On Some Functional Equations with Borel Summable Solutions

By

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Abstract. A functional equation \( \sum_{i=1}^{m} a_i\phi_i(z) = f(z) \) is considered, where \( \{\phi_i(z)\}_{i=1}^{m} \) are holomorphic functions in a neighborhood of \( z = 0 \) with \( \phi_i(0) = 0 \) and \( f(z) \) is holomorphic in a sector with vertex \( z = 0 \). It is shown under some conditions of \( \{\phi_i(z)\}_{i=1}^{m}, f(z) \) and \( \{a_i\}_{i=1}^{m} \) that the equation has a formal power series solution that is Borel summable.

Key Words and Phrases. Borel summable, Asymptotic expansion, Functional equation.

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0. Introduction

Formal power series appear in many mathematical analysis, in particular differential equations and singular perturbation, and play important roles. We have formal power series solutions in the theory of ordinary or partial differential equations. It is an important problem to show the existence of a genuine solution which behaves asymptotically as the formal solution. For ordinary differential equations there are many investigations for this problem (for example see Wasow [15]). For partial differential equations it was studied in Ōuchi [10] and [11].

There are very important notions in the theory of asymptotic expansions, that is, Borel summability and multisummability of formal power series. Multisummability is a generalization of Borel summability (see Balser [2] and [3]). These notions were not referred in [10], [11] and [15]. Multisummability of formal solutions implies the existence of genuine solutions in much stronger sense. It holds that formal power series solutions of ordinary differential equations are multisummable (see Balser, Braaksma, Ramis and Sibuya [5], Braaksma [6] and [7]). Multisummability of formal solutions are studied for some class of partial differential equations in Balser [4], Lutz, Miyake and Schäfke [9], Ōuchi [12], [13], [14] and others. As for singular perturbations of differential equations the Borel summation method is also applicable to exact WKB theory (Kawai and Takei [8]).
In this paper we treat some class of functional equations, not differential equations,

\[ \sum_{i=1}^{m} a_i u(\varphi_i(z)) = f(z). \]  

(0.1)

Here we assume \( \varphi_i(z) \ (1 \leq i \leq m) \) are different holomorphic functions in a neighborhood of \( z = 0 \) with \( \varphi_i(0) = 0 \) and

\[ \varphi'_i(0) = \ldots = \varphi''_i(0) = \ldots = \varphi'''_m(0) \neq 0. \]  

(0.2)

As for \( f(z) \), we assume it is holomorphic in a neighborhood of \( z = 0 \), or holomorphic in a sector with vertex \( z = 0 \) with asymptotic expansion. We show under some conditions that there exists a formal power series solution \( \tilde{u}(z) \) of (0.1). Our main aim is to study its analytical meaning, that is, we show that it is Borel summable, which means there exists a genuine solution \( u(z) \) of (0.1) holomorphic in some sector with \( u(z) \sim \tilde{u}(z) \) asymptotically.

Hence we obtain new examples having Borel summable formal power series solutions, which are different from differential equations.

In § 1 we give notations and definitions. In § 2 we give conditions imposed on (0.1) and main results. There are two cases, (1) \( \sum_{i=1}^{m} a_i \neq 0 \) and (2) \( \sum_{i=1}^{m} a_i = 0 \). For the latter case (2) we set conditions of the derivatives of \( f(z) \) at \( z = 0 \). The proofs are given in the following sections. We show the existence of formal solutions in § 3 and their Borel summability is studied in § 4. In § 5 we study the case \( \sum_{i=1}^{m} a_i = 0 \) without the conditions on \( f(z) \). We find solutions in a wider class than \( C[[z]] \) or Borel summable functions. As an example we apply the result to Abel’s functional equation. In § 6 we study functional equations in more general situations, which are reduced to ones treated in the previous sections. We give a comment on (0.2). If (0.2) does not hold, that is, \( \{\varphi'_i(0)\}_{i=1}^{m} \) are not necessarily equal, then the situations are different (see Remark 2.5 and Appendix).

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1. Notations and definitions

In this section we give notations and definitions. The set of all integers is denoted by \( \mathbb{Z}, \mathbb{Z}_+ = \{ n \in \mathbb{Z}; n \geq 0 \} \) and \( z, \zeta \in \mathbb{C} \). The set of all holomorphic functions on the domain \( U \subset \mathbb{C} \) is denoted by \( \mathcal{O}(U) \) and \( \mathcal{O}(0) \) is the set of all holomorphic functions at \( z = 0 \). The convex hull of \( A \subset \mathbb{C} \) is denoted by \( \mathcal{A} \). Let us give some notations of sectors. Let \( \theta \in \mathbb{R} \) and \( R, \delta > 0 \). Put \( S(\theta, \delta, R) = \{ z; 0 < |z| < R, |\arg z - \theta| < \delta \} \) that is a sector in \( z \)-space and \( S^*(\theta, \delta, R) = \{ \zeta; 0 < |\zeta| < R, |\arg \zeta - \theta| < \delta \} \) that is a sector in \( \zeta \)-space. Put
Definition 1.1. Let $s \geq 0$. A subspace $C_{\{s\}}[[z]]$ of $C[[z]]$ is the set of all $\tilde{f}(z) = \sum_{n=0}^{\infty} f_n z^n \in C[[z]]$ with the following bounds: There exist positive constants $F$ and $C$ such that for all $n \in \mathbb{Z}_+$

\[ |f_n| \leq FC^n \Gamma(sn + 1). \]

If $s = 0$, then $\tilde{f}(z) = \sum_{n=0}^{\infty} f_n z^n$ converges and defines a holomorphic function in a neighborhood of $z = 0$. The following is one of the most important facts about $C_{\{s\}}[[z]]$.

Proposition 1.2. Let $s > 0$, $\tilde{f}(z) = \sum_{n=0}^{\infty} f_n z^n \in C_{\{s\}}[[z]]$ and $0 < \delta < s \pi / 2$. Then there exists a function $f(z)$, holomorphic in $S(\theta, \delta)$ with the following estimate: There exist positive constants $C_0$ and $K$ such that for any $N \geq 1$

\[ |f(z) - \sum_{n=0}^{N-1} f_n z^n| \leq C_0 K^N \Gamma(sN + 1)|z|^N \]

holds for $z \in S(\theta, \delta)$.

Let $f(z) \in \mathcal{O}(S(\theta, \delta, R))$. If there exists $\tilde{f}(z) \in C_{1/s}[[z]]$ ($\gamma = s^{-1}$) such that (1.2) holds in $S = S(\theta, \delta, R)$, we denote

\[ f(z) \sim_{\gamma} \tilde{f}(z) \quad \text{in} \quad S. \]

Definition 1.3. Let $s > 0$ and $\gamma = 1/s$. We say that $\tilde{f}(z) \in C_{\{s\}}[[z]]$ is $\gamma$-Borel summable in a direction $\theta$, if there exist a sector $S(\theta, \delta, R)$ with $\delta > \pi / 2\gamma$ ($= s \pi / 2$) and $f(z) \in \mathcal{O}(S(\theta, \delta, R))$ such that (1.2) holds for $z \in S(\theta, \delta, R)$.

