Higher Order Asymptotic Expansion for the Heat Equation with a Nonlinear Boundary Condition

By

Tatsuki KAWAKAMI

(Osaka Prefecture University, Japan)

Abstract. We consider the heat equation with a nonlinear boundary condition: (P) $\partial_t u = \Delta u$ in $\mathbf{R}_+^N \times (0, \infty)$, $\partial_\nu u = \kappa |u|^{p-1} u$ on $\partial \mathbf{R}_+^N \times (0, \infty)$, $u(x, 0) = \varphi(x)$ in $\mathbf{R}_+^N$, where $\mathbf{R}_+^N = \{x = (x', x_N) \in \mathbf{R}^N : x_N > 0\}$, $N \geq 2$, $\partial_t = \partial/\partial t$, $\partial_\nu = -\partial/\partial x_N$, $\kappa \in \mathbf{R}$, and $p > 1 + 1/N$. Let u be a solution of (P) satisfying $\sup_{t>0} (1+t)^{(N/2)(1-1/q)} [\|u(t)\|_{L^q(\mathbf{R}_+^N)} + t^{1/(2q)} \|u(t)\|_{L^q(\partial \mathbf{R}_+^N)}] < \infty$, $q \in [1, \infty]$. In this paper, under suitable assumptions of the initial function φ , we establish the method of obtaining higher order asymptotic expansions of the solution u as $t \rightarrow \infty$.

Key Words and Phrases. Nonlinear boundary condition, Asymptotic expansion, Large time behavior.

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1. Introduction

In this paper we consider the large time behavior of the solution of the heat equation in the half space $\mathbf{R}_+^N = \{x = (x', x_N) \in \mathbf{R}^N : x_N > 0\}$ with a nonlinear boundary condition,

$$(1.1) \quad \begin{cases} \partial_t u = \Delta u & \text{in } \mathbf{R}_+^N \times (0, \infty), \\ \partial_\nu u = \kappa |u|^{p-1} u & \text{on } \partial \mathbf{R}_+^N \times (0, \infty), \\ u(x, 0) = \varphi(x) & \text{in } \mathbf{R}_+^N, \end{cases}$$

where $N \geq 2$, $\partial_t = \partial/\partial t$, $\partial_\nu = -\partial/\partial x_N$, $p > 1$, and

$$(1.2) \quad \varphi \in X_K(\mathbf{R}_+^N) := L^\infty(\mathbf{R}_+^N) \cap \left\{ f \in L^1(\mathbf{R}_+^N) : \int_{\mathbf{R}_+^N} (1 + |x|)^K |f(x)| dx < \infty \right\}$$

for some $K \geq 0$. Nonlinear boundary value problem (1.1) can be physically interpreted as a nonlinear radiation law, and has been studied in many papers (see [1]–[5], [7], [10]–[12], and references therein). For the case $\kappa > 0$, $\varphi \geq 0$, and $\varphi \not\equiv 0$, it is well known that if $1 < p \leq 1 + 1/N$, then the problem (1.1) does not have any positive global in time solutions, and if $p > 1 + 1/N$, then, for some initial datum φ , the problem (1.1) has a positive global in time solution (see, for example, [3] and [5]). In particular, for the case $p > 1 + 1/N$, in [10],

the author of this paper proved that, if the initial datum φ satisfies (1.2) with $K = 0$ and $\|\varphi\|_{L^1(\mathbf{R}_+^N)}\|\varphi\|_{L^\infty(\mathbf{R}_+^N)}^{N(p-1)-1}$ is sufficiently small, then the solution u of (1.1) exists globally in time, and satisfies

$$(1.3) \quad \sup_{t>0} (1+t)^{(N/2)(1-1/q)} [\|u(t)\|_{L^q(\mathbf{R}_+^N)} + t^{1/(2q)} \|u(t)\|_{L^q(\partial\mathbf{R}_+^N)}] < \infty, \quad q \in [1, \infty].$$

Furthermore he proved that, if the solution u satisfies (1.3), then it holds that

$$(1.4) \quad \begin{cases} \int_{\mathbf{R}_+^N} u(x, t) dx \text{ converges to a constant } M \text{ as } t \rightarrow \infty \text{ and} \\ \lim_{t \rightarrow \infty} \|u(\cdot, t) - M\Gamma(\cdot, 0, 1+t)\|_{L^q(\mathbf{R}_+^N)} / \|\Gamma(\cdot, 0, 1+t)\|_{L^q(\mathbf{R}_+^N)} = 0 \\ \text{for any } q \in [1, \infty]. \end{cases}$$

Here $\Gamma = \Gamma(x, y, t)$ is the Green function for the heat equation on $\mathbf{R}_+^N$ with the homogeneous Neumann boundary condition, that is,

$$(1.5) \quad \Gamma(x, y, t) = G_N(x - y, t) + G_N(x - y_*, t), \quad x, y \in \mathbf{R}_+^N, t > 0,$$

where $y_* = (y', -y_N)$ for $y = (y', y_N) \in \mathbf{R}_+^N$ and G_N is the N -dimensional Gauss kernel, that is,

$$G_N(x, t) = (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

On the other hand, under condition (1.2) with $K = 0$, if $\kappa = 0$, then we can easily obtain (1.3) and (1.4), and if $\kappa < 0$ with $p > 1 + 1/N$, then, applying the comparison principle and modifying the argument in [10], we see that there exists a solution u satisfying (1.3) and (1.4).

Assertion (1.4) gives the large time behavior, in particular the first term of asymptotic expansions, of the solution u of (1.1) satisfying (1.3). The large time behavior of the solutions for nonlinear parabolic problem including the problem (1.1) is influenced by the behavior of the initial datum at the spatial infinity and the nonlinearity, and has been studied in many papers (see [6]–[11], [13], and references therein). In fact, for the problem

$$(1.6) \quad \begin{cases} \partial_t u = \Delta u & \text{in } \mathbf{R}_+^N \times (0, \infty), \\ \partial_\nu u = u^p & \text{on } \partial\mathbf{R}_+^N \times (0, \infty), \\ u(x, 0) = \varphi(x) \geq 0 & \text{in } \mathbf{R}_+^N \end{cases}$$

with $p > 1 + 1/N$, in [11], applying the entropy dissipation method, the author of this paper proved that, under condition (1.2) with $K = 2$, if the solution u of (1.6) satisfies (1.3), then, for any $q \in [1, \infty]$,

$$\begin{aligned}
(1.7) \quad & t^{(N/2)(1-1/q)} \|R(t)\|_{L^q(\mathbf{R}_+^N)} \\
&= \begin{cases} O(t^{-1/2}) + O(t^{-(N/4)(p-1-1/N)}) & \text{if } N(p-1-1/N) \neq 2, \\ O(t^{-1/2}(\log t)^{1/2}) & \text{if } N(p-1-1/N) = 2, \end{cases}
\end{aligned}$$

as $t \rightarrow \infty$, where

$$(1.8) \quad R(x, t) = u(x, t) - M\Gamma(x, 0, 1+t), \quad (x, t) \in \mathbf{R}_+^N \times (0, \infty).$$

Furthermore, in [7], the author of this paper and Ishige showed that, under condition $\varphi \in L^\infty(\mathbf{R}_+^N) \cap L^2(\mathbf{R}_+^N, e^{|x|^2/4} dx)$ and $(N-2)p < N$, if the solution u of (1.6) satisfies (1.3), then, for any $q \in [1, \infty]$,

$$(1.9) \quad t^{(N/2)(1-1/q)} \|R(t)\|_{L^q(\mathbf{R}_+^N)} = O(t^{-1/2}) + O(t^{-(N/2)(p-1-1/N)})$$

as $t \rightarrow \infty$. These results imply that, if the initial datum φ belongs to some suitable spaces like $X_2(\mathbf{R}_+^N)$ or $L^\infty(\mathbf{R}_+^N) \cap L^2(\mathbf{R}_+^N, e^{|x|^2/4} dx)$, then we can obtain more precise estimate on the difference between the solution and its asymptotic profiles than (1.4). However, from these results, we can not obtain the relationship between the decay rate of $\|R(t)\|_{L^q(\mathbf{R}_+^N)}$ and the behavior of the initial datum φ at the spatial infinity. Actually, although the initial conditions are different, for sufficiently large p , these two inequalities (1.7) and (1.9) yield the same estimate, that is, $t^{(N/2)(1-1/q)} \|R(t)\|_{L^q(\mathbf{R}_+^N)} = O(t^{-1/2})$ as $t \rightarrow \infty$. Furthermore, as far as we know, there are no results treating higher order asymptotic expansions of the solution of (1.1) even if $\varphi \in C_0^\infty(\mathbf{R}_+^N)$.

Recently, for the Cauchy problem of the semilinear heat equation

$$(1.10) \quad \partial_t u = \Delta u + \kappa |u|^{p-1} u \quad \text{in } \mathbf{R}^N \times (0, \infty), \quad u(x, 0) = \varphi(x) \quad \text{in } \mathbf{R}^N$$

with $\kappa \in \mathbf{R}$ and $p > 1 + 2/N$, in [9], improving the argument in [6], the author of this paper and Ishige established the useful method which derives higher order asymptotic expansions of the solutions for the Cauchy problem (1.10). The argument in [9] is applicable to the large class of the nonlinear parabolic equations in the whole space (see [8]). However it is not applicable directly to nonlinear boundary value problem (1.1) because the integral equation has a different kernel and the nonlinear term appears on the boundary (see Definition 2.1). In this paper, as an extension of [9], under condition (1.2) and $p > 1 + 1/N$, we consider the initial-boundary value problem (1.1), and study the large time behavior of the solutions satisfying (1.3). In particular, modifying the arguments in [8] and [9], we give higher order asymptotic expansions of the solution u of (1.1). Throughout this paper we write

$$A_p := \frac{N}{2}(p-1)$$

for simplicity. We recall that $A_p > 1/2$ under condition $p > 1 + 1/N$.

The rest of this paper is organized as follows. In Section 2 we give some notation and introduce some operators. Furthermore we state the main results of this paper. In Section 3 we give some preliminaries. Section 4 is devoted to the proof of main results.

2. Notation and main results

In this section we give some notation and the definition of the solution of (1.1). Furthermore we introduce some operators $P_K(t)$ and $Q_K(t)$, and state the main results of this paper.

2.1. Notation

We first introduce some notation. Let $N_0 = N \cup \{0\}$ and $\mathcal{O}$ be the set of odd numbers. For any $k \in \mathbf{R}$, let $[k]$ be an integer such that $k - 1 < [k] \leq k$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in N_0^N$, we put

$$|\alpha| := \sum_{i=1}^N |\alpha_i|, \quad \alpha! := \prod_{i=1}^N \alpha_i!, \quad x^\alpha := \prod_{i=1}^N x_i^{\alpha_i}, \quad \partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}},$$

$$J(\alpha) := \{\rho = (\rho_1, \dots, \rho_N) \in N_0^N \setminus \{\alpha\} : \rho_i \leq \alpha_i \text{ for all } i = 1, \dots, N\},$$

$$g_\alpha(x, t) := \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha \Gamma)(x, 0, 1 + t),$$

where $\Gamma(x, y, t)$ is the function given in (1.5). In particular, we write $g(x, t) = g_0(x, t)$ for simplicity. Furthermore, for any multi-index $\tilde{\alpha} \in N_0^{N-1}$, we put

$$\tilde{g}_{\tilde{\alpha}}(x', t) := \frac{(-1)^{|\tilde{\alpha}|}}{\tilde{\alpha}!} (\partial_{x'}^{\tilde{\alpha}} G_{N-1})(x', 1 + t).$$

In particular, we write $\tilde{g}(x, t) = \tilde{g}_0(x, t)$ for simplicity. We denote by $S(t)\varphi$ the unique bounded solution of the heat equation on $\mathbf{R}_+^N$ with the homogeneous Neumann boundary condition and the initial datum φ , that is,

$$(2.1) \quad (S(t)\varphi)(x) := \int_{\mathbf{R}_+^N} \Gamma(x, y, t) \varphi(y) dy, \quad x \in \mathbf{R}_+^N, t > 0.$$

Furthermore we denote by $e^{tA'}\psi$ the unique bounded solution of the heat equation on $\mathbf{R}^{N-1}$ with the initial datum ψ , that is,

$$(2.2) \quad (e^{tA'}\psi)(x') := \int_{\mathbf{R}^{N-1}} G_{N-1}(x' - y', t) \psi(y') dy', \quad x' \in \mathbf{R}^{N-1}, t > 0.$$

For any two nonnegative functions f_1 and f_2 defined in a subset D of $[0, \infty)$, we say $f_1(t) \preceq f_2(t)$ for all $t \in D$ if there exists a positive constant C such that $f_1(t) \leq Cf_2(t)$ for all $t \in D$. For any $m \geq 0$, we denote by L_m^1 and $L_{m, \mathbf{R}^{N-1}}^1$ the function spaces $L^1(\mathbf{R}_+^N, (1 + |x|)^m dx)$ and $L^1(\mathbf{R}^{N-1}, (1 + |x'|)^m dx')$ respectively. In what follows, we write

$$\begin{aligned} \|\cdot\|_q &= \|\cdot\|_{L^q(\mathbf{R}_+^N)}, & \|\cdot\|_m &= \|\cdot\|_{L^1(\mathbf{R}_+^N, (1+|x|)^m dx)}, \\ \|\cdot\|_{q, \partial\mathbf{R}_+^N} &= \|\cdot\|_{L^q(\partial\mathbf{R}_+^N)}, & \|\cdot\|_{m, \partial\mathbf{R}_+^N} &= \|\cdot\|_{L^1(\partial\mathbf{R}_+^N, (1+|x|)^m d\sigma)}, \\ \|\cdot\|_{q, \mathbf{R}^{N-1}} &= \|\cdot\|_{L^q(\mathbf{R}^{N-1})}, & \|\cdot\|_{m, \mathbf{R}^{N-1}} &= \|\cdot\|_{L^1(\mathbf{R}^{N-1}, (1+|x'|)^m dx')}, \end{aligned}$$

for simplicity, where $q \in [1, \infty]$ and $m \geq 0$.

