Global Existence and Decay Properties of Solutions for Coupled Degenerate Dissipative Hyperbolic Systems of Kirchhoff Type

By

Kosuke ONO

(The University of Tokushima, Japan)

Abstract. Consider the initial-boundary value problem for coupled degenerate dissipative hyperbolic systems of Kirchhoff type: $\rho u_{tt} - (\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2)^\gamma \Delta u + u_t = 0$, $\rho v_{tt} - (\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2)^\gamma \Delta v + v_t = 0$, with homogeneous Dirichlet boundary condition and $\rho > 0$ and $\gamma > 0$. When either the coefficient ρ or the initial data are appropriately small, we prove the global existence theorem by using several identities and the energy decay. Moreover, under the same assumption for ρ and the initial data, we derive the decay estimates of the solutions and their second order derivatives.

Key Words and Phrases. Dissipative wave equation, Degenerate, Kirchhoff type, Decay estimates.

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1. Introduction

In this paper we consider the global existence and decay rates of solutions to the initial-boundary value problem for the following coupled degenerate dissipative hyperbolic system:

$$(1.1) \quad \rho u_{tt} - (\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2)^\gamma \Delta u + u_t = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$(1.2) \quad \rho v_{tt} - (\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2)^\gamma \Delta v + v_t = 0 \quad \text{in } \Omega \times (0, \infty)$$

with

$$(1.3) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad u_t(x, 0) = u_1(x), \quad v_t(x, 0) = v_1(x),$$

$$(1.4) \quad u(x, t) = v(x, t) = 0 \quad \text{on } \partial\Omega \times (0, \infty)$$

where $u = u(x, t)$ and $v = v(x, t)$ are unknown real value functions, Ω is an open boundary domain in N -dimensional Euclidean space $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $\Delta = \nabla \cdot \nabla = \sum_{j=1}^N \partial^2 / \partial x_j^2$ is the Laplacian, $\|\cdot\| = \|\cdot\|_{L^2}$ is the usual norm of $L^2(\Omega)$, and ρ and γ are positive constants. The coupled hyperbolic system (1.3)–(1.4) will be useful for the research of amplitude vibrations of two kinds of elastic stretched strings (see [19]).

The coupled system (1.1)–(1.4) traces back to the single Kirchhoff type wave equation:

$$(1.5) \quad \rho u_{tt} - \left(\mu + \int_{\Omega} |\nabla u(x, t)|^2 dx \right)^{\gamma} \Delta u + \delta u_t = 0 \quad \text{in } \Omega \times (0, \infty)$$

with $u(0) = u_0$, $u_t(0) = u_1$, and $u|_{\partial\Omega} = 0$, which is called a non-degenerate equation when $\mu > 0$ and a degenerate one when $\mu = 0$. The single equation (1.5) was introduced by Kirchhoff [7] (also see [5], [6]).

In the non-degenerate case $\mu > 0$ in (1.5), the existence of a unique global solution was proved by Brito [3], Yamada [26], Nishihara [16], Ghisi and Gobbino [10] and the references cited there), and moreover, it is well-known that the energies have exponential decay rates.

In the degenerate case $\mu = 0$ in (1.5), Nishihara and Yamada [18] have shown the existence and uniqueness of the global solutions when the initial data sufficiently small and $\gamma \geq 1$. Ghisi [9] has treated the case of $0 < \gamma < 1$, and Ghisi and Gobbino [10] have extended the existence results to generalized nonlinear equations when $0 < \rho \ll 1$ is sufficiently small. The decay estimates of the solutions and the energy have been studied several authors (see [10], [12], [13], [18], [20], [21], [25], [26]).

In particular, Mizumachi [12] proved the lower decay estimate $\|u(t)\|^2 \geq C(1+t)^{-1}$ when $\gamma = 1$ (also see [20]).

Ghisi and Gobbino [11] derived the lower decay estimate $\|\nabla u(t)\|^2 \geq C(1+t)^{-1/\gamma}$ for $\gamma > 0$ when $0 < \rho \ll 1$ is sufficiently small (see also [17]). In previous paper [21], we obtained the lower decay estimate $\|u(t)\|^2 \geq C(1+t)^{-1/\gamma}$ for $\gamma > 0$ when either the coefficient $\rho > 0$ or the initial data are appropriately small, and derived the decay estimates of the second order derivatives.

On the other hand, when the hyperbolic system (1.1)–(1.4) is the non-degenerate case Park and Bae [19] have studied the existence and uniqueness of global solutions. In the special case $\gamma = 1$ in the system (1.1)–(1.4), we have derived the decay estimates of the solutions and their derivatives in [22].

The purpose of this paper is to prove the global existence theorem for the system (1.1)–(1.4) for any $\gamma > 0$ when either the coefficient $\rho > 0$ or the initial data are appropriately small, and moreover, to derive the decay estimates for the solutions and their derivatives as in [21]. In particular, we will obtain lower decay estimates of the solutions under the same assumption.

In order to derive the global existence of the solutions and the decay rate of the energy we will use the following energy and functional associated with (1.1)–(1.2):

$$(1.6) \quad E(u, v, u_t, v_t) \equiv \rho(\|u_t\|^2 + \|v_t\|^2) + \frac{1}{\gamma+1}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+1}$$

and

$$(1.7) \quad H(u, v, u_t, v_t) \equiv \rho \frac{\|\nabla u_t\|^2 + \|\nabla v_t\|^2}{(\|\nabla u\|^2 + \|\nabla v\|^2)^{\gamma+[1-\gamma]^+}} + \frac{\|\Delta u\|^2 + \|\Delta v\|^2}{(\|\nabla u\|^2 + \|\nabla v\|^2)^{[1-\gamma]^+}}$$

where $[a]^+ = \max\{0, a\}$, and we often write $E(t) \equiv E(u(t), v(t), u_t(t), v_t(t))$ and $H(t) \equiv H(u(t), v(t), u_t(t), v_t(t))$ for simplicity. In particular, we will use the following notations related with the initial data $\{u_0, v_0, u_1, v_1\}$:

$$E(0) \equiv \rho(\|u_1\|^2 + \|v_1\|^2) + \frac{1}{\gamma+1}(\|\nabla u_0\|^2 + \|\nabla v_0\|^2)^{\gamma+1},$$

$$H(0) \equiv \rho \frac{\|\nabla u_1\|^2 + \|\nabla v_1\|^2}{(\|\nabla u_0\|^2 + \|\nabla v_0\|^2)^{\gamma+[1-\gamma]^+}} + \frac{\|\Delta u_0\|^2 + \|\Delta v_0\|^2}{(\|\nabla u_0\|^2 + \|\nabla v_0\|^2)^{[1-\gamma]^+}},$$

and

$$I(0) \equiv 2\rho \frac{\|\nabla u_1\|^2 + \|\nabla v_1\|^2}{(\|\nabla u_0\|^2 + \|\nabla v_0\|^2)^{\gamma+1}} + \frac{\|\Delta u_0\|^2 + \|\Delta v_0\|^2}{\|\nabla u_0\|^2 + \|\nabla v_0\|^2}.$$

Moreover, we denote the Poincaré constant by c_* , that is, $c_* = \sup\{\|\phi\|/\|\nabla\phi\| \mid \phi \in H_0^1(\Omega), \phi \neq 0\}$, and then, it holds that $\|\phi\| \leq c_*\|\nabla\phi\|$.

