Global Existence and Decay Estimates for Quasilinear Wave Equations with Nonuniform Dissipative Term

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Abstract. In this paper, we study a Cauchy problem for quasilinear wave equations with dissipative term in Sobolev space $H^L \times H^{L-1}$ ($L \geq [d/2] + 3$). The coefficients of the dissipative term depends on space variables and may vanish in some compact region. In order to control the derivatives of the dissipative coefficients, we introduce a rescaling argument. Using the argument, we obtain a global existence theorem and decay estimates with additional assumptions for the initial data.

Key Words and Phrases. Dissipative quasilinear wave equations, Space variable coefficients, Time decay estimates.

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1. Introduction

In this paper, we consider the following Cauchy problem for quasilinear wave equations with nonuniform dissipative term in $\mathbf{R}^d$ ($d \geq 1$):

$$(DW) \quad \begin{cases} (\partial_t^2 - \Delta + B(x)\partial_t)u(t, x) = N[u, u](t, x), & (t, x) \in [0, \infty) \times \mathbf{R}^d, \\ u(0, x) = u_0(x), \partial_t u(0, x) = u_1(x), & x \in \mathbf{R}^d, \end{cases}$$

where $u = (u^1, u^2, \dots, u^d)$ is a vector valued function, $\Delta u(t, x) = (\Delta u^1, \Delta u^2, \dots, \Delta u^d)$, $\partial_t u = (\partial_t u^1, \partial_t u^2, \dots, \partial_t u^d)$ and $\partial_t^2 u = (\partial_t^2 u^1, \partial_t^2 u^2, \dots, \partial_t^2 u^d)$. The initial data (u_0, u_1) in (DW) belong to $H^L \times H^{L-1}$, where H^L is the Sobolev space in $\mathbf{R}^d$. The coefficient function $B(x)$ is $d \times d$ symmetric matrix-valued function and quasilinear term $N[u, v](t, x)$ is defined by

$$N[u, v] = (N[u, v]^i)_{i=1,2,\dots,d} = \left(\sum_{j,k,l,m,n=1}^d N_{lmn}^{ijk} \partial_l (\partial_m u^j \partial_n v^k) \right)_{i=1,2,\dots,d}.$$

For B and N , we make the following assumptions:

(B0) $B(x) = (B_{pq}(x))_{p,q=1,2,\dots,d}$ is $d \times d$ symmetric matrix-valued function whose components belong to $\mathcal{B}^\infty$, where $\mathcal{B}^\infty$ is a function space of smooth functions with bounded derivatives.

(B1) $B(x)$ is nonnegative, i.e.

$$\sum_{p,q=1}^d B_{pq}(x) \eta_p \eta_q \geq 0 \quad (\eta, x \in \mathbf{R}^d).$$

(B2) There exist $b_0 > 0$ and $R > 0$ such that

$$\sum_{p,q=1}^d B_{pq}(x) \eta_p \eta_q \geq b_0 |\eta|^2 \quad (|x| \geq R, \eta \in \mathbf{R}^d).$$

(N0) $N_{lmn}^{ijk} \in \mathbf{R}$ ($i, j, k, l, m, n = 1, 2, \dots, d$).

(N1) $N_{lmn}^{ijk} = N_{mln}^{jik} = N_{lmn}^{ikj}$ ($i, j, k, l, m, n = 1, 2, \dots, d$).

If we assume $B = \text{const} > 0$, there are many results (see [3], [7], [9], [10] etc.). For general quasilinear version including N , Racke [13] shows that there exists a unique global solution and decay estimates when the initial data are sufficiently smooth and small. On the other hand, the global existence is not obvious when B is not positive constant. Indeed if B vanishes, (DW) becomes the quasilinear wave equations. Then it is well known that no matter how small the initial data, there does not exist globally defined smooth solution in general (e.g. [6]). However if the nonlinear term N has “Null Condition”, then (DW) has a global smooth solution for sufficiently smooth and small initial data (e.g. [1], [8], [14]).

There are some results for nonuniform dissipative term which satisfies **(B1)–(B2)** ([2], [5], [11], [12] etc.). In particular, Nakao [12] obtains a global existence when the nonlinear term have second order derivatives. He assume that the order of the nonlinear term p satisfies $p \geq 3$. In this paper, we treat the case of $p = 2$, i.e. the quasilinear term $N[u, u]$ is a quadratic function. In order to treat the quadratic case, we introduce a rescaling argument and energy norms for controlling higher order derivatives. These norms are used for the nonlinear wave equations.

We prove the following global existence theorem:

Theorem 1.1. *Let $L \geq L_0 = [d/2] + 3$. Then there exists a small constant $\hat{\delta} > 0$ such that if the initial data $(u_0, u_1) \in H^L \times H^{L-1}$ satisfies*

$$(1) \quad \|u_0\|_{H^{L_0}}^2 + \|u_1\|_{H^{L_0-1}}^2 \leq \hat{\delta},$$

then there exists a unique solution to (DW) in $\bigcap_{j=0}^L C^j([0, \infty); H^{L-j})$.

We note that $L \geq L_0 = [d/2] + 3$ is reasonable. Because we use the assumption to obtain a local solution in usual sense (see [10], [12], [16]). Furthermore the assumption is necessary to estimate the quasilinear term for the

purpose of extending the solution. Indeed using product estimate (proposition 2.2) and embedding (proposition 2.1), we calculate as

$$\begin{aligned} \|\partial_l(\partial_m u \partial_n u)\|_{H^{L-2}} &\leq C\{\|\nabla u\|_{H^{L-2}}\|\nabla^2 u\|_\infty + \|\nabla u\|_\infty\|\nabla^2 u\|_{H^{L-2}}\} \\ &\leq C\|\nabla u\|_{H^{L-1}}\|\nabla u\|_{H^{L_0-1}} \leq C\|\nabla u\|_{H^{L-1}}^2 \end{aligned}$$

(see lemma 5.1).

We also remark that the smallness of $\|u_0\|_{L^2}$ is needed to estimate the higher order derivatives of nonuniform dissipative term as $\nabla^\alpha(B(x)u)$ (see lemma 5.2). Indeed if $B = \text{Const} > 0$, we can prove theorem 1.1 under the assumption $\|\nabla u_0\|_{H^{L_0-1}}^2 + \|u_1\|_{H^{L_0-1}}^2 \leq \hat{\delta}$ instead of (1).

We show the decay estimates with the additional assumptions as follow:

Theorem 1.2. *In addition to the assumptions in theorem 1.1, we assume that one of the following (H1)–(H3) is held:*

- (H1) $d \geq 3$ and there exists $1 \leq p \leq 2d/(d+2)$ such that $Bu_0 + u_1 \in L^p$,
- (H2) $d \geq 3$ and $|\cdot| \{Bu_0 + u_1\} \in L^2$,
- (H3) $d = 1$ or 2 , $|\cdot| \{Bu_0 + u_1\} \in L^1$ and

$$\int_{\mathbf{R}^d} B(x)u_0(x) + u_1(x)dx = 0.$$

Then for any μ ($0 \leq \mu \leq L - L_0$), there exists a constant $E_0 > 0$ depending on (u_0, u_1) such that the global solution u to (DW) satisfies following estimates:

$$(2) \quad \|\partial_t^\mu u(t)\|_{H^{L-\mu}}^2 + \|\partial_t^{\mu+1} u(t)\|_{H^{L-\mu-1}}^2 \leq E_0(1+t)^{-2\mu-1},$$

$$(3) \quad \|\nabla \partial_t^\mu u(t)\|_2^2 + \|\partial_t^{\mu+1} u(t)\|_2^2 \leq E_0(1+t)^{-2\mu-2},$$

$$(4) \quad \|\partial_t^\mu u(t)\|_\infty^2 \leq E_0(1+t)^{-2\mu-1}.$$

Furthermore if $L > L_0$, it holds that

$$(5) \quad \|\Delta u(t)\|_2^2 \leq E_0(1+t)^{-3}.$$

Ikehata [5] treats the linear or semilinear cases and obtains the result corresponding to theorem 1.2 with the assumption (H2). Thus theorem 1.2 means that even for the quasilinear case, the same type decay properties as in [5] are held. For the proof, we use the method which is introduced in [5]. However, to estimate the quasilinear term, we use the time weighted higher order estimates (see §4).

In this paper, the rescaling argument plays an important role (for the definition of the scaling, see §2). We have to control terms include $\nabla^\beta B(x)$ to calculate the higher order energy estimates. Using a usual way, we need the smallness of $\|\nabla^\beta B(x)\|_\infty$ or delicate calculation. However, using the rescaling,

we can control the smallness of $\|\nabla^\beta B(x)\|_\infty$ by a scale parameter. Therefore we can expect to prove the higher order energy estimates with simple argument.

2. Preliminaries

Throughout this paper, $\|\cdot\|_p$ and $\|\cdot\|_{H^l}$ stand for the usual $L^p(\mathbf{R}^d)$ -norm and $H^l(\mathbf{R}^d)$ -norm. Furthermore, we adopt

$$\langle g, f \rangle = \sum_{i=1}^d \int_{\mathbf{R}^d} f^i(x) g^i(x) dx$$

as the usual $L^2(\mathbf{R}^d)$ -inner product.

We consider the rescaling to (DW). Let u be the solution to (DW). We define $v(t, x) = \lambda^{-1}u(\lambda t, \lambda x)$ ($\lambda > 0$), then v satisfies

$$\begin{aligned} \partial_t^2 v(t, x) - \Delta v(t, x) &= \lambda \{ \partial_t^2 u(\lambda t, \lambda x) - \Delta u(\lambda t, \lambda x) \} \\ &= -\lambda B(\lambda x) \partial_t u(\lambda t, \lambda x) + \lambda N[u, u](\lambda t, \lambda x) \\ &= -\lambda B(\lambda x) \partial_t v(t, x) + N[v, v](t, x). \end{aligned}$$

Thus v is the solution to the following Cauchy problem $(\text{DW})_\lambda$:

$$(\text{DW})_\lambda \quad \begin{cases} (\partial_t^2 - \Delta + B_\lambda(x) \partial_t) v(t, x) = N[v, v](t, x), & (t, x) \in [0, \infty) \times \mathbf{R}^d, \\ v(0, x) = v_0(x), \partial_t v(0, x) = v_1(x), & x \in \mathbf{R}^d, \end{cases}$$

where $B_\lambda(x) = \lambda B(\lambda x)$, $v_0(x) = \lambda^{-1}u_0(\lambda x)$, $v_1(x) = u_1(\lambda x)$. Now B_λ satisfies

(B1) $_\lambda$ $B_\lambda(x)$ is nonnegative.

(B2) $_\lambda$ There exist $b_0 > 0$ and $R > 0$ such that

$$\sum_{p, q=1}^d (B_\lambda)_{pq}(x) \eta_p \eta_q = \sum_{p, q=1}^d \lambda B_{pq}(\lambda x) \eta_p \eta_q \geq \lambda b_0 |\eta|^2 \quad \left(|x| \geq \frac{R}{\lambda}, \eta \in \mathbf{R}^d \right)$$

instead of **(B1)** and **(B2)**. Furthermore B_λ satisfies

$$\mathbf{(B3)}_\lambda \quad \|\nabla^b B_\lambda\|_\infty \leq \lambda^{|b|+1} \|\nabla^b B\|_\infty.$$

We consider $(\text{DW})_\lambda$ for $\lambda \leq 1$ instead of (DW).

We introduce the known results. First, we use the following two lemmas for estimating the nonlinearity terms. For the proof, see for example [16] section 13.

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$\|f\|_\infty \leq C \|f\|_{H^{[d/2]+1}}.$$

Lemma 2.2. *Let $k \in \mathbf{Z}$ and $b, c \in \mathbf{Z}_+^d$ satisfy $|b| + |c| = k$. There exists $C > 0$ such that*

$$\begin{aligned} \|\nabla^b f \nabla^c g\|_2 &\leq C \|f\|_\infty \|\nabla^{b+c} g\|_2 + C \|\nabla^{b+c} f\|_2 \|g\|_\infty \\ &\leq C \|f\|_\infty \|g\|_{H^k} + C \|f\|_{H^k} \|g\|_\infty. \end{aligned}$$

Next, we prepare the Poincare type inequality of B_λ for the proof of global existence.

Lemma 2.3 (Poincare type inequality). *There exists a constant $C_1 \geq 1/4$ such that*

$$(6) \quad \|f\|_2^2 \leq \frac{C_1}{\lambda} \langle f, B_\lambda f \rangle + \frac{C_1}{\lambda^2} \|\nabla f\|_2^2 \quad (\lambda > 0).$$

Proof. We define $U_r = \{x \in \mathbf{R}^d \mid |x| \leq r\}$. Using Poincare inequality (see [4]), we obtain the following estimate:

$$\int_{U_r} |f(x)|^2 dx \leq r^2 \int_{U_r} |\nabla f(x)|^2 dx.$$

Let $\rho \in C_0^\infty(\mathbf{R}^d)$ be a function satisfying $0 \leq \rho \leq 1$, $\rho(x) = 1$ ($|x| \leq 1$), $\rho(x) = 0$ ($|x| \geq 3/2$). For any $f \in H^1(\mathbf{R}^d)$ and $\lambda > 0$ we define $\rho_\lambda(x) = \rho(\lambda x/R)$, then because of $\rho_\lambda f \in H_0^1(U_{2R/\lambda})$ we have

$$\begin{aligned} \|f\|_2^2 &= \int_{\mathbf{R}^d} |\rho_\lambda(x) f(x)|^2 dx + \int_{\mathbf{R}^d} (1 - |\rho_\lambda(x)|^2) |f(x)|^2 dx \\ &= \int_{|x| \leq 2R/\lambda} |\rho_\lambda(x) f(x)|^2 dx + \int_{|x| \geq R/\lambda} (1 - |\rho_\lambda(x)|^2) |f(x)|^2 dx \\ &\leq \frac{4R^2}{\lambda^2} \int_{|x| \leq 2R/\lambda} |\nabla(\rho_\lambda(x) f(x))|^2 dx + \int_{|x| \geq R/\lambda} |f(x)|^2 dx \\ &\leq \frac{4R^2}{\lambda^2} \int_{R/\lambda \leq |x| \leq 2R/\lambda} |\nabla \rho_\lambda(x)|^2 |f(x)|^2 dx \\ &\quad + \frac{4R^2}{\lambda^2} \int_{|x| \leq 2R/\lambda} |\rho_\lambda(x)|^2 |\nabla f(x)|^2 dx + \int_{|x| \geq R/\lambda} |f(x)|^2 dx \\ &\leq (4R^2 \|\nabla \rho\|_\infty^2 + 1) \int_{|x| \geq R/\lambda} |f(x)|^2 dx + \frac{4R^2}{\lambda^2} \int_{\mathbf{R}^d} |\nabla f(x)|^2 dx \\ &\leq \frac{4R^2 \|\nabla \rho\|_\infty^2 + 1}{b_0 \lambda} \langle f, B_\lambda f \rangle + \frac{4R^2}{\lambda^2} \|\nabla f\|_2^2. \end{aligned}$$

Hence we get (6). □

Finally, we introduce Hardy inequality and Gagliardo-Nirenberg inequality. For the proof, see for example [15] appendix A.