Next we give the definition of the solution of (1.1).

Definition 2.1. Let $\varphi \in X_0(\mathbf{R}_+^N)$. Then the function $u \in C(\overline{\mathbf{R}_+^N} \times (0, \infty)) \cap L^\infty(0, \infty : L^\infty(\mathbf{R}_+^N))$ is said to be a solution of (1.1) if

$$\begin{aligned} (2.3) \quad u(x, t) &= \int_{\mathbf{R}_+^N} \Gamma(x, y, t) \varphi(y) dy \\ &\quad + \kappa \int_0^t \int_{\partial\mathbf{R}_+^N} \Gamma(x, y, t-s) |u(y, s)|^{p-1} u(y, s) d\sigma_y ds \end{aligned}$$

holds for all $(x, t) \in \mathbf{R}_+^N \times (0, \infty)$. Here Γ is the Green function given by (1.5) and $d\sigma_y$ is the $(N-1)$ -dimensional Lebesgue measure on $\partial\mathbf{R}_+^N = \mathbf{R}^{N-1}$.

It is known that, under the above definition, for any initial datum $\varphi \in X_0(\mathbf{R}_+^N)$, the problem (1.1) has a unique classical solution (see, for example, [7]). Furthermore, combining (1.5), and (2.1)–(2.3) we have

$$u(x, t) = (S(t)\varphi)(x) + 2\kappa \int_0^t G_1(x_N, t-s) (e^{(t-s)A'} \gamma[|u(s)|^{p-1} u(s)]) ds$$

for all $(x, t) \in \mathbf{R}_+^N \times (0, \infty)$, where $\gamma[f](x') = f(x', 0)$ for all $x' \in \mathbf{R}^{N-1}$.

The end of this subsection we introduce some operators $P_K(t)$ and $Q_K(t)$. Following [6] and [9], for any $k \geq 0$ and $t > 0$, we introduce a linear operator $P_k(t)$ on L_k^1 by

$$(2.4) \quad [P_k(t)f_1](x) := f_1(x) - \sum_{|\alpha| \leq k, \alpha_N \notin \mathcal{O}} M_\alpha(f_1, t) g_\alpha(x, t),$$

where $f_1 \in L_k^1$ and $M_\alpha(f, t)$ is a constant defined inductively (in α) by

$$(2.5) \quad M_\alpha(f_1, t) := \int_{\mathbf{R}_+^N} x^\alpha f_1(x) dx - \sum_{\rho \in J(\alpha), \rho_N \notin \mathcal{O}} M_\rho(f_1, t) \int_{\mathbf{R}_+^N} x^\rho g_\rho(x, t) dx.$$

Furthermore, for any $k \geq 0$ and $t > 0$, we put

$$(2.6) \quad [Q_k(t)f_2](x') := f_2(x') - \sum_{|\tilde{\alpha}| \leq k} \tilde{M}_{\tilde{\alpha}}(f_2, t) \tilde{g}_{\tilde{\alpha}}(x', t),$$

where $f_2 \in L^1_{k, \mathbf{R}^{N-1}}$ and $\tilde{M}_{\tilde{\alpha}}(f, t)$ is a constant defined inductively (in $\tilde{\alpha}$) by

$$(2.7) \quad \tilde{M}_{\tilde{\alpha}}(f_2, t) := \int_{\mathbf{R}^{N-1}} (x')^{\tilde{\alpha}} f_2(x') dx' - \sum_{\rho \in J(\tilde{\alpha})} \tilde{M}_{\rho}(f_2, t) \int_{\mathbf{R}^{N-1}} (x')^{\tilde{\alpha}} \tilde{g}_{\rho}(x', t) dx'.$$

Then, it is easy to see that the operators $P_k(t)$ and $Q_k(t)$ satisfy

$$(2.8) \quad \int_{\mathbf{R}_+^N} x^{\alpha} [P_k(t)f_1](x) dx = 0, \quad |\alpha| \leq k \text{ with } \alpha_N \notin \mathcal{O},$$

$$(2.9) \quad \int_{\mathbf{R}^{N-1}} (x')^{\tilde{\alpha}} [Q_k(t)f_2](x') dx' = 0, \quad |\tilde{\alpha}| \leq k,$$

for all $t > 0$ (see [6]). This operators are key of our proof, in particular (2.8) and (2.9) are crucial properties in our analysis. Furthermore, if $\varphi \in X_K(\mathbf{R}_+^N)$ with $K \geq 0$, then, applying the operator $P_K(t)$ to $S(t)\varphi$, we have

$$\begin{aligned} P_K(t)S(t)\varphi &= S(t)\varphi - \sum_{|\alpha| \leq K, \alpha_N \notin \mathcal{O}} M_{\alpha}(S(t)\varphi, t) g_{\alpha}(x, t) \\ &= S(t)\varphi - \sum_{|\alpha| \leq K, \alpha_N \notin \mathcal{O}} M_{\alpha}(\varphi, 0) g_{\alpha}(x, t) = S(t)[P_K(0)\varphi] \end{aligned}$$

for all $t > 0$ (see Lemma 3.2 (ii)). This together with (2.8) yields

$$(2.10) \quad \begin{aligned} & t^{(N/2)(1-1/q)} \left\| S(t)\varphi - \sum_{|\alpha| \leq K, \alpha_N \notin \mathcal{O}} M_{\alpha}(\varphi, 0) g_{\alpha}(t) \right\|_q \\ &= \begin{cases} o(t^{-K/2}) & \text{if } K = [K], \\ O(t^{-K/2}) & \text{if } K > [K], \end{cases} \end{aligned}$$

as $t \rightarrow \infty$, for any $q \in [1, \infty]$, which gives higher order asymptotic expansion of $S(t)\varphi$. See also Proposition 3.1.

2.2. Main results

In this subsection we give the main theorem, which yields higher order asymptotic expansions of the solutions of (1.1) satisfying (1.3). Furthermore, as

a corollary, we give the relationship between the decay rate of $\|R(t)\|_q$ and the integrability of the initial datum φ .

Before treating our main theorem, modifying [9], we introduce the functions $U_n = U_n(x, t)$ defined inductively by

$$(2.11) \quad \begin{aligned} U_0(x, t) &:= \sum_{|z| \leq K, \alpha_N \notin \mathcal{O}} \left(M_\alpha(u(t), t) - \int_0^t N_\alpha(\gamma[f(s)], s) ds \right) g_\alpha(x, t) \\ &\quad + 2 \sum_{|\tilde{\alpha}| \leq K} \int_0^t G_1(x_N, t-s) \tilde{M}_{\tilde{\alpha}}(\gamma[f(s)], s) ds \tilde{g}_{\tilde{\alpha}}(x', t), \\ U_n(x, t) &:= U_0(x, t) + 2 \int_0^t G_1(x_N, t-s) (e^{(t-s)A'} \mathcal{Q}_K(s) \gamma[F_{n-1}(s)]) ds, \end{aligned}$$

for $n = 1, 2, \dots$, where $f(x, t) = \kappa |u(x, t)|^{p-1} u(x, t)$, $F_{n-1}(x, t) = \kappa |U_{n-1}(x, t)|^{p-1} U_{n-1}(x, t)$, and

$$(2.12) \quad \begin{aligned} N_\alpha(\gamma[f(t)], t) &:= \int_{\partial \mathbf{R}_+^N} x^\alpha \gamma[f(t)](x') d\sigma \\ &\quad - \sum_{\rho \in J(\alpha), \rho_N \notin \mathcal{O}} N_\rho(\gamma[f(t)], t) \int_{\mathbf{R}_+^N} x^\alpha g_\rho(x, t) dx. \end{aligned}$$

Now we are ready to state the main theorem of this paper.

Theorem 2.1. *Consider the initial-boundary value problem (1.1) under condition (1.2) for some $K \geq 0$ and $A_p > 1/2$. Let u be a unique solution of (1.1) satisfying (1.3), and let $n = 0, 1, 2, \dots$. Then there holds the following:*

(i) *The function U_n defined by (2.11) satisfies*

$$(2.13) \quad \sup_{t>0} (1+t)^{(N/2)(1-1/q)} [\|U_n(t)\|_q + t^{1/(2q)} \|U_n(t)\|_{q, \partial \mathbf{R}_+^N}] < \infty,$$

$$(2.14) \quad \sup_{t>0} (1+t)^{-l/2} [\|U_n(t)\|_l + t^{1/2} \|U_n(t)\|_{l, \partial \mathbf{R}_+^N}] < \infty,$$

for any $q \in [1, \infty]$ and $l \in [0, K]$;

(ii) *For any $q \in [1, \infty]$,*

$$(2.15) \quad \begin{aligned} &t^{(N/2)(1-1/q)} [\|u(t) - U_n(t)\|_q + t^{1/(2q)} \|u(t) - U_n(t)\|_{q, \partial \mathbf{R}_+^N}] \\ &\preceq \begin{cases} (1+t)^{-K/2} + (1+t)^{-(n+1)(A_p-1/2)} & \text{if } (n+1)(2A_p-1) \neq K, \\ (1+t)^{-K/2} \log(2+t) & \text{if } (n+1)(2A_p-1) = K, \end{cases} \end{aligned}$$

for all $t > 0$;

(iii) If $(n+1)(2A_p-1) > K$, then, for any $q \in [1, \infty]$,

$$(2.16) \quad t^{(N/2)(1-1/q)} [\|u(t) - U_n(t)\|_q + t^{1/(2q)} \|u(t) - U_n(t)\|_{q, \partial R_+^N}]$$

$$= \begin{cases} o(t^{-K/2}) & \text{if } K = [K], \\ O(t^{-K/2}) & \text{if } K > [K], \end{cases}$$

as $t \rightarrow \infty$;

(iv) For any $l \in [0, K]$ and $\sigma > 0$,

$$(2.17) \quad (1+t)^{-l/2} [\|u(t) - U_n(t)\|_l + t^{1/2} \|u(t) - U_n(t)\|_{l, \partial R_+^N}]$$

$$\preceq (1+t)^{-K/2+\sigma} + (1+t)^{-(n+1)(A_p-1/2)}, \quad t > 0.$$

By Theorem 2.1 we see that the functions U_0 and $\{U_n\}_{n=1}^\infty$ give a linear approximation and nonlinear approximations to the solution u , respectively.

Remark 2.1. (i) Since, for any multi-index $\alpha \in N_0^N$, it holds that

$$(2.18) \quad \frac{1}{\alpha!} \partial_y^\alpha \Gamma(x, y, 1+t) \Big|_{y=0} = \begin{cases} g_\alpha(x, t) & \text{if } \alpha_N \notin \mathcal{O}, \\ 0 & \text{if } \alpha_N \in \mathcal{O}, \end{cases}$$

the first term of U_0 is represented as a linear combination of $\{\partial_y^\alpha \Gamma(x, y, 1+t)|_{y=0}\}_{|\alpha| \leq K}$, and plays a role of projection of the solution onto the space spanned by $\{\partial_y^\alpha \Gamma(x, y, 1+t)|_{y=0}\}_{|\alpha| \leq K}$.

(ii) If $(n+1)(2A_p-1) > K$, then the decay estimate of $\|u(t) - U_n(t)\|_q$ as $t \rightarrow \infty$ in (2.16) is the same as in (2.10).

(iii) U_n ($n = 1, 2, \dots$) gives the $([K] + 2)$ -th order asymptotic expansion of the solution u and is determined systematically by the function U_0 .

As a corollary of Theorem 2.1, we can obtain the following result, which gives a precise estimate on the difference between the solution and its first asymptotic profiles. Compare with (1.7) and (1.9).