Our main result is as follows.

Theorem 1.1. *Let the initial data $\{u_0, v_0, u_1, v_1\}$ belong to $(H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2$. Suppose that $u_0 \neq 0$ (or $v_0 \neq 0$) and*

$$(1.8) \quad ((\gamma+2)c_*^{\gamma-1})^2 \rho H(0)^\gamma < 1 \quad \text{if } \gamma \geq 1;$$

$$(1.9) \quad (\gamma+2)^2 \rho (H(0) + E(0)^{(\gamma+4)/(\gamma+1)} B(0)) < 1 \quad \text{if } 0 < \gamma < 1$$

with $B(0) = 2^6(1-\gamma)^2(1+\gamma^{-1})(4\rho+1)^2 I(0)^2 (E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)^2$. Then the problem (1.1)–(1.4) admits a unique global solution $(u(t), v(t))$ in the class $(C^0([0, \infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, \infty); H_0^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega)))^2$. Moreover, it holds that

$$(1.10) \quad C^{-1}(1+t)^{-1/\gamma} \leq \|\nabla^j u(t)\|^2 + \|\nabla^j v(t)\|^2 \leq C(1+t)^{-1/\gamma}, \quad j = 0, 1, 2,$$

$$(1.11) \quad \|\nabla^k u_t(t)\|^2 + \|\nabla^k v_t(t)\|^2 \leq C(1+t)^{-2-1/\gamma}, \quad k = 0, 1,$$

$$(1.12) \quad \|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 \leq C(1+t)^{-3-1/\gamma}$$

for $t \geq 0$ with some positive constant $C > 0$.

Theorem 1.1 follows from Theorem 3.2 and Theorem 4.6 in the continuing sections.

The notations we use in this paper are standard. The symbol $(\cdot, \cdot)$ means the inner product in the Hilbert space $L^2(\Omega)$ or some times duality between the space X and its dual X' . We put $[a]^+ = \max\{0, a\}$ where $1/[a]^+ = \infty$ if $[a]^+ = 0$. Positive constants will be denoted by C and will change from line to line.

2. Preliminaries

First we introduce the following functions which we will use through this paper:

$$\begin{aligned} K(t) &\equiv \|u(t)\|^2 + \|v(t)\|^2, & L(t) &\equiv \|u_t(t)\|^2 + \|v_t(t)\|^2, \\ M(t) &\equiv \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2, & X(t) &\equiv \|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2, \\ Y(t) &\equiv \|\nabla u_t(t)\|^2 + \|\nabla v_t(t)\|^2, & Z(t) &\equiv \|Au(t)\|^2 + \|Av(t)\|^2, \end{aligned}$$

and then, we see from (1.6) and (1.7) that

$$(2.1) \quad E(t) = \rho L(t) + \frac{1}{\gamma+1} M(t)^{\gamma+1} \quad \text{and} \quad H(t) = \rho \frac{Y(t)}{M(t)^{\gamma+[1-\gamma]^+}} + \frac{Z(t)}{M(t)^{[1-\gamma]^+}}.$$

By applying the Banach contraction mapping theorem to the problem (1.1)–(1.4), we obtain the following local existence theorem. The proof is standard and we omit it here (see [1], [2], [4], [20], [23], [24]).

Proposition 2.1. *Let the initial data $\{u_0, v_0, u_1, v_1\}$ belong to $(H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2$. Suppose that $M(0) > 0$. Then the problem (1.1)–(1.4) admits a unique global solution $(u(t), v(t))$ in the class $(C^0([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, T]; H_0^1(\Omega)) \cap C^2([0, T]; L^2(\Omega)))^2$ for some $T = T(\|u_0\|_{H^2}, \|v_0\|_{H^2}, \|u_1\|_{H^1}, \|v_1\|_{H^1}) > 0$. Moreover, if $M(t) > 0$ and $\|u(t)\|_{H^2} + \|v(t)\|_{H^2} + \|u_t(t)\|_{H^1} + \|v_t(t)\|_{H^1} < \infty$ for $t \geq 0$, then we can take $T = \infty$.*

By simple calculations, we have the following identities associated with (1.1)–(1.2).

Proposition 2.2. *Let the functions $u(t), v(t)$ satisfy the system (1.1)–(1.2). Let $k \geq 0$. Suppose that $M(t) > 0$. Then*

$$(2.2) \quad \frac{d}{dt} \rho L(t) + M(t)^\gamma \frac{d}{dt} M(t) + 2L(t) = 0,$$

$$(2.3) \quad \frac{d}{dt} \left(\rho \frac{L(t)}{M(t)^{\gamma+k}} + \frac{1}{M(t)^{k-1}} \right) + 2 \left(1 + \frac{\gamma+k}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{L(t)}{M(t)^{\gamma+k}} = -k \frac{M'(t)}{M(t)^k},$$

$$\begin{aligned}
 (2.4) \quad & \frac{d}{dt} \left(\rho \frac{Y(t)}{M(t)^{\gamma+k}} + \frac{Z(t)}{M(t)^k} \right) + 2 \left(1 + \frac{\gamma+k}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{Y(t)}{M(t)^{\gamma+k}} \\
 & = -k \frac{M'(t)}{M(t)^{k+1}} Z(t),
 \end{aligned}$$

$$\begin{aligned}
 (2.5) \quad & \frac{d}{dt} \left(\rho \frac{X(t)}{M(t)^{\gamma+k}} + \frac{Y(t)}{M(t)^k} \right) + 2 \left(1 + \frac{\gamma+k}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{X(t)}{M(t)^{\gamma+k}} \\
 & = -k \frac{M'(t)}{M(t)^{k+1}} Y(t) + 2\gamma \frac{M'(t)}{M(t)^{k+1}} ((\Delta u(t), u_{tt}) + (\Delta v(t), v_{tt})),
 \end{aligned}$$

and

$$(2.6) \quad \frac{d}{dt} \left(\frac{Z(t)}{M(t)} + \rho J(t) \right) = -2 \left(1 + \frac{\gamma+2}{2} \rho \frac{M'(t)}{M(t)} \right) J(t)$$

where

$$(2.7) \quad J(t) \equiv \frac{1}{M(t)^{\gamma+2}} \left(M(t) Y(t) - \frac{1}{4} |M'(t)|^2 \right) \quad (\geq 0).$$

Proof. Multiplying (1.1) and (1.2) by $2u_t$ and $2v_t$, respectively, and integrating them over Ω , and adding the resulting equations, we have (2.2). Moreover, multiplying (2.2) by $M(t)^{-\gamma-k}$, we obtain (2.3).