Lemma 2.4. *Let $d \geq 3$. There exists a constant $C > 0$ such that any $f \in H^1$ satisfies*

$$(7) \quad \left\| \frac{f}{|\cdot|} \right\|_2 \leq C \|\nabla f\|_2.$$

Lemma 2.5. *Assume $1 \leq q < d$ and $1/p = 1/q - 1/d$. Then there exists a constant $C > 0$ depend on p, q, d such that*

$$(8) \quad \|g\|_p \leq C \|\nabla g\|_q.$$

3. Global existence

In this section we prove theorem 1.1. First we define some notations. For any $h, g : \mathbf{R}^d \rightarrow \mathbf{R}^d$, we define $[h; \nabla g] : \mathbf{R}^d \rightarrow \mathbf{R}^d$ as follows:

$$([h; \nabla g])^i(x) = h(x) \cdot \nabla g^i(x) \quad (i = 1, 2, \dots, d).$$

The energy $E(u(t))$ and higher order energies $E_{\bar{L}}(u(t))$ of u are defined by

$$(9) \quad E(u(t)) = \frac{1}{2} \{ \|\partial_t u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \}$$

and

$$(10) \quad E_{\bar{L}}(u(t)) = \sum_{|a| \leq \bar{L}-1} E(\nabla^a u(t)).$$

Moreover we define

$$(11) \quad \tilde{N}[u, v, w](t) = \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_{\mathbf{R}^d} \partial_l u^i(t, x) \partial_m v^j(t, x) \partial_n w^k(t, x) dx$$

and

$$(12) \quad \tilde{E}_{\bar{L}, \mu}(u(t)) = E_{\bar{L}-\mu}(\partial_t^\mu u(t)) + \sum_{|a| \leq \bar{L}-\mu-1} \tilde{N}[\partial_t^\mu \nabla^a u, \partial_t^\mu \nabla^a u, u](t).$$

The function spaces $X_{\delta, T}$ and X_δ are defined by

$$(13) \quad X_{\delta, T} = \{u \in \bigcap_{j=0}^L C^j([0, T]; H^{L-j}) \mid E_{L_0}(u(t)) \leq \delta^2 \quad (0 \leq t \leq T)\}$$

and

$$(14) \quad X_\delta = \{u \in \bigcap_{j=0}^L C^j([0, \infty); H^{L-j}) \mid E_{L_0}(u(t)) \leq \delta^2 \quad (0 \leq t < \infty)\}.$$

Let $L_0 \leq \bar{L} \leq L$, $\mu \leq L - L_0$ and $\lambda > 0$, we define $G_{\bar{L},\mu}(v(t))$ below.

$$(15) \quad G_{\bar{L},\mu}(v(t)) = \frac{C_0}{\lambda} \tilde{E}_{\bar{L},\mu}(v(t)) + \frac{b_0(2d-1)}{4} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), \partial_t^{\mu+1} \nabla^a v(t) \rangle \\ + \frac{b_0(2d-1)}{8} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t) \rangle \\ + \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v(t), [h; \nabla \partial_t^\mu \nabla^a v](t) \rangle,$$

where

$$(16) \quad C_0 = \max \left\{ \left(b_0 R d^2 + \frac{C_1 b_0 (2d-1)}{2} \right) \times 4, d, 2 \|B\|_\infty b_0^2 R^2 \times \frac{8}{b_0} \right\},$$

and

$$(17) \quad \phi(r) = \begin{cases} b_0, & (r \leq R/\lambda) \\ b_0 R/(\lambda r), & (r \geq R/\lambda) \end{cases}, \quad h(x) = x\phi(|x|).$$

We need the energy estimate to prove the global existence.

Lemma 3.1. *Let $L_0 \leq \bar{L} \leq L$ and $\mu \leq \bar{L} - L_0$. There exists a constant $C > 0$ such that for any $\delta, \lambda, T > 0$ and a local solution $v \in X_{\delta,T}$ to $(\mathbf{DW})_\lambda$ satisfy*

$$(18) \quad \frac{d}{dt} G_{\bar{L},\mu}(v(t)) + \frac{b_0}{2} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \\ \leq \lambda C E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)) + \frac{2 \|B\|_\infty b_0^2 R^2}{C_0} E_{\bar{L}-\mu}(\partial_t^\mu v(t)),$$

where $C_0 > 0$ is the constant give in (16), and

$$(19) \quad D_{\bar{L},\mu}(v(t)) = E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) \cdot \left(\sum_{v=0}^{\mu} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{\bar{L}-v}^{1/2}(\partial_t^v v(t)) \right).$$

The proof of the lemma 3.1 is given in section 5. In what follows, assuming lemma 3.1, we derive the energy estimates for $(\mathbf{DW})_\lambda$.

3.1. Energy estimate

We prove the higher order energy estimates of $(\mathbf{DW})_\lambda$.

Lemma 3.2. *Let $L_0 \leq \bar{L} \leq L$ and $\mu \leq \bar{L} - L_0$. There exists $C > 0$ such that if $\delta, \lambda > 0$ are sufficiently small, then the local solution $v \in X_{\delta,T}$ to $(\mathbf{DW})_\lambda$ satisfies*

$$\begin{aligned}
(20) \quad \frac{1}{C} \left\{ \lambda \|\partial_t^\mu v(t)\|_2^2 + \frac{1}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \right\} &\leq G_{\bar{L},\mu}(\partial_t^\mu v(t)) \\
&\leq C \left\{ \lambda \|\partial_t^\mu v(t)\|_2^2 + \frac{1}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \right\}.
\end{aligned}$$

Proof. Let $v \in X_{\delta,T}$. First it holds that

$$(21) \quad \frac{1}{2} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \leq \tilde{E}_{\bar{L},\mu}(v(t)) \leq \frac{3}{2} E_{\bar{L}-\mu}(\partial_t^\mu v(t)).$$

Because

$$\begin{aligned}
\left| \sum_{|a| \leq \bar{L}-\mu-1} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, v](t) \right| &\leq C \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla v(t)\|_\infty \|\nabla \partial_t^\mu \nabla^a v(t)\|_2^2 \\
&\leq C \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla v(t)\|_{H^{[d/2]+1}} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2^2 \\
&\leq \delta C E_{\bar{L}-\mu}(\partial_t^\mu v(t)),
\end{aligned}$$

we choose δ sufficiently small depend on d and N , then we get (21). It follows from lemma 2.3 that

$$\begin{aligned}
(22) \quad &\left| \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), \partial_t^{\mu+1} \nabla^a v(t) \rangle \right| \\
&\leq \sum_{|a| \leq \bar{L}-\mu-1} \left\{ \frac{\lambda}{4C_1} \|\partial_t^\mu \nabla^a v(t)\|_2^2 + \frac{C_1}{\lambda} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2^2 \right\} \\
&\leq \sum_{|a| \leq \bar{L}-\mu-1} \left\{ \frac{1}{4} \langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t) \rangle \right. \\
&\quad \left. + \frac{1}{4\lambda} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2^2 + \frac{C_1}{\lambda} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2^2 \right\} \\
&\leq \frac{1}{4} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t) \rangle + \frac{2C_1}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)).
\end{aligned}$$

Using

$$(23) \quad \|h\|_\infty \leq \frac{b_0 R}{\lambda} \quad \text{and} \quad \|\nabla h\|_\infty \leq 2b_0,$$

we get

$$\begin{aligned}
(24) \quad & \left| \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v(t), [h; \nabla \partial_t^\mu \nabla^a v](t) \rangle \right| \\
& \leq \sum_{|a| \leq \bar{L}-\mu-1} \sum_{k,j=1}^d \left| \int_{\mathbf{R}^d} \partial_t^{\mu+1} \nabla^a v^k(t) \partial_j \partial_t^\mu \nabla^a v^k(t) h^j dx \right| \\
& \leq \sum_{|a| \leq \bar{L}-\mu-1} \sum_{k,j=1}^d \|\partial_t^{\mu+1} \nabla^a v^k(t)\|_2 \|\partial_j \partial_t^\mu \nabla^a v^k(t)\|_2 \|h^j\|_\infty \\
& \leq d^2 \sum_{|a| \leq \bar{L}-\mu-1} \left\{ \frac{1}{2} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2^2 + \frac{1}{2} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2^2 \right\} \|h\|_\infty \\
& \leq \frac{b_0 R d^2}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)).
\end{aligned}$$

Using (21), (22), (24) and (16) we have

$$\begin{aligned}
G_{\bar{L},\mu}(v(t)) & \geq \frac{1}{\lambda} \left(\frac{C_0}{2} - b_0 R d^2 - \frac{C_1 b_0 (2d-1)}{2} \right) E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \\
& \quad + \frac{b_0 (2d-1)}{16} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t) \rangle \\
& \geq \frac{1}{\lambda} \left(b_0 R d^2 + \frac{C_1 b_0 (2d-1)}{2} \right) E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \\
& \quad + \frac{b_0 (2d-1)}{16} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t) \rangle.
\end{aligned}$$

So there exists a constant C such that v satisfies

$$(25) \quad \frac{1}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t) \rangle \leq C G_{\bar{L},\mu}(v(t)).$$

Furthermore using (25) and lemma 2.3, we have

$$\begin{aligned}
(26) \quad & \|\partial_t^\mu v(t)\|_2^2 \leq \frac{C_1}{\lambda} \langle B_\lambda \partial_t^\mu v(t), \partial_t^\mu v(t) \rangle + \frac{C_1}{\lambda^2} \|\nabla \partial_t^\mu v\|_2^2 \\
& \leq \frac{C_1}{\lambda} \sum_{|a| \leq \bar{L}-\mu-1} \langle B_\lambda \partial_t^\mu \nabla^a v(t), \partial_t^\mu \nabla^a v(t) \rangle + \frac{2C_1}{\lambda^2} E_{\bar{L}-\mu}(\partial_t^\mu v(t))
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2C_1}{\lambda} \left(\sum_{|a| \leq \bar{L}-\mu-1} \langle B_\lambda \partial_t^\mu \nabla^a v(t), \partial_t^\mu \nabla^a v(t) \rangle + \frac{1}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \right) \\
&\leq \frac{2C_1 C}{\lambda} G_{\bar{L},\mu}(v(t)).
\end{aligned}$$

From (25) and (26), there exists a constant $C > 0$ such that v satisfies

$$(27) \quad \frac{1}{C} \left\{ \lambda \|\partial_t^\mu v(t)\|_2^2 + \frac{1}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \right\} \leq G_{\bar{L},\mu}(v(t)).$$

On the other hand using (21), (22) and (24), we get

$$\begin{aligned}
(28) \quad G_{\bar{L},\mu}(v(t)) &\leq \frac{1}{\lambda} \left(\frac{3C_0}{2\lambda} + \frac{C_1 b_0 (2d-1)}{2} + b_0 R d^2 \right) E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \\
&\quad + \frac{3b_0 (2d-1)}{16} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t) \rangle \\
&\leq \frac{C}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + C \|B_\lambda\|_\infty \sum_{|a| \leq \bar{L}-\mu-1} \|\partial_t^\mu \nabla^a v(t)\|_2^2 \\
&\leq \frac{C}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + \lambda C \|\partial_t^\mu v(t)\|_2^2.
\end{aligned}$$

So combining (27) and (28), we get (20). This completes the proof of lemma 3.2. $\square$

Lemma 3.3. *Let $L_0 \leq \bar{L} \leq L$. There exist sufficiently small constants λ and δ such that if $v \in X_{\delta,T}$ is a local solution to $(\text{DW})_\lambda$ then v satisfies*

$$(29) \quad \frac{d}{dt} G_{\bar{L},0}(v(t)) + \frac{b_0}{8} E_{\bar{L}}(v(t)) \leq 0.$$

Proof. Let $v \in X_{\delta,T}$ be a local solution to $(\text{DW})_\lambda$. From (16), C_0 satisfies $C_0 \geq d$. Thus using lemma 3.1 for $\mu = 0$ and

$$D_{\bar{L},0}(v(t)) = E_{L_0}^{1/2}(\partial_t^\mu v(t)) E_{\bar{L}}(\partial_t^\mu v(t)) \leq \delta E_{\bar{L}}(\partial_t^\mu v(t)),$$

we have

$$\frac{d}{dt} G_{\bar{L},0}(v(t)) + \frac{b_0}{2} E_{\bar{L}}(v(t)) \leq C \left(\lambda + \frac{\delta}{\lambda} \right) E_{\bar{L}}(v(t)) + \frac{2\|B\|_\infty b_0^2 R^2}{C_0} E_{\bar{L}}(v(t)).$$

From (16), we also obtain

$$\frac{2\|B\|_\infty b_0^2 R^2}{C_0} \leq \frac{b_0}{8}.$$

We choose sufficiently small constants λ and δ such that

$$C\left(\lambda + \frac{\delta}{\lambda}\right) \leq \frac{b_0}{4},$$

then we have

$$\frac{d}{dt}G_{\bar{L},0}(v(t)) + \frac{b_0}{8}E_{\bar{L}}(v(t)) \leq 0.$$

This completes the proof of lemma 3.3. $\square$

Corollary 3.4. *Let $L_0 \leq \bar{L} \leq L$, and λ and δ be sufficiently small constants in lemma 3.3. Then there exists a constant C^* depending on λ such that for any $T > 0$ and a local solution $v \in X_{\delta,T}$ to $(\text{DW})_\lambda$ satisfy*

$$(30) \quad \|v(t)\|_2^2 + E_{\bar{L}}(v(t)) + \int_0^t E_{\bar{L}}(v(s))ds \leq C^*\{\|v(0)\|_2^2 + E_{\bar{L}}(v(0))\}$$

and

$$(31) \quad \|v(t)\|_{H^{L_0}}^2 + \|\partial_t v(t)\|_{H^{L_0-1}}^2 \leq C^*\{\|v(0)\|_{H^{L_0}}^2 + \|\partial_t v(0)\|_{H^{L_0-1}}^2\}.$$

Proof. The estimate (31) is verified by (30). Thus we should prove (30). Integrating (29) over $[0, t]$ we, get

$$(32) \quad G_{\bar{L},0}(v(t)) + \frac{b_0}{8} \int_0^t E_{\bar{L}}(v(s))ds \leq G_{\bar{L},\mu}(v(0)).$$

Then using lemma 3.2, there exists a constant $C > 0$ such that

$$\frac{1}{C} \left\{ \lambda \|v(t)\|_2^2 + \frac{1}{\lambda} E_{\bar{L}}(v(t)) \right\} + \frac{b_0}{8} \int_0^t E_{\bar{L}}(v(s))ds \leq C \left\{ \lambda \|v(0)\|_2^2 + \frac{1}{\lambda} E_{\bar{L}}(v(0)) \right\}.$$

We rearrange coefficient and define C^* depend on λ , it holds that (30). $\square$

3.2. Global existence

The following local existence theorem is known (see [10], [12], [16]).

