Functional Equations for Appell’s $F_1$ Arising from Transformations of Elliptic Curves

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Abstract. We give a functional equation for Appell’s hypergeometric function $F_1$, which arises from transformations of elliptic curves. As an application, we give an efficient algorithm for computing incomplete elliptic integrals of the first kind. We also give a reduction formula that simplifies Lauricella’s hypergeometric function $F_D$ of five variables to $F_1$.

Key Words and Phrases. Hypergeometric functions, Transformation formulas, Incomplete elliptic integrals of the first kind.

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1. Introduction

It is classically known that Gauss’ hypergeometric function $F(a, b; c; z)$ satisfies the transformation formula:

$$(1 + 2z)^{2p} \cdot F\left(\frac{p}{2}, \frac{p - q + 1}{2}, q + \frac{1}{2}; z^2\right) = F\left(p, q, 2q; \frac{4z}{(1 + z)^2}\right).$$

By this transformation formula and an integral representation of $F(a, b; c; z)$ of Euler type, we can express the arithmetic-geometric mean of $(a, b) \in (R_{>0})^2$ by a complete elliptic integral of the first kind, where $R_{>0}$ is the set of positive real numbers. Transformation formulas for other hypergeometric functions are also applied to the study of iterations of several means of several terms. For example, it is studied in [4] that a transformation formula for Appell’s hypergeometric function $F_1$ implies three means of three terms and that the triple of sequences defined by the iteration of these means converges and has a common limit expressed by an incomplete elliptic integral of the first kind.

In this paper, we find a new transformation formula for Appell’s hypergeometric function $F_1$ by considering transformations of elliptic curves. Our main theorem (Theorem 3.1) is as follows:
We prove this formula by using the integration by substitution corresponding to the isogeny map. We apply our theorem to the computation of incomplete elliptic integrals. By our transformation formula, we define a map \((R^3, 0) \to (R^3, 0, 0)\), which implies a triple \((a_n, b_n, c_n)_{n \in N}\) of sequences by the iteration. It turns out that the sequences converge and satisfy

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n \neq \lim_{n \to \infty} c_n
\]

for general initial terms. An incomplete elliptic integral of the first kind can be expressed in terms of their limits. Since the convergence of them is quadratic, we obtain an efficient algorithm for computing incomplete elliptic integrals. As has been mentioned, there are several extensions and analogies of the arithmetic-geometric mean; each of them is a common limit of a multiple of sequences. This example suggests to us the study of the iteration of a map, even if the induced multiple of sequences does not have a common limit.

The contents of this paper are as follows. Firstly, we describe transformations of elliptic curves in terms of the theta functions by using results in [3]. We give convenient expressions of the isogeny and the twice map for our study. Next, we prove the main theorem by using the expression of the isogeny, and explain the algorithm for computing incomplete elliptic integrals of the first kind. Finally, we consider a triple of sequences given by the transformation formula in [4]:

\[
\frac{z_1 + z_2}{2} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - z_1^2, 1 - z_2^2 \right)
= F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - \frac{z_1(1 + z_2)}{z_1 + z_2}, 1 - \frac{z_2(1 + z_1)}{z_1 + z_2} \right)
\]

(equivalent to Proposition 5.3). Calculating an elliptic integral by a substitution arising from the map of multiplication by 2, we give another proof of this formula and a reduction formula of Lauricella’s hypergeometric function \(F_D\) of five variables to Appell’s hypergeometric function \(F_1\).

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2. Elliptic curves and complex tori

We review some results in [3] in this section.

2.1. Abel-Jacobi map

We consider the elliptic curves
\[ C(\lambda) : y^2 = x(1 - x)(1 - \lambda x), \quad \lambda \in \mathbb{C} - \{0, 1\}, \]
which are double coverings of the complex projective line \( P^1 \). We denote four ramification points \((x, y) = (0, 0), (1, 0), (1/\lambda, 0)\) and the point at infinity in \( C(\lambda) \) by \( P_0, P_1, P_\lambda \) and \( P_\infty \), respectively. We choose a symplectic basis \( A, B \in H_1(C(\lambda), \mathbb{Z}) \) so that \( A \cdot A = B \cdot B = 0 \), \( B \cdot A = 1 \), and that
\[
\int_A \frac{dx}{y} = 2 \int_1^{1/i} \frac{dx}{\sqrt{x(1 - x)(1 - \lambda x)}} \in i\mathbb{R}_{>0},
\]
\[
\int_B \frac{dx}{y} = 2 \int_0^1 \frac{dx}{\sqrt{x(1 - x)(1 - \lambda x)}} \in \mathbb{R}_{>0},
\]
when \( \lambda \) is in the open interval \((0, 1)\). We set
\[
\tau_A := \int_A \frac{dx}{y}, \quad \tau_B := \int_B \frac{dx}{y}, \quad \tau := \tau_A/\tau_B;
\]
note that \( \tau \) belongs to the upper half plane \( \mathbb{H} \). Let \( L(\tau) \) be the lattice \( \mathbb{Z}\tau + \mathbb{Z} \); then the complex torus \( E(\tau) := C/L(\tau) \) is isomorphic to \( C(\lambda) \) by the Abel-Jacobi map
\[
\Phi : C(\lambda) \rightarrow E(\tau); \quad P \mapsto \frac{1}{\tau_B} \int_{P_\lambda}^P \frac{dx}{y} \mod L(\tau).
\]
We represent the inverse map of \( \Phi \) by the theta functions with half integral characteristics. For \( a, b \in \{0, 1\} \), the theta function is defined by
\[
\eta_{ab}(z, \tau) := \sum_{n \in \mathbb{Z}} \exp \left( \pi i \left( n + \frac{a}{2} \right)^2 \tau + 2\pi i \left( n + \frac{a}{2} \right) \left( z + \frac{b}{2} \right) \right),
\]
where \( z \in \mathbb{C} \) and \( \tau \in \mathbb{H} \). We denote \( \eta_{ab}(0, \tau) \) by \( \eta_{ab}(\tau) \).