By the same way, multiplying (1.1) and (1.2) by $-2\Delta u_t$ and $-2\Delta v_t$, respectively, and integrating them over Ω , and adding the resulting equations, we have

$$\frac{d}{dt} \rho Y(t) + M(t)^\gamma \frac{d}{dt} Z(t) + 2Y(t) = 0.$$

And, multiplying this equation by $M(t)^{-\gamma-k}$, we obtain (2.4).

Multiplying (1.1) and (1.2) differentiated once with respect to t by $2u_{tt}$ and $2v_{tt}$, respectively, and integrating them over Ω , and adding the resulting equations, we have

$$\rho \frac{d}{dt} X(t) + M(t)^\gamma \frac{d}{dt} Y(t) + 2X(t) = 2\gamma M(t)^{\gamma-1} M'(t) ((\Delta u, u_{tt}) + (\Delta v, v_{tt})).$$

Adding, multiplying this equation by $M(t)^{-\gamma-k}$, we obtain (2.5).

Since we observe from (1.1) and (1.2) that

$$\begin{aligned}
 M(t)^\gamma Z'(t) &= 2((M(t)^\gamma \Delta u, \Delta u_t) + (M(t)^\gamma \Delta v, \Delta v_t)) \\
 &= -2Y(t) - \rho Y'(t)
 \end{aligned}$$

and

$$\begin{aligned}
 M(t)^\gamma Z(t) &= (M(t)^\gamma \Delta u, \Delta u) + (M(t)^\gamma \Delta v, \Delta v) \\
 &= -\frac{1}{2} M'(t) + \rho((u_{tt}, \Delta u) + (v_{tt}, \Delta v)) \\
 &= -\frac{1}{2} M'(t) + \rho \left(Y(t) - \frac{1}{2} M''(t) \right),
 \end{aligned}$$

we have

$$\begin{aligned}
 (2.8) \quad \frac{d}{dt} \frac{Z(t)}{M(t)} &= \frac{1}{M(t)^{\gamma+2}} (M(t)^\gamma Z'(t) M(t) - M(t)^\gamma Z(t) M'(t)) \\
 &= -2J(t) - \rho R(t),
 \end{aligned}$$

where $J(t)$ is defined by (2.7) and it is easy to see that $J(t)$ is non-negative, and we put

$$R(t) \equiv \frac{1}{M(t)^{\gamma+2}} \left(M(t) Y'(t) + M'(t) Y(t) - \frac{1}{2} M'(t) M''(t) \right).$$

On the other hand, we observe

$$(2.9) \quad \frac{d}{dt} J(t) = -(\gamma + 2) \frac{M'(t)}{M(t)} J(t) + R(t).$$

Thus, multiplying (2.9) by ρ and adding (2.8), we obtain the identity (2.6). $\square$

The energy $E(t)$ has the following decay estimate.

Proposition 2.3. *Let the functions $u(t)$, $v(t)$ satisfy the system (1.1)–(1.4) on $[0, T)$. Then, the energy $E(t)$ satisfies*

$$\begin{aligned}
 (2.10) \quad E(t) &\leq \left(E(0)^{-\gamma/(\gamma+1)} + \frac{\gamma}{\gamma+1} (2(4\rho+1))^2 \right. \\
 &\quad \left. \times (E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)^2)^{-1} [t-1]^+ \right)^{-(\gamma+1)/\gamma},
 \end{aligned}$$

that is,

$$(2.11) \quad M(t) \leq C(1+t)^{-1/\gamma} \quad \text{and} \quad L(t) \leq C(1+t)^{-1-1/\gamma} \quad \text{for } 0 \leq t < T.$$

Proof. The identity (2.2) yields

$$(2.12) \quad \frac{d}{dt} E(t) + 2L(t) = 0$$

and

$$(2.13) \quad E(t) + 2 \int_0^t L(s) ds = E(0).$$

In what follows, we assume that $T > 1$. Integrating (2.14) over $[t, t+1]$, $0 \leq t < T-1$, we have

$$(2.14) \quad 2 \int_t^{t+1} L(s) ds = E(t) - E(t+1) \quad (\equiv 2D(t)^2).$$

There exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$L(t_j) \leq 4D(t)^2 \quad \text{for } j = 1, 2.$$

On the other hand, multiplying (1.1) and (1.2) by u and v , respectively, and integrating them over Ω , and adding the resulting equations, we have

$$M(t)^{\gamma+1} = \rho L(t) - \frac{\rho}{2} \frac{d}{dt} K'(t) - \frac{1}{2} K'(t),$$

and integrating it over $[t_1, t_2]$, we obtain from (2.13) and (2.14) that

$$(2.15) \quad \begin{aligned} \int_{t_1}^{t_2} M(t)^{\gamma+1} ds &\leq \rho \int_{t_1}^{t_2} L(s) ds + \frac{\rho}{2} \sum_{j=1}^2 |K'(t_j)| + \frac{1}{2} \int_{t_1}^{t_2} |K'(s)| ds \\ &\leq \rho \int_t^{t+1} L(s) ds + \rho \sum_{j=1}^2 c_* M(t_j)^{1/2} L(t_j)^{1/2} \\ &\quad + \int_t^{t+1} c_* M(s)^{1/2} L(s)^{1/2} ds \\ &\leq \rho D(t)^2 + (4\rho + 1) c_* D(t) \sup_{t \leq s \leq t+1} M(s)^{1/2}, \end{aligned}$$

where we used the fact that $|K'(t)| \leq 2K(t)^{1/2} L(t)^{1/2} \leq 2c_* M(t)^{1/2} L(t)^{1/2}$. From (2.12), (2.14), and (2.15), we have

$$\begin{aligned} E(t) &= E(t_2) + 2 \int_t^{t_2} L(s) ds \\ &\leq 2 \int_{t_1}^{t_2} E(s) ds + 2 \int_t^{t+1} L(s) ds \\ &\leq 2(\rho + 1) \int_t^{t+1} L(s) ds + \frac{2}{\gamma + 1} \int_{t_1}^{t_2} M(s)^{\gamma+1} ds \\ &\leq 2(2\rho + 1) D(t)^2 + 2(4\rho + 1) c_* D(t) E(t)^{1/\{2(\gamma+1)\}}, \end{aligned}$$

where we used the facts that $M(t) \leq ((\gamma + 1)E(t))^{1/(\gamma+1)}$ and $E(t)$ is non-increasing.