Corollary 2.1. Assume the same conditions as in Theorem 2.1 and $K > 0$. Let R be the function defined by (1.8). Then there holds the following:

(i) If $K \in (0, 1]$, then, for any $q \in [1, \infty]$,

$$(2.19) \quad t^{(N/2)(1-1/q)} \|R(t)\|_q$$

$$= \begin{cases} O(t^{-K/2}) + O(t^{-(A_p-1/2)}) & \text{if } 2A_p - 1 \neq K, \\ O(t^{-K/2} \log t) & \text{if } 2A_p - 1 = K, \end{cases} \quad \text{as } t \rightarrow \infty;$$

(ii) If $K > 1$, then, for any $q \in [1, \infty]$,

$$(2.20) \quad t^{(N/2)(1-1/q)} \|R(t)\|_q = O(t^{-1/2}) + O(t^{-(A_p-1/2)}) \quad \text{as } t \rightarrow \infty.$$

Remark 2.2. Consider the initial-boundary value problem (1.1) with the nonlinear term $\kappa|u|^{p-1}u$ replaced by $F(x, t, u)$ satisfying

$$|F(x, t, u(x, t))| \leq C(1+t)^{-A}|u(x, t)|, \quad (x, t) \in \overline{\mathbf{R}}_+^N \times (0, \infty),$$

for some constant $C > 0$ and $A > 1/2$. Then, modifying the argument in Subsection 3.2 (see also [8] and [10]), we can prove that there exists a solution satisfying (1.3). Furthermore if the nonlinear term F satisfies

$$\left\{ \begin{array}{l} \text{(i)} \quad F(x, t, 0) = 0 \text{ for all } (x, t) \in \overline{\mathbf{R}}_+^N \times (0, \infty); \\ \text{(ii)} \quad \text{For any } v_1 \text{ and } v_2 \in \mathcal{S}, \text{ there exists a constant } C \text{ such that} \\ \quad |F(x, t, v_1(x, t)) - F(x, t, v_2(x, t))| \leq C(1+t)^{-A}|v_1(x, t) - v_2(x, t)| \\ \quad \text{for all } (x, t) \in \overline{\mathbf{R}}_+^N \times (0, \infty), \end{array} \right.$$

then there holds the conclusion of Theorem 2.1. Here

$$\begin{aligned} \mathcal{S} := & \left\{ v \in C(\overline{\mathbf{R}}_+^N \times (0, \infty)) \cap L^\infty(0, \infty : L^\infty(\mathbf{R}_+^N)) : \sup_{t>0} (1+t)^{(N/2)(1-1/q)} \right. \\ & \left. \times [\|v(t)\|_q + t^{1/(2q)}\|v(t)\|_{q, \partial\mathbf{R}_+^N}] < \infty \text{ for any } q \in [1, \infty] \right\}. \end{aligned}$$

3. Preliminary results

In this section we recall some properties on the solution of the heat equation and operators $P_K(t)$ and $Q_K(t)$. Furthermore we give some weighted L^1 estimates for the solution u of (1.1) satisfying (1.3).

3.1. Preliminaries

In this subsection we recall some preliminary results of $S(t)\varphi$ and $e^{t\Delta'}\psi$ given in (2.1) and (2.2) respectively. Furthermore we give some preliminary results on the operators $P_K(t)$ and $Q_K(t)$ defined by (2.4) and (2.6).

For any $\alpha \in \mathcal{N}_0^N$ and $j \in \mathcal{N}_0$, then there exists a constant C_1 such that

$$(3.1) \quad |\partial_t^j \partial_x^\alpha G_N(x, t)| \leq C_1 t^{-(N+|\alpha|+2j)/2} \left[1 + \left(\frac{|x|}{t^{1/2}} \right) \right]^{| \alpha | + 2j} \exp\left(-\frac{|x|^2}{4t}\right)$$

for all $(x, t) \in \mathbf{R}^N \times (0, \infty)$. This inequality together with (1.5) yields the inequalities

$$(3.2) \quad \begin{aligned} \|g_\alpha(t)\|_q &\preceq (1+t)^{-(N/2)(1-1/q)-|\alpha|/2}, \\ \int_{\mathbf{R}_+^N} |x|^m |g_\alpha(x, t)| dx &\preceq (1+t)^{(m-|\alpha|)/2}, \end{aligned}$$

$$\begin{aligned}
\|g_\alpha(t)\|_{q, \partial \mathbf{R}_+^N} &\preceq (1+t)^{-(N/2)(1-1/q)-1/(2q)-|\alpha|/2}, \\
\int_{\partial \mathbf{R}_+^N} |x|^m |g_\alpha(x, t)| d\sigma &\preceq (1+t)^{(m-|\alpha|-1)/2}, \\
\|\tilde{g}_{\tilde{\alpha}}(t)\|_{q, \mathbf{R}^{N-1}} &\preceq (1+t)^{-((N-1)/2)(1-1/q)-|\tilde{\alpha}|/2}, \\
\int_{\mathbf{R}^{N-1}} |x'|^m |\tilde{g}_{\tilde{\alpha}}(x', t)| dx' &\preceq (1+t)^{(m-|\tilde{\alpha}|)/2},
\end{aligned}$$

for all $t > 0$ and any $q \in [1, \infty]$ and $m \geq 0$. Furthermore, by (2.1) and (2.2) we have the following (see also Lemma 2.1 in [6]):

(S1) There exists a constant C_2 such that, for any $1 \leq r \leq q \leq \infty$,

$$\|S(t)\varphi\|_q \leq C_2 t^{-(N/2)(1/r-1/q)} \|\varphi\|_r, \quad t > 0.$$

Furthermore, for any $l \geq 0$ and $\delta > 0$, there exists a constant C_3 such that

$$\int_{\mathbf{R}_+^N} |x|^l |(S(t)\varphi)(x)| dx \leq (1+\delta) \int_{\mathbf{R}_+^N} |x|^l |\varphi(x)| dx + C_3 t^{l/2} \int_{\mathbf{R}_+^N} |\varphi(x)| dx, \quad t > 0.$$

This inequality implies that

$$\| \|S(t)\varphi\|_l \leq (1+\delta) \|\varphi\|_l + C_4 (1+t^{l/2}) \|\varphi\|_1, \quad t > 0,$$

for some constant C_4 ;

(S2) There exists a constant C_5 such that, for any $q \in [1, \infty]$,

$$\|S(t)\varphi\|_{q, \partial \mathbf{R}_+^N} \leq C_5 t^{-(N/2)(1-1/q)-1/(2q)} \|\varphi\|_1, \quad t > 0.$$

Furthermore, for any $l \geq 0$ and $\delta > 0$, there exists a constant C_6 such that

$$\begin{aligned}
\int_{\mathbf{R}^{N-1}} |x'|^l |(S(t)\varphi)(x', 0)| dx &\leq (1+\delta) t^{-1/2} \int_{\mathbf{R}_+^N} |x|^l |\varphi(x)| dx \\
&\quad + C_6 t^{(l-1)/2} \int_{\mathbf{R}_+^N} |\varphi(x)| dx, \quad t > 0.
\end{aligned}$$

This inequality implies that

$$\| \|S(t)\varphi\|_{l, \partial \mathbf{R}_+^N} \leq (1+\delta) t^{-1/2} \|\varphi\|_l + C_7 t^{-1/2} (1+t^{l/2}) \|\varphi\|_1, \quad t > 0,$$

for some constant C_7 ;

(S3) There exists a constant C_8 such that, for any $1 \leq r \leq q \leq \infty$,

$$\|e^{tA'} \psi\|_{q, \mathbf{R}^{N-1}} \leq C_8 t^{-(N-1)/2(1/r-1/q)} \|\psi\|_{r, \mathbf{R}^{N-1}}, \quad t > 0.$$

Furthermore, for any $l \geq 0$ and $\delta > 0$, there exists a constant C_9 such that

$$\begin{aligned} \int_{\mathbf{R}^{N-1}} |x'|^l |(e^{tD'} \psi)(x')| dx' &\leq (1 + \delta) \int_{\mathbf{R}^{N-1}} |x'|^l |\psi(x')| dx' \\ &+ C_9 t^{l/2} \int_{\mathbf{R}^{N-1}} |\psi(x')| dx', \quad t > 0. \end{aligned}$$

This inequality implies that

$$\|e^{tD'} \psi\|_{l, \mathbf{R}^{N-1}} \leq (1 + \delta) \|\psi\|_{l, \mathbf{R}^{N-1}} + C_{10} (1 + t^{l/2}) \|\psi\|_{1, \mathbf{R}^{N-1}}, \quad t > 0,$$

for some constant C_{10} .

Applying a similar argument as in the proof of [6, Lemmata 2.2 and 2.5] with (2.18), we have the following lemma.

Lemma 3.1. *Let $k \geq 0$. Assume $\varphi \in L_k^1$ and $\psi \in L_{k, \mathbf{R}^{N-1}}^1$. Then there holds the following:*

(i) *If it holds that*

$$\int_{\mathbf{R}_+^N} x^\alpha \varphi(x) dx = 0, \quad |\alpha| \leq k \text{ with } \alpha_N \notin \mathcal{O},$$

then, for any $l \in [0, k - [k]]$, there exists a constant C_1 such that

$$\int_{\mathbf{R}_+^N} |x|^l |(S(t)\varphi)(x)| dx \leq C_1 t^{-(k-l)/2} \int_{\mathbf{R}_+^N} |x|^k |\varphi(x)| dx, \quad t > 0.$$

In particular, if $k = [k]$, then $\lim_{t \rightarrow \infty} t^{k/2} \|S(t)\varphi\|_1 = 0$.

(ii) *If it holds that*

$$\int_{\mathbf{R}^{N-1}} (x')^{\tilde{\alpha}} \psi(x') dx' = 0, \quad |\tilde{\alpha}| \leq k,$$

then, for any $l \in [0, k - [k]]$, there exists a constant C_3 such that

$$\int_{\mathbf{R}^{N-1}} |x'|^l |(e^{tD'} \psi)(x')| dx' \leq C_3 t^{-(k-l)/2} \int_{\mathbf{R}^{N-1}} |x'|^k |\psi(x')| dx', \quad t > 0.$$

In particular, if $k = [k]$, then $\lim_{t \rightarrow \infty} t^{k/2} \|e^{tD'} \psi\|_{1, \mathbf{R}^{N-1}} = 0$.

Next we give the following lemma on the operator $P_k(t)$ (see also [6, Lemma 2.3]).

Lemma 3.2. *Let $k \geq 0$ and $f = f(x, t) \in C(\overline{\mathbf{R}_+^N} \times (0, \infty))$ be a bounded function such that $\sup_{0 < \tau < t} \tau^{1/2} \|f(\tau)\|_{k, \partial \mathbf{R}_+^N} < \infty$ for all $t > 0$. Let $\varphi \in X_k(\mathbf{R}_+^N)$ and u be a solution of the initial-boundary value problem*

$$\partial_t u = \Delta u \quad \text{in } \mathbf{R}_+^N \times (0, \infty), \quad \partial_\nu u = f(x, t) \quad \text{on } \partial \mathbf{R}_+^N \times (0, \infty),$$

$$u(x, 0) = \varphi(x) \quad \text{in } \mathbf{R}_+^N.$$

Then there holds the following:

(i) The function $v = [P_k(t)u(t)](x)$ satisfies

$$(3.3) \quad \begin{cases} \partial_t v = \Delta v - \sum_{|\alpha| \leq k, \alpha_N \notin \mathcal{O}} N_\alpha(\gamma[f(t)], t) g_\alpha(x, t) & \text{in } \mathbf{R}_+^N \times (0, \infty), \\ \partial_\nu v = f(x, t) & \text{on } \partial \mathbf{R}_+^N \times (0, \infty), \\ v(x, 0) = (P_k(0)u(0))(x) & \text{in } \mathbf{R}_+^N; \end{cases}$$

(ii) For any $|\alpha| \leq K$ with $\alpha_N \notin \mathcal{O}$,

$$(3.4) \quad \frac{d}{dt} M_\alpha(u(t), t) = N_\alpha(\gamma[f(t)], t), \quad t > 0.$$

Proof. Following the idea of the proof of Lemma 2.3 in [6], we prove this lemma. We first prove (3.4). For $|\alpha| \leq 1$ with $\alpha_N = 0$, by (2.5) we have

$$M_\alpha(u(t), t) = \int_{\mathbf{R}_+^N} (x')^\alpha u(x, t) dx.$$

Since $\nabla(x')^\alpha \cdot \nu = 0$ for $\nu = (0, \dots, 0, 1)$, we obtain

$$\begin{aligned} \frac{d}{dt} M_{\alpha, 2\lambda}(u(t), t) &= \int_{\mathbf{R}_+^N} (x')^\alpha u_t(x, t) dx = \int_{\mathbf{R}_+^N} (x')^\alpha \Delta u(x, t) dx \\ &= \int_{\partial \mathbf{R}_+^N} (x')^\alpha f(x, t) d\sigma - \int_{\mathbf{R}_+^N} \nabla(x')^\alpha \cdot \nabla u(x, t) dx = N_\alpha(\gamma[f(t)], t). \end{aligned}$$

So we have (3.4) for the cases $|\alpha| \leq 1$ with $\alpha_N = 0$.