Proposition 2.1 ([3]). The inverse map of \( \Phi \) is expressed as follows:
\[
\Phi^{-1}([z]) = \left( \frac{\eta_{00}(\tau)^2 \eta_{01}(z, \tau)^2 \eta_{10}(z, \tau)^2}{\eta_{11}(z, \tau)^2 \eta_{10}(\tau)^2 \eta_{00}(z, \tau) \eta_{01}(z, \tau) \eta_{10}(z, \tau) \eta_{11}(z, \tau)} \right).
\]
By Proposition 2.1, maps, where the second equality is followed from (6) and (7). Via the Abel-Jacobi formula, the element of \(E(\tau)\) represented by \(z\) is expressed as
\[
\lambda = \frac{\mathcal{g}_{10}(\tau)^4}{\mathcal{g}_{00}(\tau)^4} = 1 - \frac{\mathcal{g}_{01}(\tau)^4}{\mathcal{g}_{00}(\tau)^4}.
\]

### 2.2. Maps between elliptic curves

We use the following formulas from [2] and [5].

**Fact 2.2.**

\((1)\) \(\mathcal{g}_{00}(\tau)^3 \mathcal{g}_{00}(2z, \tau) = \mathcal{g}_{00}(z, \tau)^4 + \mathcal{g}_{11}(z, \tau)^4\),
\((2)\) \(\mathcal{g}_{01}(\tau)^3 \mathcal{g}_{01}(2z, \tau) = \mathcal{g}_{01}(z, \tau)^4 - \mathcal{g}_{11}(z, \tau)^4\),
\((3)\) \(\mathcal{g}_{10}(\tau)^3 \mathcal{g}_{10}(2z, \tau) = \mathcal{g}_{10}(z, \tau)^4 - \mathcal{g}_{11}(z, \tau)^4\),
\((4)\) \(\mathcal{g}_{00}(\tau)^2 \mathcal{g}_{01}(\tau) \mathcal{g}_{01}(2z, \tau) = \mathcal{g}_{00}(z, \tau)^2 \mathcal{g}_{01}(z, \tau)^2 + \mathcal{g}_{10}(z, \tau)^2 \mathcal{g}_{11}(z, \tau)^2\),
\((5)\) \(\mathcal{g}_{00}(\tau) \mathcal{g}_{01}(\tau) \mathcal{g}_{10}(\tau) \mathcal{g}_{11}(2z, \tau) = 2 \mathcal{g}_{00}(z, \tau) \mathcal{g}_{01}(z, \tau) \mathcal{g}_{10}(z, \tau) \mathcal{g}_{11}(z, \tau)\),
\((6)\) \(2 \mathcal{g}_{00}(2\tau) \mathcal{g}_{00}(2z, 2\tau) = \mathcal{g}_{00}(z, \tau)^2 + \mathcal{g}_{01}(z, \tau)^2\),
\((7)\) \(\mathcal{g}_{01}(2\tau) \mathcal{g}_{01}(2z, 2\tau) = \mathcal{g}_{00}(z, \tau) \mathcal{g}_{01}(z, \tau)\).

We consider the isogeny and the translation by \(\tau/2\):
\[
pr : E(2\tau) \rightarrow E(\tau); \quad z \mod L(2\tau) \mapsto z \mod L(\tau),
\]
\[
T_{\tau/2} : E(\tau) \rightarrow E(\tau); \quad z \mod L(\tau) \mapsto z + \frac{\tau}{2} \mod L(\tau).
\]

By Proposition 2.1, \(E(2\tau)\) is isomorphic to the elliptic curve \(C(\lambda')\) with
\[
\lambda' = 1 - \frac{\mathcal{g}_{01}(2\tau)^4}{\mathcal{g}_{00}(2\tau)^4} = 1 - \frac{\mathcal{g}_{00}(\tau)^2 \mathcal{g}_{01}(\tau)^2}{(\mathcal{g}_{00}(\tau)^2 + \mathcal{g}_{01}(\tau)^2)^2} = \left(\frac{\mathcal{g}_{00}(\tau)^2 - \mathcal{g}_{01}(\tau)^2}{\mathcal{g}_{00}(\tau)^2 + \mathcal{g}_{01}(\tau)^2}\right)^2,
\]
where the second equality is followed from (6) and (7). Via the Abel-Jacobi maps, \(pr\) and \(T_{\tau/2}\) induce \(\tilde{pr} : C(\lambda') \rightarrow C(\lambda)\) and \(\tilde{T}_{\tau/2} : C(\lambda) \rightarrow C(\lambda)\), respectively.

**Proposition 2.3 ([3]).** We have
\((i)\)
\[
\tilde{pr}(x', y') = \left(\frac{\sqrt{\lambda'} x' + 1}{4\sqrt{\lambda'} x'}, \frac{\sqrt{\lambda'}(1 + \sqrt{1 - \lambda'})}{8} \left(1 - \frac{1}{\lambda' x'^2}\right) y'\right),
\]
where
\[
\sqrt{\lambda'} = \frac{\eta_{10}(2\tau)^2}{\eta_{00}(2\tau)^2} = \frac{\eta_{00}(2\tau)^2 - \eta_{01}(2\tau)^2}{\eta_{00}(2\tau)^2 + \eta_{01}(2\tau)^2}, \\
\sqrt{1 - \lambda} = \frac{\eta_{01}(\tau)^2}{\eta_{00}(\tau)^2},
\]

(ii)
\[
\overline{T}_{1/2}(x,y) = \left( \frac{1}{\lambda x'}, -\frac{y'}{\lambda x'}^2 \right),
\]

(iii)
\[
\overline{T}_{1/2} \circ \overline{pr}(x',y') = \left( \frac{4\sqrt{\lambda'} x'}{\lambda(\sqrt{\lambda'} x' + 1)^2}, \frac{-2\sqrt{\lambda'}(\sqrt{\lambda'} x' - 1)y'}{(1 - \sqrt{\lambda'}(\sqrt{\lambda'} x' + 1)^3} \right).
\]

We consider the map \( \psi : C(\lambda) \to C(\lambda) \) induced from
\[
E(\tau) \to E(\tau); \quad z \text{ mod } L(\tau) \mapsto z \text{ mod } L(\tau)
\]
via the Abel-Jacobi map \( \Phi \). The following proposition is in some texts of elliptic curves (e.g., [6]). However we give our proof using the theta functions, because this representation of \( \lambda \) is key of the study in Section 5.