Moreover, since $D(t) \leq E(0)^{\gamma/\{2(\gamma+1)\}} E(t)^{1/\{2(\gamma+1)\}}$ by (2.13) and (2.14), we observe from the Young inequality that

$$\begin{aligned} E(t) &\leq 2(4\rho + 1)(E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)D(t)E(t)^{1/\{2(\gamma+1)\}} \\ &\leq \frac{2\gamma + 1}{2(\gamma + 1)}(2(4\rho + 1)(E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)D(t))^{2(\gamma+1)/(2\gamma+1)} + \frac{1}{2(\gamma + 1)}E(t) \end{aligned}$$

and

$$E(t) \leq (2(4\rho + 1)(E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)D(t))^{2(\gamma+1)/(2\gamma+1)}.$$

From (2.14) we observe

$$\begin{aligned} (2.16) \quad E(t)^{1+\gamma/(\gamma+1)} &= E(t)^{(2\gamma+1)/(\gamma+1)} \leq (2(4\rho + 1)(E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)D(t))^2 \\ &\leq 2(4\rho + 1)^2(E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)^2(E(t) - E(t+1)). \end{aligned}$$

Applying Lemma 2.4 below to (2.16), we obtain the desired estimate (2.10). $\square$

In order to get the decay rate of the energy $E(t)$, we used the following Nakao's inequality in the proof of Proposition 2.3 (see [13], [14], [15] for the proof).

Lemma 2.4. *Let $\phi(t)$ be a non-negative and non-increasing function on $[0, \infty)$ and satisfy*

$$\phi(t)^{1+\alpha} \leq k(\phi(t) - \phi(t+1))$$

with certain constants $k \geq 0$ and $\alpha > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \leq (\phi(0)^{-\alpha} + \alpha k^{-1}[t-1]^+)^{-1/\alpha} \quad \text{for } t \geq 0.$$

3. Global existence

Proposition 3.1. *In addition to the assumption of Theorem 1.1 with (1.8) and (1.9), suppose that $M(t) > 0$ for $0 \leq t < T$. Then, it holds that*

$$(3.1) \quad H(t) \leq \begin{cases} H(0) & \text{if } \gamma \geq 1; \\ H(0) + E(0)^{(\gamma+4)/(\gamma+1)} B(0) & \text{if } 0 < \gamma < 1 \end{cases}$$

for $0 \leq t < T$. Moreover, we have

$$(3.2) \quad \rho \frac{|M'(t)|}{M(t)} \leq \delta < \frac{2}{\gamma + 2}$$

with $\delta = 2\rho^{1/2}(H(0) + E(0)^{(\gamma+4)/(\gamma+1)}B(0))^{1/2}$ and

$$(3.3) \quad \frac{Z(t)}{M(t)} \leq \frac{Z(0)}{M(0)} + J(0) \leq I(0)$$

for $0 \leq t < T$.

Proof. (1) Let $\gamma \geq 1$. It follows from (2.4) with $k = 0$ that

$$(3.4) \quad \frac{d}{dt}H(t) + 2\left(1 + \frac{\gamma}{2}\rho\frac{M'(t)}{M(t)}\right)\frac{Y(t)}{M(t)^\gamma} = 0$$

where $H(t) \equiv \rho Y(t)/M(t)^\gamma + Z(t)$ for $\gamma \geq 1$.

We observe that

$$(3.5) \quad \rho\frac{|M'(t)|}{M(t)} \leq 2\rho^{1/2}c_*^{\gamma-1}\left(\rho\frac{Y(t)}{M(t)^\gamma}Z(t)^{\gamma-1}\right)^{1/2} \leq 2c_*^{\gamma-1}(\rho H(t)^\gamma)^{1/2}.$$

Thus, if $(\gamma c_*^{\gamma-1})^2 \rho H(0)^\gamma < 1$, then there exists $T_1 \in (0, T]$ such that

$$(3.6) \quad (\gamma c_*^{\gamma-1})^2 \rho H(t)^\gamma \leq 1$$

for $0 \leq t \leq T_1$, and we have from (3.4)–(3.6) that

$$(3.7) \quad \frac{d}{dt}H(t) \leq 0 \quad \text{or} \quad H(t) \leq H(0)$$

for $0 \leq t \leq T_1$. Then

$$(\gamma c_*^{\gamma-1})^2 \rho H(t)^\gamma \leq (\gamma c_*^{\gamma-1})^2 \rho H(0)^\gamma < 1.$$

Therefore we see that (3.7) holds true for $0 \leq t < T$.

Moreover, if $((\gamma + 2)c_*^{\gamma-1})^2 \rho H(0)^\gamma < 1$, then we obtain from (3.5) that

$$\left(\frac{\gamma + 2}{2}\rho\frac{|M'(t)|}{M(t)}\right)^2 \leq ((\gamma + 2)c_*^{\gamma-1})^2 \rho H(t)^\gamma \leq ((\gamma + 2)c_*^{\gamma-1})^2 \rho H(0)^\gamma < 1$$

for $0 \leq t < T$, and from (2.6) and (2.7) that

$$\frac{d}{dt}\left(\frac{Z(t)}{M(t)} + \rho J(t)\right) = -2\left(1 + \frac{\gamma + 2}{2}\rho\frac{M'(t)}{M(t)}\right)J(t) \leq 0$$

or

$$\frac{Z(t)}{M(t)} + \rho J(t) \leq \frac{Z(0)}{M(0)} + \rho J(0) \leq I(0)$$

for $0 \leq t < T$.

(2) Let $0 < \gamma < 1$. It follows from (2.4) with $k = 1 - \gamma$ that

$$\begin{aligned} \frac{d}{dt}H(t) + 2\left(1 + \frac{\rho M'(t)}{2M(t)}\right) \frac{Y(t)}{M(t)} &= -(1-\gamma)M'(t)M(t)^\gamma Z(t) \\ &\leq (1-\gamma)\left(\frac{Y(t)}{M(t)}\right)^{1/2} \frac{Z(t)}{M(t)} M(t)^{\gamma+2} \end{aligned}$$

where $H(t) \equiv \rho Y(t)/M(t) + Z(t)/M(t)^{1-\gamma}$ for $0 < \gamma < 1$. From the Young inequality we observe

$$\begin{aligned} (3.8) \quad \frac{d}{dt}H(t) + \left(1 + \rho \frac{M'(t)}{M(t)}\right) \frac{Y(t)}{M(t)} &\leq \left((1-\gamma) \frac{Z(t)}{M(t)} M(t)^{\gamma+2}\right)^2 \\ &\leq \left(2^2(1-\gamma) \frac{Z(t)}{M(t)} E(0)^{2/(\gamma+1)} E(t)^{\gamma/(\gamma+1)}\right)^2 \end{aligned}$$

where we used the fact that $M(t)^{\gamma+2} \leq ((\gamma+1)E(t))^{(\gamma+2)/(\gamma+1)} \leq 2^2 E(0)^{2/(\gamma+1)} E(t)^{\gamma/(\gamma+1)}$ at the last inequality.