We note that $f(z)$ is uniquely determined for $\gamma$-Borel summable $\tilde{f}(z)$. Hence we say that $f(z)$ is the $\gamma$-Borel sum of $\tilde{f}(z)$. We also say that $f(z)$ is $\gamma$-Borel summable in a direction $\theta$ or $f(z)$ has $\gamma$-strong asymptotic expansion in a direction $\theta$.

Let $\tilde{f}(z) = \sum_{n=1}^{\infty} f_n z^n \in C_{\{1/s\}}[[z]]$ ($f_0 = 0$). Then we define formal $\gamma$-Borel transform $(\mathcal{B}_\gamma \tilde{f})(\xi)$ of $\tilde{f}$ by

\[ (\mathcal{B}_\gamma \tilde{f})(\xi) = \sum_{n=1}^{\infty} \frac{f_n}{\Gamma(n/\gamma)} \xi^{n-\gamma}. \]
It follows from (1.1) that there exists a constant $R > 0$ such that $(\hat{A}_f f)(\xi)$ converges in $\{0 < |\xi| < R\}$. We note the relations between $(\hat{A}_f f)(\xi)$ and the $\gamma$-Borel summability of $f(z)$. A formal series $\hat{f}(z) \in C_{\{1/2\}}[[z]]$ $(f_0 = 0)$ is $\gamma$-Borel summable in a direction $\theta_0$, if and only if $(\hat{A}_f f)(\xi)$ is holomorphically extensible to an infinite sector $S^*(\theta_0, \delta_0)$ for some $\delta_0 > 0$ and there exist constants $M, c > 0$ such that

$$\text{(1.5)} \quad |(\hat{A}_f f)(\xi)| \leq M \exp(c|\xi|^{\gamma}) \quad \text{for} \quad \xi \in S^*(\theta_0, \delta_0) \cap \{|\xi| \geq 1\}. $$

Let $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n \in C_{\{1/2\}}[[z]]$ be $\gamma$-Borel summable in a direction $\theta_0$. Let $\hat{g}(z) = \sum_{n=1}^{\infty} f_n z^n \in C_{\{1/2\}}[[z]]$. Then $\tilde{f}(z) = f_0 + \hat{g}(z)$. Put $\tilde{g}(\xi) = (\hat{A} \hat{g})(\xi)$ and define

$$\text{(1.6)} \quad f(z) := f_0 + \int_0^{\infty} \exp\left(-\left(\frac{\xi}{2}\right)^\gamma\right) \tilde{g}(\xi) d\xi \quad (|\theta - \theta_0| < \delta_0).$$

Then $f(z) \in C(S_{\theta_0}(\theta_0, \pi/(2\gamma) + \delta_0))$ and for any $\delta_1$ with $0 < \delta_1 < \delta_0$ there are positive constants $R_1, C_1$ and $K_1$ such that for any positive integer $N$

$$\text{(1.7)} \quad \left|f(z) - \sum_{n=0}^{N-1} f_n z^n\right| \leq C_1 K_1^N \left|\frac{N}{\gamma} + 1\right| |z|^N$$

holds in $S(\theta_0, \pi/(2\gamma) + \delta_1, R_1)$. Now let us define Laplace transform and Borel transform. Let $\gamma > 0$ be a constant. For simplicity we set $S^* := S^*(\theta, \delta)$ and $S^*_{\theta_0} := S_{\theta_0}^*(\theta, \delta)$ for given $\theta$ and $\delta > 0$.

**Definition 1.4.** The set of all $\phi(\xi) \in C(S^*)$ satisfying

$$\text{(1.8)} \quad |\phi(\xi)| \leq A \exp(c|\xi|^{\gamma}) \quad \text{on} \quad \{\xi \in S^*; |\xi| \geq 1\}$$

for some positive constants $A$ and $c$ is denoted by $Exp(\gamma, S^*)$.

Let $\phi(\xi) \in Exp(\gamma, S^*)$ with

$$\text{(1.9)} \quad |\phi(\xi)| \leq C|\xi|^{\xi-\gamma} \quad \text{on} \quad \{\xi \in S^*; 0 < |\xi| \leq 1\}$$

for some $\epsilon > 0$. Then we can define the $\gamma$-Laplace transform $\mathcal{L}_{\gamma, \theta} \phi$ by

$$\text{(1.10)} \quad (\mathcal{L}_{\gamma, \theta} \phi)(z) = \int_0^{\infty} \exp\left(-\left(\frac{\xi}{2}\right)^\gamma\right) \phi(\xi) d\xi, \quad d\xi = \gamma \xi^{-\gamma-1} d\xi.$$  

This $\mathcal{L}_{\gamma, \theta} \phi$ is holomorphic in $S_{\theta_0}(\theta, \pi/(2\gamma) + \delta)$. Let $\psi(z)$ be a holomorphic function on $S_{\theta_0}(\theta, \pi/(2\gamma) + \delta)$ $(0 < \delta < \pi/(2\gamma))$ with $|\psi(z)| \leq C|z|^c$ for some $c > 0$. Take $\delta'$ such that $\delta_0 < \delta' < \delta$ for given $\delta_0$ with $0 < \delta_0 < \delta$. Let
be a contour in \( S_{(0)}(\theta, \pi/(2\gamma) + \delta) \) from 0 \( \exp(i(\theta + \pi/(2\gamma) + \delta')) \) to 0 \( \exp(i(\theta - \pi/(2\gamma) - \delta')) \). Then we define the \( \gamma \)-Borel transform \( \mathcal{B}_{\gamma, \psi} \) by
\[
(\mathcal{B}_{\gamma, \psi})(\xi) = \frac{1}{2\pi i} \int_{\gamma} \left( \exp \left( \frac{\xi}{z} \right) \right) \psi(z) dz^{-\gamma},
\]
which is in \( \text{Exp}(\gamma, S^*(\theta, \delta_0)) \).

Let \( \phi_i(\xi) \in C(S_{(0)}) \) \((i = 1, 2)\) with \( |\phi_i(\xi)| \leq C|\xi|^{\epsilon - \gamma} \). Then \( \gamma \)-convolution of \( \phi_1(\xi) \) and \( \phi_2(\xi) \) is defined by
\[
(\phi_1 \ast_\gamma \phi_2)(\xi) = \int_0^\xi \phi_1((\xi - \eta)^{1/\gamma}) \phi_2(\eta) d\eta^\gamma \quad (\xi \in S_{(0)}).
\]
The following relations hold between \( \gamma \)-Laplace transform, \( \gamma \)-Borel transform and \( \gamma \)-convolution.