**Proposition 2.4.** The map \( \psi : C(\lambda) \to C(\lambda) \) is represented as follows:
\[
\psi(x',y') = \left( \frac{(1 - \lambda'^2)}{4\lambda x'(1 - x')(1 - \lambda' x')}, \frac{(\lambda x'^2 - 1)(\lambda x'^2 - 2x' + 1)(\lambda x'^2 - 2\lambda x' + 1)}{8\lambda y'^3} \right).
\]

**Proof.** Put \( \psi(x',y') = (x,y) \), then we have
\[
x = \frac{\eta_{00}(\tau)^2 \eta_{01}(2\tau,\tau)^2}{\eta_{10}(\tau)^2 \eta_{11}(2\tau,\tau)^2},
\]
\[
= \frac{1}{4} \left( \frac{\eta_{00}(\tau,\tau)^2 \eta_{01}(\tau,\tau)^2}{\eta_{10}(\tau,\tau)^2 \eta_{11}(\tau,\tau)^2} + 2 \right) + \frac{\eta_{10}(\tau,\tau)^2 \eta_{11}(\tau,\tau)^2}{\eta_{00}(\tau,\tau)^2 \eta_{01}(\tau,\tau)^2}
\]
\[
= \frac{1}{4} \left( \frac{\lambda x'(1 - x')}{1 - \lambda x'} + 2 + \frac{1 - \lambda x'}{\lambda x'(1 - x')} \right) = \frac{1}{4} \cdot \frac{(1 - \lambda'^2)^2}{\lambda x'(1 - x')(1 - \lambda' x')},
\]
by (4) and (5). Similarly, we also obtain the expression of \( y \) by applying (1), (2) and (3). \( \square \)

3. Transformation formula for Appell’s hypergeometric function \( F_1 \)

Appell’s hypergeometric function \( F_1 \) of two variables \( z_1, z_2 \) with parameters \( \alpha, \beta_1, \beta_2, \gamma \) is defined as
\[
F_1(\alpha,\beta_1,\beta_2,\gamma; z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} \frac{(\alpha, n_1 + n_2)(\beta_1, n_1)(\beta_2, n_2)}{(\gamma, n_1 + n_2)(1, n_1)(1, n_2)} z_1^{n_1} z_2^{n_2},
\]
where \( z_j \)'s satisfy \( |z_j| < 1, \gamma \neq 0, -1, -2, \ldots \), and \((x, n) = x(x + 1) \cdots (x + n - 1) = \Gamma(x + n)/\Gamma(x) \). This function admits an integral representation of Euler type:

\[
F_1(x, \beta_1, \beta_2, \gamma; z_1, z_2) = \frac{\Gamma(\gamma)}{\Gamma(x)\Gamma(\gamma - x)} \int_0^1 t^x (1 - t)^{\gamma-x} (1 - z_1 t)^{-\beta_1} (1 - z_2 t)^{-\beta_2} \, dt / t(1 - t).
\]

**Theorem 3.1.** We have a transformation formula for \( F_1 \):

\[
(8) \quad F_1 \left( \frac{1}{2}, p, p + 1; 1 - z_1^2, 1 - z_2^2 \right) = \frac{1}{z_1} F_1 \left( 1, p, p + 1; 1 - w_1, 1 - w_2 \right),
\]

where \((z_1, z_2)\) is in a small neighborhood of \((1, 1)\).

**Remark 3.2.** If we choose another branch of \(\sqrt{(1 - z_1^2)(z_1^2 - z_2^2)}\), then \(w_1\) and \(w_2\) interchange. By \(F_1(x, \beta, \gamma; z_1, z_2) = F_1(x, \beta, \gamma; z_2, z_1)\), the right-hand side of (8) is independent of the choice of the branch of the square root.

**Proof of Theorem 3.1.** Replace \(1 - z_1^2\) and \(1 - z_2^2\) with \(z_1\) and \(z_2\), respectively, and use the integral representation for \(F_1\). Then it is sufficient to show that

\[
(9) \quad \int_0^1 (1 - t)^{p-1} (1 - z_1 t)^{-1/2} (1 - z_2 t)^{-p} \, dt
\]

for \(z_1, z_2 \in \mathbb{R}\) satisfying \(0 < z_1 < z_2 < 1\), where

\[
w_1 = 1 - \sqrt{1 - z_1^2} + 1 - z_2 + \sqrt{z_2(z_2 - z_1)},
\]

\[
w_2 = 1 - \sqrt{1 - z_1^2} + 1 - z_2 - \sqrt{z_2(z_2 - z_1)}.
\]

To prove the identity (9), we use three kinds of substitutions. By the first substitution

\[
t = \frac{1 - z_2}{z_2} x + 1,
\]
we have
\[
\frac{dt}{dx} = \frac{1 - z_2}{z_2}, \quad 1 - t = -\frac{1 - z_2}{z_2}x,
\]
\[
1 - z_1 t = (1 - z_1)\left(1 - \frac{(1 - z_2)z_1}{(1 - z_1)z_2}x\right), \quad 1 - z_2 t = (1 - z_2)(1 - x),
\]
\[
(10) \int_0^1 (1 - t)^{\mu - 1}(1 - z_1 t)^{-1/2}(1 - z_2 t)^{-\mu} dt
\]
\[
= \frac{(1 - z_1)^{-1/2}}{z_2 \cdot (-z_2)^{\mu - 1}} \int_{R_1}^0 x^{\mu - 1}(1 - x)^{-\mu}(1 - \lambda x)^{-1/2} dx,
\]
where
\[
\lambda = \frac{(1 - z_2)z_1}{(1 - z_1)z_2} = 1 - \frac{z_2 - z_1}{(1 - z_1)z_2}, \quad R_1 = -\frac{z_2}{1 - z_2}.
\]
We set
\[
\lambda' = \left(1 - \sqrt{1 - \lambda}\right)^2 = \left(1 - \sqrt{(z_2 - z_1)/(1 - z_1)z_2}\right)^2
\]
and consider the integral in the right-hand side of (10) by the second substitution
\[
x = \frac{4\sqrt{\lambda} \lambda'}{\lambda'(\sqrt{\lambda'} \lambda' + 1)^2}
\]
in Proposition 2.3 (iii). If \(x = 0\), then \(x' = 0\). On the other hand, the equation
\[
R_1 = \frac{4\sqrt{\lambda} \lambda'}{\lambda'(\sqrt{\lambda'} \lambda' + 1)^2}
\]
has two solutions
\[
x' = R_2^\pm := -\frac{\lambda R_1 + 2(1 + \sqrt{1 - \lambda} R_1)}{\lambda R_1}.
\]
Since \(R_1 < 0\), the inequality \(R_2^+ < R_2^- < 0\) holds. Hence the integral interval \([R_1, 0]\) for \(x\) is changed to the integral interval \([R_2^-, 0]\) for \(x'\). We have
\[
\frac{dx}{dx'} = \frac{4\sqrt{\lambda'}(1 - \sqrt{\lambda'} x')}{\lambda(\sqrt{\lambda'} x' + 1)^3},
\]
\[
1 - x = \frac{\lambda(\sqrt{\lambda'}x' + 1)^2 - 4\sqrt{\lambda'}x'}{\lambda(\sqrt{\lambda'}x' + 1)^2} \\
= \frac{1}{(\sqrt{\lambda'}x' + 1)^2} \left(1 + \frac{2(\lambda - 2)}{(1 + \sqrt{1 - \lambda})^2} x' + \lambda'x'^2\right) \\
= \frac{1}{(\sqrt{\lambda'}x' + 1)^2} (1 - x')(1 - \lambda'x'), \\
1 - \lambda x = \frac{\lambda(\sqrt{\lambda'}x' + 1)^2 - 4\lambda\sqrt{\lambda'}x'}{\lambda(\sqrt{\lambda'}x' + 1)^2} = \left(\frac{\sqrt{\lambda'}x' - 1}{\sqrt{\lambda'}x' + 1}\right)^2.
\]