Let $k \geq 2$ and $n \in \{0, \dots, [k] - 1\}$, and assume

$$(3.5) \quad \frac{d}{dt} M_\alpha(u(t), t) = N_\alpha(\gamma[f(t)], t), \quad |\alpha| \leq n \text{ with } \alpha_N \notin \mathcal{O}.$$

We recall that, by (2.4) and (2.8), for any $|\alpha| \leq k$ with $\alpha_N \notin \mathcal{O}$ and $\zeta \in X_k(\mathbf{R}_+^N)$, it holds that

$$\begin{aligned} &\int_{\mathbf{R}_+^N} (\Delta x^\alpha) \zeta(x) dx - \sum_{\rho \in J(\alpha), \rho_N \notin \mathcal{O}} M_\rho(\zeta, t) \int_{\mathbf{R}_+^N} (\Delta x^\alpha) g_\rho(t) dx \\ &= \int_{\mathbf{R}_+^N} (\Delta x^\alpha) \left[\zeta(x) - \sum_{|\rho| \leq |\alpha| - 1, \rho_N \notin \mathcal{O}} M_\rho(\zeta, t) g_\rho(t) \right] dx = 0, \quad t > 0. \end{aligned}$$

This together with (2.5), (2.12), and (3.5) implies that, for $|\alpha| = n + 1$ with $\alpha_N \notin \mathcal{O}$,

$$\begin{aligned}
\frac{d}{dt} M_\alpha(u(t), t) &= \int_{\mathbf{R}_+^N} x^\alpha u_t(x, t) dx - \frac{d}{dt} \sum_{\rho \in J(\alpha), \rho_N \notin \mathcal{O}} M_\rho(u(t), t) \int_{\mathbf{R}_+^N} x^\alpha g_\rho(x, t) dx \\
&= \int_{\partial \mathbf{R}_+^N} x^\alpha f(x, t) d\sigma - \sum_{\rho \in J(\alpha), \rho_N \notin \mathcal{O}} \frac{d}{dt} M_\rho(u(t), t) \int_{\mathbf{R}_+^N} x^\alpha g_\rho(x, t) dx \\
&\quad + \int_{\mathbf{R}_+^N} (\Delta x^\alpha) u(x, t) dx - 2 \sum_{\rho \in J(\alpha), \rho_N \notin \mathcal{O}} M_\rho(u(t), t) \int_{\mathbf{R}_+^N} (\Delta x^\alpha) g_\rho(x, t) dx \\
&= \int_{\partial \mathbf{R}_+^N} x^\alpha f(x, t) d\sigma - \sum_{\rho \in J(\alpha), \rho_N \notin \mathcal{O}} N_\rho(\gamma[f(t)], t) \int_{\mathbf{R}_+^N} x^\alpha g_\rho(x, t) dx \\
&\quad + \int_{\mathbf{R}_+^N} (\Delta x)^\alpha \left(u(x, t) - \sum_{|\rho| \leq n, \rho_N \notin \mathcal{O}} M_\rho(u(t), t) g_\rho(x, t) \right) dx \\
&= N_\alpha(\gamma[f(t)], t)
\end{aligned}$$

for all $t > 0$. Therefore, by the inductive argument with respect to n we obtain (3.4) for all $|\alpha| \leq k$ with $\alpha_N \notin \mathcal{O}$.

Next we prove (3.3). By (2.4) and (3.4) we have

$$\partial_t v - \Delta v = - \sum_{|\alpha| \leq k, \alpha_N \notin \mathcal{O}} \frac{d}{dt} M_\alpha(u(t), t) g_\alpha(x, t) = - \sum_{|\alpha| \leq k, \alpha_N \notin \mathcal{O}} N_\alpha(\gamma[f(t)], t) g_\alpha(x, t)$$

for all $(x, t) \in \mathbf{R}_+^N \times (0, \infty)$. Furthermore we put $h(x) = e^{-cx^2}$ for some positive constant c . Then, for any $n \in \mathbf{N}$, we have

$$h^{(n+1)}(x) = -2cnh^{(n-1)}(x) - 2cxh^{(n)}(x), \quad x \in \mathbf{R}.$$

Therefore, since $\partial_{x_N} g_\alpha(x, t) = 0$ on $\partial \mathbf{R}_+^N \times (0, \infty)$ for any multi-index $\alpha \in \mathbf{N}_0^N$ with $\alpha_N = 0$, we obtain

$$\partial_v v = f(x, t) - \sum_{|\alpha| \leq k, \alpha_N \notin \mathcal{O}} M_\alpha(u(t), t) (-\partial_{x_N}) g_\alpha(x, t) = f(x, t)$$

for all $(x, t) \in \partial \mathbf{R}_+^N \times (0, \infty)$. Thus we have (3.3), and the proof of Lemma 3.2 is complete. $\square$

By Lemmata 3.1 and 3.2 we obtain the asymptotic expansions of the solutions of the heat equation on $\mathbf{R}_+^N$ with the homogeneous Neumann boundary condition.

Proposition 3.1. *Let $\varphi \in X_K(\mathbf{R}_+^N)$ for some $K \geq 0$ and $u(x, t) = (S(t)\varphi)(x)$. Put*

$$U_{lin}(x, t) = \sum_{|\alpha| \leq K, \alpha_N \notin \mathcal{O}} M_\alpha(\varphi, 0) g_\alpha(x, t).$$

Then, for any $q \in [1, \infty]$,

$$\begin{aligned} & t^{(N/2)(1-1/q)} (\|u(t) - U_{lin}(t)\|_q + t^{1/(2q)} \|u(t) - U_{lin}(t)\|_{q, \partial \mathbf{R}_+^N}) \\ &= \begin{cases} o(t^{-K/2}) & \text{if } K = [K], \\ O(t^{-K/2}) & \text{if } K > [K], \end{cases} \end{aligned}$$

as $t \rightarrow \infty$.

Proof. Put

$$v(x, t) := P_K(t)u(t) = u(x, t) - \sum_{|\alpha| \leq K, \alpha_N \notin \mathcal{O}} M_\alpha(u(t), t) g_\alpha(x, t).$$

Then, by Lemma 3.2 (i) and (2.8) we see that the function v is a solution of the heat equation in $\mathbf{R}_+^N$ with the homogeneous Neumann boundary condition such that

$$\int_{\mathbf{R}_+^N} x^\alpha v(x, t) dx = 0, \quad |\alpha| \leq K \text{ with } \alpha_N \notin \mathcal{O},$$

for all $t \geq 0$. This together with Lemma 3.1 implies

$$(3.6) \quad \|v(t)\|_1 = o(t^{-K/2}) \quad \text{if } K = [K], \quad \|v(t)\|_1 = O(t^{-K/2}) \quad \text{if } K > [K],$$

as $t \rightarrow \infty$. On the other hand, by Lemma 3.2 (ii) with $f \equiv 0$ we have

$$(3.7) \quad v(x, t) = u(x, t) - U_{lin}(x, t) \quad \text{in } \overline{\mathbf{R}_+^N} \times (0, \infty).$$

Therefore, since $v(2t) = S(t)v(t)$, Proposition 3.1 follows from (S1), (S2), (3.6), and (3.7). $\square$

We put

$$(3.8) \quad D[\psi](x, t) = 2G_1(x_N, t)(e^{t\Delta'}\psi)(x'), \quad x \in \mathbf{R}_+^N, t > 0.$$

Then, for any $q \in [1, \infty]$,

$$(3.9) \quad \|D[\psi](t)\|_q + t^{1/(2q)} \|D[\psi](t)\|_{q, \partial \mathbf{R}_+^N} \leq t^{-(N/2)(1-1/q)} \|\psi\|_{1, \mathbf{R}^{N-1}}, \quad t > 0.$$

Furthermore, by (S3), for any $l \in [0, k]$, we have

$$\begin{aligned}
(3.10) \quad \int_{\mathbf{R}_+^N} |x|^l |D[\psi](x, t)| dx &\leq \int_{\mathbf{R}^{N-1}} \left(\int_0^\infty (|x'|^l + x_N^l) G_1(x_N, t) dx_N \right) (e^{tD'} \psi)(x') dx' \\
&\leq \int_{\partial \mathbf{R}^{N-1}} |x'|^l |\psi(x')| dx' + t^{l/2} \|\psi\|_{1, \mathbf{R}^{N-1}}, \quad t > 0.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
(3.11) \quad \int_{\partial \mathbf{R}_+^N} |x|^l |D[\psi](x, t)| d\sigma &\leq t^{-1/2} \int_{\partial \mathbf{R}^{N-1}} |x'|^l |\psi(x')| dx' \\
&\quad + t^{(l-1)/2} \|\psi\|_{1, \mathbf{R}^{N-1}}, \quad t > 0.
\end{aligned}$$

These inequalities yield

$$\begin{aligned}
(3.12) \quad &\|D[\psi](t)\|_l + t^{1/2} \|D[\psi](t)\|_{l, \partial \mathbf{R}_+^N} \\
&\leq \|\psi\|_{l, \mathbf{R}^{N-1}} + (1 + t^{l/2}) \|\psi\|_{1, \mathbf{R}^{N-1}}, \quad t > 0.
\end{aligned}$$

At the end of this section, we give the following lemma on the operator $Q_k(t)$ (see also [9, Lemma 2.3]).

Lemma 3.3. *Let $k \geq 0$ and $f = f(x, t) \in C(\overline{\mathbf{R}_+^N} \times (0, \infty))$ be a bounded function such that $\sup_{0 < \tau < t} \tau^{1/2} \|f(\tau)\|_{k, \partial \mathbf{R}_+^N} < \infty$ for all $t > 0$. Then there holds the following:*

(i) *Assume that there exists a constant $\mu \geq 0$ such that*

$$(3.13) \quad \sup_{t>0} (1+t)^{-l/2+\mu} t^{1/2} \|f(t)\|_{l, \partial \mathbf{R}_+^N} < \infty$$

for all $l \in [0, k]$. Then, for any $|\tilde{\alpha}| \leq k$, there exists a constant C_1 such that

$$(3.14) \quad |\tilde{M}_{\tilde{\alpha}}(\gamma[f(t)], t)| \leq C_1 (1+t)^{|\tilde{\alpha}|/2-\mu} t^{-1/2}, \quad t > 0.$$

Furthermore

$$(3.15) \quad \sup_{t>0} (1+t)^{-l/2+\mu} t^{1/2} \|Q_K(t) \gamma[f(t)]\|_{l, \mathbf{R}^{N-1}} < \infty$$

for any $l \in [0, K]$ and $q \in [1, \infty]$;

(ii) *Put*

$$(3.16) \quad \mathcal{D}(x, t) = \int_0^t D[Q_K(s) \gamma[f(s)]](t-s) ds, \quad x \in \mathbf{R}_+^N, t > 0,$$

where $D[\cdot]$ is the function given in (3.8). If there exists a constant $\mu' \geq 0$ such that

$$(3.17) \quad \sup_{t>0} [t^{(N/2)(1-1/q)+\mu'+1/(2q)} \|f(t)\|_{q, \partial \mathbf{R}_+^N} + (1+t)^{-l/2+\mu'} t^{1/2} \|f(t)\|_{l, \partial \mathbf{R}_+^N}] < \infty$$

for all $l \in [0, K]$ and $q \in [1, \infty]$, then

$$(3.18) \quad t^{(N/2)(1-1/q)} (\|\mathcal{D}(t)\|_q + t^{1/(2q)} \|\mathcal{D}(t)\|_{q, \partial \mathbf{R}_+^N}) \\ \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\mu'} s^{-1/2} ds, \quad t > 0.$$

Proof. We first prove assertion (i). By (2.7) we can obtain (3.14) for the case $l \in \{0, 1\}$ directly. Furthermore, by (2.5) and (3.2) we can prove (3.14) inductively. On the other hand, by (2.6), (3.2), (3.13), and (3.14) we have

$$\begin{aligned} \|\|\| \mathcal{Q}_K(t) [\gamma[f(t)]] \|\|\|_{l, \mathbf{R}^{N-1}} &\leq \|\|\| f(t) \|\|\|_{l, \partial \mathbf{R}_+^N} + \sum_{|\tilde{\alpha}| \leq K} |\tilde{\mathbf{M}}_{\tilde{\alpha}}(\gamma[f(t)], t)| \|\|\| \tilde{g}_{\tilde{\alpha}}(t) \|\|\|_{l, \mathbf{R}^{N-1}} \\ &\preceq \|\|\| f(t) \|\|\|_{l, \partial \mathbf{R}_+^N} + \sum_{|\tilde{\alpha}| \leq K} (1+t)^{|\tilde{\alpha}|/2-\mu} t^{-1/2} (1+t)^{(l-|\tilde{\alpha}|)/2} \\ &\preceq (1+t)^{l/2-\mu} t^{-1/2} \end{aligned}$$

for all $t > 0$. This implies (3.15), and we have assertion (i).