On the other hand, we observe that

$$(3.9) \quad \rho \frac{|M'(t)|}{M(t)} \leq 2\rho^{1/2} \left(\rho \frac{Y(t)}{M(t)}\right)^{1/2} \leq 2(\rho H(t))^{1/2},$$

and from (2.10) that

$$\begin{aligned} (3.10) \quad &\int_0^t E(s)^{2\gamma/(\gamma+1)} ds \\ &\leq \int_0^t \left(E(0)^{-\gamma/(\gamma+1)} + \frac{\gamma}{\gamma+1} (2(4\rho+1))^2 \right. \\ &\quad \left. \times (E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)^2)^{-1} [s-1]^+ \right)^{-2} ds \\ &\leq E(0)^{2\gamma/(\gamma+1)} + 2(1+\gamma^{-1})(4\rho+1)^2 E(0)^{\gamma/(\gamma+1)} (E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)^2 \\ &\leq 2^2(1+\gamma^{-1})(4\rho+1)^2 E(0)^{\gamma/(\gamma+1)} (E(0)^{\gamma/\{2(\gamma+1)\}} + c_*)^2 \quad (\equiv K(0)). \end{aligned}$$

If $(\gamma+2)^2 \rho(H(0) + E(0)^{(\gamma+4)/(\gamma+1)} B(0)) < 1$, there exists $T_2 \in (0, T]$ such that

$$(3.11) \quad (\gamma+2)^2 \rho H(t) \leq 1$$

for $0 \leq t \leq T_2$. Since

$$(3.12) \quad \frac{\gamma+2}{2} \rho \frac{|M'(t)|}{M(t)} \leq (\gamma+2)(\rho H(t))^{1/2} \leq 1,$$

we observe from (2.6) and (2.7) that

$$\frac{d}{dt} \left(\frac{Z(t)}{M(t)} + \rho J(t) \right) = -2 \left(1 + \frac{\gamma+2}{2} \rho \frac{M'(t)}{M(t)} \right) J(t) \leq 0$$

or

$$(3.13) \quad \frac{Z(t)}{M(t)} + \rho J(t) \leq \frac{Z(0)}{M(0)} + \rho J(0) \leq I(0)$$

for $0 \leq t \leq T_2$. Moreover, we have from (3.8) that

$$\frac{d}{dt} H(t) \leq (2^2(1-\gamma)I(0)E(0)^{2/(\gamma+1)})^2 E(t)^{2\gamma/(\gamma+1)}$$

and from (3.10) that

$$(3.14) \quad \begin{aligned} H(t) &\leq H(0) + (2^2(1-\gamma)I(0)E(0)^{2/(\gamma+1)})^2 K(0) \\ &= H(0) + E(0)^{(\gamma+4)/(\gamma+1)} B(0) \end{aligned}$$

for $0 \leq t \leq T_2$. Then,

$$(\gamma+2)^2 \rho H(t) \leq (\gamma+2)^2 \rho (H(0) + E(0)^{(\gamma+4)/(\gamma+1)} B(0)) < 1.$$

Therefore, we see that

$$\frac{\gamma+2}{2} \rho \frac{|M'(t)|}{M(t)} \leq (\gamma+2)(\rho H(t))^{1/2} \leq \frac{\gamma+2}{2} \delta < 1$$

for $0 \leq t < T$, and (3.13) and (3.14) hold true for $0 \leq t < T$. $\square$

Theorem 3.2. Suppose that the assumption of Theorem 1.1 is fulfilled. Then, the problem (1.1)–(1.4) admits a unique solution $(u(t), v(t))$ is the class $(C^0([0, \infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, \infty); H_0^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega)))^2$, and it holds that $M(t) > 0$,

$$(3.15) \quad E(t) \equiv \rho L(t) + \frac{1}{\gamma+1} M(t)^{\gamma+1} \leq C(1+t)^{-1-1/\gamma},$$

$$(3.16) \quad \rho \frac{|M'(t)|}{M(t)} \leq \delta < \frac{2}{\gamma+2},$$

$$(3.17) \quad \frac{Z(t)}{M(t)} \leq I(0) \quad \text{and} \quad Z(t) \leq C(1+t)^{-1/\gamma} \quad \text{for } t \geq 0.$$

Proof. Since $M(0) > 0$, putting $T \equiv \sup\{t \in [0, \infty) \mid M(s) > 0 \text{ for } 0 \leq s < t\}$, we see that $T > 0$ and $M(t) > 0$ for $0 \leq t < T$. Moreover, it follows from (3.1) that

$$(3.18) \quad H(t) \leq C < \infty \quad \text{for } 0 \leq t < T.$$

If $T < \infty$, then $M(T) = 0$, and hence, we observe from (3.18) that $\lim_{t \rightarrow T} Y(t) = 0$ and $\lim_{t \rightarrow T} E(t) = \lim_{t \rightarrow T} E(u(t), v(t), u_t(t), v_t(t)) = 0$.

We perform the change of variable $s = T - t$ or $t = T - s$, then the functions $U(s) = u(T - t)$ and $V(s) = v(T - t)$ satisfy the following system

$$\begin{cases} U_{ss} + (\|\nabla U(s)\|^2 + \|\nabla V(s)\|^2)^\gamma \Delta U - U_s = 0, \\ V_{ss} + (\|\nabla U(s)\|^2 + \|\nabla V(s)\|^2)^\gamma \Delta V - V_s = 0. \end{cases}$$

Multiplying the above equations by $2U_s$ and $2V_s$, respectively, and integrating them over Ω , and adding the resulting equations, we have

$$\begin{aligned} \frac{d}{ds} E(U(s), V(s), U_s(s), V_s(s)) &= (\|U_s(s)\|^2 + \|V_s(s)\|^2) \\ &\leq \frac{2}{\rho} E(U(s), V(s), U_s(s), V_s(s)). \end{aligned}$$

Since $E(U(0), V(0), U_s(0), V_s(0)) = \lim_{t \rightarrow T} E(u(t), v(t), u_t(t), v_t(t)) = 0$, we observe

$$E(U(s), V(s), U_s(s), V_s(s)) \leq \frac{2}{\rho} \int_0^s E(U(\tau), V(\tau), U_s(\tau), V_s(\tau)) d\tau.$$

Applying the Gronwall inequality to the above inequality, we see that $E(U(s), V(s), U_s(s), V_s(s)) = 0$ on $[0, T]$ or $E(u(t), v(t), u_t(t), v_t(t)) = 0$ on $[0, T]$, which contradicts $M(0) \neq 0$, and hence, we obtain that $T = \infty$ and $M(t) > 0$ for $t \geq 0$.

Thus, using (3.18) we observe that $Y(t) + Z(t) \leq C$ for $t \geq 0$. Therefore, by the second statement of Proposition 2.1, we see that the problem (1.1)–(1.4) admits a unique solution $(u(t), v(t))$. Moreover, from Propositions 2.3–3.1, we obtain the desired estimates (3.15)–(3.17). $\square$

4. Decay rates

First we will improve the decay rate of $L(t)$ given by (3.15).