**Lemma 1.5.** Let \( \phi_i(\xi) \in \text{Exp}(\gamma, S^*) \) with \( |\phi_i(\xi)| \leq C|\xi|^{\epsilon - \gamma} \) \((\epsilon > 0)\) on \( \{\xi \in S^*; 0 < |\xi| \leq 1\} \) \((i = 0, 1, 2)\). Then
\[
\mathcal{B}_{\gamma, \psi} L_{\gamma, \psi} \psi_0 = \psi_0,
\]
\[
(L_{\gamma, \psi} \psi_1)(L_{\gamma, \psi} \psi_2) = L_{\gamma, \psi}(\psi_1 \ast_\gamma \psi_2).
\]

2. **Functional equation**

Let us consider a functional equation
\[
s_{m} \sum_{i=1}^{m} a_i u(\phi_i(z)) = f(z),
\]
where \( a_i \neq 0 \) \((m \geq 2)\) and \( f(z) \in C[[z]] \). We assume \( \{\phi_i(z)\}_{i=1}^{m} \) are different holomorphic functions in a neighborhood of \( z = 0 \) such that \( \phi_i(0) = 0 \) and
\[
\phi_i'(0) = \cdots = \phi_i'(0) = \cdots = \phi_m'(0) \neq 0.
\]
We may assume \( \phi_i'(0) = 1 \), by changing the coordinate. Let \( \{p_i\}_{i=1}^{m} \) be positive integers such that
\[
\phi_i(z) = z(1 + b_{i, p_i} z^{p_i} + b_{i, p_i+1} z^{p_i+1} + \cdots) \quad \text{with} \quad b_{i, p_i} \neq 0.
\]
If \( \phi_i(z) = z \), we put \( p_i = +\infty \). By putting \( b_{i,0} = 1 \) and \( b_{i,j} = 0 \) \((1 \leq j < p_i)\), we sometimes write
\[
\phi_i(z) = z \sum_{j=0}^{\infty} b_{i,j} z^j.
\]
We assume \( \varphi_1(z) = z \) and \( +\infty = p_1 > p_2 \geq p_3 \geq \cdots \geq p_m \geq 1 \) and set \( p = p_m \).

From the above notations we have

\[
(2.5) \quad \sum_{\{i, p_i = p\}} a_ib_{i,p} = \sum_{i=1}^{m} a_ib_{i,p} = \sum_{i=2}^{m} a_ib_{i,p}
\]

which is often used in the following. Our problems are

(i) the existence of a formal solution \( \tilde{u}(z) \in \mathcal{C}[[z]] \),

(ii) the Borel summability of \( \tilde{u}(z) \).

As for problem (i) we have

**Theorem 2.1.** (1) Suppose \( \sum_{i=1}^{m} a_i \neq 0 \). Then there exists a unique solution \( \tilde{u}(z) \in \mathcal{C}[[z]] \) of (2.1).

(2) Suppose \( \sum_{i=1}^{m} a_i = 0 \), \( f(0) = f'(0) = \cdots = f^{(p)}(0) = 0 \) (\( p = p_m \)) and \( \sum_{\{i, p_i = p\}} a_ib_{i,p} \neq 0 \). Then there exists a solution \( \tilde{u}(z) \in \mathcal{C}[[z]] \) of (2.1). It is unique, provided \( \tilde{u}(0) = 0 \).

In the following Theorems 2.2 and 2.3 we assume that either (1) or (2) in Theorem 2.1 holds and \( \tilde{u}(z) \in \mathcal{C}[[z]] \) is that in Theorem 2.1.

**Theorem 2.2.** If \( f(z) \in \mathcal{C}(1/p)[[z]] \), then \( \tilde{u}(z) \in \mathcal{C}(1/p)[[z]] \).

The proofs of Theorems 2.1 and 2.2 are given in §3. In order to answer problem (ii) we impose a condition on \( \{\varphi_i(z)\}_{i=1}^{m} \),

\[
(2.6) \quad p_2 = p_3 = \cdots = p_m.
\]

Assume (2.6), and set \( p = p_2 = p_3 = \cdots = p_m \). We set \( b_1 = 0 \) and \( b_i = b_{i,p} \) for \( i \geq 2 \). Let \( B = \{b_i; 1 \leq i \leq m\} \). We have

**Theorem 2.3.** Further assume (2.6) and that \( b_1 = 0 \) is the vertex of the convex hull \( \bar{B} \) of \( B \). Then there exists a direction \( \theta_0 \) such that \( \tilde{u}(z) \) is \( p \)-Borel summable in the direction \( \theta_0 \), provided \( f(z) \) is \( p \)-Borel summable in the direction \( \theta_0 \).

Theorem 2.3 means that there exists uniquely a \( p \)-Borel summable function \( u(z) \) with \( u(z) \sim_p \tilde{u}(z) \) such that \( u(z) \) satisfies (2.1). The proof of Theorem 2.3 is given in §4.

**Remark 2.4.** Suppose \( \sum_{i=1}^{m} a_i \neq 0 \). Set \( u(z) = c + v(z), \ c = f(0)/(\sum_{i=1}^{m} a_i) \). Then the equation becomes \( \sum_{i=1}^{m} a_i v(\varphi_i(z)) = f(z) - f(0) \). Hence we may assume \( f(0) = 0 \) in the proof of Theorems 2.1, 2.2 and 2.3.

**Remark 2.5.** Suppose that (2.2) does not necessarily hold. Set \( \lambda_i = \varphi_i'(0) \) and assume \( |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m| \). If \( |\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m| \), \( f(z) \) is holomorphic and a formal solution \( \tilde{u}(z) \in \mathcal{C}[[z]] \) exists, then \( \tilde{u}(z) \) converges (see Appendix).
If $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_m|$ for some $2 \leq m' \leq m$, then the situations are complicated and small divisors problems may happen.

3. Existence of formal solutions

In this section we give proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. By Remark 2.4 we may assume $f(0) = 0$. Put $\tilde{u}(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathbb{C}[[z]]$. Then, by (2.4), we obtain

$$
\tilde{u}(\varphi_1(z)) = \sum_{k=0}^{\infty} c_k z^k \left( \sum_{j=0}^{\infty} b_{i,j} z^j \right)^k
$$

$$
= c_0 + \sum_{k=1}^{\infty} c_k z^k \left( \sum_{\ell=0}^{\infty} \left( \sum_{(j_1,j_2,\ldots,j_k) \atop j_1+j_2+\cdots+j_k=\ell} b_{i,j_1} b_{i,j_2} \cdots b_{i,j_k} \right) z^\ell \right)
$$

$$
= c_0 + \sum_{n=1}^{\infty} \left( \sum_{k+\ell=n \atop k \geq 1} c_k B_{k,\ell}(i) \right) z^n,
$$

where $B_{k,\ell}(i) = \sum_{(j_1,j_2,\ldots,j_k) \atop j_1+j_2+\cdots+j_k=\ell} b_{i,j_1} b_{i,j_2} \cdots b_{i,j_k}$, $B_{k,0}(i) = 1$. We note that $B_{k,\ell}(i) = 0$ for $1 \leq \ell < p_i$ and $B_{k,p_i}(i) = k b_{i,p_i}$ for $i \geq 2$. From $\varphi_1(z) = z$ we have $B_{k,0}(1) = 1$, $B_{k,\ell}(1) = 0$ for $\ell \geq 1$ and $\tilde{u}(\varphi_1(z)) = \tilde{u}(z)$. Thus we have

$$
\sum_{i=1}^{m} a_i \tilde{u}(\varphi_1(z)) = \left( \sum_{i=1}^{m} a_i \right) c_0 + \sum_{n=1}^{\infty} \left( \sum_{k+\ell=n \atop k \geq 1} a_k c_k B_{k,\ell}(i) \right) z^n.
$$