Note that if \( R_2^- < x' < 0 \), then \( \sqrt{\lambda'}x' + 1 > 0 \) and \( \sqrt{\lambda'}x' - 1 < 0 \). Thus the identity (10) is equivalent to

\[
(11) \quad \int_0^1 (1 - t)^{p-1}(1 - z_1t)^{-1/2}(1 - z_2t)^{-p}dt \\
= \frac{(1 - z_1)^{-1/2}}{z_2 \cdot (-z_2)^{p-1} (1 + \sqrt{(z_2 - z_1)/(1 - z_1)})^2} 2^{2p} \\
\cdot \int_R^0 x'^{p-1}(1 - x')^{-p}(1 - \lambda'x')^{-p}dx'.
\]

Finally, we consider the integral in the right-hand side of (11) by the third substitution

\[x' = -R_2^- t' + R_2^- .\]

Then it follows that

\[
\frac{dx'}{dt'} = -R_2^- , \quad x' = R_2^- (1 - t'), \quad 1 - x' = (1 - R_2^- ) \left(1 - \frac{-R_2^-}{1 - R_2^-} t'\right), \\
1 - \lambda'x' = (1 - \lambda'R_2^- ) \left(1 - \frac{-\lambda'R_2^-}{1 - \lambda'R_2^-} t'\right).
\]

Using \( \lambda R_1 = -z_1/(1 - z_1) \), we calculate \( \sqrt{\lambda'} \) and \( R_2^- \):

\[
\sqrt{\lambda'} = \frac{\sqrt{(1 - z_1)z_2 - \sqrt{z_2 - z_1}}}{\sqrt{(1 - z_1)z_2 + \sqrt{z_2 - z_1}}} = \frac{-z_1 + 2z_2 - z_1z_2 - 2\sqrt{z_2(1 - z_1)(z_2 - z_1)}}{(1 - z_2)z_1}, \\
R_2^- = \frac{1}{\sqrt{\lambda'}} \left(-1 + 2 \cdot \frac{1 - 1/\sqrt{1 - z_1}}{-z_1/(1 - z_1)}\right) = \frac{-z_1 - 2(1 - z_1 - \sqrt{1 - z_1})}{\sqrt{\lambda'}z_1} \\
= \frac{z_1 - 2 + 2\sqrt{1 - z_1}}{\sqrt{\lambda'}z_1}.
\]
This implies that
\[
1 - \frac{-R_2}{1 - R_2} = \frac{1}{1 - R_2} = \frac{\sqrt{z_1}}{\sqrt{z_1} - z_1 + 2 - 2\sqrt{1 - z_1}}
\]
\[
= \frac{1}{2\sqrt{1 - z_1}} \cdot \frac{(\sqrt{1 - z_1} - \sqrt{z_2(z_2 - z_1)})^2 - (1 - z_2)^2}{\sqrt{1 - z_1} - \sqrt{z_2(z_2 - z_1)} - (1 - z_2)}
\]
\[
= \frac{\sqrt{1 - z_1} - \sqrt{z_2(z_2 - z_1)} + 1 - z_2}{2\sqrt{1 - z_1}} = 1 - w_2,
\]
\[
1 - \frac{-\lambda' R_2}{1 - \lambda' R_2} = \frac{1}{1 - \lambda' R_2} = \frac{z_1}{z_1 - \sqrt{\lambda'(z_1 - 2 + 2\sqrt{1 - z_1})}}
\]
\[
= \frac{\sqrt{1 - z_1} + \sqrt{z_2(z_2 - z_1)} + 1 - z_2}{2\sqrt{1 - z_1}} = 1 - w_1.
\]
Thus we have
\[
\int_0^1 (1 - t)^{\rho - 1}(1 - z_1 t)^{-1/2}(1 - z_2 t)^{-p} dt
\]
\[
= \frac{(1 - z_1)^{-1/2}}{z_2 \cdot (-z_2)^{\rho - 1}} \cdot \frac{2^{2\rho}}{(1 + \sqrt{(z_2 - z_1)}/\{(1 - z_1)z_2\})^{2\rho}} (-R_2^\rho)(R_2^\rho)
\]
\[
\cdot (1 - R_2^\rho)^{-\rho}(1 - \lambda' R_2^\rho)^{-\rho} \int_0^1 (1 - t')^{\rho - 1}(1 - w_1 t')^{-p}(1 - w_2 t')^{-p} dt',
\]
and hence, to conclude the identity (9), it is sufficient to show the following:
\[
(12) \quad z_2^{-\rho} \left( \frac{2}{1 + \sqrt{(z_2 - z_1)}/\{(1 - z_1)z_2\}} \right)^{2\rho} (-R_2^\rho)(1 - R_2^\rho)^{-\rho}(1 - \lambda' R_2^\rho)^{-\rho} = 1.
\]
By these calculations, we obtain
\[
\frac{-R_2}{(1 - R_2)(1 - \lambda' R_2)} = \frac{2 - z_1 - 2\sqrt{1 - z_1}}{\sqrt{\lambda' z_1}} \cdot \frac{(\sqrt{1 - z_1} + 1 - z_2)^2 - z_2(z_2 - z_1)}{4(1 - z_1)}
\]
\[
= \frac{(1 - z_2)z_1}{4(1 - z_1)\sqrt{\lambda'}} = \frac{(1 - z_2)z_1}{4(1 - z_1)\sqrt{\lambda'}} \frac{\sqrt{(1 - z_1)z_2} + \sqrt{z_2 - z_1}}{\sqrt{(1 - z_1)z_2} - \sqrt{z_2 - z_1}}
\]
\[
= \left( \frac{\sqrt{(1 - z_1)z_2} + \sqrt{z_2 - z_1}}{2\sqrt{1 - z_1}} \right)^2
\]
\[
= z_2 \left( \frac{1 + \sqrt{(z_2 - z_1)}/\{(1 - z_1)z_2\}}{2} \right)^2,
\]
which implies (12).
4. Triple of sequences and its application to computing elliptic integrals