Proposition 4.1. *The function $L(t)$ satisfies*

$$(4.1) \quad \frac{L(t)}{M(t)^\gamma} \leq C(1+t)^{-1-1/\gamma} \quad \text{and} \quad L(t) \leq C(1+t)^{-2-1/\gamma} \quad \text{for } t \geq 0.$$

Proof. The identity (2.3) with $k = 1$ yields

$$\rho \frac{d}{dt} \frac{L(t)}{M(t)^{\gamma+1}} + 2 \left(1 + \frac{\gamma+1}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{L(t)}{M(t)^{\gamma+1}} = - \frac{M'(t)}{M(t)},$$

and we observe from the Young inequality and (3.15)–(3.17) that

$$\rho \frac{d}{dt} \frac{L(t)}{M(t)^{\gamma+1}} + \frac{2}{\gamma+2} \frac{L(t)}{M(t)^{\gamma+1}} \leq 2 \left(\frac{L(t)}{M(t)^{\gamma+1}} \right)^{1/2} \left(\frac{Z(t)}{M(t)} M(t)^\gamma \right)^{1/2}$$

and

$$\rho \frac{d}{dt} \frac{L(t)}{M(t)^{\gamma+1}} + \frac{1}{\gamma+2} \frac{L(t)}{M(t)^{\gamma+1}} \leq C \frac{Z(t)}{M(t)} M(t)^\gamma \leq C(1+t)^{-1},$$

and hence, $\rho L(t)/M(t)^{\gamma+1} \leq C(1+t)^{-1}$ which gives the desired estimate (4.1). $\square$

Next we will the lower decay estimate of $M(t)$.

Proposition 4.2. *The function $M(t)$ satisfies*

$$(4.2) \quad M(t) \geq C'(1+t)^{-1/\gamma} \quad \text{for } t \geq 0$$

with some positive constant $C' > 0$.

Proof. (1) Let $\gamma \geq 1$. The identity (2.3) with $k = 2$ yields

$$\begin{aligned} & \frac{d}{dt} \left(\rho \frac{L(t)}{M(t)^{\gamma+2}} + \frac{1}{M(t)} \right) + 2 \left(1 + \frac{\gamma+2}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{L(t)}{M(t)^{\gamma+2}} \\ &= - \frac{M'(t)}{M(t)^2} \leq 4 \left(\frac{L(t)}{M(t)^{\gamma+2}} \right)^{1/2} \left(\frac{Z(t)}{M(t)} M(t)^{\gamma-1} \right)^{1/2}, \end{aligned}$$

and we observe from the Young inequality and (3.15)–(3.17) that

$$\frac{d}{dt} \left(\rho \frac{L(t)}{M(t)^{\gamma+2}} + \frac{1}{M(t)} \right) \leq C \frac{Z(t)}{M(t)} M(t)^{\gamma-1} \leq C(1+t)^{-1+1/\gamma},$$

and hence,

$$\rho \frac{L(t)}{M(t)^{\gamma+2}} + \frac{1}{M(t)} \leq C(1+t)^{1/\gamma}$$

which gives the desired estimate (4.2) for $\gamma \geq 1$.

(2) Let $0 < \gamma < 1$. The identity (2.3) with $k = \gamma + 1$ yields

$$\begin{aligned} & \frac{d}{dt} \left(\rho \frac{L(t)}{M(t)^{2\gamma+1}} + \frac{1}{M(t)^\gamma} \right) + 2 \left(1 + \frac{2\gamma+1}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{L(t)}{M(t)^{2\gamma+1}} \\ &= -(\gamma+1) \frac{M'(t)}{M(t)^{\gamma+1}} \leq 2(\gamma+1) \left(\frac{L(t)}{M(t)^{2\gamma+1}} \right)^{1/2} \left(\frac{Z(t)}{M(t)} \right)^{1/2}, \end{aligned}$$

and we observe from the Young inequality and (3.16)–(3.17) that

$$\frac{d}{dt} \left(\rho \frac{L(t)}{M(t)^{2\gamma+1}} + \frac{1}{M(t)^\gamma} \right) \leq C \frac{Z(t)}{M(t)} \leq C,$$

and hence,

$$\rho \frac{L(t)}{M(t)^{2\gamma+1}} + \frac{1}{M(t)^\gamma} \leq C(1+t)$$

which gives the desired estimate (4.2) for $0 < \gamma < 1$. □

We will derive the decay estimates of $Y(t)$ and $X(t)$.

Proposition 4.3. *The functions $Y(t)$ and $X(t)$ satisfy*

$$(4.3) \quad Y(t) \leq C(1+t)^{-2-1/\gamma} \quad \text{and} \quad X(t) \leq C(1+t)^{-3-1/\gamma} \quad \text{for } t \geq 0.$$

Proof. The identity (2.5) with $k = 0$ yields

$$(4.4) \quad \begin{aligned} \frac{d}{dt} F(t) + 2 \left(1 + \frac{\gamma}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{X(t)}{M(t)^\gamma} &= 2\gamma \frac{M'(t)}{M(t)} ((\Delta u, u_t) + (\Delta v, v_t)) \\ &\leq 4\gamma \left(\frac{X(t)}{M(t)^\gamma} \right)^{1/2} \frac{Z(t)}{M(t)} (L(t)M(t)^\gamma)^{1/2} \end{aligned}$$

where we put

$$F(t) \equiv \rho \frac{X(t)}{M(t)^\gamma} + Y(t).$$

We observe from the Young inequality and (3.15)–(3.17) that

$$(4.5) \quad \frac{d}{dt} F(t) + b \frac{X(t)}{M(t)^\gamma} \leq C \left(\frac{Z(t)}{M(t)} \right)^2 L(t)M(t)^\gamma \leq C(1+t)^{-3-1/\gamma}$$

with $b = 2/(\gamma + 2) > 0$, and hence, $F(t) \leq C$ for $t \geq 0$.