From $\sum_{i=1}^{m} a_i \tilde{u}(\varphi_1(z)) = f(z)$ and $f(z) = \sum_{n=1}^{\infty} f_n z^n \in \mathbb{C}[[z]]$ we have

(3.1) $$
\left( \sum_{i=1}^{m} a_i \right) c_0 = 0,
$$

$$
\sum_{i=1}^{m} \sum_{k+\ell=n \atop k \geq 1} a_k c_k B_{k,\ell}(i) = f_n \quad (n \geq 1),
$$

and hence

(3.2) $$
\left( \sum_{i=1}^{m} a_i \right) c_n + \sum_{i=1}^{m} \sum_{k+\ell=n \atop 1 \leq k < n} a_k c_k B_{k,\ell}(i) = f_n \quad (n \geq 1).
$$
(1) Suppose \( \sum_{i=1}^{m} a_i \neq 0 \). Then \( c_0 = 0 \), and from (3.2) \( c_n (n \geq 1) \) are inductively and uniquely determined.

(2) Suppose \( \sum_{i=1}^{m} a_i = 0 \) and \( \sum_{\{i,p=p\}} a_i \) are not \( \neq 0 \). As for \( f(z) \) we assume \( f_n = 0 \) for \( 0 \leq n \leq p \). Let \( \bar{u}(z) = \sum_{k=0}^\infty c_k z^k \) with some constants \( c_k \). Then we have from (3.2) and \( B_{k,\ell}(i) = 0 \) for \( 1 \leq \ell < p \).

\[
(3.3) \quad \left( \sum_{\{i,p=p\}} a_i B_{n-p,p}(i) \right) c_{n-p} + \sum_{i=1}^{m} \sum_{k \geq 1, \ell > p} a_i c_k B_{k,\ell}(i) = f_n \quad (n \geq p + 1).
\]

Note \( \sum_{\{i,p=p\}} a_i B_{n-p,p}(i) = (n - p)(\sum_{\{i,p=p\}} a_i) \neq 0 \) for \( n \geq p + 1 \) and (3.3) does not contain \( c_0 \). Therefore \( c_0 \) is arbitrary and \( \{c_n\}_{n \geq 1} \) are uniquely determined.

Next let us proceed to give the proof of Theorem 2.2, that is, estimate of \( \{c_n\}_{n \geq 1} \). For this purpose we give a lemma.

**Lemma 3.1.** Let \( \psi(z) \) be holomorphic in a neighborhood of \( z = 0 \) such that \( \psi(z) = 1 + O(z^p) \) for some \( p \geq 1 \), that is, \( \psi(z) = 1 + \sum_{j=p}^\infty b_j z^j \). Let \( \{B_{k,\ell}\} \) be constants defined by \( \psi(z)^k = \sum_{j=0}^k B_{k,\ell} z^\ell \) for each \( k \geq 1 \). Then there exists a positive constant \( B_0 \) such that

\[
|B_{k,\ell}| \leq B_0 \frac{(k + \ell)(k + \ell)p}{k!p \ell p} \quad (k, \ell \geq 1).
\]

**Proof.** Firstly we note

\[
(3.5) \quad B_{k,\ell} = \frac{1}{2\pi i} \int_{|z| = \varepsilon} \frac{\psi(z)^k}{z^{\ell+1}} dz.
\]

Let \( \psi(z)^p = 1 + \sum_{j=p}^\infty b_j z^j \) be the expansion at \( z = 0 \) and \( b_1 > 1 \) with \( |b_j| \leq b_1 \). Set \( d = \sum_{j=0}^{p-1} b_j \) and \( c = bd \). Then it follows from \( b^{pk} \leq (bd)^p = e^{pk} \) that for \( |z| < 1/c < 1 \)

\[
|\psi(z)|^p \leq 1 + \sum_{j=p}^\infty b_j |z|^j = 1 + \sum_{k=1}^\infty \left( \sum_{j=pk}^{p(k+1)-1} b_j |z|^j \right)
\]

\[
= 1 + \sum_{k=1}^\infty b^{pk} |z|^{pk} \left( \sum_{j=0}^{p-1} b_j \right) \leq 1 + \sum_{k=1}^\infty b^{pk} |z|^{pk} \left( \sum_{j=0}^{p-1} b_j \right)
\]

\[
= 1 + d \sum_{k=1}^\infty b^{pk} |z|^{pk} \leq 1 + \sum_{k=1}^\infty e^{pk} |z|^{pk} = \frac{1}{1 - (c|z|)^p}.
\]
Hence
\[(3.6) \quad |\psi(z)| \leq \frac{1}{(1 - (c|z|)^{p})^{1/p}}.\]

From (3.6)
\[|B_{k,\ell}| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\psi(re^{i\theta})} {r^{\ell}} d\theta \leq \frac{1}{(1 - (cr)^{p})^{k/p^{\ell/p}}}.
\]

We choose \(r\) so that \((cr)^{p} = \ell/(k + \ell)\) holds, then
\[\frac{1}{(1 - (cr)^{p})^{k/p^{\ell/p}}} = c^{\ell} \frac{(k + \ell)^{(k+\ell)/p}} {k^{k/p^{\ell/p}}},\]

which proves (3.4).