We apply Theorem 3.1 to an efficient algorithm for computing incomplete elliptic integrals of the first kind. We consider a triple of sequences \((a_n, b_n, c_n)\) where

\[
(a_0, b_0, c_0) = (a, b, c), \quad a \geq b \geq c > 0,
\]

\[
a_{n+1} := \sqrt{a_n b_n},
\]

\[
b_{n+1} := \frac{c_n + \sqrt{a_n b_n} + \sqrt{(a_n - c_n)(b_n - c_n)}}{2},
\]

\[
c_{n+1} := \frac{c_n + \sqrt{a_n b_n} - \sqrt{(a_n - c_n)(b_n - c_n)}}{2}.
\]

**Lemma 4.1.** (i) The sequences \(\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}\) and \(\{c_n\}_{n \in \mathbb{N}}\) converge.

(ii) \(\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n\).

(iii) \(\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n \Leftrightarrow b = c\).

(iv) If \(b > c\), then \(\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}\) and \(\{c_n\}_{n \in \mathbb{N}}\) converge quadratically.

**Proof.** If we assume \(a_n \geq b_n \geq c_n > 0\), then we have

\[
a_n - a_{n+1} = \sqrt{a_n} (\sqrt{a_n} - \sqrt{b_n}) \geq 0,
\]

\[
a_{n+1} - b_{n+1} = \frac{\sqrt{a_n b_n} - c_n - \sqrt{(a_n - c_n)(b_n - c_n)}}{2} = \frac{c_n (a_n + b_n - c_n - \sqrt{a_n b_n} - \sqrt{(a_n - c_n)(b_n - c_n)})}{2(\sqrt{a_n b_n} + \sqrt{(a_n - c_n)(b_n - c_n)})} \geq \frac{c_n (a_n + b_n - c_n - \frac{a_n + b_n}{2} - \frac{a_n - c_n + b_n - c_n}{2})}{2(\sqrt{a_n b_n} + \sqrt{(a_n - c_n)(b_n - c_n)})} = 0,
\]

\[
b_{n+1} - b_n = \frac{\sqrt{b_n} (\sqrt{a_n} - \sqrt{b_n}) + \sqrt{b_n - c_n} (\sqrt{a_n - c_n} - \sqrt{b_n - c_n})}{2} \geq 0,
\]

\[
c_{n+1} - c_n = a_{n+1} - b_{n+1} \geq 0.
\]

It follows that

\[
a \geq a_n \geq a_{n+1} \geq b_{n+1} \geq b_n \geq c_n \geq c_{n-1} \geq c \quad (n \geq 1),
\]

which implies (i). By \(a_{n+1} = \sqrt{a_n b_n}\), we have (ii). Inequalities

\[
b_{n+1} - c_{n+1} = \sqrt{(a_n - c_n)(b_n - c_n)} \geq b_n - c_n \geq b - c \quad (n \in \mathbb{N})
\]
show (iii). Since (iii) and

\[ a_{n+1} - b_{n+1} = c_{n+1} - c_n \]

\[ = \left( \sqrt{a_n} - \sqrt{c_n} \right) \left( \sqrt{b_n} + \sqrt{c_n} \right) - \frac{4}{\left( \sqrt{a_n} + \sqrt{c_n} \right)^2} - \left( \sqrt{a_n} + \sqrt{c_n} \right) \left( \sqrt{b_n} - \sqrt{c_n} \right)^2, \]

there exists \( M > 0 \) such that

\[ a_{n+1} - b_{n+1} \leq M(a_n - b_n)^2, \quad c_{n+1} - c_n \leq M(c_n - c_{n-1})^2. \]

These inequalities mean (iv). \( \square \)

Example 4.2. Let \((a, b, c) = (1, 0.5, 0.3)\). The values of \((a_n, b_n, c_n)\) and \([-\log_{10}(a_n - b_n)]\) are computed by Maple14 as in Table 1, where \([d]\) means the largest integer not greater than \(d\). Note that the growth of \([-\log_{10}(a_n - b_n)]\) means rapidity of the convergence, because \(a_n\) and \(b_n\) are in agreement till the \([-\log_{10}(a_n - b_n)]\)-th decimal place. Comparing to Table 2, we notice that this triple of sequences converges much faster than that in Section 5.