For $t > 0$, integrating (4.5) over $[t, t+1]$, we have

$$(4.6) \quad b \int_t^{t+1} \frac{X(s)}{M(s)^\gamma} ds \leq F(t) - F(t+1) + C(1+t)^{-3-1/\gamma} \quad (\equiv bD(t)^2).$$

There exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$(4.7) \quad \frac{X(t_j)}{M(t_j)^\gamma} \leq 4D(t)^2 \quad \text{for } j = 1, 2,$$

and there exists $t_* \in [t_1, t_2]$ such that

$$(4.8) \quad F(t_*) \leq 2 \int_{t_1}^{t_2} F(s) ds.$$

On the other hand, multiplying (1.1) and (1.2) differentiated once with respect to t by u_t and v_t , respectively, and integrating them over Ω , and adding the resulting equations, we have

$$M(t)^\gamma Y(t) + \frac{\gamma}{2} M(t)^{\gamma-1} |M'(t)|^2 = \rho X(t) - \frac{d}{dt} \frac{\rho}{2} L'(t) - \frac{1}{2} L'(t),$$

and multiplying it by $M(t)^{-\gamma}$, we have

$$Y(t) + \frac{\gamma}{2} \frac{|M'(t)|^2}{M(t)} = \rho \frac{X(t)}{M(t)^\gamma} - \frac{\rho}{2} \frac{d}{dt} \frac{L'(t)}{M(t)^\gamma} - \left(1 + \frac{\gamma}{2} \rho \frac{M'(t)}{M(t)}\right) \frac{L'(t)}{M(t)^\gamma},$$

and integrating it over $[t_1, t_2]$, we observe from (3.16), (4.1), (4.6), and (4.7) that

$$\begin{aligned} (4.9) \quad & \int_{t_1}^{t_2} \left(Y(s) + \frac{\gamma}{2} \frac{|M'(s)|^2}{M(s)} \right) ds \\ & \leq \rho \int_t^{t+1} \frac{X(s)}{M(s)^\gamma} ds + \rho \sum_{j=1}^2 \left(\frac{L(t_j)}{M(t_j)^\gamma} \frac{X(t_j)}{M(t_j)^\gamma} \right)^{1/2} \\ & \quad + C \int_t^{t+1} \left(\frac{L(s)}{M(s)^\gamma} \frac{X(s)}{M(s)^\gamma} \right)^{1/2} ds \\ & \leq CD(t)^2 + C(1+t)^{-1/2-1/(2\gamma)} D(t), \end{aligned}$$

and from (4.6) and (4.9) that

$$(4.10) \quad \int_{t_1}^{t_2} F(s) ds \leq CD(t)^2 + C(1+t)^{-1/2-1/(2\gamma)} D(t).$$

For $\tau \in [t, t+1]$, integrating (4.4) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (3.16) that

$$\begin{aligned} F(\tau) &= F(t_*) + \int_\tau^{t_*} \left(2 \left(1 + \frac{\gamma}{2} \rho \frac{M'(s)}{M(s)} \right) \frac{X(s)}{M(s)^\gamma} - 2\gamma \frac{M'(s)}{M(s)} ((\Delta u, u_t) + (\Delta v, v_t)) \right) ds \\ &\leq F(t_*) + \int_t^{t+1} \left(C \frac{X(s)}{M(s)^\gamma} + C \left(\frac{X(s)}{M(s)^\gamma} \right)^{1/2} \frac{Z(s)}{M(s)} (L(s) M(s)^\gamma)^{1/2} \right) ds \\ &\leq F(t_*) + C \int_t^{t+1} \frac{X(s)}{M(s)^\gamma} ds + C \int_t^{t+1} \left(\frac{Z(s)}{M(s)} \right)^2 L(s) M(s)^\gamma ds \end{aligned}$$

and from (3.15), (3.17), (4.1), (4.7), (4.8), and (4.10) that

$$\sup_{t \leq \tau \leq t+1} F(\tau) \leq CD(t)^2 + C(1+t)^{-1/2-1/(2\gamma)} D(t) + C(1+t)^{-3-1/\gamma}$$

and from (4.6) that

$$\begin{aligned}
\sup_{t \leq s \leq t+1} F(s)^2 &\leq C(D(t)^2 + (1+t)^{-1-1/\gamma})D(t)^2 + C(1+t)^{-6-2/\gamma} \\
&\leq C(F(t) + (1+t)^{-1-1/\gamma})(F(t) - F(t+1) + C(1+t)^{-3-1/\gamma}) \\
&\quad + C(1+t)^{-6-2/\gamma} \\
&\leq C(F(t) + (1+t)^{-1-1/\gamma})(F(t) - F(t+1)) \\
&\quad + CF(t)(1+t)^{-3-1/\gamma} + C(1+t)^{-4-2/\gamma}
\end{aligned}$$

and from the Young inequality that

$$\begin{aligned}
(4.11) \quad \sup_{t \leq s \leq t+1} F(s)^2 &\leq C(F(t) + (1+t)^{-1-1/\gamma})(F(t) - F(t+1)) \\
&\quad + C(1+t)^{-4-2/\gamma}.
\end{aligned}$$

Therefore, applying Lemma 4.4 below to (4.11), we obtain

$$F(t) \equiv \rho \frac{X(t)}{M(t)^\gamma} + Y(t) \leq C(1+t)^{-2-1/\gamma}$$

which gives the desired estimates. $\square$

In order to get the decay rate of the function $F(t)$, we used the following generalized Nakao type inequality in the proof of Proposition 4.3 (see [14], [8], [21] for the proof).

Lemma 4.4. *Let $\phi(t)$ be a non-negative function on $[0, \infty)$ and satisfy*

$$\sup_{t \leq s \leq t+1} \phi(s)^{1+\alpha} \leq (k_0 \phi(t)^\alpha + k_1(1+t)^{-\beta})(\phi(t) - \phi(t+1)) + k_2(1+t)^{-\gamma}$$

with certain constants $k_0, k_1, k_2 \geq 0$, $\alpha > 0$, $\beta \geq 0$, and $\gamma > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \leq C_0(1+t)^{-\theta}, \quad \theta = \min \left\{ \frac{1+\beta}{\alpha}, \frac{\gamma}{1+\alpha} \right\}$$

for $t \geq 0$ with some positive constant C_0 depending on $\phi(0)$.

Finally, we will derive the lower decay estimate of $K(t)$.

Proposition 4.5. *The function $K(t)$ satisfies*

$$(4.12) \quad K(t) \geq C'(1+t)^{-1/\gamma} \quad \text{for } t \geq 0$$

with some positive constant $C' > 0$.