Lemma 3.5. *Let $L \geq L_0 = [d/2] + 3$, $\lambda > 0$ and $(v_0, v_1) \in H^L \times H^{L-1}$. For any sufficiently small constant $\varepsilon > 0$ there exist constants $0 < t_0$ and $0 < \eta \leq 1$ such that if*

$$(33) \quad \|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2 \leq \eta \varepsilon^2$$

then $(\text{DW})_\lambda$ has a unique local solution $v \in \bigcap_{j=0}^L C^j([0, t_0]; H^{L-j})$ and the v satisfies

$$(34) \quad E_{L_0}(v(t)) \leq \varepsilon^2 \quad (0 \leq t \leq t_0).$$

Using lemma 3.5 and corollary 3.4, we can prove a global existence theorem for $(\text{DW})_\lambda$.

Theorem 3.6. *Let $L \geq L_0 = [d/2] + 3$, λ and δ are sufficiently small constants. There exists a small constant $\delta^* > 0$ such that if the initial data $(v_0, v_1) \in H^L \times H^{L-1}$ satisfy*

$$(35) \quad \|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2 \leq \delta^*,$$

then $(\text{DW})_\lambda$ has a unique global solution $v \in X_\delta$.

Proof. Let λ and $\delta > 0$ are sufficiently small constants for which corollary 3.4 holds. Furthermore let $0 < \varepsilon \leq \delta$, $0 < t_0$ and $0 < \eta \leq 1$ be the constants given in lemma 3.5. Now we define

$$\delta^* = \min \left\{ \eta \varepsilon^2, \frac{\eta \varepsilon^2}{C^*} \right\},$$

where the constant C^* is given in corollary 3.4. We assume that $(v_0, v_1) \in H^L \times H^{L-1}$ satisfy

$$\|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2 \leq \delta^*.$$

Lemma 3.5 yields that there exists $v \in \bigcap_{j=0}^L C^j([0, t_0]; H^{L-j})$ such that v is a unique local solution to $(\text{DW})_\lambda$ and satisfies

$$E_{L_0}(v(t)) \leq \varepsilon^2 \leq \delta^2 \quad (0 \leq t \leq t_0).$$

Because of $v \in X_{\delta, t_0}$ and corollary 3.4, it holds that

$$\|v(t)\|_{H^{L_0}}^2 + \|\partial_t v(t)\|_{H^{L_0-1}}^2 \leq C^* \{\|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2\} \leq \eta \varepsilon^2 \quad (t \in [0, t_0]).$$

Thus we can use lemma 3.5 in $t = t_0$. The solution v is uniquely extended to $\bigcap_{j=0}^L C^j([0, 2t_0]; H^{L-j})$ and satisfies

$$E_{L_0}(v(t)) \leq \varepsilon^2 \leq \delta^2 \quad (0 \leq t \leq 2t_0).$$

Because of $v \in X_{\delta, 2t_0}$ we can use corollary 3.4 again. Then v satisfies

$$\|v(t)\|_{H^{L_0}}^2 + \|\partial_t v(t)\|_{H^{L_0-1}}^2 \leq C^* \{\|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2\} \leq \eta \varepsilon^2 \quad (t \in [0, 2t_0]).$$

Thus we can use lemma 3.5 in $t = 2t_0$.

Repeating this argument, we can uniquely extend v to a global solution to $(\text{DW})_\lambda$, furthermore the v satisfies

$$E_{L_0}(v(t)) \leq \delta^2 \quad (t \in [0, \infty)).$$

This completes the proof of theorem 3.6. $\square$

Proof of theorem 1.1. Let λ and δ^* are the small constants in theorem 3.6. We define $\hat{\delta} = \lambda^{d+1}\delta^*$ and assume the initial data $(u_0, u_1) \in H^L \times H^{L-1}$ satisfy

$$\|u_0\|_{H^{L_0}}^2 + \|u_1\|_{H^{L_0-1}}^2 \leq \hat{\delta}.$$

Now we define

$$v_0(x) = \frac{1}{\lambda} u_0(\lambda x), \quad v_1(x) = u_1(\lambda x),$$

then (v_0, v_1) satisfy

$$\begin{aligned} & \|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2 \\ &= \sum_{|a| \leq L_0} \lambda^{2(|a|-1)} \int_{\mathbf{R}^d} |\nabla^a u_0(\lambda x)|^2 dx + \sum_{|a| \leq L_0-1} \lambda^{2|a|} \int_{\mathbf{R}^d} |\nabla^a u_1(\lambda x)|^2 dx \\ &= \sum_{|a| \leq L_0} \lambda^{2(|a|-1)-d} \int_{\mathbf{R}^d} |\nabla^a u_0(x)|^2 dx + \sum_{|a| \leq L_0-1} \lambda^{2|a|-d} \int_{\mathbf{R}^d} |\nabla^a u_1(x)|^2 dx \\ &\leq \lambda^{-d-1} \left\{ \sum_{|a| \leq L_0} \int_{\mathbf{R}^d} |\nabla^a u_0(x)|^2 dx + \sum_{|a| \leq L_0-1} \int_{\mathbf{R}^d} |\nabla^a u_1(x)|^2 dx \right\} \\ &= \lambda^{-d-1} \{ \|u_0\|_{H^{L_0}}^2 + \|u_1\|_{H^{L_0-1}}^2 \} \leq \delta^*. \end{aligned}$$

From theorem 3.6, there exists a unique global solution v to $(\text{DW})_\lambda$ in $\bigcap_{j=0}^L C^j([0, \infty); H^{L-j})$. We define

$$u(t, x) = \lambda v\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right),$$

then $u \in \bigcap_{j=0}^L C^j([0, \infty); H^{L-j})$ and the u satisfies (DW) .

As regard to uniqueness, if u and u' are solutions to (DW) then rescaling functions u_λ and u'_λ are solutions to $(\text{DW})_\lambda$. From theorem 3.6 we got the uniqueness of $(\text{DW})_\lambda$, so we obtain $u_\lambda = u'_\lambda$, thus $u = u'$. $\square$

4. Decay estimates

The goal of this section is to show theorem 4.1. We say that f satisfies the property **(H1)'**, **(H2)'** or **(H3)'** if and only if

- (H1)' $d \geq 3$ and there exists $1 \leq p \leq 2d/(d+2)$ such that $f \in L^p$,
 (H2)' $d \geq 3$ and $|\cdot|f \in L^2$,
 (H3)' $d = 1$ or 2 , $|\cdot|f \in L^1$ and

$$\int_{\mathbf{R}^d} f(x) dx = 0.$$

We prove the decay estimates for $(\text{DW})_\lambda$ as follow:

Theorem 4.1. *In addition to the assumptions in theorem 3.6, we assume that $B_\lambda v_0 + v_1$ satisfies one of the (H1)'–(H3)'. Then for any i ($0 \leq i \leq L - L_0$), there exists a constant E_0 depending on λ , v_0 and v_1 such that the global solution $v \in X_\delta$ to $(\text{DW})_\lambda$ satisfies*

$$(36) \quad (1+t)^{2i+1} \{ \|\partial_t^i v(t)\|_2^2 + E_{L-i}(\partial_t^i v(t)) \} + \int_0^t (1+s)^{2i+1} E_{L-i}(\partial_t^i v(s)) ds \leq E_0$$

and

$$(37) \quad (1+t)^{2i+2} E(\partial_t^i v(t)) + \int_0^t (1+s)^{2i+2} \langle \partial_t^{i+1} v(s), B_\lambda \partial_t^{i+1} v(s) \rangle \leq E_0.$$

Using theorem 4.1, we can prove theorem 1.2.

Proof of theorem 1.2. We assume that theorem 4.1 is true. From theorem 1.1 there exists a constant $\hat{\delta}$ such that if the initial data (u_0, u_1) satisfies

$$\|u_0\|_{H^{L_0}}^2 + \|u_1\|_{H^{L_0-1}}^2 \leq \hat{\delta},$$

then (DW) has a unique global solution $u \in \bigcap_{j=0}^L C^j([0, \infty); H^{L-j})$. Now we define $v = \lambda^{-1}u(\lambda t, \lambda x)$ then v satisfies

$$(\text{DW})_\lambda \quad \begin{cases} (\partial_t^2 - \Delta + B_\lambda(x) \partial_t) v(t, x) = N[v, v](t, x), & (t, x) \in [0, \infty) \times \mathbf{R}^d, \\ v(0, x) = v_0(x), \partial_t v_\lambda(0, x) = v_1, & x \in \mathbf{R}^d, \end{cases}$$

where $v_0(x) = \lambda^{-1}u_0(\lambda x)$ and $v_1(x) = u_1(\lambda x)$. If $Bu_0 + u_1$ satisfies (H1), (H2) or (H3), then $B_\lambda v_0 + v_1$ satisfies (H1)', (H2)' or (H3)'. Thus we can use theorem 4.1. For any μ ($0 \leq \mu \leq L - L_0$), there exists a constant E_0 depending on λ , v_0 and v_1 such that v satisfies

$$\|\partial_t^\mu v(\tau)\|_2^2 + E_{L-\mu}(\partial_t^\mu v(\tau)) \leq E_0(1+\tau)^{-2\mu-1}$$

and

$$E(\partial_t^\mu v(\tau)) \leq E_0(1+\tau)^{-2\mu-2}.$$

Replacing $v(t, x) = \lambda^{-1}u(\lambda t, \lambda x)$ and $t = \lambda\tau$, we get (2) and (3).

Using (2) and lemma 2.1, the estimate (4) is clear. Thus we prove (5). Let $L_0 < L$. The global solution u to (DW) satisfies

$$(38) \quad \begin{aligned} \|\Delta u(t)\|_2 &= \|\partial_t^2 u(t) + \partial_t u(t) - N[u, u](t)\|_2 \\ &\leq \|\partial_t^2 u(t)\|_2 + \|\partial_t u(t)\|_2 + \|N[u, u](t)\|_2. \end{aligned}$$

Because of $L_0 < L$, we can use (2) and (3) to $\mu = 1$. Then we obtain

$$\begin{aligned} \|\partial_t u\|_2^2 &\leq E_0(1+t)^{-3}, \\ \|\partial_t^2 u\|_2^2 &\leq E_0(1+t)^{-4}. \end{aligned}$$

Furthermore using (2) and (3), we obtain

$$\begin{aligned} \|N[u, u](t)\|_2^2 &\leq \sum_{i,j,k,l,m,n=1}^d |N_{lmn}^{ijk}| \|\partial_l(\partial_m u^j \partial_n u^k)\|_2^2 \\ &\leq C \|\nabla^2 u(t)\|_\infty^2 \|\nabla u(t)\|_2^2 \\ &\leq CE_{L_0}(u(t))E(u(t)) \\ &\leq CE_0(1+t)^{-3}. \end{aligned}$$

Thus the estimate (5) holds from (38). $\square$

4.1. Proof of theorem 4.1

Let v be the global solution to $(\text{DW})_\lambda$ and define

$$(39) \quad w(t, x) = \int_0^t v(s, x) ds.$$

Then w satisfies

$$(40) \quad \begin{cases} \partial_t^2 w(t, x) - \Delta w(t, x) + B_\lambda(x) \partial_t w(t, x) \\ \quad = \int_0^t N[v, v](\tau, x) d\tau + B_\lambda(x) v_0(x) + v_1(x), & (t, x) \in [0, \infty) \times \mathbf{R}^d, \\ w(0, x) = 0, \partial_t w(0, x) = v_0(x), & x \in \mathbf{R}^d. \end{cases}$$

We remark $\partial_t w = v$ and $E_{L_0-1}(w(t))$ are well-defined in $[0, \infty)$ because of corollary 3.4.

We prove the energy estimate of w under the assumption which $B_\lambda v_0 + v_1$ satisfies one of the (H1)'–(H3)'. We prepare the next lemma.

Lemma 4.2. *Let $f \in H^{L_0-1}$ satisfies one of the **(H1)'**–**(H3)'**. Then for any $g \in H^{L_0-1}$, there exists a constant E_0 depending on f such that*

$$(41) \quad \langle g, f \rangle \leq E_0 \|\nabla g\|_{H^{L_0-2}}.$$

Proof. First we assume that f satisfies **(H1)'**. Then $f \in L^{2d/(d+2)}$ because $f \in H^{L_0-1} \subset L^\infty$ and $f \in L^p$ ($1 \leq p \leq 2d/(d+2)$). On the other hand, it holds from lemma 2.5 that

$$\|g\|_{L^{2d/(d-2)}} \leq C \|\nabla g\|_2.$$

Thus using Hölder inequality, we get

$$\langle g, f \rangle \leq \|f\|_{L^{2d/(d+2)}} \|g\|_{L^{2d/(d-2)}} \leq C \|f\|_{L^{2d/(d+2)}} \|\nabla g\|_2 \leq C \|f\|_{L^{2d/(d+2)}} \|\nabla g\|_{H^{L_0-2}}.$$

This means (41).

Next we assume that f satisfies **(H2)'**. Then using lemma 2.4, we get

$$\langle g, f \rangle \leq \left\| |\cdot| f \right\|_2 \left\| \frac{g}{|\cdot|} \right\|_2 \leq C \left\| |\cdot| f \right\|_2 \|\nabla g\|_2 \leq C \left\| |\cdot| f \right\|_2 \|\nabla g\|_{H^{L_0-2}}.$$

Thus we have (41).