Next we prove assertion (ii). By (3.16) we put

$$(3.19) \quad \mathcal{D}(x, t) = \left(\int_0^{t/2} + \int_{t/2}^t \right) D[\mathcal{Q}_K(s) [\gamma[f(s)]]] (t-s) ds =: I_1(x, t) + I_2(x, t).$$

By (2.9) we can apply Lemma 3.1 (ii), and by (3.9), (3.15) with $\mu = \mu'$, and (3.17), for any $q \in [1, \infty]$, we have

$$(3.20) \quad t^{(N/2)(1-1/q)} \|I_1(t)\|_q \preceq \int_0^{t/2} \|e^{((t-s)/2)A'} [\mathcal{Q}_K(s) [\gamma[f(s)]]]\|_{1, \mathbf{R}^{N-1}} ds \\ \preceq \int_0^{t/2} (t-s)^{-K/2} \|\|\| \mathcal{Q}_K(s) [\gamma[f(s)]] \|\|\|_{K, \mathbf{R}^{N-1}} ds \\ \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\mu'} s^{-1/2} ds, \quad t > 0.$$

Similarly, applying the above argument, for any $q \in [1, \infty]$, it holds that

$$(3.21) \quad t^{(N/2)(1-1/q)+1/(2q)} \|I_1(t)\|_{q, \partial \mathbf{R}_+^N} \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\gamma'} s^{-1/2} ds, \quad t > 0.$$

On the other hand, by (3.9) and (3.15) with $(l, \mu) = (0, \mu')$ we have

$$(3.22) \quad \|I_2(t)\|_1 \preceq \int_{t/2}^t \|\|\| \mathcal{Q}_K(s) [\gamma[f(s)]] \|\|\|_{1, \mathbf{R}^{N-1}} ds \\ \preceq \int_{t/2}^t (1+s)^{-\mu'} s^{-1/2} ds \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\mu'} s^{-1/2} ds, \quad t > 0.$$

Similarly we obtain

$$(3.23) \quad t^{1/2} \|I_2(t)\|_{1, \partial \mathbf{R}_+^N} \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\gamma'} s^{-1/2} ds, \quad t > 0.$$

Furthermore, by (2.6), (3.2), (3.14) with $\mu = \mu'$, and (3.17) we see that

$$\begin{aligned} \|\mathcal{Q}_K(s)[\gamma[f(s)]]\|_{\infty, \mathbf{R}^{N-1}} &\leq \|f(s)\|_{\infty, \partial \mathbf{R}_+^N} + \sum_{|\tilde{\alpha}| \leq K} |\tilde{\mathcal{M}}_{\tilde{\alpha}}(\gamma[f(s)], s)| \|\tilde{g}_{\tilde{\alpha}}(s)\|_{\infty, \mathbf{R}^{N-1}} \\ &\preceq s^{-N/2-\mu'} + (1+s)^{-(N-1)/2-\mu'} s^{-1/2} \preceq s^{-N/2-\mu'} \end{aligned}$$

for all $s > 0$. Therefore, by (3.8) we have

$$\begin{aligned} t^{N/2} |I_2(x, t)| &\preceq t^{N/2} \int_{t/2}^t (t-s)^{-1/2} \|\mathcal{Q}_K(s)[\gamma[f(s)]]\|_{\infty, \mathbf{R}^{N-1}} ds \\ &\preceq t^{1/2-\mu'} \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\mu'} s^{-1/2} ds \end{aligned}$$

for all $(x, t) \in \overline{\mathbf{R}_+^N} \times (0, \infty)$. This implies that

$$(3.24) \quad t^{N/2} (\|I_2(t)\|_{\infty} + \|I_2(t)\|_{\infty, \partial \mathbf{R}_+^N}) \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\mu'} s^{-1/2} ds, \quad t > 0.$$

Combing (3.22)–(3.24), for any $q \in [1, \infty]$, we have

$$t^{(N/2)(1-1/q)} (\|I_2(t)\|_q + t^{1/(2q)} \|I_2(t)\|_{q, \partial \mathbf{R}_+^N}) \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\mu'} s^{-1/2} ds, \quad t > 0.$$

This together with (3.19)–(3.21) yields (3.18); thus Lemma 3.3 follows. $\square$

3.2. Weighted L^1 estimate for the solution of (1.1)

In this subsection we consider the initial-boundary value problem (1.1) with (1.2) and $A_p > 1/2$. Applying a similar argument as in [8, Section 3], we give some weighted L^1 estimates of the solution u satisfying (1.3). By using approximate solutions of (1.1) we prove the following lemma, which is a local boundedness of the solution u in the weighted L^1 space.

Lemma 3.4. *Assume condition (1.2) with some $K \geq 0$ and $A_p > 1/2$. Let u be a unique solution of (1.1) satisfying (1.3). Then, for any $T > 0$ and $l \in [0, K]$,*

$$(3.25) \quad \sup_{0 < t \leq T} (\|u(t)\|_l + t^{1/2} \|u(t)\|_{l, \partial \mathbf{R}_+^N}) < \infty.$$

Proof. Let u be a unique solution of (1.1) satisfying (1.3). Put

$$(3.26) \quad u_1(x, t) := (S(t)\varphi)(x), \quad u_{n+1}(x, t) := u_1(x, t) + \int_0^t D[\gamma[f_n(s)]](x, t-s)ds,$$

for $(x, t) \in \mathbf{R}_+^N \times (0, \infty)$, where $n = 1, 2, \dots$. Here $D[\cdot]$ is the function given in (3.8) and $f_n(x, t) = \kappa|u(x, t)|^{p-1}u_n(x, t)$. Then, applying the standard parabolic theorem (see, for example, [7, Lemma 2.6]), we see that $u_n \in C(\overline{\mathbf{R}_+^N} \times (0, \infty)) \cap L^\infty(0, \infty; L^\infty(\mathbf{R}_+^N))$ for any $n = 1, 2, \dots$, and

$$(3.27) \quad \lim_{n \rightarrow \infty} u_n(x, t) = u(x, t)$$

for all $(x, t) \in \mathbf{R}_+^N \times (0, \infty)$. Let $l \in [0, K]$, and put

$$w_n(t) = \|u_n(t)\|_l + t^{1/2} \|u_n(t)\|_{l, \partial \mathbf{R}_+^N}.$$

Then, by (S1), (S2), and (3.26) we have

$$(3.28) \quad \sup_{0 < t < 1} w_1(t) \leq C_1 \|\varphi\|_l$$

for some constant C_1 . Furthermore, by (3.26) we obtain

$$(3.29) \quad \begin{aligned} w_2(t) &\leq \int_{\mathbf{R}_+^N} (1 + |x|)^l |(S(t)\varphi)(x)| dx + t^{1/2} \int_{\partial \mathbf{R}_+^N} (1 + |x|)^l |(S(t)\varphi)(x)| d\sigma \\ &\quad + \int_0^t \int_{\mathbf{R}_+^N} (1 + |x|)^l D[\gamma[f_n(s)]](x, t-s) dx ds \\ &\quad + t^{1/2} \int_0^t \int_{\partial \mathbf{R}_+^N} (1 + |x|)^l D[\gamma[f_n(s)]](x, t-s) d\sigma ds \\ &=: I_1(t) + I_2(t) + I_3(t), \quad t > 0. \end{aligned}$$

Let T_1 be a sufficiently small constant to be chosen later such that $0 < T_1 < 1$. Then, since $I_1(t) = w_1(t)$, by (3.28) we have

$$(3.30) \quad \sup_{0 < t \leq T_1} I_1(t) \leq C_1 \|\varphi\|_l.$$

On the other hand, by (1.3), (3.12), and (3.28) we obtain

$$(3.31) \quad \begin{aligned} I_2(t) &\leq C_2 \int_0^t (1+s)^{-A_p} (1+(t-s)^{l/2}) \|u_1(s)\|_{l, \partial \mathbf{R}_+^N} ds \\ &\leq C_3 \int_0^t s^{-1/2} w_1(s) ds \leq C_1 C_4 T_1^{1/2} \|\varphi\|_l \end{aligned}$$

for all $0 < t \leq T_1 < 1$, where C_2 , C_3 , and C_4 are constants. Applying a similar argument as above, we see that there exist constants C_5 , C_6 , and C_7 such that

$$\begin{aligned}
(3.32) \quad I_3(t) &\leq C_5 T_1^{1/2} \int_0^t (1+s)^{-A_p} (t-s)^{-1/2} (1+(t-s)^{1/2}) \|u_1(s)\|_{l, \partial \mathbf{R}_+^N} ds \\
&\leq C_6 \int_0^t (t-s)^{-1/2} s^{-1/2} w_1(s) ds \leq C_1 C_7 T_1^{1/2} \|\varphi\|_l
\end{aligned}$$

for all $0 < t \leq T_1 < 1$. Combining (3.29)–(3.32) and taking a sufficiently small $T_1 > 0$ so that $(C_4 + C_7)T_1^{1/2} \leq 2^{-1}$, we have

$$\sup_{0 < t \leq T_1} w_2(t) \leq C_1 [1 + (C_4 + C_7)T_1^{1/2}] \|\varphi\|_l \leq C_1 (1 + 2^{-1}) \|\varphi\|_l.$$

Repeating the above argument, we obtain

$$\begin{aligned}
(3.33) \quad \sup_{0 < t \leq T_1} w_n(t) &\leq C_1 (1 + 2^{-1} + \dots + 2^{-(n-1)}) \|\varphi\|_l \\
&\leq 2C_1 \|\varphi\|_l, \quad n = 1, 2, \dots
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
(3.34) \quad u_1(x, t) &= [S(t - T)u_1(T)](x), \\
u_{n+1}(x, t) &= [S(t - T)u_{n+1}(T)](x) + \int_T^t D[\gamma[f_n(s)]](x, t - s) ds,
\end{aligned}$$

for all $(x, t) \in \mathbf{R}_+^N \times (T, \infty)$ and all $T > 0$, applying the same argument to (3.34) with $T = T_1/2$, by (3.33) we have

$$\begin{aligned}
&\sup_{T_1/2 < t \leq 3T_1/2} [\|u_n(t)\|_l + (t - T_1/2)^{1/2} \|u_n(t)\|_{l, \partial \mathbf{R}_+^N}] \\
&\leq 2C_1 \|u_n(T_1/2)\|_l \leq (2C_1)^2 \|\varphi\|_l
\end{aligned}$$

for all $n = 1, 2, \dots$. This relation together with (3.33) implies

$$\sup_{0 < t \leq 3T_1/2} w_n(t) \leq \sup_{0 < t \leq T_1} w_n(t) + \sup_{T_1 < t \leq 3T_1/2} w_n(t) \leq C_8 \|\varphi\|_l < \infty, \quad n = 1, 2, \dots,$$

for some constant C_8 . Repeating this argument, for any $T > 0$, we obtain

$$\sup_{n \geq 1} \sup_{0 < t \leq T} w_n(t) < \infty.$$

This together with (3.27) yields (3.25); thus Lemma 3.4 follows. $\square$

By Lemma 3.4 we give the main lemma of this section, which is weighted L^1 estimates of the solution u .

Lemma 3.5. *Assume the same conditions as in Lemma 3.4. Then, for any $l \in [0, K]$,*

$$(3.35) \quad \sup_{0 < t < \infty} (1+t)^{-l/2} (\|u(t)\|_l + t^{1/2} \|u(t)\|_{l, \partial \mathbf{R}_+^N}) < \infty.$$

Proof. For any $l \in [0, K]$, we put

$$\mathcal{U}_l(t) := \int_{\mathbf{R}_+^N} |x|^l |u(x, t)| dx + t^{1/2} \int_{\partial \mathbf{R}_+^N} |x|^l |u(x, t)| d\sigma.$$

Let T be a sufficiently large constant to be chosen later such that $T > 1$. We put $f(x, t) = \kappa |u(x, t)|^{p-1} u(x, t)$. Since

$$u(x, t) = [S(t - T)u(T)](x) + \int_T^t D[\gamma[f(s)]](x, t - s) ds$$

for all $(x, t) \in \mathbf{R}_+^N \times (T, \infty)$, we have

$$\begin{aligned} (3.36) \quad \mathcal{U}_l(t) &\leq \int_{\mathbf{R}_+^N} |x|^l |[S(t - T)u(T)](x)| dx \\ &\quad + t^{1/2} \int_{\partial \mathbf{R}_+^N} |x|^l |[S(t - T)u(T)](x)| d\sigma \\ &\quad + \int_T^t \int_{\mathbf{R}_+^N} |x|^l D[\gamma[f(s)]](x, t - s) dx ds \\ &\quad + t^{1/2} \int_T^t \int_{\partial \mathbf{R}_+^N} |x|^l D[\gamma[f(s)]](x, t - s) d\sigma ds \\ &=: I_1(t) + I_2(t) + I_3(t) \end{aligned}$$

for all $t > T$. By (1.3), (S1), (S2), and (3.25) we obtain

$$\begin{aligned} (3.37) \quad I_1(t) &\preceq \left(\int_{\mathbf{R}_+^N} |x|^l |u(x, T)| dx + (t - T)^{l/2} \|u(T)\|_1 \right) \\ &\quad + t^{1/2} \left((t - T)^{-1/2} \int_{\mathbf{R}_+^N} |x|^l |u(x, T)| dx + (t - T)^{(l-1)/2} \|u(T)\|_1 \right) \\ &\preceq t^{l/2} \end{aligned}$$

for all $t > 2T$. Furthermore, it follows from (1.3) and (3.10) that

$$\begin{aligned} (3.38) \quad I_2(t) &\preceq \int_T^t s^{-A_p} \left(\int_{\partial \mathbf{R}_+^N} |x|^l |u(x, s)| d\sigma + (t - s)^{l/2} \|u(s)\|_{1, \partial \mathbf{R}_+^N} \right) ds \\ &\preceq \int_T^t s^{-A_p} (s^{-1/2} \mathcal{U}_l(s) + (t - s)^{l/2} s^{-1/2}) ds \\ &\preceq T^{-A_p+1/2} t^{l/2} \left(\sup_{T < s < t} s^{-l/2} \mathcal{U}_l(s) \right) + t^{l/2} \end{aligned}$$

for all $t > 2T$. Applying the similar argument as above with (3.11), we have

$$I_3(t) \leq T^{-A_p+1/2} t^{l/2} \left(\sup_{T < s < t} s^{-1/2} \mathcal{U}_l(s) \right) + t^{l/2}$$

for all $t > 2T$. This together with (3.36)–(3.38) implies that there exists a constant C such that

$$\sup_{2T < s < t} s^{-l/2} \mathcal{U}_l(s) \leq CT^{-A_p+1/2} \sup_{T < s < t} s^{-l/2} \mathcal{U}_l(s) + C$$

for all $t > 2T \geq 2$. Then, since $A_p > 1/2$, we can take a sufficiently large T so that $CT^{-A_p+1/2} \leq 1/2$ if necessary, and we have

$$\sup_{2T < s < \infty} s^{-l/2} \mathcal{U}_l(s) \leq 2 \sup_{T < s < t} s^{-l/2} \mathcal{U}_l(s) + 2C.$$

