_\Omega G(z, u(z)) dz, \quad \forall u \in W_n^{1,p}(\Omega),$$

$$\varphi(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{p} \|u\|_p^p, \quad \forall u \in W_n^{1,p}(\Omega).$$

Also we consider the C^1 -Banach manifold M of codimension 1 in $W_n^{1,p}(\Omega)$, defined by

$$M = \{u \in W_n^{1,p}(\Omega) : p\phi(u) = \eta^p\}$$

(the sphere of radius η). By a solution of $(P)_\lambda$, we mean a pair $(\lambda, u) \in (0, +\infty) \times (W_n^{1,p}(\Omega) \setminus \{0\})$ which satisfies $(P)_\lambda$.

Problem $(P)_\lambda$ was investigated by Tehrani [25] and Li-Liu [20] with Dirichlet boundary condition and $p = 2$ (semilinear case). An extension to the Dirichlet p -Laplacian can be found in the work of Chen-Chen-Li [4], who considered a reaction $g : \bar{\Omega} \times \mathbf{R} \rightarrow \mathbf{R}$ which is locally Lipschitz, nondecreasing in $\zeta \in \mathbf{R}$ and impose some technical restrictions on p . However, they provide precise sign information for all three solutions they obtain. Their approach is completely different and instead of a result like Theorem 2.1 here, they exploit the invariance properties of the constrained negative pseudogradient flow, in the spirit of the unconstrained work of Bartsch-Liu [2].

First we produce two constant sign smooth solutions.

Proposition 3.2. *If hypotheses H hold, then problem $(P)_\lambda$ has at least two constant sign smooth solutions:*

$$(\lambda_1, u_0) \in (0, +\infty) \times (\text{int } C_+), \quad (\lambda_2, v_0) \in (0, +\infty) \times (-\text{int } C_+)$$

and u_0, v_0 are both local minimizers of $\psi|_M$.

Proof. We consider the positive truncation of $g(z, \zeta)$, defined by

$$g_+(z, \zeta) = g(z, \zeta^+)$$

(here and in what follows, for every $\zeta \in \mathbf{R}$, we denote $\zeta^\pm = \max\{\pm\zeta, 0\}$). This is a Carathéodory function. We set

$$G_+(z, \zeta) = \int_0^\zeta g_+(z, s) ds$$

and consider the C^1 -functional $\psi_+ : W_n^{1,p}(\Omega) \rightarrow \mathbf{R}$, defined by

$$\psi_+(u) = - \int_\Omega G_+(z, u(z)) dz, \quad \forall u \in W_n^{1,p}(\Omega).$$

We consider the following minimization problem:

$$(3.1) \quad \inf_{u \in M} \psi_+(u) = m_+.$$

It is clear from hypotheses H that $-\infty < m_+ < 0$. Let $\{u_n\}_{n \geq 1} \subseteq M$ be a minimizing sequence for problem (3.1). We may assume that

$$(3.2) \quad u_n \rightarrow u_0 \quad \text{weakly in } W_n^{1,p}(\Omega),$$

$$(3.3) \quad u_n \rightarrow u_0 \quad \text{in } L^r(\Omega).$$

Hypothesis $H(i)$ and the Krasnoselskii theorem (see e.g., Gasiński-Papageorgiou [7, p. 407]) imply that ψ_+ is continuous on $L^r(\Omega)$. Hence

$$\psi_+(u_n) \rightarrow \psi_+(u_0)$$

(see (3.3)). Also, since the norm in a Banach space is sequentially weakly lower semicontinuous, we have

$$\varphi(u_0) \leq \liminf_{n \rightarrow +\infty} \varphi(u_n)$$

(see (3.2)), so

$$(3.4) \quad \varphi(u_0) \leq \frac{1}{p} \eta^p.$$

From the definition of g_+ , we see that

$$\psi_+(u^+) = \psi_+(u), \quad \forall u \in W_n^{1,p}(\Omega)$$

and so we may assume that $u_n \geq 0$ for all $n \geq 1$. It follows that $u_0 \geq 0$ and since

$$\psi_+(u_0) = m_+ < 0 = \psi_+(0),$$

we infer that $u_0 \neq 0$. We consider the function $\sigma_+ : [0, +\infty)$, defined by

$$\sigma_+(t) = \psi_+(tu_0), \quad \forall t \geq 0.$$

Evidently $\sigma_+ \in C^1((0, +\infty))$ and

$$(3.5) \quad \sigma'_+(t) = - \int_{\Omega} g_+(z, tu_0(z)) u_0(z) dz, \quad \forall t > 0.$$