<table>
<thead>
<tr>
<th>Table 1: Fast convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>

| \([-\log_{10}(a_n - b_n)]\) | 1 | 4 | 10 | 22 | 45 | 92 | 185 | 371 | 744 | 1490 |

Theorem 4.3. For \(0 < z_1 < z_2 < 1\), we consider the triple of sequences \((a_n, b_n, c_n)\) with \((a, b, c) = (1, 1 - z_1, 1 - z_2)\) and put

\[ x := \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n, \quad \gamma := \lim_{n \to \infty} c_n. \]

Then we have

\[ \int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}} = \frac{a}{x \sqrt{1 - \gamma/x}} \left( \log \left( \frac{l}{x} \right) - 2 \log \left( 1 - \sqrt{1 - \frac{\gamma}{x}} \right) \right). \]
Proof. We set \( z_1 = \sqrt{b_n/a_n} \), \( z_2 = \sqrt{c_n/a_n} \) and \( p = 1/2 \) in Theorem 3.1; then we have
\[
\int_0^1 \frac{dt}{\sqrt{(1-t)(1-B_1t)(1-C_1t)}} = \sqrt{\frac{a_n}{b_n}} \int_0^1 \frac{dt'}{\sqrt{(1-t')(1-B_2t')(1-C_2t')}} ,
\]
\[
B_1 = 1 - \frac{b_n}{a_n} , \quad C_1 = 1 - \frac{c_n}{a_n} ,
\]
\[
B_2 = 1 - \frac{c_n + \sqrt{a_n b_n} + \sqrt{(a_n - c_n)(b_n - c_n)}}{2\sqrt{a_n b_n}} = 1 - \frac{b_{n+1}}{a_{n+1}} ,
\]
\[
C_2 = 1 - \frac{c_n + \sqrt{a_n b_n} - \sqrt{(a_n - c_n)(b_n - c_n)}}{2\sqrt{a_n b_n}} = 1 - \frac{c_{n+1}}{a_{n+1}} .
\]
This implies that the function
\[
\mu(p,q,r) := \frac{p}{\sqrt{(1-t)(1-(1-q/p)t)(1-(1-r/p)t)}}
\]
satisfies \( \mu(a_n,b_n,c_n) = \mu(a_{n+1},b_{n+1},c_{n+1}) \) for all \( n \in \mathbb{N} \). Then we obtain
\[
\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}} = \int_0^1 \frac{dt}{\sqrt{(1-t)(1-(1-b/a)t)(1-(1-c/a)t)}}
\]
\[
= \frac{a}{\mu(a,b,c)} = \frac{a}{\mu(\alpha,\alpha,\gamma)} = \frac{a}{\alpha} \int_0^1 \frac{dt}{\sqrt{(1-t)(1-(1-\gamma/z)t)}}
\]
\[
= \frac{a}{\alpha} \sqrt{1-\gamma/z} \left( \log \left( \frac{\gamma}{z} \right) - 2 \log \left( 1 - \sqrt{1 - \frac{\gamma}{z}} \right) \right) .
\]

Theorem 4.3 and Lemma 4.1 (iv) imply an efficient algorithm for computing incomplete elliptic integrals of the first kind:

**Algorithm 4.4.** To approximate
\[
\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}} \quad (0 < z_1 < z_2 < 1),
\]
we evaluate \((a_N,b_N,c_N)\) in Theorem 4.3 by the recurrence relation (13), where \( N \) is sufficiently large. Thus \( a_N \) and \( c_N \) approximate \( \alpha \) and \( \gamma \), respectively, and hence an approximation of the integral (14) is evaluated as
\[
\frac{a}{a_N \sqrt{1-c_N/a_N}} \left( \log \left( \frac{c_N}{a_N} \right) - 2 \log \left( 1 - \sqrt{1 - \frac{c_N}{a_N}} \right) \right) .
\]
Remark 4.5. Actually, $N$ does not have to be so large, since the convergence of $(a_n, b_n, c_n)$ is quadratic by Lemma 4.1 (iv). For example, to evaluate the integral (14) for $z_1 = 0.5$, $z_2 = 0.7$, we approximate $x$ and $y$ as $a_{10}$ and $c_{10}$, respectively, then $|a_{10} - x|, |c_{10} - y| < 10^{-1000}$ by Example 4.2.

5. Triple of sequences in [4]

5.1. Triple of sequences and their common limit

We define a triple of sequences $(a_n, b_n, c_n)$ by

$$(a_0, b_0, c_0) = (a, b, c), \quad a \geq b \geq c > 0,$$

$$(a_{n+1}, b_{n+1}, c_{n+1}) = \left(\frac{a_n}{2}(\sqrt{b_n + \sqrt{c_n}}), \frac{b_n}{2}(\sqrt{a_n + \sqrt{c_n}}), \frac{c_n}{2}(\sqrt{a_n + \sqrt{b_n}})\right).$$

Fact 5.1 ([4]). The sequences $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ and $\{c_n\}_{n \in \mathbb{N}}$ converge and satisfy

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n.$$  

This common limit of the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ is denoted by $m_N^\infty(a, b, c)$.

Theorem 5.2 ([4]). The common limit of the triple of sequences can be expressed as

$$m_N^\infty(a, b, c) = \frac{2a}{\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1t)(1-z_2t)}}},$$

where $z_1 = 1 - b/a$, $z_2 = 1 - c/a$.

To prove this theorem, we use the following proposition which we prove in the next subsection.

Proposition 5.3 ([4]). If $a \geq b \geq c > 0$, then we have

$$\int_0^1 \frac{dt'}{\sqrt{(1-t')(1-w_1t')(1-w_2t')}} = \frac{\sqrt{ab} + \sqrt{ac}}{2a} \int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1t)(1-z_2t)}},$$

where

$$z_1 = 1 - \frac{b}{a}, \quad z_2 = 1 - \frac{c}{a}, \quad w_1 = 1 - \frac{\sqrt{ab} + \sqrt{bc}}{\sqrt{ab} + \sqrt{ac}}, \quad w_2 = 1 - \frac{\sqrt{ac} + \sqrt{bc}}{\sqrt{ab} + \sqrt{ac}}.$$
Proof of Theorem 5.2 (Refer to [4]). Let \(\mu(a, b, c)\) be the right-hand side of (15). Proposition 5.3 implies that
\[
\mu(a_n, b_n, c_n) = \mu(a_{n+1}, b_{n+1}, c_{n+1})
\]
for all \(n \in \mathbb{N}\). Hence we have
\[
\mu(a, b, c) = \lim_{n \to \infty} \mu(a_n, b_n, c_n) = \mu(m_{\infty}(a, b, c), m_{\infty}(a, b, c), m_{\infty}(a, b, c)) = \frac{2m_{\infty}(a, b, c)}{\int_0^1 \frac{dt}{\sqrt{1-t}}} = m_{\infty}(a, b, c).
\]

Remark 5.4. By this triple of sequences, we can also compute an incomplete elliptic integral of the first kind. However, the convergence of \((a_n, b_n, c_n)\) is not rapid. For example, the values of \((a_n, b_n, c_n)\) and \([\log_{10}(a_n - b_n)]\) with \((a, b, c) := (1, 0.5, 0.3)\) are computed by Maple14 as in Table 2.