Finally we assume that f satisfies **(H3)'**. Because of $g \in H^{L_0-1} \subset C^1(\mathbf{R}^d)$, we have

$$|g(x) - g(0)| = \left| \int_0^1 \frac{d}{d\theta} g(\theta x) d\theta \right| = \left| \int_0^1 x \cdot \nabla g(\theta x) d\theta \right| \leq |x| \|\nabla g\|_\infty \quad \text{a.e. } x.$$

Then using $\int_{\mathbf{R}^d} f(x) dx = 0$, we obtain

$$\begin{aligned} |\langle g, f \rangle| &= \left| \int_{\mathbf{R}^d} (g(x) - g(0)) f(x) dx \right| \leq \int_{\mathbf{R}^d} |g(x) - g(0)| |f(x)| dx \\ &\leq \|\nabla g\|_\infty \int_{\mathbf{R}^d} |x| |f(x)| dx \leq C \|\nabla g\|_{H^{L_0-2}} \| |\cdot| f \|_1. \end{aligned}$$

Thus we get (41). $\square$

We need the estimate $\int_0^\infty \|v(t)\|_2^2 dt < \infty$. In order to prove this property, we use the idea in Ikehata [5].

Proposition 4.3. *In addition to the assumptions theorem 3.6, we assume $B_\lambda v_0 + v_1$ satisfies one of the **(H1)'**–**(H3)'**. Then there exists a constant E_0 depending on λ , v_0 and v_1 such that the global solution $v \in X_\delta$ to **(DW)** $_\lambda$ satisfies*

$$(42) \quad \int_0^t \|v(s)\|_2^2 ds \leq E_0, \quad (t \in [0, \infty)).$$

Proof. Let v be the global solution to **(DW)** $_\lambda$ and w be defined by (39). Using (40), we have

$$\begin{aligned}
\frac{d}{dt} E_{L_0-1}(w(t)) &= \sum_{|a| \leq L_0-2} \langle \partial_t \nabla^a w(t), \partial_t^2 \nabla^a w(t) \rangle - \sum_{|a| \leq L_0-2} \langle \partial_t \nabla^a w(t), \Delta \nabla^a w(t) \rangle \\
&= - \sum_{|a| \leq L_0-2} \langle \partial_t \nabla^a w(t), \nabla^a (B_\lambda \partial_t w(t)) \rangle \\
&\quad + \sum_{|a| \leq L_0-2} \left\langle \partial_t \nabla^a w(t), \int_0^t \nabla^a N[v, v] d\tau \right\rangle \\
&\quad + \sum_{|a| \leq L_0-2} \langle \partial_t \nabla^a w(t), \nabla^a (B_\lambda v_0 + v_1) \rangle \\
&= - \sum_{|a| \leq L_0-2} \langle \partial_t \nabla^a w(t), B_\lambda \partial_t \nabla^a w(t) \rangle \\
&\quad - \sum_{1 \leq |a| \leq L_0-2} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \langle \partial_t \nabla^a w(t), \nabla^b B_\lambda \partial_t \nabla^{a-b} w(t) \rangle \\
&\quad + \sum_{|a| \leq L_0-2} \frac{d}{dt} \left\langle \nabla^a w(t), \int_0^t \nabla^a N[v, v] d\tau \right\rangle \\
&\quad - \sum_{|a| \leq L_0-2} \langle \nabla^a w(t), \nabla^a N[v, v](t) \rangle \\
&\quad + \sum_{|a| \leq L_0-2} \frac{d}{dt} \langle \nabla^a w(t), \nabla^a (B_\lambda v_0 + v_1) \rangle.
\end{aligned}$$

Then integrating it over $[0, t]$, we get

$$\begin{aligned}
(43) \quad E_{L_0-1}(w(t)) &+ \sum_{|a| \leq L_0-2} \int_0^t \langle \nabla^a v(s), B_\lambda \nabla^a v(s) \rangle ds \\
&\leq \frac{1}{2} \|v_0\|_{H^{L_0-2}}^2 - \sum_{1 \leq |a| \leq L_0-2} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \int_0^t \langle \nabla^a v(s), \nabla^b B_\lambda \nabla^{a-b} v(s) \rangle ds \\
&\quad + \sum_{|a| \leq L_0-2} \left\langle \nabla^a w(t), \int_0^t \nabla^a N[v, v](s) ds \right\rangle \\
&\quad - \sum_{|a| \leq L_0-2} \int_0^t \langle \nabla^a w(s), \nabla^a N[v, v](s) \rangle ds \\
&\quad + \sum_{|a| \leq L_0-2} \langle \nabla^a w(t), \nabla^a (B_\lambda v_0 + v_1) \rangle \\
&= \frac{1}{2} \|v_0\|_{H^{L_0-2}}^2 + A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

We estimate from A_1 to A_4 . Using lemma 2.3, $(\mathbf{B3})_\lambda$ and smallness of λ , we get

$$\begin{aligned}
|A_1| &\leq \sum_{1 \leq |a| \leq L_0-2} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \int_0^t \|\nabla^a v(s)\|_2 \|\nabla^b B_\lambda\|_\infty \|\nabla^{a-b} v(s)\|_2 ds \\
&\leq \lambda^2 C \sum_{1 \leq |a| \leq L_0-2} \sum_{\substack{b \leq a \\ b \neq 0}} \int_0^t \|\nabla^a v(s)\|_2 \|\nabla^{a-b} v(s)\|_2 ds \\
&\leq C \sum_{1 \leq |a| \leq L_0-2} \left(\int_0^t \|\nabla^a v(s)\|_2^2 ds + \lambda^4 C \sum_{\substack{b \leq a \\ b \neq 0}} \int_0^t \|\nabla^{a-b} v(s)\|_2^2 ds \right) \\
&\leq C \int_0^t E_{L_0-2}(v(s)) ds + \lambda^2 C \sum_{1 \leq |a| \leq L_0-2} \sum_{\substack{b \leq a \\ b \neq 0}} \left\{ \int_0^t \langle \nabla^{a-b} v(s), B_\lambda \nabla^{a-b} v(s) \rangle ds \right. \\
&\quad \left. + \int_0^t \|\nabla \nabla^{a-b} v(s)\|_2^2 ds \right\} \\
&\leq C \int_0^t E_{L_0}(v(s)) ds + \frac{1}{4} \sum_{|a| \leq L_0-2} \int_0^t \langle \nabla^a v(s), B_\lambda \nabla^a v(s) \rangle ds.
\end{aligned}$$

Next we define

$$(44) \quad M(t) = \sup_{0 \leq s \leq t} E_{L_0-1}(w(s)).$$

Using $M(t)$ and lemma 2.2, we have

$$\begin{aligned}
|A_2| &= \sum_{|a| \leq L_0-2} \left| \left\langle \nabla^a w(t), \int_0^t \nabla^a N[v, v](s) ds \right\rangle \right| \\
&= \sum_{|a| \leq L_0-2} \left| \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_0^t \int_{\mathbf{R}^d} \nabla^a w^i(t) \partial_l \nabla^a (\partial_m v^j(s) \partial_n v^k(s)) dx ds \right| \\
&= \sum_{|a| \leq L_0-2} \left| \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_0^t \int_{\mathbf{R}^d} \partial_l \nabla^a w^i(t) \nabla^a (\partial_m v^j(s) \partial_n v^k(s)) dx ds \right| \\
&\leq C \sum_{|a| \leq L_0-2} \int_0^t \|\nabla \nabla^a w(t)\|_2 \sum_{j,k,m,n=1}^d \|\nabla^a (\partial_m v^j(s) \partial_n v^k(s))\|_2 ds \\
&\leq C(M(t))^{1/2} \int_0^t E_{L_0}(v(s)) ds
\end{aligned}$$

and

$$\begin{aligned}
|A_3| &= \sum_{|a| \leq L_0-2} \left| \int_0^t \langle \nabla^a w(s), \nabla^a N[v, v](s) ds \rangle \right| \\
&\leq \sum_{|a| \leq L_0-2} \left| \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_0^t \int_{\mathbf{R}^d} \nabla^a w^i(s) \partial_l \nabla^a (\partial_m v^j(s) \partial_n v^k(s)) dx ds \right| \\
&= \sum_{|a| \leq L_0-2} \left| \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_0^t \int_{\mathbf{R}^d} \partial_l \nabla^a w^i(s) \nabla^a (\partial_m v^j(s) \partial_n v^k(s)) dx ds \right| \\
&\leq C \sum_{|a| \leq L_0-2} \int_0^t \|\nabla \nabla^a w(s)\|_2 \sum_{j,k,m,n=1}^d \|\nabla^a (\partial_m v^j(s) \partial_n v^k(s))\|_2 ds \\
&\leq C(M(t))^{1/2} \int_0^t E_{L_0}(v(s)) ds.
\end{aligned}$$

Using lemma 4.2, we get

$$\begin{aligned}
A_4 &= \langle w(t), B_\lambda v_0 + v_1 \rangle + \sum_{1 \leq |a| \leq L_0-2} \langle \nabla^a w(t), \nabla^a (B_\lambda v_0 + v_1) \rangle \\
&\leq E_0 \|\nabla w(t)\|_{H^{L_0-2}} + \sum_{1 \leq |a| \leq L_0-2} \|\nabla^a w(t)\|_2 \|\nabla^a (B_\lambda v_0 + v_1)\|_2 \\
&\leq E_0 E_{L_0-1}^{1/2}(w(t)) \leq E_0^2 + \frac{1}{4} E_{L_0-1}(w(t)),
\end{aligned}$$

where E_0 depend on λ , v_0 and v_1 . Corollary 3.4 implies that there exists a constant $C^* > 0$ such that

$$\int_0^t E_{L_0}(v(s)) ds \leq C^* \{\|v(0)\|_2^2 + E_{L_0}(v(0))\}.$$

From (43) and the estimates of A_1 – A_4 , it holds that

$$(45) \quad E_{L_0-1}(w(t)) + \sum_{|a| \leq L_0-2} \int_0^t \langle \nabla^a v(s), B_\lambda \nabla^a v(s) \rangle ds \leq E_0 + E_0(M(t))^{1/2}.$$

From (45) we have

$$M(t) \leq E_0 + E_0(M(t))^{1/2} \quad (t \in [0, \infty)).$$

It means that $M(t)$ is bounded in $[0, \infty)$. Thus it holds from (45) that

$$(46) \quad \int_0^t \langle v(s), B_\lambda v(s) \rangle ds \leq E_0.$$

Finally using (46), lemma 2.3 and Corollary 3.4, we have

$$\int_0^t \|v(s)\|_2^2 ds \leq \frac{C}{\lambda} \int_0^t \langle v(s), B_\lambda v(s) \rangle ds + \frac{C}{\lambda^2} \int_0^t \|\nabla v(s)\|_2^2 ds \leq \frac{E_0}{\lambda} + \frac{E_0}{\lambda^2}.$$

Thus we get (42). $\square$

Let prove theorem 4.1 by the induction. First we prove for $i = 0$.

Theorem 4.4. *In addition to the assumption theorem 3.6, we assume that $B_\lambda v_0 + v_1$ satisfies one of the **(H1)'**–**(H3)'**. Then there exists constant E_0 depending on λ , v_0 and v_1 such that the global solution $v \in X_\delta$ to $(\mathbf{DW})_\lambda$ satisfies*

$$(47) \quad (1+t)\{\|v(t)\|_2^2 + E_L(v(t))\} + \int_0^t (1+s)E_L(v(s))ds \leq E_0$$

and

$$(48) \quad (1+t)^2 E(v(t)) + \int_0^t (1+s)^2 \langle \partial_t v(s), B_\lambda \partial_t v(s) \rangle \leq E_0.$$

Proof. First we prove (47). Using lemma 3.3 for $\bar{L} = L$, we obtain

$$\begin{aligned} \frac{d}{dt} \{(1+t)G_{L,0}(v(t))\} &= G_{L,0}(v(t)) + (1+t) \frac{d}{dt} G_{L,0}(v(t)) \\ &\leq G_{L,0}(v(t)) - \frac{b_0(1+t)}{8} E_L(v(t)). \end{aligned}$$

Integrating it over $[0, t]$ and using lemma 3.2, we get

$$\begin{aligned} \frac{(1+t)}{C} \left\{ \lambda \|v(t)\|_2^2 + \frac{1}{\lambda} E_L(v(t)) \right\} &+ \frac{b_0}{8} \int_0^t (1+s)E_L(v(s))ds \\ &\leq C \left\{ \lambda \|v(0)\|_2^2 + \frac{1}{\lambda} E_L(v(0)) + \lambda \int_0^t \|v(s)\|_2^2 ds + \frac{1}{\lambda} \int_0^t E_L(v(s))ds \right\}. \end{aligned}$$

Then there exists a constant E_0 which depends on λ , v_0 and v_1 such that

$$(49) \quad \begin{aligned} (1+t)\{\|v(t)\|_2^2 + E_L(v(t))\} &+ \int_0^t (1+s)E_L(v(s))ds \\ &\leq E_0 + E_0 \int_0^t \|v(s)\|_2^2 ds + E_0 \int_0^t E_L(v(s))ds. \end{aligned}$$

Using proposition 4.3 and corollary 3.4, we get

$$\begin{aligned}
& (1+t)\{\lambda\|v(t)\|_2^2 + E_L(v(t))\} + \int_0^t (1+s)E_L(v(s))ds \\
& \leq E_0 + E_0^2 + E_0 C^* \{\|v(0)\|_2^2 + E_L(v(0))\}.
\end{aligned}$$

Rearranging E_0 if we need, we get (47).