This together with (3.25) implies (3.35), and the proof of Lemma 3.5 is complete. $\square$

4. Proof of main results

In this section we prove Theorem 2.1 and Corollary 2.1. Let u be a unique solution of the initial-boundary value problem (1.1) under condition (1.2) for some $K \geq 0$ and $A_p > 1/2$. Assume (1.3). Then, by Lemma 3.5 we see that the solution u satisfies (3.35). Therefore, for any $|\alpha| \leq K$ with $\alpha_N \notin \mathcal{O}$, we can define $M_\alpha(u(t), t)$ for all $t \geq 0$, and it holds that

$$(4.1) \quad |M_\alpha(u(t), t)| \leq (1+t)^{|\alpha|/2}, \quad t \geq 0$$

(see (2.5)). Furthermore we put

$$F(x, t) := \kappa |u(x, t)|^{p-1} u(x, t)$$

for simplicity. Then, by (1.3) and (3.35) we have

$$(4.2) \quad \|F(t)\|_{q, \partial \mathbf{R}_+^N} \leq (1+t)^{-A_p} \|u(t)\|_{q, \partial \mathbf{R}_+^N} \leq (1+t)^{-A_p-(N/2)(1-1/q)} t^{-1/(2q)},$$

$$\|F(t)\|_{l, \partial \mathbf{R}_+^N} \leq (1+t)^{-A_p} \|u(t)\|_{l, \partial \mathbf{R}_+^N} \leq (1+t)^{-A_p+l/2} t^{-1/2},$$

for all $t > 0$, where $q \in [1, \infty]$ and $l \in [0, K]$. This together with Lemma 3.3 (i) implies that

$$(4.3) \quad \tilde{M}_{\tilde{\alpha}}(\gamma[F(t)], t) \leq (1+t)^{-A_p+|\tilde{\alpha}|/2} t^{-1/2}, \quad t > 0,$$

for any $|\tilde{\alpha}| \leq K$. Furthermore, by (2.12), (3.2), and (4.2) we have

$$(4.4) \quad N_\alpha(\gamma[F(t)], t) \leq (1+t)^{-A_p+|\alpha|/2} t^{-1/2}, \quad t > 0,$$

for any $|\alpha| \leq K$ with $\alpha_N \notin \mathcal{O}$. Therefore, applying Lemma 3.2 (ii), it holds that

$$|M_\alpha(u(t), t) - M_\alpha(u(t_0), t_0)| = \left| \int_{t_0}^t N_\alpha(\gamma[F(s)], s) ds \right| \leq \int_{t_0}^t (1+s)^{-A_p+|\alpha|/2} s^{-1/2} ds$$

for all $t \geq t_0 \geq 0$. This implies the following:

(i) For any $|\alpha| \leq K$ with $\alpha_N \notin \mathcal{O}$, if $2A_p > |\alpha| + 1$, there exists a constant M_α such that

$$(4.5) \quad |M_\alpha(u(t), t) - M_\alpha| \leq (1+t)^{-A_p+(|\alpha|+1)/2} \quad \text{for all } t > 0;$$

(ii) For any $|\alpha| \leq K$ with $\alpha_N \notin \mathcal{O}$, if $1 < 2A_p \leq |\alpha| + 1$, then

$$(4.6) \quad M_\alpha(u(t), t) = \begin{cases} O(t^{-A_p+(|\alpha|+1)/2}) & \text{if } 2A_p < |\alpha| + 1, \\ O(\log t) & \text{if } 2A_p = |\alpha| + 1, \end{cases} \quad \text{as } t \rightarrow \infty.$$

Now we are ready to prove our main theorem. We first prove assertions (i), (ii), and (iv) of Theorem 2.1.

Proof of assertions (i), (ii), and (iv) of Theorem 2.1. Substituting (4.1), (4.3), and (4.4) to the function U_0 (see (2.11)) we have

$$(4.7) \quad |U_0(x, t)| \leq \sum_{|\alpha| \leq K, \alpha_N \notin \mathcal{O}} \left(|M_\alpha(u(t), t)| + \int_0^t |N_\alpha(\gamma[f(s)], s)| ds \right) |g_\alpha(x, t)| \\ + 2 \sum_{|\tilde{\alpha}| \leq K} \int_0^t G_1(x_N, t-s) |\tilde{M}_{\tilde{\alpha}}(\gamma[f(s)], s)| ds |\tilde{g}_{\tilde{\alpha}}(x', t)| \\ \leq \sum_{|\alpha| \leq K, \alpha_N \notin \mathcal{O}} \left((1+t)^{|\alpha|/2} + \int_0^t (1+s)^{-A_p+|\alpha|/2} s^{-1/2} ds \right) |g_\alpha(x, t)| \\ + \sum_{|\tilde{\alpha}| \leq K} \int_0^t (1+s)^{-A_p+|\tilde{\alpha}|/2} s^{-1/2} G_1(x_N, t-s) ds |\tilde{g}_{\tilde{\alpha}}(x', t)|$$

for all $(x, t) \in \overline{\mathbf{R}}_+^N \times (0, \infty)$. Since $A_p > 1/2$ and

$$\int_0^t (t-s)^{-1/2} (1+s)^{-A_p} s^{-1/2} ds \leq t^{-1/2}, \quad t > 0,$$

combining (3.2) and (4.7) we obtain (2.13) and (2.14) for the case $n = 0$, and assertion (i) follows for the case $n = 0$.

Let $n = -1, 0, 1, 2, \dots$. We assume, without loss of generality, that $\sigma \in (0, A_p - 1/2)$. Put

$$\sigma_n = \begin{cases} \sigma & \text{if } n(2A_p - 1) \geq K, \\ (K/2) - n(A_p - 1/2) & \text{if } n(2A_p - 1) < K, \end{cases} \quad \mu_n = A_p + \frac{K}{2} - \sigma_n.$$

Let $U_{-1} \equiv 0$ in $\overline{\mathbf{R}_+^N} \times (0, \infty)$. Then (2.11) holds for $n = 0, 1, 2, \dots$. Furthermore, since the solution u satisfies (1.3) and (3.35), assertions (i), (ii), and (iv) hold with $n = -1$ and $\sigma = \sigma_0$.

Assume that there exists a number $n_* \in \{-1, 0, 1, 2, \dots\}$ such that assertions (i), (ii), and (iv) hold with $n = n_*$ and $\sigma = \sigma_{n_*+1}$. We first prove assertion (i) for $n = n_* + 1$. Since assertion (i) holds with $n = n_*$ and $F_n(x, t) = \kappa|U_n(x, t)|^{p-1}U_n(x, t)$, we obtain

$$\sup_{t>0} (1+t)^{A_p} \left[t^{(N/2)(1-1/q)+1/(2q)} \|F_{n_*}(t)\|_{q, \partial\mathbf{R}_+^N} + (1+t)^{-1/2} t^{1/2} \|F_{n_*}(t)\|_{l, \partial\mathbf{R}_+^N} \right] < \infty,$$

for any $q \in [1, \infty]$ and $l \in [0, K]$. This together with Lemma 3.3 (i) implies that

$$(4.8) \quad \sup_{t>0} (1+t)^{A_p-l/2} t^{1/2} \|Q_K(t) \gamma[F_{n_*}(t)]\|_{l, \mathbf{R}^{N-1}} < \infty$$

for any $l \in [0, K]$. Since $A_p > 1/2$, by (2.11), (2.14) with $n = 0$, (3.12), and (4.8) we have

$$\begin{aligned} (4.9) \quad \|U_{n_*+1}(t)\|_l &\leq \|U_0(t)\|_l + \left\| \int_0^t D[Q_K(s) \gamma[F_{n_*}(s)]](t-s) ds \right\|_l \\ &\preceq (1+t)^{l/2} + \int_0^t \|Q_K(s) \gamma[F_{n_*}(s)]\|_{l, \mathbf{R}^{N-1}} ds \\ &\quad + \int_0^t (1+(t-s)^{l/2}) \|Q_K(s) \gamma[F_{n_*}(s)]\|_{1, \mathbf{R}^{N-1}} ds \\ &\preceq (1+t)^{l/2} + \int_0^t (1+s)^{-A_p+l/2} s^{-1/2} ds \\ &\quad + \int_0^t (1+(t-s)^{l/2}) (1+s)^{-A_p} s^{-1/2} ds \preceq (1+t)^{l/2} \end{aligned}$$

for all $t > 0$. Furthermore, applying a similar argument as above, we obtain

$$(4.10) \quad t^{1/2} \|U_{n_*+1}(t)\|_{l, \partial\mathbf{R}_+^N}, \quad t > 0. \preceq (1+t)^{l/2}$$

On the other hand, by (2.14) with $n = 0$, (S3), and (3.9) we have

$$\begin{aligned} |U_{n_*+1}(x, t)| &\leq |U_0(x, t)| + \int_0^t |D[Q_K(s) \gamma[F_{n_*}(s)]](t-s)| ds \\ &\preceq t^{-N/2} + \int_0^{t/2} (t-s)^{-N/2} \|Q_K(s) \gamma[F_{n_*}(s)]\|_{1, \mathbf{R}^{N-1}} ds \\ &\quad + \int_{t/2}^t (t-s)^{-1/2} s^{-(N-1)/2} \|Q_K(s) \gamma[F_{n_*}(s)]\|_{1, \mathbf{R}^{N-1}} ds \end{aligned}$$

$$\begin{aligned}
&\preceq t^{-N/2} \left(1 + \int_0^{t/2} (1+s)^{-A_p} s^{-1/2} ds + \int_{t/2}^t (t-s)^{-1/2} (1+s)^{-A_p} ds \right) \\
&\preceq t^{-N/2}
\end{aligned}$$

for all $(x, t) \in \overline{\mathbf{R}_+^N} \times (0, \infty)$. This together with (4.9) and (4.10) implies that assertion (i) holds with $n = n_* + 1$.

Next we prove that assertions (ii) and (iv) hold with $n = n_* + 1$ and $\sigma = \sigma_{n_*+2}$. Since the solution u satisfies (1.3) and (3.25), due to assertion (i) with $n = n_* + 1$, it suffices to prove that (2.15) and (2.17) hold with $n = n_* + 1$ and $\sigma = \sigma_{n_*+2}$ for all sufficiently large t . Put $z(t) := u(t) - U_{n_*+1}(t)$. Then, by (2.4), (2.11), and the semigroup properties of $S(t)$ and $e^{tA'}$ we have

$$\begin{aligned}
z(x, t) &= P_K(t)u(t) + \sum_{|\alpha| \leq K, \alpha_N \notin \mathcal{O}} \int_0^t S(t-s) N_\alpha(\gamma[F(s)], s) g_\alpha(s) ds \\
&\quad - 2 \int_0^t G_1(x_N, t-s) e^{(t-s)A'} \left(\sum_{|\tilde{\alpha}| \leq K} \tilde{M}_{\tilde{\alpha}}(\gamma[F(s)], s) \tilde{g}_{\tilde{\alpha}}(s) + Q_K(s) \gamma[F_{n_*}(s)] \right) ds.
\end{aligned}$$

Then, by Lemma 3.2 (i) we obtain

$$\begin{cases} \partial_t z = Az & \text{in } \mathbf{R}_+^N \times (0, \infty), \\ \partial_\nu z = Q_K(t)(\gamma[\tilde{F}_{n_*}(t)]) & \text{on } \partial \mathbf{R}_+^N \times (0, \infty), \\ z(x, 0) = (P_K(0)u(0))(x) & \text{in } \mathbf{R}_+^N, \end{cases}$$

where $\tilde{F}_n(t) = F(t) - F_n(t)$. This implies that

$$(4.11) \quad z(t) = S(t-t_0)z(t_0) + \int_{t_0}^t D[Q_K(s)(\gamma[\tilde{F}_{n_*}(s)])] ds$$

for all $t > t_0 \geq 0$. Let $q \in [1, \infty]$. By (S1) we have

$$\begin{aligned}
(4.12) \quad t^{(N/2)(1-1/q)} \|S(t)z(0)\|_q &= t^{(N/2)(1-1/q)} \|S(t/2)S(t/2)z(0)\|_q \\
&\preceq \|S(t/2)z(0)\|_1, \quad t > 0.
\end{aligned}$$