5.2. Another proof of Proposition 5.3

In [4], Proposition 5.3 is proved as a consequence of the transformation formula for Appell’s hypergeometric function \(F_1\), which is obtained by the calculation of connection matrices of integrable Pfaffian systems. Here we give our proof using integration by substitution.

We consider two elliptic curves
\[
C : s^2 = (1-t)(1-z_1t)(1-z_2t),
\]
\[
C' : s'^2 = (1-t')(1-w_1t')(1-w_2t'),
\]

Table 2: Slow convergence

<table>
<thead>
<tr>
<th>n</th>
<th>(a_n)</th>
<th>(b_n)</th>
<th>(c_n)</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>1.000000000000000</td>
<td>0.500000000000000</td>
<td>0.300000000000000</td>
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<tr>
<td>1</td>
<td>0.627414669345856</td>
<td>0.547202557903644</td>
<td>0.467510446062953</td>
</tr>
<tr>
<td>2</td>
<td>0.563765287089548</td>
<td>0.545863514844305</td>
<td>0.523691167084954</td>
</tr>
<tr>
<td>3</td>
<td>0.54905054905967</td>
<td>0.544702305679079</td>
<td>0.539010662167320</td>
</tr>
<tr>
<td>4</td>
<td>0.545439775683462</td>
<td>0.544360558450349</td>
<td>0.54292827999162</td>
</tr>
<tr>
<td>5</td>
<td>0.544541226396508</td>
<td>0.544271910695070</td>
<td>0.543913237208964</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>20</td>
<td>0.544242076130621</td>
<td>0.544242076130370</td>
<td>0.544242076130036</td>
</tr>
</tbody>
</table>

\([-\log_{10}(a_n - b_n)]\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... | 20

...
where \( z_1, z_2, w_1 \) and \( w_2 \) are as in Proposition 5.3. Both of these curves are isomorphic to

\[
C(\lambda) : y^2 = x(1-x)(1-\lambda x), \quad \lambda = \frac{(1-z_2)z_1}{(1-z_1)z_2} = \frac{(1-w_2)w_1}{(1-w_1)w_2} = \frac{(a-b)c}{(a-c)b},
\]

Then there is an isomorphism

\[
C \ni (t,s) \mapsto \left( \frac{(1-w_1)z_1}{(1-z_1)w_1} t + \frac{w_1-z_1}{(1-z_1)w_1}, \frac{1-w_1}{1-z_1} \sqrt{\frac{(1-w_2)z_1}{(1-z_2)w_1}} \cdot s \right) \in C',
\]

which maps the branched points 1, 1/z_1 and 1/z_2 of \( C \to P^1 \) to 1, 1/w_1 and 1/w_2 of \( C' \to P^1 \), respectively. We calculate the integral

\[
\int_0^1 \frac{dt'}{\sqrt{(1-t')(1-w_1 t')(1-w_2 t')}}
\]

by the substitution

\[
t' = \frac{(1-w_1)z_1}{(1-z_1)w_1} t + \frac{w_1-z_1}{(1-z_1)w_1}.
\]

Then we have

\[
\int_0^1 \frac{dt'}{\sqrt{(1-t')(1-w_1 t')(1-w_2 t')}} = \frac{\sqrt{ab} + \sqrt{ac}}{a} \int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}}, \quad t_0 = \frac{z_1-w_1}{(1-w_1)z_1}.
\]

Comparing to Proposition 5.3, we have to show that

\begin{equation}
(16) \quad \int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}} = 2 \int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}}.
\end{equation}

**Claim 5.5.** The equation (16) corresponds to the twice map via the Abel-Jacobi map sending \((1,0) \in C\) to the origin of the complex torus. More precisely, 
\((t_0, \sqrt{(1-t_0)(1-z_1 t_0)(1-z_2 t_0)}) \in C\) multiplied by 2 is \((0,1) \in C\).

Then we should remake a substitution considering the twice map. We define an isomorphism by

\[
\rho : C \to C(\lambda);
\]

\[
(t,s) \mapsto \left( \frac{1-z_1}{z_1} \frac{1}{t-1}, \frac{1}{z_1} \sqrt{\frac{1-z_1}{-z_2} \frac{s}{(t-1)^2}} \right),
\]

which maps \((1,0) \in C\) to the point at infinity of \( C(\lambda) \) (the isomorphism \( \rho' : C' \to C(\lambda) \) is also given similarly). Via \( \rho \) and the Abel-Jacobi map for
\( C(\lambda), (0, 1) \in C \) corresponds to the origin of the complex torus \( E(\tau) \). Let \( \psi \) be as Proposition 2.4 and \((t, s)\) be \( p^{-1} \circ \psi \circ p'(t', s')\), then we obtain

\[
(17) \quad t = 1 - 4 \cdot \frac{(1 - z_1)(1 - w_2)w_1(1 - t')(1 - w_1t') - (1 - w_2t')}{z_1(w_1w_2t'^2 - 2w_1w_2t' + w_1 + w_2 - 1)^2}.
\]

**Proof of Proposition 5.3.** We prove Proposition 5.3 by making the substitution (17). Then we have

\[
\frac{dt}{dt'} = -4 \cdot \frac{(1 - z_1)(1 - w_2)w_1}{z_1} \cdot \frac{(w_1w_2t'^2 - 2w_1t' + w_1 - w_2 + 1)(w_1w_2t'^2 - 2w_2t' - w_1 + w_2 + 1)}{(w_1w_2t'^2 - 2w_1w_2t' + w_1 + w_2 - 1)^3}.
\]

For simplicity, we set

\[ f_1(t') = w_1w_2t'^2 - 2w_1t' + w_1 - w_2 + 1, \quad f_2(t') = w_1w_2t'^2 - 2w_2t' - w_1 + w_2 + 1, \]
\[ f_3(t') = w_1w_2t'^2 - 2w_1w_2t' + w_1 + w_2 - 1. \]