Next we prove (48). It holds that

$$\begin{aligned}
(50) \quad & \frac{d}{dt}\{(1+t)^2 E(v(t))\} \\
& = 2(1+t)E(v(t)) + (1+t)^2 \frac{d}{dt}E(v(t)) \\
& = 2(1+t)E(v(t)) + (1+t)^2 \{\langle \partial_t v(t), \partial_t^2 v(t) \rangle - \langle \partial_t v(t), \Delta v(t) \rangle\} \\
& = 2(1+t)E(v(t)) - (1+t)^2 \langle \partial_t v(t), B_\lambda \partial_t v(t) \rangle \\
& \quad + (1+t)^2 \langle \partial_t v(t), N[v, v](t) \rangle.
\end{aligned}$$

Using (50) and

$$\langle \partial_t v(t), N[v, v](t) \rangle \leq C E^{1/2}(v(t)) E_L(v(t)),$$

we get

$$\begin{aligned}
(51) \quad & \frac{d}{dt}\{(1+t)^2 E(v(t))\} + (1+t)^2 \langle \partial_t v(t), B_\lambda \partial_t v(t) \rangle \\
& \leq 2(1+t)E(v(t)) + C(1+t)^2 E^{1/2}(v(t)) E_L(v(t)).
\end{aligned}$$

Now we define

$$(52) \quad M_0(t) = \sup_{0 \leq s \leq t} (1+s)^2 E(v(s)).$$

Integrating (51) over $[0, \infty)$ and using (47), we obtain

$$\begin{aligned}
(53) \quad & (1+t)^2 E(v(t)) + \int_0^t (1+s)^2 \langle \partial_t v(s), B_\lambda \partial_t v(s) \rangle ds \\
& \leq E(v(0)) + 2 \int_0^t (1+s)E(v(s))ds + C \int_0^t (1+s)^2 E^{1/2}(v(s)) E_L(v(s))ds \\
& \leq E(v(0)) + 2 \int_0^t (1+s)E(v(s))ds + C(M_0(t))^{1/2} \int_0^t (1+s)E_L(v(s))ds \\
& \leq E_0 + E_0(M_0(t))^{1/2}.
\end{aligned}$$

From (53) we have

$$M_0(t) \leq E_0 + E_0(M_0(t))^{1/2},$$

which means that $M_0(t)$ is bounded in $[0, \infty)$. Therefore it holds from (53) that

$$(1+t)^2 E(v(t)) + \int_0^t (1+s)^2 \langle \partial_t v(s), B_\lambda \partial_t v(s) \rangle \leq E_0.$$

Thus we get (48). $\square$

Next assuming the decay estimate of $\partial_t^i v$ for $0 \leq i \leq \mu - 1$, we show the decay estimate of $\partial_t^\mu v$. For the purpose, we need the following lemma:

Lemma 4.5. *In addition to the assumption theorem 3.6, we assume that $B_\lambda v_0 + v_1$ satisfies one of the (H1)'–(H3)'. Let $1 \leq \mu \leq L - L_0$ and assume that for any $0 \leq i \leq \mu - 1$ estimates (36) and (37) in theorem 4.1 hold. Then there exists a constant E_0 depending on λ , v_0 and v_1 such that the global solution $v \in X_\delta$ to (DW) $_\lambda$ satisfies*

$$(54) \quad \frac{d}{dt} G_{L,\mu}(v(t)) + \frac{b_0}{8} E_{L-\mu}(\partial_t^\mu v(t)) \leq E_0 \sum_{v=1}^{\mu} E_{L-v}(\partial_t^v v(t)) (1+t)^{-2(\mu-v)-1},$$

if λ and δ in theorem 3.6 are chosen small enough.

Proof. From (18) in lemma 3.1 and the assumption of induction, it follows that

$$\begin{aligned} D_{L,\mu}(v(t)) &= E_{L-\mu}^{1/2}(\partial_t^\mu v(t)) \cdot \left(\sum_{v=0}^{\mu} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{L-v}^{1/2}(\partial_t^v v(t)) \right) \\ &\leq \delta E_{L-\mu}(\partial_t^\mu v(t)) + E_{L-\mu}^{1/2}(\partial_t^\mu v(t)) \cdot \left(\sum_{v=0}^{\mu-1} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{L-v}^{1/2}(\partial_t^v v(t)) \right) \\ &\leq 2\delta E_{L-\mu}(\partial_t^\mu v(t)) + \frac{1}{4\delta} \left(\sum_{v=0}^{\mu-1} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{L-v}^{1/2}(\partial_t^v v(t)) \right)^2 \\ &\leq 2\delta E_{L-\mu}(\partial_t^\mu v(t)) + \frac{\mu}{4\delta} \sum_{v=0}^{\mu-1} E_{L_0}(\partial_t^{\mu-v} v(t)) E_{L-v}(\partial_t^v v(t)) \\ &\leq 2\delta E_{L-\mu}(\partial_t^\mu v(t)) + \frac{\mu E_0}{4\delta} \sum_{v=0}^{\mu-1} E_{L-(\mu-v)}(\partial_t^{\mu-v} v(t)) (1+t)^{-2v-1} \\ &= 2\delta E_{L-\mu}(\partial_t^\mu v(t)) + \frac{\mu E_0}{4\delta} \sum_{v=1}^{\mu} E_{L-v}(\partial_t^v v(t)) (1+t)^{-2(\mu-v)-1}. \end{aligned}$$

The above estimate and lemma 3.1 imply that

$$\begin{aligned}
 (55) \quad & \frac{d}{dt} G_{L,\mu}(v(t)) + \frac{b_0}{2} E_{L-\mu}(\partial_t^\mu v(t)) \\
 & \leq \lambda C E_{L-\mu}(\partial_t^\mu v(t)) + \frac{C}{\lambda} D_{L,\mu}(v(t)) + \frac{2\|B\|_\infty b_0^2 R^2}{C_0} E_{L-\mu}(\partial_t^\mu v(t)) \\
 & \leq \left(\lambda C + \frac{2\delta C}{\lambda} + \frac{2\|B\|_\infty b_0^2 R^2}{C_0} \right) E_{L-\mu}(\partial_t^\mu v(t)) \\
 & \quad + \frac{C E_0}{\lambda \delta} \sum_{v=1}^{\mu} E_{L-v}(\partial_t^v v(t)) (1+t)^{-2(\mu-v)-1}.
 \end{aligned}$$

From (16), it follows that

$$\frac{2\|B\|_\infty b_0^2 R^2}{C_0} \leq \frac{b_0}{8}.$$

Choosing λ and $\delta > 0$ in theorem 3.6 sufficiently small enough if necessary, we obtain

$$\lambda C + \frac{2\delta C}{\lambda} \leq \frac{b_0}{4}.$$

From these estimates and (55), it holds that there exists a constant E_0 such that

$$\frac{d}{dt} G_{L,\mu}(v(t)) + \frac{b_0}{8} E_{L-\mu}(\partial_t^\mu v(t)) \leq E_0 \sum_{v=1}^{\mu} E_{L-v}(\partial_t^v v(t)) (1+t)^{-2(\mu-v)-1}.$$

This means (54). Hence we obtain lemma 4.5. $\square$

We complete to prove theorem 4.1. When $i = 0$ we already proved (theorem 4.4), thus we assume $1 \leq \mu \leq L - L_0$ and for any $0 \leq i \leq \mu - 1$ satisfy theorem 4.1. The goal is to show (36) and (37) to $i = \mu$.

First we prove (36) for $i = \mu$. Lemma 4.5 yields that

$$\begin{aligned}
 (56) \quad & \frac{d}{dt} \{(1+t)^{2\mu+1} G_{L,\mu}(v(t))\} \\
 & = (2\mu+1)(1+t)^{2\mu} G_{L,\mu}(v(t)) + (1+t)^{2\mu+1} \frac{d}{dt} G_{L,\mu}(v(t)) \\
 & \leq (2\mu+1)(1+t)^{2\mu} G_{L,\mu}(v(t)) - (1+t)^{2\mu+1} \frac{b_0}{8} E_{L-\mu}(\partial_t^\mu v(t)) \\
 & \quad + E_0 \sum_{v=1}^{\mu} E_{L-v}(\partial_t^v v(t)) (1+t)^{2v}.
 \end{aligned}$$

Integrating (56) over $[0, t]$, we get

$$\begin{aligned}
 (57) \quad & (1+t)^{2\mu+1} G_{L,\mu}(v(t)) + \frac{b_0}{8} \int_0^t (1+s)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(s)) ds \\
 & \leq G_{L,\mu}(v(0)) + (2\mu+1) \int_0^t (1+s)^{2\mu} G_{L,\mu}(v(s)) ds \\
 & \quad + E_0 \sum_{v=1}^{\mu} \int_0^t (1+s)^{2v} E_{L-v}(\partial_t^v v(s)) ds.
 \end{aligned}$$

From (20) and (56), there exists a constant E_0 depending on λ , v_0 and v_1 such that

$$\begin{aligned}
 (58) \quad & (1+t)^{2\mu+1} \{ \|\partial_t^\mu v(t)\|_2^2 + E_{L-\mu}(\partial_t^\mu v(t)) \} + \int_0^t (1+s)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(s)) ds \\
 & \leq E_0 + E_0 \int_0^t (1+s)^{2\mu} \|\partial_t^\mu v(s)\|_2^2 ds + E_0 \int_0^t (1+t)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s)) ds \\
 & \quad + E_0 \sum_{v=1}^{\mu-1} \int_0^t (1+s)^{2v} E_{L-v}(\partial_t^v v(s)) ds.
 \end{aligned}$$

We estimate the right-side of (58). Using lemma 2.3 and (37) for $i = \mu - 1$, we get

$$\begin{aligned}
 (59) \quad & \int_0^t (1+s)^{2\mu} \|\partial_t^\mu v(s)\|_2^2 ds \\
 & \leq \frac{C_1}{\lambda} \int_0^t (1+s)^{2\mu} \langle \partial_t^\mu v(s), B_\lambda \partial_t^\mu v(s) \rangle ds + \frac{C_1}{\lambda^2} \int_0^t (1+s)^{2\mu} \|\nabla \partial_t^\mu v(s)\|_2^2 ds \\
 & \leq \frac{C_1 E_0}{\lambda} + \frac{2C_1}{\lambda^2} \int_0^t (1+s)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s)) ds.
 \end{aligned}$$

From the assumption of induction, it follows that (36) for $i = v$ with $v \leq \mu - 1$ hold, which yields that

$$(60) \quad \sum_{v=1}^{\mu-1} \int_0^t (1+s)^{2v} E_{L-v}(\partial_t^v v(s)) ds \leq (\mu-1) E_0.$$

Using (58), (59) and (60), we obtain

$$\begin{aligned}
 (61) \quad & (1+t)^{2\mu+1} \{ \|\partial_t^\mu v(t)\|_2^2 + E_{L-\mu}(\partial_t^\mu v(t)) \} + \int_0^t (1+s)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(s)) ds \\
 & \leq E_0 + E_0 \int_0^t (1+t)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s)) ds.
 \end{aligned}$$

We choose a constant t^* such that $2E_0(1+t^*)^{-1} \leq 1$ then we get

$$\begin{aligned}
 (62) \quad & E_0 \int_0^t (1+s)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s)) ds \\
 &= E_0 \int_0^{t^*} (1+s)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s)) ds \\
 &\quad + E_0 \int_{t^*}^t (1+s)^{2\mu+1} (1+s)^{-1} E_{L-\mu}(\partial_t^\mu v(s)) ds \\
 &\leq \frac{E_0(1+t^*)^{2\mu+1}}{2\mu+1} \sup_{0 \leq s \leq t^*} E_{L-\mu}(\partial_t^\mu v(s)) + \frac{1}{2} \int_0^t (1+t)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(s)) ds.
 \end{aligned}$$

Now

$$\frac{E_0(1+t^*)^{2\mu+1}}{2\mu+1} \sup_{0 \leq s \leq t^*} E_{L-\mu}(\partial_t^\mu v(s))$$

can include to E_0 because the constant is depended on λ , u_0 and u_1 . Therefore using (61) and (62), we get (36) to $i = \mu$.

Next we prove (37) to $i = \mu$. For the solution v to $(\text{DW})_\lambda$ holds that

$$\begin{aligned}
 & \frac{d}{dt} \{(1+t)^{2\mu+2} E(\partial_t^\mu v(t))\} \\
 &= (2\mu+2)(1+t)^{2\mu+1} E(\partial_t^\mu v(t)) + (1+t)^{2\mu+2} \frac{d}{dt} E(\partial_t^\mu v(t)) \\
 &\leq (2\mu+2)(1+t)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(t)) - (1+t)^{2\mu+2} \langle \partial_t^{\mu+1} v(t), B_\lambda \partial_t^{\mu+1} v(t) \rangle \\
 &\quad + (1+t)^{2\mu+2} \langle \partial_t^{\mu+1} v(t), \partial_t^\mu N[v, v](t) \rangle.
 \end{aligned}$$

Using the above estimate and

$$\begin{aligned}
 \langle \partial_t^{\mu+1} v(t), \partial_t^\mu N[v, v](t) \rangle &\leq C \|\partial_t^{\mu+1} v(t)\|_2 \sum_{v=0}^{\mu} \binom{\mu}{v} \|\partial_t^v \nabla^2 v(t)\|_\infty \|\partial_t^{\mu-v} \nabla v(t)\|_2 \\
 &\leq C E^{1/2}(\partial_t^\mu v(t)) \sum_{v=0}^{\mu} E_{L-v}^{1/2}(\partial_t^v v(t)) E_{L-(\mu-v)}^{1/2}(\partial_t^{\mu-v} v(t)),
 \end{aligned}$$

we obtain

$$\begin{aligned}
 (63) \quad & \frac{d}{dt} \{(1+t)^{2\mu+2} E(\partial_t^\mu v(t))\} + (1+t)^{2\mu+2} \langle \partial_t^{\mu+1} v(t), B_\lambda \partial_t^{\mu+1} v(t) \rangle \\
 &\leq C(1+t)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(t)) \\
 &\quad + C(1+t)^{2\mu+2} E^{1/2}(\partial_t^\mu v(t)) \sum_{v=0}^{\mu} E_{L-v}^{1/2}(\partial_t^v v(t)) E_{L-(\mu-v)}^{1/2}(\partial_t^{\mu-v} v(t)).
 \end{aligned}$$

Now we define

$$M_\mu(t) = \sup_{0 \leq s \leq t} (1+s)^{2\mu+2} E(\partial_t^\mu v(s)).$$

Integrating (63) over $[0, t]$ and using (36) for $i = v$, we obtain

$$\begin{aligned}
 (64) \quad & (1+t)^{2\mu+2} E(\partial_t^\mu v(t)) + \int_0^t (1+s)^{2\mu+2} \langle \partial_t^{\mu+1} v(s), B_\lambda \partial_t^{\mu+1} v(s) \rangle ds \\
 & \leq E(\partial_t^\mu v(0)) + C \int_0^t (1+s)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(s)) ds \\
 & \quad + C \int_0^t (1+s)^{2\mu+2} E^{1/2}(\partial_t^\mu v(s)) \sum_{v=0}^{\mu} E_{L-v}^{1/2}(\partial_t^v v(s)) E_{L-(\mu-v)}^{1/2}(\partial_t^{\mu-v} v(s)) ds \\
 & \leq E(\partial_t^\mu v(0)) + C \int_0^t (1+s)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(s)) ds + \frac{C}{2} (M_\mu(t))^{1/2} \\
 & \quad \times \sum_{v=0}^{\mu} \left\{ \int_0^t (1+s)^{2v+1} E_{L-v}(\partial_t^v v(s)) ds \right. \\
 & \quad \left. + \int_0^t (1+s)^{2(\mu-v)+1} E_{L-(\mu-v)}(\partial_t^{\mu-v} v(s)) ds \right\} \\
 & \leq E(\partial_t^\mu v(0)) + CE_0 + 2(\mu+1)CE_0(M_\mu(t))^{1/2}.
 \end{aligned}$$

From (64), we obtain

$$\begin{aligned}
 (65) \quad & (1+t)^{2\mu+2} E(\partial_t^\mu v(t)) + \int_0^t (1+s)^{2\mu+2} \langle \partial_t^{\mu+1} v(s), B_\lambda \partial_t^{\mu+1} v(s) \rangle ds \\
 & \leq E_0 + E_0(M_\mu(t))^{1/2}
 \end{aligned}$$

Thus we have

$$M_\mu(t) \leq E_0 + E_0(M_\mu(t))^{1/2}.$$

It means that $M_\mu(t)$ is bounded in $[0, \infty)$. Thus it holds from (65) that

$$(1+t)^{2\mu+2} E(\partial_t^\mu v(t)) + \int_0^t (1+s)^{2\mu+2} \langle \partial_t^{\mu+1} v(s), B_\lambda \partial_t^{\mu+1} v(s) \rangle ds \leq E_0.$$

This means (37) for $i = \mu$. This completes the proof of theorem 4.1. $\square$

5. Proof of lemma 3.1

We prove lemma 3.1. First we prepare the estimates of the nonlinear term.