Proof. From the system (1.1)–(1.2), it follows that

$$\begin{aligned}
 (4.13) \quad \frac{d}{dt} \frac{M(t)}{K(t)} &= \frac{-2}{K(t)} \left(\left(\Delta u + \frac{M(t)}{K(t)} u, u_t \right) + \left(\Delta v + \frac{M(t)}{K(t)} v, v_t \right) \right) \\
 &= \frac{2\rho}{K(t)} \left(\left(\Delta u + \frac{M(t)}{K(t)} u, u_{tt} \right) + \left(\Delta v + \frac{M(t)}{K(t)} v, v_{tt} \right) \right) \\
 &\quad - \frac{2M(t)^\gamma}{K(t)} \left(\left(\Delta u + \frac{M(t)}{K(t)} u, \Delta u \right) + \left(\Delta v + \frac{M(t)}{K(t)} v, \Delta v \right) \right).
 \end{aligned}$$

Since we observe

$$\left(\Delta u + \frac{M(t)}{K(t)} u, \Delta u \right) + \left(\Delta v + \frac{M(t)}{K(t)} v, \Delta v \right) = \left\| \Delta u + \frac{M(t)}{K(t)} u \right\|^2 + \left\| \Delta v + \frac{M(t)}{K(t)} v \right\|^2,$$

we have from (4.13) that

$$\begin{aligned}
 \frac{d}{dt} \frac{M(t)}{K(t)} + \frac{2M(t)^\gamma}{K(t)} &\left(\left\| \Delta u + \frac{M(t)}{K(t)} u \right\|^2 + \left\| \Delta v + \frac{M(t)}{K(t)} v \right\|^2 \right) \\
 &= \frac{2\rho}{K(t)} \left(\left(\Delta u + \frac{M(t)}{K(t)} u, u_{tt} \right) + \left(\Delta v + \frac{M(t)}{K(t)} v, v_{tt} \right) \right) \\
 &\leq 2\rho \left(\frac{2M(t)^\gamma}{K(t)} \right)^{1/2} \left(\left\| \Delta u + \frac{M(t)}{K(t)} u \right\|^2 + \left\| \Delta v + \frac{M(t)}{K(t)} v \right\|^2 \right)^{1/2} \left(\frac{X(t)}{K(t)M(t)^\gamma} \right)^{1/2},
 \end{aligned}$$

and from the Young inequality that

$$\frac{d}{dt} \frac{M(t)}{K(t)} \leq C \frac{X(t)}{K(t)M(t)^\gamma} = C \frac{M(t)}{K(t)} \frac{X(t)}{M(t)^{\gamma+1}},$$

and from the Gronwall inequality and (4.2) and (4.3) that

$$\begin{aligned}
 \frac{M(t)}{K(t)} &\leq \frac{M(0)}{K(0)} \exp \left(C \int_0^t \frac{X(s)}{M(s)^{\gamma+1}} ds \right) \\
 &\leq \frac{M(0)}{K(0)} \exp \left(C \int_0^t (1+s)^{-2} ds \right) \leq C
 \end{aligned}$$

and hence, we obtain from (4.2) that

$$K(t) \geq C' M(t) \geq C'(1+t)^{-1/\gamma}$$

which implies the desired estimate (4.12). $\square$

Summing up Propositions 4.1–4.5 in this section, we arrive at the following theorem.

Theorem 4.6. *Under the assumption of Theorem 1.1, it holds that*

$$\begin{aligned} M(t) &\geq C'(1+t)^{-1/\gamma}, & K(t) &\geq C'(1+t)^{-1/\gamma}, \\ Y(t) &\leq C(1+t)^{-2-1/\gamma}, & X(t) &\leq C(1+t)^{-3-1/\gamma} \quad \text{for } t \geq 0 \end{aligned}$$

with some positive constants $C, C' > 0$.

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References

- [1] Arosio, A. and Garavaldi, S., On the mildly degenerate Kirchhoff string, *Math. Methods Appl. Sci.*, **14** (1991), 177–195.
- [2] Arosio, A. and Panizzi, S., On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.*, **348** (1996), 305–330.
- [3] de Brito, E. H., The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability, *Applicable Anal.*, **13** (1982), 219–233.
- [4] Crippa, H. R., On local solutions of some mildly degenerate hyperbolic equations, *Non-linear Analysis*, **21** (1993), 565–574.
- [5] Carrier, G. F., On the non-linear vibration problem of the elastic string, *Quart. Appl. Math.*, **3** (1945), 157–165.
- [6] Dickey, R. W., Infinite systems of nonlinear oscillation equations with linear damping, *SIAM J. Appl. Math.*, **19** (1970), 208–214.
- [7] Kirchhoff, G., *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.
- [8] Kawashima, S., Nakao, M. and Ono, K., On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, *J. Math. Soc. Japan*, **47** (1995), 617–653.
- [9] Ghisi, M., Global solutions for dissipative Kirchhoff strings with non-Lipschitz nonlinear term, *J. Differential Equations*, **230** (2006), 128–139.
- [10] Ghisi, M. and Gobbino, M., Global existence and asymptotic behaviour for a mildly degenerate dissipative hyperbolic equation of Kirchhoff type, *Asymptot. Anal.*, **40** (2004), 25–36.
- [11] Ghisi, M. and Gobbino, M., Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: time-decay estimates, *J. Differential Equations*, **245** (2008), 2979–3007.
- [12] Mizumachi, T., Decay properties of solutions to degenerate wave equations with dissipative terms, *Adv. Differential Equations*, **2** (1997), 573–592.
- [13] Nakao, M., Decay of solutions of some nonlinear evolution equations, *J. Math. Anal. Appl.*, **60** (1977), 542–549.
- [14] Nakao, M., A difference inequality and its application to nonlinear evolution equations, *J. Math. Soc. Japan*, **30** (1978), 747–762.

- [15] Nakao, M. and Ono, K., Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, *Math. Z.*, **214** (1993), 325–342.
- [16] Nishihara, K., Global existence and asymptotic behaviour of the solution of some quasilinear hyperbolic equation with linear damping, *Funkcial. Ekvac.*, **32** (1989), 343–355.
- [17] Nishihara, K., Decay properties of solutions of some quasilinear hyperbolic equations with strong damping, *Nonlinear Anal.*, **21** (1993), 17–21.
- [18] Nishihara, K. and Yamada, Y., On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms, *Funkcial. Ekvac.*, **33** (1990), 151–159.
- [19] Park, J. Y. and Bae, J. J., On existence of solutions of nondegenerate wave equations with nonlinear damping terms, *Nihonkai Math. J.*, **9** (1998), 27–46.
- [20] Ono, K., On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation, *Math. Methods Appl. Sci.*, **20** (1997), 151–177.
- [21] Ono, K., On sharp decay estimates of solutions for mildly degenerate dissipative wave equations of Kirchhoff type, *Math. Methods Appl. Sci.*, **34** (2011), 1339–1352.
- [22] Ono, K., Global Existence and Decay Rate for a Coupled Degenerate Hyperbolic System with Dissipation, *Sci. Math. Jpn.*, **74** (2011), 1–13.
- [23] Strauss, W. A., *Nonlinear wave equations*, CBMS Regional Conference Series in Mathematics, 73, American Mathematical Society, Providence, RI., 1989.
- [24] Temam, R., *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, 68, Springer-Verlag, New York, 1988.
- [25] Yamada, Y., On the decay of solutions for some nonlinear evolution equations of second order, *Nagoya Math. J.*, **73** (1979), 67–98.
- [26] Yamada, Y., On some quasilinear wave equations with dissipative terms, *Nagoya Math. J.*, **87** (1982), 17–39.

nuna adreso:

Department of Mathematical Sciences

The University of Tokushima

Tokushima 770-8502

Japan

E-mail: ono@ias.tokushima-u.ac.jp

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