**Proposition 3.2.** Let \(\tilde{u}(z) = \sum_{n=1}^{\infty} c_{n}z^{n} \in C[[z]]\) be a unique formal solution of (2.1) assured in (1) or (2) in Theorem 2.1. If \(f(z) \in C_{(1/p)}[[z]]\), then there exist positive constants \(M\) and \(C\) such that
\[(3.7) \quad |c_{n}| \leq MC^{n}n^{n/p}.
\]

**Proof.** We firstly give a proof for the case (1), that is, \(A_{0} = \sum_{i=1}^{m} a_{i} \neq 0\). It follows from \(f(z) = \sum_{n=1}^{\infty} f_{n}z^{n} \in C_{(1/p)}[[z]]\) that there exist constants \(F, C_{1}\) such that \(|f_{n}| \leq FC_{1}n^{n/p}\). We show (3.7) by induction. Let \(M > 2F/|A_{0}|\) and \(C \geq C_{1}\). Since \(c_{1} = f_{1}/A_{0}\), we have (3.7) for \(n = 1\). Assume that (3.7) holds for \(1 \leq k \leq n - 1\). For \(k, \ell \geq 1\) with \(k + \ell = n\), it follows from Lemma 3.1 that
\[|c_{k}B_{k,\ell}(i)| \leq MC^{k}k^{k/p}B_{0}^{\ell} \frac{(\ell + k)^{(k+\ell)/p}} {k^{k/p^{\ell/p}}} = MC^{k}B_{0}^{\ell}n^{n/p} \frac{n^{n/p}} {\ell^{\ell/p}}.
\]

Let \(A_{1} = \sum_{i=1}^{m} |a_{i}|\). Take \(C > B_{0}\) so large that \(A_{1}B_{0}/(C - B_{0}) \leq |A_{0}|/2\) holds. Then we have
\[
\sum_{i=1}^{m} \sum_{k=1}^{n} |a_{i}| |c_{k}B_{k,\ell}(i)| \leq MA_{1}C^{n}n^{n/p} \sum_{\ell=1}^{n-1} \frac{B_{0}^\ell}{C^{\ell/\ell/p}} \frac{1}{\ell^{\ell/p}}
\leq MA_{1}B_{0} \frac{C^{n}n^{n/p}} {C - B_{0}} \leq \frac{M|A_{0}|}{2} C^{n}n^{n/p}.
\]

Hence
\[|c_{n}| \leq M \frac{C^{n}n^{n/p}}{2} + \frac{F}{|A_{0}|} C^{n}n^{n/p} \leq M \frac{(C^{n} + C_{1}^{n})n^{n/p}}{2} \leq MC^{n}n^{n/p}.
\]
Next let us consider the case (2), that is, $\sum_{i=1}^{m} a_i = 0$, $A_0' = \sum_{i=p}^{m} a_i b_i \neq 0$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n \in C([1/p])$ with $|f_{n+p}| \leq FC_n n^{n/p+1}$ ($n \geq 1$). We show (3.7) by induction. Let $M > 2F/|A_0'|$ and $C \geq C_1$. Since $c_1 = f_{p+1}/A_0'$, we have (3.7) for $n = 1$. Assume that (3.7) holds for $1 \leq k \leq n - 1$. The coefficients $\{c_n\}_{n \geq 1}$ are determined by the following relation (see (3.3))

$$n \left( \sum_{i=p}^{m} a_i b_i \right) c_n + \sum_{i=1}^{m} \sum_{k \geq 1, \ell > p} a_i c_k B_{k, \ell} \langle i \rangle = f_{n+p}. \tag{3.8}$$

We have

$$|c_k| |B_{k, \ell} (i)| \leq MC^k k^{k/p} B_0^\ell \frac{(\ell + k)^{(k+\ell)/p}}{k^{k/p} \ell^{\ell/p}} = MC^k B_0^\ell \frac{(n+p)^{(n+p)/p}}{\ell^{\ell/p}}
= MC^n n^{1+p/n} \left( \frac{B_0}{C} \right)^{\ell/p} \frac{B_0^{\ell/p}}{\ell^{\ell/p}} \frac{(n+p)^{1+n/p}}{n}
\leq MA_2 C^n n^{1+p/n} B_0^\ell \left( \frac{B_0}{C} \right)^{\ell/p} \frac{1}{\ell^{\ell/p}}
(A_2 = \sup_{n} (1 + p/n)^{1+n/p}).$$

Let $A_1 = \sum_{i=1}^{m} |a_i|$. Take $C > B_0$ so large that $A_1 A_2 B_0^p/(C - B_0) \leq |A_0'|/2$ holds. Then we have

$$\sum_{i=1}^{m} \sum_{k \geq 1, \ell > p} |a_i| |c_k B_{k, \ell} (i)| \leq MA_1 A_2 C^n n^{1+p/n} B_0^\ell \sum_{\ell=p+1}^{p} \left( \frac{B_0}{C} \right)^{\ell/p} \frac{1}{\ell^{\ell/p}}
\leq \frac{MA_1 A_2 C^n n^{1+p/n} B_0^{p+1}}{C - B_0} \leq \frac{M |A_0'|}{2} C^n n^{n/p+1}.$$

Hence from (3.8)

$$|c_n| \leq \frac{M}{2} C^n n^{n/p} + \frac{F}{|A_0'|} C^n n^{n/p} \leq \frac{M}{2} (C^n + C^n) n^{n/p} \leq MC^n n^{n/p}. \quad \square$$

We have Theorem 2.2 from Proposition 3.2 and Stirling’s formula of the Gamma function.

4. Borel summability-I

In the previous sections we study the existence of formal solutions of the functional equation (2.1) with $f(0) = 0$. The purpose of this section is to show
that formal solutions are \( p \)-Borel summable, provided \( f(z) \) is \( p \)-Borel summable. For this purpose we give a few assumptions and definitions. In the following of this section we assume \( p = p_2 = p_3 = \cdots = p_m \). We set

\[
(4.1) \quad b_1 = 0, \quad b_i = b_{i,p} \neq 0 \quad (i \geq 2).
\]

Let \( B = \{b_i; 1 \leq i \leq m\} \) that is a finite set in the complex plane \( \mathbb{C} \). We give a condition on \( B \).

**Condition \( \mathcal{B} \):** \( b_1 = 0 \) is a vertex of the convex hull \( \mathcal{B} \) of \( B \).

Let \( \tilde{u}(z) = \sum_{n=1}^{\infty} c_n z^n \in \mathcal{C}_{1/p}(\mathbb{C}) \) be a unique solution of (2.1) with \( \tilde{u}(0) = 0 \). In order to show Theorem 2.3 we construct a solution \( u(z) \) with \( u(z) \sim_p \tilde{u}(z) \) of (2.1) of the form

\[
(4.2) \quad u(z) = \int_0^{\infty} \exp \left( -\left( \frac{\xi}{z} \right)^p \right) \tilde{u}(\xi) d\xi^p,
\]

where \( \theta \) will be appropriately chosen later. Let us proceed to find an equation \( \hat{u}(\xi) \) satisfies. For this purpose we give a lemma.