Lemma 5.1 (Estimates of the nonlinear term). *Let $L_0 \leq \bar{L} \leq L$, $0 \leq \mu \leq \bar{L} - L_0$. Then there exists a constant $C > 0$ such that for any $T, \delta > 0$, $0 < \lambda \leq 1$ and the local solution $v \in X_{\delta, T}$ to $(\text{DW})_\lambda$ satisfy*

$$(66) \quad \sum_{|a| \leq \bar{L} - \mu - 1} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, \partial_t v](t) \leq CD_{\bar{L}, \mu}(v(t)),$$

$$(67) \quad \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v(t), N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v](t) \rangle \leq CD_{\bar{L}, \mu}(v(t)),$$

$$(68) \quad \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{\substack{b+c=a \\ b, c \neq 0}} \langle \partial_t^{\mu+1} \nabla^a v(t), \partial_t^\mu N[\nabla^b v, \nabla^c v](t) \rangle \leq CD_{\bar{L}, \mu}(v(t)),$$

$$(69) \quad \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{b+c=a} \binom{a}{b} \langle \partial_t^\mu \nabla^a v(t), \partial_t^\mu N[\nabla^b v, \nabla^c v](t) \rangle \leq CD_{\bar{L}, \mu}(v(t))$$

and

$$(70) \quad \sum_{|a| \leq \bar{L} - \mu - 1} \langle \partial_t^\mu \nabla^a N[v, v](t), [h; \nabla \partial_t^\mu \nabla^a v(t)] \rangle \leq \frac{C}{\lambda} D_{\bar{L}, \mu}(v(t)),$$

where $\tilde{N}$ and $D_{\bar{L}, \mu}$ are defined by (11) and (19).

Proof. First, we prove (66). Using lemma 2.1, we have

$$\begin{aligned} & \sum_{|a| \leq \bar{L} - \mu - 1} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, \partial_t v](t) \\ & \leq C \sum_{|a| \leq \bar{L} - \mu - 1} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2 \|\nabla \partial_t^\mu \nabla^a v(t)\|_2 \|\nabla \partial_t v(t)\|_\infty \\ & \leq C \sum_{|a| \leq \bar{L} - \mu - 1} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2 \|\nabla \partial_t^\mu \nabla^a v(t)\|_2 \|\partial_t v(t)\|_{H^{[d/2]+2}} \\ & \leq CE_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) E_{L_0}^{1/2}(v(t)) \\ & \leq CE_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) \sum_{v=0}^{\mu} E_{\bar{L}-v}^{1/2}(\partial_t^v v(t)) E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) = CD_{\bar{L}, \mu}(v(t)). \end{aligned}$$

Next, we prove (67). We remark that if $|a| \leq \bar{L} - \mu - 1$ and $0 \leq v \leq \mu - 1$ then $|a| + 1 \leq \bar{L} - v - 1$. Hence it follows that

$$\begin{aligned}
& \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v(t), N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v](t) \rangle \\
& \leq C \sum_{|a| \leq \bar{L} - \mu - 1} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2 \sum_{v=0}^{\mu-1} \sum_{j,k,l,m,n=1}^d \|\partial_l (\partial_m \partial_t^v \nabla^a v^j(t) \partial_n \partial_t^{\mu-v} v^k(t))\|_2 \\
& \leq C \sum_{|a| \leq \bar{L} - \mu - 1} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2 \\
& \quad \times \sum_{v=0}^{\mu-1} \sum_{j,k,l,m,n=1}^d \{ \|\partial_l \partial_m \partial_t^v \nabla^a v^j(t)\|_2 \|\partial_n \partial_t^{\mu-v} v^k(t)\|_\infty \\
& \quad + \|\partial_m \partial_t^v \nabla^a v^j(t)\|_2 \|\partial_l \partial_n \partial_t^{\mu-v} v^k(t)\|_\infty \} \\
& \leq C E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) \sum_{v=0}^{\mu-1} \{ E_{\bar{L}-v}^{1/2}(\partial_t^v v(t)) E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) \\
& \quad + E_{\bar{L}-v}^{1/2}(\partial_t^v v(t)) E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) \} \\
& \leq C E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) \sum_{v=0}^{\mu} E_{\bar{L}-v}^{1/2}(\partial_t^v v(t)) E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) = C D_{\bar{L},\mu}(v(t)).
\end{aligned}$$

Next, we prove (68). For any $|a| \leq \bar{L} - \mu - 1$, $b + c = a$ and $b, c \neq 0$ it hold that

$$(71) \quad \|\partial_t^\mu N[\nabla^b v, \nabla^c v](t)\|_2 \leq C \sum_{v=0}^{\mu} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{\bar{L}-v}^{1/2}(\partial_t^v v(t)).$$

Because if $|a| \leq \bar{L} - \mu - 1$, $b + c = a$ and $b, c \neq 0$ then we can decompose c of the form: $c = c' + c''$, $|c'| = |c| - 1$, $|c''| = 1$. Then using lemma 2.1 and lemma 2.2 we obtain

$$\begin{aligned}
& \|\partial_t^\mu (\partial_l \partial_m \nabla^b v^i(t) \partial_n \nabla^c v^k(t))\|_2 \\
& \leq C \sum_{v=0}^{\mu} \|\partial_t^v \partial_l \partial_m \nabla^b v^j(t) \partial_t^{\mu-v} \partial_n \nabla^{c''} \nabla^{c'} v^k(t)\|_2 \\
& \leq C \sum_{v=0}^{\mu} \{ \|\nabla^2 \partial_t^v v(t)\|_\infty \|\nabla^2 \nabla^{b+c'} \partial_t^{\mu-v} v(t)\|_2 \\
& \quad + \|\nabla^2 \nabla^{b+c'} \partial_t^v v(t)\|_2 \|\nabla^2 \partial_t^{\mu-v} v(t)\|_\infty \}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{v=0}^{\mu} \{ \|\nabla^2 \partial_t^v v(t)\|_{H^{[d/2]+1}} \|\nabla^2 \nabla^{b+c'} \partial_t^{\mu-v} v(t)\|_2 \\
&\quad + \|\nabla^2 \nabla^{b+c'} \partial_t^v v(t)\|_2 \|\nabla^2 \partial_t^{\mu-v} v(t)\|_{H^{[d/2]+1}} \} \\
&\leq C \sum_{v=0}^{\mu} \|\nabla^2 \partial_t^{\mu-v} v(t)\|_{H^{[d/2]+1}} \|\nabla^2 \nabla^{b+c'} \partial_t^v v(t)\|_2 \\
&\leq C \sum_{v=0}^{\mu} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{L-v}^{1/2}(\partial_t^v v(t)).
\end{aligned}$$

Similarly we obtain

$$\|\partial_t^\mu (\partial_m \nabla^b v^i \partial_l \partial_n \nabla^c v^k)\|_2 \leq C \sum_{v=0}^{\mu} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{L-v}^{1/2}(\partial_t^v v(t)).$$

Then we get (71). It follows from (71) that

$$\begin{aligned}
&\sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c \neq 0}} \langle \partial_t^{\mu+1} \nabla^a v(t), \partial_t^\mu N[\nabla^b v, \nabla^c v](t) \rangle \\
&\leq \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c \neq 0}} \|\partial_t \partial_t^\mu \nabla^a v(t)\|_2 \|\partial_t^\mu N[\nabla^b v, \nabla^c v](t)\|_2 \\
&\leq C E_{L-\mu}^{1/2}(\partial_t^\mu v(t)) \sum_{v=0}^{\mu} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{L-v}^{1/2}(\partial_t^v v(t)) = C D_{\bar{L},\mu}(v(t)).
\end{aligned}$$

Next, we prove (69). Using lemma 2.1 and lemma 2.2 we have

$$\begin{aligned}
&\sum_{|a| \leq \bar{L}-\mu-1} \sum_{b+c=a} \binom{a}{b} \langle \partial_t^\mu \nabla^a v(t), \partial_t^\mu N[\nabla^b v, \nabla^c v](t) \rangle \\
&\leq C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{b+c=a} \left| \sum_{i,j,k,l,m,n=1}^d \int_{\mathbf{R}^d} \partial_t^\mu \nabla^a v^i \partial_l^\mu \partial_l (\partial_m \nabla^b v^j \partial_n \nabla^c v^k) dx \right| \\
&= C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{b+c=a} \left| \sum_{i,j,k,l,m,n=1}^d \int_{\mathbf{R}^d} \partial_l \partial_t^\mu \nabla^a v^i \partial_l^\mu (\partial_m \nabla^b v^j \partial_n \nabla^c v^k) dx \right| \\
&\leq C \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2 \\
&\quad \times \sum_{v=0}^{\mu} \{ \|\nabla \partial_t^{\mu-v} v(t)\|_\infty \|\nabla \nabla^a \partial_t^v v(t)\|_2 + \|\nabla \nabla^a \partial_t^{\mu-v} v(t)\|_2 \|\nabla \partial_t^v v(t)\|_\infty \}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2 \sum_{v=0}^{\mu} \|\nabla \partial_t^{\mu-v} v(t)\|_{H^{[d/2]+1}} \|\nabla \nabla^a \partial_t^v v(t)\|_2 \\
&\leq C E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) \sum_{v=0}^{\mu} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{\bar{L}-\mu}^{1/2}(\partial_t^v v) \leq C D_{\bar{L},\mu}(v(t)).
\end{aligned}$$

Finally, we prove (70).

$$\begin{aligned}
&\sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a N[v, v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
&= \sum_{|a| \leq \bar{L}-\mu-1} \sum_{b+c=a} \binom{a}{b} \langle \partial_t^\mu N[\nabla^b v, \nabla^c v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
&= \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b, c \neq 0}} \binom{a}{b} \langle \partial_t^\mu N[\nabla^b v, \nabla^c v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
&\quad + 2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \sum_{p=1}^d \int_{\mathbf{R}^d} \partial_t^\mu \partial_l (\partial_m \nabla^a v^j \partial_n v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i \, dx \\
&= \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b, c \neq 0}} \binom{a}{b} \langle \partial_t^\mu N[\nabla^b v, \nabla^c v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
&\quad + 2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \\
&\quad \times \sum_{p=1}^d \sum_{v=0}^{\mu-1} \binom{\mu}{v} \int_{\mathbf{R}^d} \partial_l (\partial_m \partial_t^v \nabla^a v^j \partial_n \partial_t^{\mu-v} v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i \, dx \\
&\quad + 2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \sum_{p=1}^d \int_{\mathbf{R}^d} \partial_l (\partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i \, dx \\
&= \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b, c \neq 0}} \binom{a}{b} \langle \partial_t^\mu N[\nabla^b v, \nabla^c v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
&\quad + 2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \\
&\quad \times \sum_{p=1}^d \sum_{v=0}^{\mu-1} \binom{\mu}{v} \int_{\mathbf{R}^d} \partial_l (\partial_m \partial_t^v \nabla^a v^j \partial_n \partial_t^{\mu-v} v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i \, dx
\end{aligned}$$

$$\begin{aligned}
& -2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \sum_{p=1}^d \int_{\mathbf{R}^d} \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k \partial_l h^p \partial_p \partial_t^\mu \nabla^a v^i \, dx \\
& -2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \sum_{p=1}^d \int_{\mathbf{R}^d} \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k h^p \partial_l \partial_p \partial_t^\mu \nabla^a v^i \, dx \\
& = \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c \neq 0}} \binom{a}{b} \langle \partial_t^\mu N[\nabla^b v, \nabla^c v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
& + 2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \\
& \times \sum_{p=1}^d \sum_{v=0}^{\mu-1} \binom{\mu}{v} \int_{\mathbf{R}^d} \partial_l (\partial_m \partial_t^v \nabla^a v^j \partial_n \partial_t^{\mu-v} v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i \, dx \\
& -2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \sum_{p=1}^d \int_{\mathbf{R}^d} \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k \partial_l h^p \partial_p \partial_t^\mu \nabla^a v^i \, dx \\
& - \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \sum_{p=1}^d \int_{\mathbf{R}^d} \partial_p (\partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i) \partial_n v^k h^p \, dx \\
& = J_1 + J_2 + J_3 + J_4,
\end{aligned}$$

where we use **(N1)**. For J_2 and J_3 , it follows from (23) that

$$\begin{aligned}
|J_2| & \leq C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d \sum_{p=1}^d \sum_{v=0}^{\mu-1} \|\partial_l (\partial_m \partial_t^v \nabla^a v^j \partial_n \partial_t^{\mu-v} v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i\|_1 \\
& \leq C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{p=1}^d \sum_{v=0}^{\mu-1} \|\nabla^2 \partial_t^v \nabla^a v\|_2 \|\nabla \partial_t^{\mu-v} v\|_\infty \|h^p\|_\infty \|\partial_p \partial_t^\mu \nabla^a v\|_2 \\
& + C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{p=1}^d \sum_{v=0}^{\mu-1} \|\nabla \partial_t^v \nabla^a v\|_2 \|\nabla^2 \partial_t^{\mu-v} v\|_\infty \|h^p\|_\infty \|\partial_p \partial_t^\mu \nabla^a v\|_2 \\
& \leq C \|h\|_\infty \sum_{v=0}^{\mu-1} E_{\bar{L}-v}^{1/2}(\partial_t^v v(t)) E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) \leq \frac{C}{\lambda} D_{\bar{L},\mu}(v(t))
\end{aligned}$$

and

$$\begin{aligned}
|J_3| &\leq C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d \sum_{p=1}^d \|\partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k \partial_l h^p \partial_p \partial_t^\mu \nabla^a v^i\|_1 \\
&\leq C \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v\|_2 \|\nabla v\|_\infty \|\nabla h\|_\infty \|\nabla \partial_t^\mu \nabla^a v\|_2 \\
&\leq C E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) E_{L_0}^{1/2}(v(t)) \leq C D_\mu(v(t)) \leq \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)).
\end{aligned}$$