Since $u_0 \geq 0$, $u_0 \neq 0$, from (3.10) and hypothesis $H(ii)$, it follows that

$$(3.6) \quad \sigma'_+(t) < 0, \quad \forall t > 0.$$

Suppose that $\varphi(u_0) < \eta^p/p$ (see (3.4)). Then we can find $t_0 > 0$, such that

$$\varphi(t_0 u_0) = \frac{1}{p} \eta^p.$$

Because of (3.6), we have

$$m_+ = \psi(u_0) > \psi_+(t_0 u_0) \geq m_+$$

(since $t_0 u_0 \in M$), a contradiction. Therefore

$$\varphi(u_0) = \frac{1}{p} \eta^p$$

(see (3.4)) and so $u_0 \in M$. By the Lagrange multiplier rule, we can find $\mu_1 \in \mathbf{R}$, such that

$$-\psi'_+(u_0) = \mu_1 \varphi'(u_0),$$

so

$$(3.7) \quad N_{g_+}(u_0) = \mu_1 A(u_0) + \mu_1 u_0^{p-1},$$

where

$$N_g(u)(\cdot) = g_+(\cdot, u(\cdot)), \quad \forall u \in W_n^{1,p}(\Omega).$$

On (3.7) we act with u_0 and obtain

$$0 < \int_{\Omega} g_+(z, u_0) u_0 \, dz = \int_{\Omega} g(z, u_0) u_0 \, dz = \mu_1 (\|\nabla u_0\|_p^p + \|u_0\|_p^p)$$

(recall that $u_0 \geq 0$, $u_0 \neq 0$ and see $H(ii)$), so

$$\mu_1 > 0.$$

Let us set

$$\lambda_1 = \frac{1}{\mu_1} > 0.$$

Then from (3.7), we have

$$(3.8) \quad \begin{cases} -A_p u_0(z) + u_0(z)^{p-1} = \lambda_1 g(z, u_0(z)) & \text{a.e. in } \Omega, \\ \frac{\partial u_0}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

(see Motreanu-Papageorgiou [22]). Nonlinear regularity theory (see Hu-Papageorgiou [15] and Lieberman [19]) implies that $u_0 \in C_+ \setminus \{0\}$. Also, from (3.8) and hypothesis $H(ii)$, we have

$$A_p u_0(z) \leq u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega,$$

so $u_0 \in \text{int } C_+$ (see Vázquez [28]).

Note that

$$\psi|_{M \cap C_+} = \psi_+|_{M \cap C_+}.$$

So, $u_0 \in \text{int } C_+$ is a local $C_n^1(\overline{\Omega})$ -minimizer of $\psi|_M$. Because $\mathcal{F}^{-1}(\varphi'(u_0)) = \|u_0\|^{p-2}u_0$, we have

$$\langle \psi'(u_0), \mathcal{F}^{-1}(\varphi'(u_0)) \rangle = -\|u_0\|^{p-2} \int_{\Omega} g(z, u_0) u_0 \, dz < 0.$$

So, we can apply Theorem 2.1 and conclude that $u_0 \in \text{int } C_+$ is a local $W_n^{1,p}(\Omega)$ -minimizer of $\psi|_M$.

Similarly, we consider

$$g_-(z, \zeta) = g(z, -\zeta^-)$$

and set

$$G_-(z, \zeta) = \int_0^{\zeta} g_-(z, s) ds$$

and introduce the C^1 -functional $\psi_- : W_n^{1,p}(\Omega) \rightarrow \mathbf{R}$, defined by

$$\psi_-(u) = - \int_{\Omega} G_-(z, u(z)) dz, \quad \forall u \in W_n^{1,p}(\Omega).$$

Reasoning as above, through the minimization of ψ_- , we obtain a second constant sign smooth solution $(\lambda_2, v_0) \in (0, +\infty) \times (-\text{int } C_+)$ of problem $(P)_{\lambda}$ with $v_0 \in -\text{int } C_+$ being a local minimizer of ψ . $\square$

Recall that $\psi|_M$ satisfies the Palais-Smale condition at level $\mathfrak{J} \in \mathbf{R}$, if the following is true:

“Every sequence $\{u_n\}_{n \geq 1} \subseteq M$, such that

$$\psi(u_n) \rightarrow \mathfrak{J} \quad \text{and} \quad (\psi|_M)'(u_n) \rightarrow 0 \quad \text{in } W_n^{1,p}(\Omega)^*$$

admits a strongly convergent subsequence.”