Similarly, by (S2) we obtain

$$\begin{aligned}
(4.13) \quad t^{(N/2)(1-1/q)+1/(2q)} \|S(t)z(0)\|_{q, \partial \mathbf{R}_+^N} &= t^{(N/2)(1-1/q)+1/(2q)} \|S(t/2)S(t/2)z(0)\|_{q, \partial \mathbf{R}_+^N} \\
&\preceq \|S(t/2)z(0)\|_1
\end{aligned}$$

for all $t > 0$. Furthermore, since it follows from (2.8) that

$$\int_{\mathbf{R}_+^N} x^\alpha z(x, 0) dx = \int_{\mathbf{R}_+^N} x^\alpha (P_K(0)u(0))(x) dx = 0, \quad |\alpha| \leq K \text{ with } \alpha_N \notin \mathcal{O},$$

we can apply Lemma 3.1 (ii) to have

$$\|S(t/2)z(0)\|_1 \preceq t^{-K/2}, \quad t > 0.$$

This together with (4.12) and (4.13) implies

$$(4.14) \quad t^{(N/2)(1-1/q)} (\|S(t)z(0)\|_q + t^{1/(2q)} \|S(t)z(0)\|_{q, \partial \mathbf{R}_+^N}) \preceq t^{-K/2}, \quad t > 0.$$

On the other hand, by (1.3) and (2.13) with $n = n_*$ we have

$$(4.15) \quad |\tilde{F}_{n_*}(x, t)| \preceq (1+t)^{-A_p} |u(x, t) - U_{n_*}(x, t)|$$

for all $(x, t) \in \overline{\mathbf{R}_+^N} \times (0, \infty)$. Then, since assertions (ii) and (iv) hold with $n = n_*$ and $\sigma = \sigma_{n_*+1}$, by (4.15) we obtain

$$(4.16) \quad \sup_{t>0} (t^{(N/2)(1-1/q)+1/(2q)+\mu_{n_*+1}} \|\tilde{F}_{n_*}(t)\|_{q, \partial \mathbf{R}_+^N} + (1+t)^{-l/2+\mu_{n_*+1}} t^{1/2} \|\tilde{F}_{n_*}(t)\|_{l, \partial \mathbf{R}_+^N}) < \infty$$

for any $q \in [1, \infty]$ and $l \in [0, K]$. This together with Lemma 3.3 (i) implies that

$$(4.17) \quad \sup_{t>0} (1+t)^{-l/2+\mu_{n_*+1}} t^{1/2} \|Q_K(t) \gamma[\tilde{F}_{n_*}(t)]\|_{l, \mathbf{R}^{N-1}} < \infty$$

for any $l \in [0, K]$. Furthermore, by (4.16) we can apply Lemma 3.3 (ii) with $\mu' = \mu_{n_*+1}$, and obtain

$$(4.18) \quad t^{(N/2)(1-1/q)} \left\| \int_0^t D[Q_K(s) \gamma[\tilde{F}_{n_*}(s)]](t-s) ds \right\|_q \\ \preceq t^{-K/2} \int_0^t (1+s)^{K/2-\mu_{n_*+1}} s^{-1/2} ds = t^{-K/2} \int_0^t (1+s)^{-A_p+\sigma_{n_*+1}} s^{-1/2} ds \\ \preceq t^{-K/2} + t^{-K/2} \int_1^t s^{-A_p+\sigma_{n_*+1}-1/2} ds \\ = \begin{cases} t^{-K/2} & \text{if } (n_*+2)(2A_p-1) > K, \\ t^{-K/2} \log t & \text{if } (n_*+2)(2A_p-1) = K, \\ t^{-(n_*+2)(A_p-1/2)} & \text{if } (n_*+2)(2A_p-1) < K, \end{cases}$$

for all sufficiently large t . Similarly, we have

$$\begin{aligned}
(4.19) \quad & t^{(N/2)(1-1/q)+1/(2q)} \left\| \int_0^t D[Q_K(s)\gamma[\tilde{F}_{n_*}(s)]](t-s)ds \right\|_{q, \partial\mathbf{R}_+^N} \\
& \leq t^{-K/2} \int_0^t (1+s)^{K/2-\mu_{n_*+1}} s^{-1/2} ds \\
& = \begin{cases} t^{-K/2} & \text{if } (n_*+2)(2A_p-1) > K, \\ t^{-K/2} \log t & \text{if } (n_*+2)(2A_p-1) = K, \\ t^{-(n_*+2)(A_p-1/2)} & \text{if } (n_*+2)(2A_p-1) < K, \end{cases}
\end{aligned}$$

for all sufficiently large t . Therefore we apply (4.14), (4.18), and (4.19) to (4.11) with $t_0 = 0$, and obtain inequality (2.15) with $n = n_* + 1$ for any sufficiently large t . Thus assertion (ii) holds with $n = n_* + 1$.

On the other hand, for any $l \in [0, K]$, we have

$$\begin{aligned}
& (1+t)^{-l/2} (\|z(t)\|_l + t^{1/2} \|z(t)\|_{l, \partial\mathbf{R}_+^N}) \\
& = \int_{\mathbf{R}_+^N} \left(\frac{1+|x|}{(1+t)^{1/2}} \right)^l |z(x, t)| dx + t^{1/2} \int_{\partial\mathbf{R}_+^N} \left(\frac{1+|x|}{(1+t)^{1/2}} \right)^l |z(x, t)| d\sigma \\
& \leq \int_{\mathbf{R}_+^N} \left[1 + \left(\frac{1+|x|}{(1+t)^{1/2}} \right)^K \right] |z(x, t)| dx \\
& \quad + t^{1/2} \int_{\partial\mathbf{R}_+^N} \left[1 + \left(\frac{1+|x|}{(1+t)^{1/2}} \right)^K \right] |z(x, t)| d\sigma \\
& = \|z(t)\|_1 + t^{1/2} \|z(t)\|_{1, \partial\mathbf{R}_+^N} \\
& \quad + (1+t)^{-K/2} (\|z(t)\|_K + t^{1/2} \|z(t)\|_{K, \partial\mathbf{R}_+^N}), \quad t > 0.
\end{aligned}$$

Then, by (2.15) with $q = 1$ and $n = n_* + 1$ we see that, if there holds (2.17) with $l = K$, then we have (2.17) for $l \in [0, K]$. Thus it suffices to prove (2.17) with $l = K$, $n = n_* + 1$, and $\sigma = \sigma_{n_*+2}$. Put $Z(t) = \|z(t)\|_K$. By (4.11) we have

$$(4.20) \quad Z(2t) \leq \|S(t)z(t)\|_K + \int_t^{2t} \|D[Q_K(s)\gamma[\tilde{F}_{n_*}(s)]](t-s)\|_K ds, \quad t > 0.$$

Let $\delta > 0$. Then, by (2.15) with $n = n_* + 1$ and (S1) we have

$$\begin{aligned}
(4.21) \quad & \|S(t)z(t)\|_K \leq (1+\delta) \|z(t)\|_K + C_1(1+t^{K/2}) \|z(t)\|_1 \\
& \leq (1+\delta) Z(t) + C_2 t^{\sigma_{n_*+2}}
\end{aligned}$$

for all $t \geq 1/2$, where C_1 and C_2 constants. Furthermore, by (3.12) and (4.17) we obtain

$$\begin{aligned}
 (4.22) \quad & \int_t^{2t} \|D[Q_K(s)\gamma[\tilde{F}_{n_*}(s)]](t-s)\|_K ds \\
 & \preceq \int_t^{2t} \|Q_K(s)\gamma[\tilde{F}_{n_*}(s)]\|_{K, \mathbf{R}^{N-1}} ds \\
 & \quad + \int_t^{2t} [1 + (2t-s)^{K/2}] \|Q_K(s)\gamma[\tilde{F}_{n_*}(s)]\|_{1, \mathbf{R}^{N-1}} ds \\
 & \preceq \int_t^{2t} (1+s)^{K/2-\mu_{n_*+1}} s^{-1/2} ds + \int_t^{2t} [1 + (2t-s)^{K/2}] (1+s)^{-\mu_{n_*+1}} s^{-1/2} ds \\
 & \preceq t^{K/2-\mu_{n_*+1}+1/2} = t^{-(A_p-1/2)+\sigma_{n_*+1}} \preceq t^{\sigma_{n_*+2}}, \quad t \geq 1/2.
 \end{aligned}$$

Therefore, by (4.20)–(4.22) we can find a constant C_3 satisfying

$$(4.23) \quad Z(2t) \leq (1+\delta)Z(t) + C_3 t^{\sigma_{n_*+2}}, \quad t \geq 1/2.$$

Furthermore, since it follows from (2.14) with $n = n_* + 1$ and (3.35) that $\sup_{0 < t < 1} Z(t) < \infty$, applying the same argument as in the proof of [6, Lemma 3.2] with the inequality (4.23), we obtain

$$(4.24) \quad Z(t) \preceq t^{\sigma_{n_*+2}}, \quad t \geq 1.$$

On the other hand, by (2.15) with $n = n_* + 1$, (S2), and (4.24) we have

$$(4.25) \quad t^{1/2} \|S(t)z(t)\|_{K, \partial \mathbf{R}_+^N} \preceq Z(t) + (1+t^{K/2}) \|z(t)\|_1 \preceq t^{\sigma_{n_*+2}}, \quad t \geq 1.$$

Furthermore, applying a similar argument as in (4.22), we obtain

$$t^{1/2} \int_t^{2t} \|D[Q_K(s)\gamma[F(s) - F_{n_*}(s)]](t-s)\|_{K, \partial \mathbf{R}_+^N} ds \preceq t^{\sigma_{n_*+2}}, \quad t \geq 1.$$

This together with (4.11) and (4.25) implies that

$$(4.26) \quad t^{1/2} \|z(t)\|_{K, \partial \mathbf{R}_+^N} \preceq t^{\sigma_{n_*+2}}, \quad t \geq 1.$$

By (4.24) and (4.26) we have inequality (2.17) with $n = n_* + 1$ with $\sigma = \sigma_{n_*+2}$ for any sufficiently large t . Therefore assertions (ii) and (iv) hold with $n = n_* + 1$ for all $t > 0$. Thus, by induction we see that (2.14), (2.15) and (2.17) hold with $\sigma = \sigma_{n+1}$ for all $n = 0, 1, 2, \dots$, and assertions (i), (ii), and (iv) of Theorem 2.1 follow. $\square$

We complete the proof of Theorem 2.1.

Proof of Theorem 2.1. It suffices to prove assertion (iii) of Theorem 2.1. Since there holds (2.16) for the case $K > [K]$ by Theorem 2.1 (ii), it suffices to prove (2.16) for the case $K = [K]$. Let $K = [K]$ and $n \in \{0, 1, 2, \dots\}$ be such that

$$(n+1)(2A_p - 1) > K.$$

Then we can take a positive constant σ so that

$$(4.27) \quad K - n(2A_p - 1) < 2\sigma < 2A_p - 1.$$

Put $z_n(t) = u(t) - U_n(t)$, $\tilde{F}_n(t) = F(t) - F_n(t)$, and $2\varepsilon := 2A_p - 1 - 2\sigma > 0$. Then, by (4.27) we apply Theorem 2.1 (ii) and (iv) to obtain

$$\begin{aligned} & t^{(N/2)(1-1/q)+1/(2q)} \|\tilde{F}_{n-1}(t)\|_{q, \partial\mathbf{R}_+^N} + (1+t)^{-1/2} t^{1/2} \|\tilde{F}_{n-1}(t)\|_{l, \partial\mathbf{R}_+^N} \\ & \preceq (1+t)^{-A_p} \{t^{(N/2)(1-1/q)+1/(2q)} \|z_{n-1}(t)\|_{q, \partial\mathbf{R}_+^N} + (1+t)^{-1/2} t^{1/2} \|z_{n-1}(t)\|_{l, \partial\mathbf{R}_+^N}\} \\ & \preceq (1+t)^{-A_p} [(1+t)^{-K/2+\sigma} + (1+t)^{-n(A_p-1/2)}] \\ & \preceq (1+t)^{-A_p-K/2+\sigma} \preceq (1+t)^{-K/2-1/2-\varepsilon}, \quad t > 0, \end{aligned}$$

where $q \in [1, \infty]$ and $l \in [0, K]$. This together with Lemma 3.3 (i) yields

$$(4.28) \quad (1+t)^{-l/2} \|\mathcal{Q}_K(t)\gamma[\tilde{F}_{n-1}(t)]\|_{l, \mathbf{R}^{N-1}} \preceq (1+t)^{-K/2-1/2-\varepsilon} t^{-1/2}, \quad t > 0.$$