It is easy to show that if \( 0 \leq t' \leq 1 \), then \( f_1(t') > 0, f_2(t') > 0 \) and \( f_3(t') < 0 \). This implies that \( dt/dt' > 0 \) when \( 0 \leq t' \leq 1 \). Since

\[ f_1(t')^2 - f_3(t')^2 = (2w_1w_2t'^2 - 2w_1(1 + w_2)t' + 2w_1)(2w_1(w_2 - 1)t' - 2w_2 + 2) = 4w_1(1 - w_2)(1 - t')(1 - w_1t')(1 - w_2t'), \]

we obtain

\[ 1 - z_1t = \frac{(1 - z_1)f_1(t')^2}{f_3(t')^3}, \quad 1 - z_2t = \frac{(1 - z_2)f_2(t')^2}{f_3(t')^3}, \]

(the latter is followed similarly by \((1 - z_1)(1 - w_2)w_1/z_1 = (1 - z_2)(1 - w_1)w_2/z_2\)). Therefore we conclude

\[
\int_0^1 \frac{dt}{\sqrt{(1 - t)(1 - z_1t)(1 - z_2t)}}
\]
\[
= \int_0^1 \frac{z_1}{4(1 - z_1)(1 - w_2)w_1} \frac{-f_3(t')}{\sqrt{(1 - t')(1 - w_1t')(1 - w_2t')}}
\]
\[
\cdot \frac{1}{\sqrt{1 - z_1}} \frac{-f_3(t')}{f_1(t')} \frac{1}{\sqrt{1 - z_2}} \frac{-f_3(t')}{f_2(t')} \left(-4 \cdot \frac{(1 - z_1)(1 - w_2)w_1}{z_1} \frac{f_1(t')f_2(t')}{f_3(t')^3}\right) dt'
\]
\[
= 2 \sqrt{(1 - w_2)w_1} \int_0^1 \frac{dt'}{(1 - z_2)z_1} \sqrt{(1 - t')(1 - w_1t')(1 - w_2t')}.
\]
This completes our proof of Proposition 5.3, since

\[
\frac{(1 - w_2)w_1}{(1 - z_2)z_1} = \frac{a^2}{(\sqrt{ab} + \sqrt{ac})^2} \frac{(\sqrt{ac} + \sqrt{bc})(\sqrt{ac} - \sqrt{bc})}{c(a - b)} = \left( \frac{a}{\sqrt{ab} + \sqrt{ac}} \right)^2. \quad \square
\]

5.3. Reduction formula

Using the substitution (17), we obtain a reduction formula from Lauricella’s \( F_D \) of five variables to Appell’s \( F_1 \). Lauricella’s hypergeometric function \( F_D \) of \( m \)-variables \( z_1, \ldots, z_m \) with parameters \( \alpha, (\beta_j) = (\beta_1, \ldots, \beta_m) \), \( \gamma \) is defined as

\[
F_D(\alpha, (\beta_j); \gamma; z_1, \ldots, z_m) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{(\alpha, \sum_{j=1}^{m} n_j) \prod_{j=1}^{m} (\beta_j, n_j) \prod_{j=1}^{m} z_j^{n_j}}{(\gamma, \sum_{j=1}^{m} n_j) \prod_{j=1}^{m} (n_j, 1, n_j)}
\]

where \( z_j \)’s satisfy \(|z_j| < 1\), \( \gamma \neq 0, -1, -2, \ldots \). Note that if we put \( m = 2 \), then

\[
F_D(\alpha, (\beta_1, \beta_2); \gamma; z_1, z_2) = F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2).
\]

The function \( F_D \) admits an integral representation:

\[
F_D(\alpha, (\beta_j); \gamma; z_1, \ldots, z_m) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^\alpha (1 - t)^{\gamma - \alpha} \left( \prod_{j=1}^{m} (1 - z_j t)^{-\beta_j} \right) \frac{dt}{t(1 - t)}.
\]

We consider the integral representation for \( F_1 \) by the substitution (17). Replace \( z_1 \) and \( z_2 \) with \( 1 - z_1^2 \) and \( 1 - z_2^2 \), respectively, then we have

\[
w_1 = 1 - \frac{z_1(1 + z_2)}{z_1 + z_2}, \quad w_2 = 1 - \frac{z_2(1 + z_1)}{z_1 + z_2}.
\]

Since calculations in Section 5.2 are valid after replacing them, and we can simplify the right-hand side of (17) as

\[
t = \frac{8z_1^2z_2^2}{(z_1 + z_2)^2} f_3(t) t' \left( 1 - \frac{1 - z_1}{2} t' \right) \left( 1 - \frac{1 - z_2}{2} t' \right) \left( 1 + \frac{(1 - z_1)(1 - z_2)}{2(z_1 + z_2)} t' \right),
\]

the following theorem is obtained.

**Theorem 5.6.** We have

\[
\left( \frac{z_1 + z_2}{2} \right)^p F_1 \left( p, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - z_1^2, 1 - z_2^2 \right) = F_D \left( p, \left( p - \frac{1}{2}, p - \frac{1}{2}, 1 - p, 1 - p, 1 - p \right), \frac{3}{2}, w_1, w_2, w_3, w_4, w_5 \right),
\]

\[
F_1 \left( p, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - z_1^2, 1 - z_2^2 \right) = F_D \left( p, \left( p - \frac{1}{2}, p - \frac{1}{2}, 1 - p, 1 - p, 1 - p \right), \frac{3}{2}, w_1, w_2, w_3, w_4, w_5 \right),
\]
\[ (w_1, w_2, w_3, w_4, w_5) = \left( 1 - \frac{z_1(1 + z_2)}{z_1 + z_2}, 1 - \frac{z_2(1 + z_1)}{z_1 + z_2}, \frac{1 - z_1}{2}, \frac{1 - z_2}{2}, -\frac{(1 - z_1)(1 - z_2)}{2(z_1 + z_2)} \right), \]

where \((z_1, z_2)\) is in a small neighborhood of \((1, 1)\).

This theorem is a generalization of Proposition 5.3, which is different from Theorem 1.1 in [4]. Indeed, put \(p = 1\), \(z_1 = \sqrt{b/a}\) and \(z_2 = \sqrt{c/a}\), then we obtain Proposition 5.3.

References


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