**Lemma 4.1.** Let \( \varphi(z) = z(1 + bz^p + O(z^{p+1})) \in \mathcal{C}_0 \) and \( v(\xi) \in \mathcal{C}(S(\theta, \delta)) \) with bound \( |v(\xi)| \leq C|\xi|^{e-p} \exp(a|\xi|^p) \) \((a, e > 0)\). Let

\[
(4.3) \quad V(z) = \int_0^{\varphi(e^0)} \exp \left( -\left( \frac{\xi}{z} \right)^p \right) v(\xi) d\xi^p.
\]

Then \( V(z) \in \mathcal{C}(S(0, \pi/(2p) + \delta)) \) and

\[
(4.4) \quad V(\varphi(z)) = \int_0^{\varphi(e^0)} \exp \left( -\left( \frac{\xi}{z} \right)^p \right) e^{ph_{\xi}^p} v(\xi) d\xi^p
\]

\[
+ \sum_{k=1}^{\infty} \int_0^{\varphi(e^0)} \exp \left( -\left( \frac{\xi}{z} \right)^p \right) \frac{1}{k!} \hat{\psi}(\xi) * \cdots * \hat{\psi}(\xi) \left( e^{ph_{\xi}^p} \frac{e^{pk}}{k!} v(\xi) \right) d\xi^p,
\]

where \( |\phi - \theta| < \delta \) and \( \hat{\psi}(\xi) \) is the Borel transform of

\[
\psi(z) = \frac{1}{z^p} - pb - \frac{1}{\varphi(z)^p}.
\]

Note that \( \psi(z) \in \mathcal{C}_0 \) and \( \psi(0) = 0 \).
Proof. Let $\zeta = re^{i\omega}$ ($|\omega - \theta| < \delta$). Then

$$V(z) = \int_0^\infty \exp\left(-\left(\frac{e^{i\omega}}{z}\right)^p\right)v(\xi)e^{ip\xi} d\xi.$$ 

It follows from the bounds of $v(\xi)$ that the integral near $r = 0$ converges. As for near $r = \infty$ the integral converges on $\{\text{Re}(e^{i\omega}/z)^p > a\}$. Hence $V(z)$ is holomorphic in

$$S_{\{0\}}(\theta, \pi/2p + \delta) = \bigcup_{|\omega - \theta| < \delta} \{z; 0 < |z| < (\cos(p(\arg z - \omega))/a)^{1/p}, |\arg z - \omega| < \pi/(2p)\}.$$

We have

$$\frac{1}{\varphi(z)^p} = \frac{1}{z^p(1 + h z^p + \cdots)^p} = \frac{1}{z^p} - pb - \psi(z),$$

where $\psi(z) \in C_0$ and $\psi(z) = O(z)$. By shrinking $S_{\{0\}}(\theta, \pi/2p + \delta)$ if necessary, we have

$$V(\varphi(z)) = \int_0^{\infty e^{i\omega}} \exp\left(-\left(\frac{\zeta}{z}\right)^p + pb\zeta^p + \psi(z)\zeta^p\right)v(\xi)d\zeta^p$$

$$= \int_0^{\infty e^{i\omega}} \exp\left(-\left(\frac{\zeta}{z}\right)^p + pb\zeta^p\right)v(\xi)d\zeta^p$$

$$+ \sum_{k=1}^{\infty} \psi(z)^k \int_0^{\infty e^{i\omega}} \exp\left(-\left(\frac{\zeta}{z}\right)^p + pb\zeta^p\right)\frac{\zeta^p}{k!} v(\xi)d\zeta^p,$$

where $|\omega - \theta| < \delta$. Hence

$$V(\varphi(z)) = \int_0^{\infty e^{i\omega}} \exp\left(-\left(\frac{\zeta}{z}\right)^p\right)e^{pb\zeta^p} v(\zeta)d\zeta^p$$

$$+ \sum_{k=1}^{\infty} \int_0^{\infty e^{i\omega}} \exp\left(-\left(\frac{\zeta}{z}\right)^p\right)\psi(\zeta)^k \hat{\psi}(\zeta)d\zeta^p.$$ 

Let us find an equation that determines $\hat{u}(\xi)$. Suppose that $f(z)$ is $p$-Borel summable in the direction $\theta$. Then

$$(4.5) \quad f(z) = \int_0^{\infty e^{i\theta}} \exp\left(-\left(\frac{\xi}{z}\right)^p\right)\hat{f}(\zeta)d\zeta^p.$$
where $\hat{f}(\xi)$ is the Borel transform of $f(z)$. We have from Lemma 4.1

\[
\begin{align*}
\quad u(\varphi(z)) &= \int_0^\infty \exp \left( -\left( \frac{x}{z} \right)^p \right) e^{\theta h \xi^p} \hat{u}(\xi) d\xi^p \\
&\quad + \sum_{k=1}^{\infty} \int_0^\infty \exp \left( -\left( \frac{x}{z} \right)^p \right) \psi_1(\xi) \ast_p \cdots \ast_p \psi_1(\xi) \\
&\quad \ast_p \left( e^{\theta h \xi^p} \frac{\xi^p}{k!} \hat{u}(\xi) \right) d\xi^p,
\end{align*}
\]

where $\psi_1(z) = O(z)$. Consequently we get from (2.1) a convolution equation, by replacing unknown $\hat{u}(\xi)$ by $v(\xi)$,

\[
\begin{align*}
(4.6) \quad u(\varphi(z)) &= \int_0^\infty \exp \left( -\left( \frac{x}{z} \right)^p \right) e^{\theta h \xi^p} \hat{u}(\xi) d\xi^p \\
&\quad + \sum_{k=1}^{\infty} \int_0^\infty \exp \left( -\left( \frac{x}{z} \right)^p \right) \psi_1(\xi) \ast_p \cdots \ast_p \psi_1(\xi) \\
&\quad \ast_p \left( e^{\theta h \xi^p} \frac{\xi^p}{k!} \hat{u}(\xi) \right) d\xi^p,
\end{align*}
\]

Let us find a solution $v(\xi)$ of (4.7) by the following iteration:

\[
\begin{align*}
(4.7) \quad (a_1 + \sum_{i=2}^m a_i e^{\theta h \xi^p}) v(\xi) \\
&\quad + \sum_{i=2}^m a_i \left( \sum_{k=1}^{\infty} \psi_1(\xi) \ast_p \cdots \ast_p \psi_1(\xi) \ast_p \left( e^{\theta h \xi^p} \frac{\xi^p}{k!} v(\xi) \right) \right) = \hat{f}(\xi).
\end{align*}
\]

Let us proceed to show the existence of $\{v_n(\xi)\}_{n=0}^\infty$ and the convergence of $\sum_{n=0}^\infty v_n(\xi)$. Set

\[
(4.8) \quad \begin{cases}
(a_1 + \sum_{i=2}^m a_i e^{\theta h \xi^p}) v_0(\xi) = \hat{f}(\xi), \\
(a_1 + \sum_{i=2}^m a_i e^{\theta h \xi^p}) v_n(\xi) \\
\quad + \sum_{i=2}^m a_i \left( \sum_{k=1}^{n} \psi_1(\xi) \ast_p \cdots \ast_p \psi_1(\xi) \ast_p \left( e^{\theta h \xi^p} \frac{\xi^p}{k!} v_{n-k}(\xi) \right) \right) = 0,
\end{cases}
\]

$(n \geq 1)$.