By (4.11), for any $L > 0$, we have

$$\begin{aligned} (4.29) \quad z_n(t) &= S(t)z_n(0) + \int_0^t D[\mathcal{Q}_K(s)\gamma[\tilde{F}_{n-1}(s)]](t-s)ds \\ &= S(t)z_n(0) + \left(\int_{t/2}^t + \int_L^{t/2} + \int_0^L \right) D[\mathcal{Q}_K(s)\gamma[\tilde{F}_{n-1}(s)]](t-s)ds \\ &=: S(t)z_n(0) + I_1(t) + I_2(t) + I_3(t), \quad t \geq 2L. \end{aligned}$$

Since $z_n(0) = P_K(0)u(0)$, by (2.8) we have

$$\int_{\mathbf{R}^N} x^\alpha z_n(0) dx = 0, \quad |\alpha| \leq [K] = K \text{ with } \alpha_N \notin \mathcal{O},$$

and by (S1), (S2), and Lemma 3.1 (ii) we obtain

$$\begin{aligned} (4.30) \quad \lim_{t \rightarrow \infty} t^{(N/2)(1-1/q)+K/2} \|S(t)z_n(0)\|_q &\preceq \lim_{t \rightarrow \infty} t^{K/2} \|S(t/2)z_n(0)\|_1 = 0, \\ \lim_{t \rightarrow \infty} t^{(N/2)(1-1/q)+1/(2q)+K/2} \|S(t)z_n(0)\|_{q, \partial\mathbf{R}_+^N} &\preceq \lim_{t \rightarrow \infty} t^{K/2} \|S(t/2)z_n(0)\|_1 = 0. \end{aligned}$$

We first give the estimate for $I_1(t)$. By (S3), (4.28), and Lemma 3.3 (i) we obtain

$$\begin{aligned} t^{N/2}|I_1(x, t)| &\preceq t^{N/2} \int_{t/2}^t (t-s)^{-1/2} s^{-(N-1)/2} \|Q_K(s) \gamma[\tilde{F}_{n-1}(s)]\|_{1, \partial \mathbf{R}_+^N} ds \\ &\preceq t^{-K/2-1/2-\varepsilon} \int_{t/2}^t (t-s)^{-1/2} ds \preceq t^{-K/2-\varepsilon} \end{aligned}$$

for all $(x, t) \in \overline{\mathbf{R}_+^N} \times (0, \infty)$. This implies that

$$(4.31) \quad t^{N/2}(\|I_1(t)\|_\infty + \|I_1\|_{\infty, \partial \mathbf{R}_+^N}) = o(t^{-K/2})$$

as $t \rightarrow \infty$. Furthermore, by (3.9) and (4.28) with $l = 0$ we obtain

$$\begin{aligned} (4.32) \quad \|I_1(t)\|_1 &\leq \int_{t/2}^t \|Q_K(s) \gamma[\tilde{F}_{n-1}(t)]\|_{1, \mathbf{R}^{N-1}} ds \\ &\preceq \int_{t/2}^t s^{-K/2-1-\varepsilon} ds \preceq t^{-K/2-\varepsilon} = o(t^{-K/2}) \end{aligned}$$

as $t \rightarrow \infty$. Similarly we have

$$t^{1/2}\|I_1(t)\|_{1, \partial \mathbf{R}_+^N} \preceq t^{-K/2-1/2-\varepsilon} \int_{t/2}^t (t-s)^{-1/2} ds \preceq t^{-K/2-\varepsilon} = o(t^{-K/2})$$

as $t \rightarrow \infty$. This together with (4.31) and (4.32) yields

$$(4.33) \quad t^{(N/2)(1-1/q)}(\|I_1(t)\|_q + t^{1/(2q)}\|I_1(t)\|_{q, \partial \mathbf{R}_+^N}) = o(t^{-K/2}) \quad \text{as } t \rightarrow \infty.$$

Next we give the estimates for $I_2(t)$ and $I_3(t)$. By Lemma 3.1 (ii), (3.9), (4.28), and the semigroup property of $e^{tA'}$ we have

$$\begin{aligned} (4.34) \quad &t^{(N/2)(1-1/q)}(\|I_2(t)\|_q + t^{1/(2q)}\|I_2(t)\|_{q, \partial \mathbf{R}_+^N}) \\ &\leq t^{(N/2)(1-1/q)} \int_L^{t/2} \|D[Q_K(s) \gamma[\tilde{F}_{n-1}(s)]](t-s)\|_q ds \\ &\quad + t^{(N/2)(1-1/q)+1/(2q)} \int_L^{t/2} \|D[Q_K(s) \gamma[\tilde{F}_{n-1}(s)]](t-s)\|_{q, \partial \mathbf{R}_+^N} ds \\ &\preceq \int_L^{t/2} \|e^{((t-s)/2)A'} Q_K(s) \gamma[\tilde{F}_{n-1}(s)]\|_{1, \mathbf{R}^{N-1}} ds \\ &\preceq \int_L^{t/2} (t-s)^{-K/2} \|Q_K(s) \gamma[\tilde{F}_{n-1}(s)]\|_{K, \mathbf{R}^{N-1}} ds \\ &\preceq t^{-K/2} \int_L^{t/2} s^{-1-\varepsilon} ds \preceq t^{-K/2} L^{-\varepsilon} \end{aligned}$$

for all sufficiently large t . Similarly we obtain

$$(4.35) \quad \begin{aligned} & t^{(N/2)(1-1/q)} (\|I_3(t)\|_q + t^{1/(2q)} \|I_3(t)\|_{q, \partial \mathbf{R}_+^N}) \\ & \preceq \int_0^L \|e^{((t-s)/2)A'} Q_K(s) \gamma[\tilde{F}_{n-1}(s)]\|_{1, \mathbf{R}^{N-1}} ds \end{aligned}$$

for all $t > 0$. On the other hand, by (2.9) and (4.28) we can apply Lemma 3.1 (ii) to have

$$\begin{aligned} & \lim_{t \rightarrow \infty} t^{K/2} \|e^{((t-s)/2)A'} Q_K(s) \gamma[\tilde{F}_{n-1}(s)]\|_1 \\ & = \lim_{t \rightarrow \infty} (t-s)^{K/2} \|e^{((t-s)/2)A'} Q_K(s) \gamma[\tilde{F}_{n-1}(s)]\|_1 = 0, \\ & \|e^{((t-s)/2)A'} Q_K(s) \gamma[\tilde{F}_{n-1}(s)]\|_1 \preceq (t-s)^{-K/2} \|Q_K(s) \gamma[\tilde{F}_{n-1}(s)]\|_{K, \mathbf{R}^{N-1}} \\ & \preceq t^{-K/2} s^{-1/2}, \quad t \geq 2L, \end{aligned}$$

for all $s \in (0, L)$. Therefore we can apply the Lebesgue dominated convergence theorem to (4.35), and obtain

$$(4.36) \quad t^{(N/2)(1-1/q)} \|I_3(t)\|_q = o(t^{-K/2})$$

as $t \rightarrow \infty$. Combining (4.29), (4.30), (4.33), (4.34), and (4.36), we see that there exists a constant C_4 such that

$$\limsup_{t \rightarrow \infty} t^{(N/2)(1-1/q)+K/2} (\|z_n(t)\|_q + t^{1/(2q)} \|z(t)\|_{q, \partial \mathbf{R}_+^N}) \leq C_4 L^{-\varepsilon}.$$

Then, since L is arbitrary, we have

$$\lim_{t \rightarrow \infty} t^{(N/2)(1-1/q)+K/2} (\|z_n(t)\|_q + t^{1/(2q)} \|z(t)\|_{q, \partial \mathbf{R}_+^N}) = 0.$$

Thus we have (2.16) for the case $K = [K]$, and the proof of Theorem 2.1 is complete. $\square$

Finally we prove Corollary 2.1.

Proof of Corollary 2.1. We first prove (2.19) with $K \in (0, 1)$. Put $\mathcal{F}(s) = \kappa|s|^{p-1}s$. By (1.8) and (2.11) we have

$$\begin{aligned} R(x, t) &= u(x, t) - U_0(x, t) - \left(M - M_0(u(t), t) + \int_0^t N_0(\gamma[\mathcal{F}(u(s))], s) ds \right) g(x, t) \\ &\quad + 2 \int_0^t G_1(x_N, t-s) \tilde{M}_0(\gamma[\mathcal{F}(u(s))], s) ds \tilde{g}(x', t) \end{aligned}$$

for all $(x, t) \in \overline{\mathbf{R}_+^N} \times (0, \infty)$. This together with (2.5), (2.7), and (2.12) yields

$$\begin{aligned}
 (4.37) \quad R(x, t) &= u(x, t) - U_0(x, t) - \int_0^\infty \int_{\partial \mathbf{R}_+^N} \mathcal{F}(u(x, s)) d\sigma ds g(x, t) \\
 &\quad + 2 \int_0^t G_1(x_N, t-s) \int_{\partial \mathbf{R}_+^N} \mathcal{F}(u(x, s)) d\sigma ds \tilde{g}(x', t) \\
 &= u(x, t) - U_0(x, t) - \int_t^\infty \int_{\partial \mathbf{R}_+^N} \mathcal{F}(u(x, s)) d\sigma ds g(x, t) \\
 &\quad - 2 \left(\int_0^{t/2} + \int_{t/2}^t \right) \left(\int_{\partial \mathbf{R}_+^N} \mathcal{F}(u(x, s)) d\sigma \right) \\
 &\quad \times (G_1(x_N, 1+t) - G_1(x_N, t-s)) ds \tilde{g}(x', t) \\
 &=: u(x, t) - U_0(x, t) - \int_t^\infty \int_{\partial \mathbf{R}_+^N} \mathcal{F}(u(x, s)) d\sigma ds g(x, t) \\
 &\quad + I_1(x, t) + I_2(x, t)
 \end{aligned}$$

for all $(x, t) \in \overline{\mathbf{R}_+^N} \times (0, \infty)$. Since it follows from (1.3) that

$$(4.38) \quad \|\mathcal{F}(u(s))\|_{1, \partial \mathbf{R}_+^N} \preceq (1+s)^{-A_p} s^{-1/2}, \quad s > 0,$$

by (3.2) we obtain

$$(4.39) \quad \|I_1(t)\|_1 \preceq \int_{t/2}^t \|\mathcal{F}(u(s))\|_{1, \partial \mathbf{R}_+^N} ds \preceq \int_{t/2}^\infty s^{-A_p-1/2} ds \preceq t^{-A_p+1/2}, \quad t > 0.$$

Similarly it holds

$$t^{N/2} \|I_1(t)\|_\infty \preceq \int_{t/2}^t s^{-A_p-1/2} ds + t^{1/2} \int_{t/2}^t (t-s)^{-1/2} s^{-A_p-1/2} ds \preceq t^{-A_p+1/2}, \quad t > 0.$$

This together with (4.39) implies that

$$(4.40) \quad t^{(N/2)(1-1/q)} \|I_1(t)\|_q \preceq t^{-A_p+1/2}, \quad t > 0,$$

for any $q \in [1, \infty]$. On the other hand, for any $s \in [0, t/2]$, applying the mean value theorem, we see that there exists a constant $s_* \in (t-s, 1+t) \subset (t/2, 1+t)$ such that

$$|G_1(x_N, 1+t) - G_1(x_N, t-s)| \preceq (1+s) |\partial_t G_1(x_N, s_*)|, \quad x_N \in \mathbf{R}_+.$$

This together with (1.3), (3.1), and (3.2) yields

$$\|I_2(t)\|_1 \preceq \int_0^{t/2} (1+s)^{-A_p+1} s^{-1/2} s_*^{-1} ds \preceq t^{-1} \int_0^{t/2} (1+s)^{-A_p+1} s^{-1/2} ds, \quad t > 0.$$

Similarly we have

$$\begin{aligned} t^{N/2} \|I_2(t)\|_\infty &\leq t^{1/2} \int_0^{t/2} (1+s)^{-A_p+1} s^{-1/2} s_*^{-3/2} ds \\ &\leq t^{-1} \int_0^{t/2} (1+s)^{-A_p+1} s^{-1/2} ds, \quad t > 0. \end{aligned}$$

Therefore, for any $q \in [1, \infty]$, we obtain

$$\begin{aligned} (4.41) \quad t^{(N/2)(1-1/q)} \|I_2(t)\|_q &\leq t^{-1} \int_0^{t/2} (1+s)^{-A_p+1} s^{-1/2} ds \\ &= \begin{cases} O(t^{-1/2}) & \text{if } A_p \geq 1, \\ O(t^{-(A_p-1/2)}) & \text{if } 1/2 < A_p < 1, \end{cases} \end{aligned}$$

as $t \rightarrow \infty$. Substituting (2.15) with $n = 0$, (3.2), (4.38), (4.40), and (4.41) to (4.37) we have (2.19) with $K \in (0, 1)$. Furthermore, applying a similar argument as in the proof of [9, Proposition 5.1] with (4.3)–(4.6), we obtain (2.19) with $K = 1$ and (2.20), and the proof of Corollary 2.1 is complete. $\square$

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nuna adreso:

Department of Mathematical Sciences

Osaka Prefecture University

1-1 Gakuen-cho, Nakaku, Sakai

Osaka 599-8531

Japan

E-mail: kawakami@ms.osakafu-u.ac.jp

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