From (23) and (71), we get

$$\begin{aligned}
|J_1| &\leq C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c \neq 0}} |\langle \partial_t^\mu N[\nabla^b v, \nabla^c v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle| \\
&\leq C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c \neq 0}} \|\langle \partial_t^\mu N[\nabla^b v, \nabla^c v] \|_2 \|h\|_\infty \|\nabla \partial_t^\mu \nabla^a v\|_2 \\
&\leq C \|h\|_\infty E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) \sum_{v=0}^{\mu} E_{L_0}^{1/2}(\partial_t^{\mu-v} v(t)) E_{\bar{L}-v}^{1/2}(\partial_t^v v(t)) \leq \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)).
\end{aligned}$$

From (71), we also have

$$\begin{aligned}
|J_4| &= \left| \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \sum_{p=1}^d \int_{\mathbf{R}^d} \partial_p (\partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i) \partial_n v^k h^p dx \right| \\
&= \left| \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \sum_{p=1}^d \int_{\mathbf{R}^d} \partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i \partial_p (\partial_n v^k h^p) dx \right| \\
&\leq C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d \sum_{p=1}^d \int_{\mathbf{R}^d} |\partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i \partial_p \partial_n v^k h^p| dx \\
&\quad + C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^d \sum_{p=1}^d \int_{\mathbf{R}^d} |\partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i \partial_n v^k \partial_p h^p| dx \\
&\leq C \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v\|_2 \|\nabla \partial_t^\mu \nabla^a v\|_2 (\|\nabla^2 v\|_\infty \|h\|_\infty + \|\nabla v\|_\infty \|\nabla h\|_\infty) \\
&\leq C \left(1 + \frac{1}{\lambda}\right) E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) E_{\bar{L}-\mu}^{1/2}(\partial_t^\mu v(t)) E_{L_0}^{1/2}(v(t)) \leq \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)).
\end{aligned}$$

Hence we obtain (70), which completes the proof of lemma 5.1. $\square$

Because of dissipation effect $(\mathbf{B2})_\lambda$, we are expected to decrease the energy in $|x| \gg 1$. Next lemma corresponds to this phenomenon.

Lemma 5.2. *Let $L_0 \leq \bar{L} \leq L$, $0 \leq \mu \leq \bar{L} - L_0$. Then there exists a constant $C > 0$ such that for any $T, \delta > 0$, $0 < \lambda \leq 1$ and the local solution $v \in X_{\delta, T}$ to $(\text{DW})_\lambda$ satisfy*

$$(72) \quad \begin{aligned} \frac{d}{dt} \tilde{E}_{\bar{L}, \mu}(v(t)) + \sum_{|a| \leq \bar{L} - \mu - 1} \langle \partial_t^{\mu+1} \nabla^a v(t), B_\lambda \partial_t^{\mu+1} \nabla^a v(t) \rangle \\ \leq C \lambda^2 E_{\bar{L} - \mu}(\partial_t^\mu v(t)) + C D_{\bar{L}, \mu}(v(t)) \end{aligned}$$

and

$$(73) \quad \begin{aligned} \frac{d}{dt} \sum_{|a| \leq \bar{L} - \mu - 1} \left\{ \langle \partial_t^\mu \nabla^a v(t), \partial_t^{\mu+1} \nabla^a v(t) \rangle + \frac{1}{2} \langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t) \rangle \right\} \\ - \sum_{|a| \leq \bar{L} - \mu - 1} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2^2 + \sum_{|a| \leq \bar{L} - \mu - 1} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2^2 \\ \leq C \lambda^2 E_{\bar{L} - \mu}(\partial_t^\mu v(t)) + C D_{\bar{L}, \mu}(v(t)). \end{aligned}$$

Proof. Let $L_0 \leq \bar{L} \leq L$ and $0 \leq \mu \leq \bar{L} - L_0$. First we show (72). We calculate

$$(74) \quad \frac{d}{dt} \tilde{E}_{\bar{L}, \mu}(v(t)) = \sum_{|a| \leq \bar{L} - \mu - 1} \frac{d}{dt} E(\partial_t^\mu \nabla^a v(t)) + \sum_{|a| \leq \bar{L} - \mu - 1} \frac{d}{dt} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, v](t).$$

For the second terms of (74), for any $|a| \leq \bar{L} - \mu - 1$, we need the following equality:

$$(75) \quad \begin{aligned} \langle \partial_t^{\mu+1} \nabla^a v, \partial_t^\mu N[\nabla^a v, v] \rangle \\ = \langle \partial_t^{\mu+1} \nabla^a v, N[\partial_t^\mu \nabla^a v, v] \rangle + \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v, N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v] \rangle \\ = \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_{\mathbf{R}^d} \partial_t^{\mu+1} \nabla^a v^i \partial_l (\partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k) dx \\ + \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v, N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v] \rangle \\ = - \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_{\mathbf{R}^d} \partial_l \partial_t^{\mu+1} \nabla^a v^i \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k dx \end{aligned}$$

$$\begin{aligned}
& + \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v, N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v] \rangle \\
& = -\frac{1}{2} \frac{d}{dt} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_{\mathbf{R}^d} \partial_l \partial_t^\mu \nabla^a v^i \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k dx \\
& \quad + \frac{1}{2} \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk} \int_{\mathbf{R}^d} \partial_l \partial_t^\mu \nabla^a v^i \partial_m \partial_t^\mu \nabla^a v^j \partial_n \partial_t v^k dx \\
& \quad + \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v, N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v] \rangle \\
& = -\frac{1}{2} \frac{d}{dt} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, v] + \frac{1}{2} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, \partial_t v] \\
& \quad + \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v, N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v] \rangle,
\end{aligned}$$

where we use **(N1)**. To handle the first terms of (74), we apply $\partial_t^\mu \nabla^a$ to $(\text{DW})_\lambda$. We get

$$(76) \quad \partial_t^{\mu+2} \nabla^a v - \triangle \partial_t^\mu \nabla^a v + \nabla^a (B_\lambda \partial_t^{\mu+1} v) = \sum_{b+c=a} \binom{a}{b} \partial_t^\mu N[\nabla^b v, \nabla^c v],$$

which yields that

$$\begin{aligned}
(77) \quad \sum_{|a| \leq \bar{L}-\mu-1} \frac{d}{dt} E(\partial_t^\mu \nabla^a v) & = - \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, \nabla^a (B_\lambda \partial_t^{\mu+1} v) \rangle \\
& \quad + \sum_{|a| \leq \bar{L}-\mu-1} \sum_{b+c=a} \binom{a}{b} \langle \partial_t^{\mu+1} \nabla^a v, \partial_t^\mu N[\nabla^b v, \nabla^c v] \rangle.
\end{aligned}$$

Combining (74), (75), (77) and lemma 5.1, we get

$$\begin{aligned}
(78) \quad \frac{d}{dt} \tilde{E}_{\bar{L},\mu}(\partial_t^\mu v) & = \sum_{|a| \leq \bar{L}-\mu-1} \frac{d}{dt} E(\partial_t^\mu \nabla^a v) + \sum_{|a| \leq \bar{L}-\mu-1} \frac{d}{dt} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, v] \\
& = - \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, \nabla^a (B_\lambda \partial_t^{\mu+1} v) \rangle \\
& \quad + \sum_{|a| \leq \bar{L}-\mu-1} \sum_{b+c=a} \binom{a}{b} \langle \partial_t^{\mu+1} \nabla^a v, \partial_t^\mu N[\nabla^b v, \nabla^c v] \rangle
\end{aligned}$$

$$\begin{aligned}
& - 2 \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, \partial_t^\mu N[\nabla^a v, v] \rangle \\
& + \sum_{|a| \leq \bar{L}-\mu-1} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, \partial_t v] \\
& + 2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v, N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v] \rangle \\
& = - \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, \nabla^a (B_\lambda \partial_t^{\mu+1} v) \rangle \\
& + \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b, c \neq 0}} \binom{a}{b} \langle \partial_t^{\mu+1} \nabla^a v, \partial_t^\mu N[\nabla^b v, \nabla^c v] \rangle \\
& + \sum_{|a| \leq \bar{L}-\mu-1} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, \partial_t v] \\
& + 2 \sum_{|a| \leq \bar{L}-\mu-1} \sum_{v=0}^{\mu-1} \binom{\mu}{v} \langle \partial_t^{\mu+1} \nabla^a v, N[\partial_t^v \nabla^a v, \partial_t^{\mu-v} v] \rangle \\
& \leq - \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, \nabla^a (B_\lambda \partial_t^{\mu+1} v) \rangle + CD_{\bar{L}, \mu}(v).
\end{aligned}$$

Since for $\lambda \leq 1$, we have

$$\begin{aligned}
& - \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, \nabla^a (B_\lambda \partial_t^{\mu+1} v) \rangle \\
& = - \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle \\
& - \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \langle \partial_t^{\mu+1} \nabla^a v, \nabla^b B_\lambda(x) \partial_t^{\mu+1} \nabla^{a-b} v \rangle \\
& \leq - \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle \\
& + C \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b \leq a \\ b \neq 0}} \|\nabla^b B_\lambda\|_\infty \|\partial_t^{\mu+1} \nabla^a v\|_2 \|\partial_t^{\mu+1} \nabla^{a-b} v\|_2 \\
& \leq - \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + C\lambda^2 E_{\bar{L}-\mu}(\partial_t^\mu v).
\end{aligned}$$

Thus we obtain (72).

Next we prove (73). Using (76) and lemma 5.1 (69), we have

$$\begin{aligned}
(79) \quad & \frac{d}{dt} \left\{ \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v, \partial_t^{\mu+1} \nabla^a v \rangle + \frac{1}{2} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v, B_\lambda \partial_t^\mu \nabla^a v \rangle \right\} \\
&= \sum_{|a| \leq \bar{L}-\mu-1} \{ \|\partial_t^{\mu+1} \nabla^a v\|_2^2 + \langle \partial_t^\mu \nabla^a v, \partial_t^{\mu+2} \nabla^a v \rangle + \langle \partial_t^\mu \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle \} \\
&\leq \sum_{|a| \leq \bar{L}-\mu-1} \|\partial_t^{\mu+1} \nabla^a v\|_2^2 - \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v\|_2^2 \\
&\quad - \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \langle \partial_t^\mu \nabla^a v, \nabla^b B_\lambda \partial_t^{\mu+1} \nabla^{a-b} v \rangle + CD_{\bar{L},\mu}(v).
\end{aligned}$$

For $\lambda \leq 1$, we have

$$\begin{aligned}
& \sum_{|a| \leq \bar{L}-\mu-1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} |\langle \partial_t^\mu \nabla^a v, \nabla^b B_\lambda \partial_t^{\mu+1} \nabla^{a-b} v \rangle| \\
&= \sum_{1 \leq |a| \leq \bar{L}-\mu-1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} |\langle \partial_t^\mu \nabla^a v, \nabla^b B_\lambda \partial_t^{\mu+1} \nabla^{a-b} v \rangle| \\
&\leq \sum_{1 \leq |a| \leq \bar{L}-\mu-1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \|\nabla^b B_\lambda\|_\infty \|\partial_t^\mu \nabla^a v\|_2 \|\partial_t^{\mu+1} \nabla^{a-b} v\|_2 \leq \lambda^2 CE_{\bar{L}-\mu}(\partial_t^\mu v),
\end{aligned}$$

which yields (73) from (79). This completes the proof of lemma 5.2. $\square$

We can't expect the effect of dissipation in $|x| \leq R/\lambda$ since $B_\lambda(x)$ may not strictly positive. So we use the local energy decay property in $(\text{DW})_\lambda$. We lead the estimate which corresponding this property by using the argument of Nakao [11] and Ikehata [5].

Lemma 5.3. *Let $L_0 \leq \bar{L} \leq L$, $0 \leq \mu \leq \bar{L} - L_0$. Then there exists a constant $C > 0$ such that for any $K, T, \delta > 0$, $0 < \lambda \leq 1$ and the local solution $v \in X_{\delta, T}$ to $(\text{DW})_\lambda$ satisfy*

$$\begin{aligned}
(80) \quad & \frac{d}{dt} \left\{ \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v(t), [h; \nabla \partial_t^\mu \nabla^a v(t)] \rangle \right\} \\
&+ \sum_{|a| \leq \bar{L}-\mu-1} \int_{\mathbf{R}^d} \left\{ \frac{d\phi + |x|\phi'}{2} \right\} |\partial_t^{\mu+1} \nabla^a v(t, x)|^2 dx \\
&+ \sum_{|a| \leq \bar{L}-\mu-1} \int_{\mathbf{R}^d} \left\{ \phi + |x|\phi' - \frac{d\phi + |x|\phi'}{2} \right\} |\nabla \partial_t^\mu \nabla^a v(t, x)|^2 dx
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda C E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)) \\
&\quad + \frac{K}{4} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v(t), B_\lambda \partial_t^{\mu+1} \nabla^a v(t) \rangle \\
&\quad + \frac{2\|B\|_\infty b_0^2 R^2}{\lambda K} E_{\bar{L}-\mu}(\partial_t^\mu v(t)),
\end{aligned}$$

where $\phi = \phi(|x|)$, h are defined by (17).