Let us proceed to show the existence of $\{v_n(\xi)\}_{n=0}^\infty$ and the convergence of $\sum_{n=0}^\infty v_n(\xi)$. Set

\[
(4.9) \quad h(\xi) := a_1 + \sum_{i=2}^m a_i e^{\theta h \xi^p}.
\]

Then $h(0) = \sum_{i=1}^m a_i$ and we have
Lemma 4.2. Assume Condition $\mathcal{B}$.

(1) If $\sum_{i=1}^{m} a_i \neq 0$, then there exist $\theta_0 \in \mathbb{R}$ and $\delta_0, d, M > 0$ such that $|e^{p b_i \xi^p}| \leq e^{-d |\xi|^p}$ ($i \geq 2$) and $|h(\xi)|^{-1} \leq M$ in $\{\xi; |\arg \xi - \theta_0| < \delta_0\}$.

(2) If $\sum_{i=1}^{m} a_i = 0$ and $\sum_{i=1}^{m} a_i b_i \neq 0$, then $\xi = 0$ is a pole of $1/h(\xi)$ with order $p$ and there exist $\theta_0 \in \mathbb{R}$ and $\delta_0, d, M, r > 0$ such that $|e^{p b_i \xi^p}| \leq e^{-d |\xi|^p}$ ($i \geq 2$) in $\{\xi; |\arg \xi - \theta_0| < \delta_0\}$ and $|h(\xi)|^{-1} \leq M_r$ in $\{\xi; |\arg \xi - \theta_0| < \delta_0, |\xi| > r\}$ for any $r > 0$.

Proof. Since the origin is a vertex of $\mathcal{B}$, there exist $\hat{h} \in \mathbb{R}$ and $c > 0$ such that $\cos(\arg b_i + \hat{h}) \leq -2c$ for $i \geq 2$. Let $\epsilon > 0$ such that $\cos(\arg b_i + \arg \xi^p) < -c$ in $\{\xi; |\arg \xi^p - \hat{h}| < \epsilon\}$. Then there exists a constant $d > 0$ with $|e^{p b_i \xi^p}| \leq e^{-d |\xi|^p}$ for $i \geq 2$ and $\lim_{|\xi| \to \infty} |h(\xi)| = |a_1| \neq 0$ in $\{\xi; |\arg \xi^p - \hat{h}| < \epsilon\}$. Hence there exists $R > 0$ such that $|h(\xi)| \geq |a_1| / 2$ in $\{\xi; |\arg \xi^p - \hat{h}| < \epsilon, |\xi| \geq R\}$. The zeros of $h(\xi)$ in $\{|\xi| \leq R\}$ is at most finite. Hence there exist $\theta_0$ and $\delta_0 > 0$ such that $h(\xi) \neq 0$ in $\{\xi \neq 0; |\arg \xi - \theta_0| < \delta_0\}$.

(1) Since $h(0) \neq 0$, there exists $M > 0$ such that $|h(\xi)|^{-1} \leq M$ in $\{\xi; |\arg \xi - \theta_0| < \delta_0\}$.

(2) From the assumptions $1/h(\xi)$ has a pole with order $p$ at $\xi = 0$. Hence there exists $M_r > 0$ such that $|h(\xi)|^{-1} \leq M_r$ in $\{\xi; |\arg \xi - \theta_0| < \delta_0, |\xi| \geq r\}$ for any $r > 0$.

In the following of this section we assume Condition $\mathcal{B}$, $\theta_0$ is that in Lemma 4.2 and $f(z)$ is $p$-Borel summable in the direction $\theta_0$. Hence it follows from $f(z) \sim_p \sum_{n=1}^{\infty} f_n z^n$ that

$$f(\xi) := \sum_{n=1}^{\infty} \frac{f_n}{\Gamma(n/p)} \xi^{n-p}$$

converges in $\{0 < |\xi| < r_0\}$ for some $r_0 > 0$. Moreover $f(\xi)$ is holomorphically extensible to an infinite sector $S^*(\theta_0, \delta) = \{\xi; |\arg \xi - \theta_0| < \delta\}$ and there exist positive constants $F$ and $c'$ such that $|f(\xi)| \leq Fe^{c' |\xi|^p}$ for $\xi \in S^*(\theta_0, \delta) \cap \{|\xi| \geq r_0\}$. Hence, by taking a large $c' > 0$, we may assume

$$|f(\xi)| \leq F |\xi|^{1-p} e^{c' |\xi|^p}$$

for $\xi \in S^*(\theta_0, \delta) \cup \{0 < |\xi| < r_0\}$.

Proposition 4.3. Suppose that either assumption of (1) or that of (2) in Theorem 2.1 holds. Then there exists a unique solution $v_n(\xi) = \sum_{\ell=-n+1}^{\infty} v_{n,\ell} \xi^{\ell-p}$ of (4.8), which converges in $\omega = \{0 < |\xi| < r_1\}$ for some $r_1 > 0$.

Proof. (1) Suppose $\sum_{i=1}^{m} a_i \neq 0$. We will show

$$v_n(\xi) = \sum_{\ell = (p+1)n+1}^{\infty} v_{n,\ell} \xi^{\ell-p}$$

for \(\ell \in \mathbb{N}\) and $\xi \in S^*(\theta_0, \delta) \cup \{0 < |\xi| < r_0\}$.
from now on. Since \( h(0) \neq 0 \), \( 1/h(\xi) \) is holomorphic in a neighborhood of \( \xi = 0 \). It follows from \( v_0(\xi) = f(\xi)/h(\xi) \) and (4.10) that it is holomorphic in \( \omega \) and the assertions are valid for \( v_0(\xi) \). We assume that (4.12) is valid for \( 0 \leq n \leq N - 1 \). Then there exist constants \( c_{l,\ell}(N, k) \ (\ell \geq (p + 1)N - k + 1) \) and \( d_{l,\ell}(k) \ (\ell \geq p) \) such that

\[
\left\{
\begin{array}{l}
eq \sum_{\ell=(p+1)N-k+1}^{\infty} c_{l,\ell}(N, k) \xi^{\ell-p}, \\
\psi_l(\xi) * \cdots * \psi_l(\xi) = \sum_{\ell=k}^{\infty} d_{l,\ell}(k) \xi^{\ell-p}.
\end{array}
\right.
\]

Hence there exist constants \( v^*_{N,\ell} \ (\ell \geq (p + 1)N + 1) \) such that

\[
\sum_{i=2}^{m} a_i \left( \sum_{k=1}^{N} \psi_l(\xi) * \cdots * \psi_l(\xi) * \left( \frac{e^{pb_1 \zeta^p}}{k!} v_{N-k}(\xi) \right) \right) = \sum_{\ell=(p+1)N+1}^{\infty} v^*_{N,\ell} \xi^{\ell-p}.
\]

Therefore \( v_N(\xi) = -(\sum_{\ell=(p+1)N+1}^{\infty} v^*_{N,\ell} \xi^{\ell-p})/h(\xi) \) is holomorphic in \( \omega = \{0 < |\xi| < r_1\} \) for some \( r_1 > 0 \). This shows our claim for \( v_N(\xi) \).