Proposition 3.3. *If hypotheses H hold, then $\psi|_M$ satisfies the Palais-Smale condition at any level $\mathfrak{J} \neq 0$.*

Proof. Let $\{u_n\}_{n \geq 1} \subseteq M$ be such that

$$(3.9) \quad \psi(u_n) \rightarrow \mathfrak{J} \neq 0 \quad \text{as } n \rightarrow +\infty$$

and

$$(3.10) \quad (\psi|_M)'(u_n) \rightarrow 0 \quad \text{in } W_n^{1,p}(\Omega)^* \quad \text{as } n \rightarrow +\infty.$$

From (3.10), we have

$$(3.11) \quad \psi'(u_n) - \lambda_n \phi'(u_n) \rightarrow 0 \quad \text{in } W_n^{1,p}(\Omega)^*,$$

with $\lambda_n \in \mathbf{R}$ for $n \geq 1$. Since $\{u_n\}_{n \geq 1} \subseteq M$, we may assume that

$$(3.12) \quad u_n \rightarrow u \quad \text{weakly in } W_n^{1,p}(\Omega),$$

$$(3.13) \quad u_n \rightarrow u \quad \text{in } L^r(\Omega).$$

From (3.11), we have

$$(3.14) \quad |\langle \psi'(u_n), y \rangle - \lambda_n \langle \phi'_n(u_n), y \rangle| \leq \varepsilon_n \|y\|, \quad \forall y \in W_n^{1,p}(\Omega),$$

with $\varepsilon_n \searrow 0$. Suppose that, at least for a subsequence, we have $\lambda_n \rightarrow 0$. In (3.14) we choose $y = u_n$ and obtain

$$(3.15) \quad |\langle \psi'(u_n), u_n \rangle - \lambda_n \eta^p| \leq \varepsilon_n \|u_n\|, \quad \forall n \geq 1,$$

so

$$(3.16) \quad - \int_{\Omega} g(z, u) u \, dz = 0$$

(see (3.12) and recall that $\lambda_n \rightarrow 0$).

On the other hand from (3.9), (3.12) and the continuity of ψ (by the Krasnoselskii theorem), we have

$$\psi(u) = \vartheta \neq 0 = \psi(0),$$

i.e., $u \neq 0$, so

$$(3.17) \quad - \int_{\Omega} g(z, u) u \, dz = 0$$

(see hypothesis $H(ii)$). Comparing (3.16) and (3.17), we reach a contradiction. This means that we can find $\xi_0 > 0$, such that

$$(3.18) \quad |\lambda_n| \geq \xi_0, \quad \forall n \geq 1.$$

From (3.15) and hypothesis $H(i)$, it follows that

$$|\lambda_n| \eta^p \leq M_3, \quad \forall n \geq 1,$$

for some $M_3 > 0$ and so the sequence $\{\lambda_n\}_{n \geq 1} \subseteq \mathbf{R}$ is bounded.

We may assume that $\lambda_n \rightarrow \lambda \neq 0$ (see (3.18)). In (3.14) we choose $y = u_n - u \in W_n^{1,p}(\Omega)$. Then passing to the limit as $n \rightarrow +\infty$ and using (3.12), we obtain

$$\lim_{n \rightarrow +\infty} \langle A(u_n), u_n - u \rangle = 0,$$

so

$$u_n \rightarrow u \quad \text{in } W_n^{1,p}(\Omega)$$

(see Barletta-Papageorgiou [1, Proposition 4.1]). □

Without any loss of generality, we may assume that

$$(3.19) \quad \psi(v_0) \leq \psi(u_0).$$

Also, we may assume that u_0 is an isolated critical point of $\psi|_M$ (otherwise we have a whole sequence of distinct pairs $(\tilde{\lambda}_n, u_n) \in (0, +\infty) \times (\text{int } C_+)$ solving problem $(P)_{\tilde{\lambda}_n}$ which converge to (λ_1, u_0)). Since $0 \notin M$, invoking Corollary 5.1.2 of Ghoussoub [13], we can state the following multiplicity theorem for problem $(P)_\lambda$.

Theorem 3.4. *If hypotheses H hold, then problem $(P)_\lambda$ has at least three smooth solutions:*

$$(\lambda_1, u_0) \in (0, +\infty) \times (\text{int } C_+), \quad (\lambda_2, v_0) \in (0, +\infty) \times (-\text{int } C_+)$$

and

$$(\lambda_3, y_0) \in (0, +\infty) \times (C_n^1(\overline{\Omega}) \setminus \{0\}).$$

Remark 3.5. We believe that our result can be extended to the case where the functional ψ includes boundary integral. We are not aware of any work in this direction.

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