Proof. Let $|a| \leq \bar{L} - \mu - 1$. We apply $\partial_t^\mu \nabla^a$ to $(\text{DW})_\lambda$ and take inner product the equation by $[h; \nabla \partial_t^\mu \nabla^a v]$ we obtain

$$\begin{aligned}
(81) \quad &\langle \partial_t^{\mu+2} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle - \langle \Delta \partial_t^\mu \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
&\quad + \langle \nabla^a (B_\lambda \partial_t^{\mu+1} v), [h; \nabla \partial_t^\mu \nabla^a v] \rangle = \langle \partial_t^\mu \nabla^a N[v, v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle.
\end{aligned}$$

Noting (81),

$$\begin{aligned}
&\langle \partial_t^{\mu+2} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
&= \frac{d}{dt} \langle \partial_t^{\mu+1} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle - \sum_{k=1}^d \sum_{i=1}^d \int_{\mathbf{R}^d} \partial_t^{\mu+1} \nabla^a v^k h^i \partial_i \partial_t^{\mu+1} \nabla^a v^k \, dx \\
&= \frac{d}{dt} \langle \partial_t^{\mu+1} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle - \frac{1}{2} \sum_{k=1}^d \sum_{i=1}^d \int_{\mathbf{R}^d} \partial_i |\partial_t^{\mu+1} \nabla^a v^k|^2 h^i \, dx \\
&= \frac{d}{dt} \langle \partial_t^{\mu+1} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle + \frac{1}{2} \int_{\mathbf{R}^d} |\partial_t^{\mu+1} \nabla^a v|^2 \operatorname{div} h \, dx
\end{aligned}$$

and

$$\begin{aligned}
&-\langle \Delta \partial_t^\mu \nabla^a v(t), [h; \nabla \partial_t^\mu \nabla^a v(t)] \rangle \\
&= \sum_{i,k=1}^d \int_{\mathbf{R}^d} \nabla \partial_t^\mu \nabla^a v^k \cdot \nabla (h^i \partial_i \partial_t^\mu \nabla^a v^k) \, dx \\
&= \sum_{i,k=1}^d \int_{\mathbf{R}^d} \nabla \partial_t^\mu \nabla^a v^k \cdot \nabla h^i \partial_i \partial_t^\mu \nabla^a v^k \, dx + \sum_{i,k=1}^d \int_{\mathbf{R}^d} \nabla \partial_t^\mu \nabla^a v^k \cdot \nabla \partial_i \partial_t^\mu \nabla^a v^k h^i \, dx \\
&= \sum_{i,j,k=1}^d \int_{\mathbf{R}^d} \partial_j \partial_t^\mu \nabla^a v^k \partial_j h^i \partial_i \partial_t^\mu \nabla^a v^k \, dx + \frac{1}{2} \sum_{i,k=1}^d \int_{\mathbf{R}^d} \partial_i |\nabla \partial_t^\mu \nabla^a v^k|^2 h^i \, dx \\
&= \sum_{i,j,k=1}^d \int_{\mathbf{R}^d} \partial_j \partial_t^\mu \nabla^a v^k \partial_j h^i \partial_i \partial_t^\mu \nabla^a v^k \, dx - \frac{1}{2} \int_{\mathbf{R}^d} |\nabla \partial_t^\mu \nabla^a v|^2 \operatorname{div} h \, dx,
\end{aligned}$$

we obtain

$$\begin{aligned}
 (82) \quad & \frac{d}{dt} \langle \partial_t^{\mu+1} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle + \frac{1}{2} \int_{\mathbf{R}^d} |\partial_t^{\mu+1} \nabla^a v|^2 \operatorname{div} h \, dx \\
 & - \frac{1}{2} \int_{\mathbf{R}^d} |\nabla \partial_t^\mu \nabla^a v|^2 \operatorname{div} h \, dx + \sum_{i,j,k=1}^d \int_{\mathbf{R}^d} \partial_j \partial_t^\mu \nabla^a v^k \partial_j h^i \partial_i \partial_t^\mu \nabla^a v^k \, dx \\
 & = -\langle \nabla^a (B_\lambda \partial_t^{\mu+1} v), [h; \nabla \partial_t^\mu \nabla^a v] \rangle + \langle \partial_t^\mu \nabla^a N[v, v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle.
 \end{aligned}$$

Now we remark that it holds that

$$(83) \quad \frac{\partial h^i}{\partial x_j} = \delta_{ij} \phi(|x|) + \phi'(|x|) \frac{x_i x_j}{|x|}$$

and

$$(84) \quad \operatorname{div} h(x) = d\phi(|x|) + |x|\phi'(|x|) \quad (x \in \mathbf{R}^d).$$

Furthermore using $\phi'(r) \leq 0$ we get

$$\begin{aligned}
 & \sum_{i,j,k=1}^d \int_{\mathbf{R}^d} \partial_j \partial_t^\mu \nabla^a v^k \partial_j h^i \partial_i \partial_t^\mu \nabla^a v^k \, dx \\
 & = \sum_{i,k=1}^d \int_{\mathbf{R}^d} \partial_i \partial_t^\mu \nabla^a v^k \phi \partial_i \partial_t^\mu \nabla^a v^k \, dx + \sum_{i,j,k=1}^d \int_{\mathbf{R}^d} \partial_j \partial_t^\mu \nabla^a v^k \phi' \frac{x_i x_j}{|x|} \partial_i \partial_t^\mu \nabla^a v^k \, dx \\
 & = \int_{\mathbf{R}^d} |\nabla \partial_t^\mu \nabla^a v|^2 \phi \, dx + \sum_{k=1}^d \int_{\mathbf{R}^d} |x \cdot \nabla \partial_t^\mu \nabla^a v^k|^2 \phi' \frac{1}{|x|} \, dx \\
 & \geq \int_{\mathbf{R}^d} \{ \phi + |x|\phi' \} |\nabla \partial_t^\mu \nabla^a v|^2 \, dx.
 \end{aligned}$$

This estimate and (82) imply that

$$\begin{aligned}
 (85) \quad & \frac{d}{dt} \langle \partial_t^{\mu+1} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle + \int_{\mathbf{R}^d} \left\{ \frac{(d\phi + |x|\phi')}{2} \right\} |\partial_t^{\mu+1} \nabla^a v|^2 \, dx \\
 & + \int_{\mathbf{R}^d} \left\{ \phi + |x|\phi' - \frac{d\phi + |x|\phi'}{2} \right\} |\nabla \partial_t^\mu \nabla^a v|^2 \, dx \\
 & \leq -\langle B_\lambda \partial_t^{\mu+1} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v(t)] \rangle \\
 & - \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \langle \nabla^b B_\lambda \partial_t^{\mu+1} \nabla^{a-b} v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
 & + \langle \partial_t^\mu \nabla^a N[v, v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle.
 \end{aligned}$$

Let estimate for the right side of (85). First, since B_λ is a nonnegative symmetric matrix, there exists a nonnegative symmetric matrix S_λ such that $S_\lambda^2 = B_\lambda$. Using **(B3)** $_\lambda$ and (23), for any $|a| \leq \bar{L} - \mu - 1$ and $K > 0$ we obtain

$$\begin{aligned}
& |\langle B_\lambda \partial_t^{\mu+1} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle| \\
&= |\langle S_\lambda \partial_t^{\mu+1} \nabla^a v, S_\lambda [h; \nabla \partial_t^\mu \nabla^a v] \rangle| \\
&\leq \frac{K}{4} \|S_\lambda \partial_t^{\mu+1} \nabla^a v\|_2^2 + \frac{1}{K} \|S_\lambda [h; \nabla \partial_t^\mu \nabla^a v]\|_2^2 \\
&= \frac{K}{4} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{1}{K} \langle [h; \nabla \partial_t^\mu \nabla^a v], B_\lambda [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
&\leq \frac{K}{4} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{1}{K} \|B_\lambda\|_\infty \|h\|_\infty^2 \|\nabla \partial_t^\mu \nabla^a v\|_2^2 \\
&\leq \frac{K}{4} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{\|B\|_\infty b_0^2 R^2}{\lambda K} \|\nabla \partial_t^\mu \nabla^a v\|_2^2.
\end{aligned}$$

So it holds that

$$\begin{aligned}
(86) \quad & - \sum_{|a| \leq \bar{L} - \mu - 1} \langle B_\lambda \partial_t^{\mu+1} \nabla^a v, [h; \nabla \nabla^a \partial_t^\mu v] \rangle \\
& \leq \sum_{|a| \leq \bar{L} - \mu - 1} \frac{K}{4} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{2\|B\|_\infty b_0^2 R^2}{\lambda K} E_{\bar{L}-\mu}(\partial_t^\mu v).
\end{aligned}$$

Second, using **(B3)** $_\lambda$ and (23), for $\lambda \leq 1$ we have

$$\begin{aligned}
(87) \quad & \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \langle \nabla^b B_\lambda \partial_t^{\mu+1} \nabla^{a-b} v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
& \leq \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \sum_{i,j,k=1}^d \left| \int_{\mathbf{R}^d} \nabla^b (B_\lambda)_{ij} \partial_t^{\mu+1} \nabla^{a-b} v^j h^k \partial_k \partial_t^\mu \nabla^a v^i dx \right| \\
& \leq \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \sum_{i,j,k=1}^d \|\nabla^b B_\lambda\|_\infty \|\partial_t^{\mu+1} \nabla^{a-b} v^j\|_2 \|h^k\|_\infty \|\partial_k \nabla^a \partial_t^\mu v^i\|_2 \\
& \leq \lambda C E_{\bar{L}-\mu}(\partial_t^\mu v).
\end{aligned}$$

We already got the estimate of $\langle \partial_t^\mu \nabla^a N[v, v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle$ in lemma 5.1. Combining estimates (85), (86), (87) and (70), we get (80). This completes the proof of lemma 5.3. $\square$

Proof of lemma 3.1. Let $K = C_0/\lambda$. Calculating $K \times (72) + b_0(2d-1)/4 \times (73) + (80)$ we get

$$\begin{aligned}
(88) \quad & \frac{d}{dt} G_{\bar{L}, \mu}(v) + \sum_{|a| \leq \bar{L} - \mu - 1} \left\{ K \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle - \frac{b_0(2d-1)}{4} \|\partial_t^{\mu+1} \nabla^a v\|_2^2 \right. \\
& \quad \left. + \int_{\mathbf{R}^d} \frac{d\phi + |x|\phi'}{2} |\partial_t^{\mu+1} \nabla^a v|^2 dx \right\} \\
& + \sum_{|a| \leq \bar{L} - \mu - 1} \int_{\mathbf{R}^d} \left(\frac{b_0(2d-1)}{4} + \phi + |x|\phi' - \frac{d\phi + |x|\phi'}{2} \right) |\nabla \partial_t^\mu \nabla^a v|^2 dx \\
& \leq C \left(\lambda^2 K + \frac{\lambda^2 b_0(2d-1)}{4} + \lambda \right) E_{\bar{L}-\mu}(\partial_t^\mu v) \\
& \quad + C \left(K + \frac{b_0(2d-1)}{4} + \frac{1}{\lambda} \right) D_{\bar{L}, \mu}(v) \\
& \quad + \frac{K}{4} \sum_{|a| \leq \bar{L} - \mu - 1} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v(t) \rangle + \frac{2\|B\|_\infty b_0^2 R^2}{\lambda K} E_{\bar{L}-\mu}(\partial_t^\mu v).
\end{aligned}$$

From (17), it holds that

$$(89) \quad r\phi'(r) = \begin{cases} 0, & (r \leq R/\lambda) \\ -\phi(r), & (r \geq R/\lambda). \end{cases}$$

Using **(B2)**_λ, (89), $K \geq d/\lambda$ and $\phi \geq 0$, we obtain

$$\begin{aligned}
(90) \quad & K \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle - \frac{b_0(2d-1)}{4} \|\partial_t^{\mu+1} \nabla^a v\|_2^2 \\
& + \int_{\mathbf{R}^d} \left\{ \frac{d\phi + |x|\phi'}{2} \right\} |\partial_t^{\mu+1} \nabla^a v|^2 dx \\
& \geq \frac{K}{2} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle \\
& + \int_{|x| \leq R/\lambda} \left(-\frac{b_0(2d-1)}{4} + \frac{db_0}{2} \right) |\partial_t^{\mu+1} \nabla^a v|^2 dx \\
& + \int_{|x| \geq R/\lambda} \left(\frac{\lambda b_0 K}{2} - \frac{b_0(2d-1)}{4} + \frac{d-1}{2} \phi \right) |\partial_t^{\mu+1} \nabla^a v|^2 dx \\
& \geq \frac{K}{2} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{b_0}{4} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2^2
\end{aligned}$$

for any $|a| \leq \bar{L} - \mu - 1$. Since we have

$$\begin{aligned}
(91) \quad & \int_{\mathbf{R}^d} \left(\frac{b_0(2d-1)}{4} + \phi + |x|\phi' - \frac{d\phi + |x|\phi'}{2} \right) |\nabla \partial_t^\mu \nabla^a v|^2 dx \\
&= \int_{|x| \leq R/\lambda} \left(\frac{b_0(2d-1)}{4} + b_0 - \frac{db_0}{2} \right) |\nabla \partial_t^\mu \nabla^a v|^2 dx \\
&\quad + \int_{|x| \geq R/\lambda} \left(\frac{b_0(2d-1)}{4} - \frac{(d-1)}{2} \frac{b_0 R}{\lambda|x|} \right) |\nabla \partial_t^\mu \nabla^a v|^2 dx \\
&\geq \frac{3b_0}{4} \int_{|x| \leq R/\lambda} |\nabla \partial_t^\mu \nabla^a v|^2 dx \\
&\quad + \int_{|x| \geq R/\lambda} \left(\frac{b_0(2d-1)}{4} - \frac{b_0(d-1)}{2} \right) |\nabla \partial_t^\mu \nabla^a v|^2 dx \\
&\geq \frac{b_0}{4} \|\nabla \partial_t^\mu \nabla^a v\|_2^2.
\end{aligned}$$

The estimates (88), (90) and (91) imply

$$\begin{aligned}
(92) \quad & \frac{d}{dt} G_{\bar{L}, \mu}(v) + \sum_{|a| \leq \bar{L}-\mu-1} \frac{K}{2} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{b_0}{2} E_{\bar{L}-\mu}(\partial_t^\mu v) \\
&\leq C \left(\lambda^2 K + \frac{\lambda^2 b_0(2d-1)}{4} + \lambda \right) E_{\bar{L}-\mu}(\partial_t^\mu v) \\
&\quad + C \left(K + \frac{b_0(2d-1)}{4} + \frac{1}{\lambda} \right) D_{\bar{L}, \mu}(v) \\
&\quad + \frac{K}{4} \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{2\|B\|_\infty b_0^2 R^2}{\lambda K} E_{\bar{L}-\mu}(\partial_t^\mu v).
\end{aligned}$$

Finally using $\lambda \leq 1$, $K = C_0/\lambda$ and $\langle \partial_t^{\mu+1} \nabla^a v(t), B_\lambda \partial_t^{\mu+1} \nabla^a v(t) \rangle \geq 0$, we get (18), which completes the proof of lemma 3.1. $\square$

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