(2) Suppose \( \sum_{i=1}^{m} a_i = 0, \sum_{i=2}^{m} a_i b_i \neq 0 \) and \( f(z) \sim_p \sum_{n=p+1}^{\infty} f_n z^n \). Then

\[
\hat{f}(\xi) = \sum_{n=p+1}^{\infty} \frac{f_n}{\Gamma(n/p)} \xi^{n-p}.
\]

From the assumptions \( \zeta^p/h(\xi) \) is holomorphic on \( \{0 < |\xi| < r_1\} \) for some \( 0 < r_1 < r_0 \). It follows from \( v_0(\xi) = \hat{f}(\xi)/h(\xi) \) and (4.13) that \( v_0(\xi) \) is holomorphic in \( \omega \) and the assertions are valid for \( n = 0 \). We assume that the assertions are valid for \( 0 \leq n \leq N - 1 \). Then there exist constants \( c_{l,\ell}(N, k) \ (\ell \geq N - k + pk + 1) \) and \( d_{l,\ell}(k) \ (\ell \geq k) \) such that

\[
\left\{
\begin{array}{l}
eq \sum_{\ell=N-k+pk+1}^{\infty} c_{l,\ell}(N, k) \xi^{\ell-p}, \\
\psi_l(\xi) * \cdots * \psi_l(\xi) = \sum_{\ell=k}^{\infty} d_{l,\ell}(k) \xi^{\ell-p}.
\end{array}
\right.
\]

Hence there exist constants \( v^*_{N,\ell} \ (\ell \geq N + p + 1) \) such that

\[
\sum_{i=2}^{m} a_i \left( \sum_{k=1}^{N} \psi_l(\xi) * \cdots * \psi_l(\xi) * \left( \frac{e^{pb_1 \zeta^p}}{k!} v_{N-k}(\xi) \right) \right) = \sum_{\ell=N+p+1}^{\infty} v^*_{N,\ell} \xi^{\ell-p}.
\]

Thus \( v_N(\xi) = -(\sum_{\ell=-N+p+1}^{\infty} v^*_{N,\ell} \xi^{\ell-p})/h(\xi) = \sum_{\ell=N+1}^{\infty} v_{N,\ell} \xi^{\ell-p} \) and the assertions for \( v_N(\xi) \) follow.
Let us proceed to study the global properties of \( \{v_n(\xi)\}_{n=0}^{\infty} \) and their estimates. We prepare a lemma.

**Lemma 4.4.** Let \( 0 < \gamma \leq \kappa \). Let \( \phi_i(\xi) \in \mathcal{O}(S_{(0)}^+(\theta, \delta)) \) \((i = 1, 2)\) with

\[
|\phi_i(\xi)| \leq \frac{C_1|\xi|^\gamma e^{c_1|\xi|^p}}{\Gamma(s_1/\gamma)} \quad \text{for} \quad \xi \in S_{(0)}^+(\theta, \delta),
\]

where \( s_1, s_2 > 0, \ c \geq 0 \). Then \((\phi_1 \ast \gamma, \phi_2)(\xi) \in \mathcal{O}(S_{(0)}^+(\theta, \delta)) \) and

\[
|((\phi_1 \ast \gamma, \phi_2)(\xi)| \leq \frac{C_1C_2|\xi|^{n+\gamma} e^{c_1|\xi|^p}}{\Gamma((s_1 + s_2)/\gamma)}.
\]

**Proof.** We have

\[
(\phi_1 \ast \gamma, \phi_2)(\xi) = \int_0^{[\xi]} |\xi| e^{i\arg \xi} \phi_1 ((\xi^\gamma - \eta^\gamma)^{1/\gamma}) \phi_2(\eta) d\eta^\gamma
\]

\[
= \int_0^{[\xi]} \phi_1 ((|\xi|^\gamma - r^\gamma)^{1/\gamma} e^{i\arg \xi}) \phi_2(r e^{i\arg \xi} e^{r^\gamma} r e^{r^\gamma} dr^\gamma.
\]

Hence, by \((|\xi|^\gamma - r^\gamma)^{1/\gamma} + r^\kappa \leq |\xi|^\kappa \) for \( 0 \leq r \leq |\xi| \)

\[
|((\phi_1 \ast \gamma, \phi_2)(\xi)| \leq \int_0^{[\xi]} |\phi_1 ((|\xi|^\gamma - r^\gamma)^{1/\gamma} e^{i\arg \xi}) \phi_2(r e^{i\arg \xi})| dr^\gamma
\]

\[
\leq \frac{C_1C_2}{\Gamma(s_1/\gamma)\Gamma(s_2/\gamma)} \int_0^{[\xi]} ((|\xi|^\gamma - r^\gamma)^{n/\gamma - 1} e^{c_1|\xi|^p - r^\gamma} r e^{r^\gamma} dr^\gamma
\]

\[
\leq \frac{C_1C_2}{\Gamma(s_1/\gamma)\Gamma(s_2/\gamma)} \int_0^{[\xi]} ((|\xi|^\gamma - r^\gamma)^{n/\gamma - 1} r^{s_2/\gamma} dr^\gamma
\]

\[
\leq \frac{C_1C_2|\xi|^{n+\gamma} e^{c_1|\xi|^p}}{\Gamma((s_1 + s_2)/\gamma)}.
\]

We remark that Lemma 4.4 also holds for \( \phi_i(\xi) \in \mathcal{O}(S^+(\theta, \delta)) \) \((i = 1, 2)\). Let us estimate \( \{v_n(\xi)\}_{n=0}^{\infty} \).

**Proposition 4.5.** Assume \( \sum_{i=1}^{n} a_i \neq 0 \). Then \( \{v_n(\xi)\}_{n=0}^{\infty} \) are holomorphically extensible to \( \Omega = S^+(\theta, \delta) \cup \{0 < |\xi| < r_1\} \) for small \( \delta, r_1 > 0 \), and there exist positive constants \( K, C \) and \( c_1 \) such that

\[
|v_n(\xi)| \leq \frac{KC^n|\xi|^{n+1-p} e^{c_1|\xi|^p}}{\Gamma((n + 1)/p)}
\]

for \( \xi \in \Omega \).
Proof. Since $1/h(\xi)$ is holomorphic in $\Omega$ for small $\delta,r_1 > 0$, $v_0(\xi) = f(\xi)/h(\xi)$ is holomorphic in $\Omega$. By Lemma 4.2 and (4.11) $|v_0(\xi)| \leq MF|\xi|^{1-p}e^{c_1|\xi|^p}$ ($c_1 \geq c'$) holds. We assume that $v_n(\xi)$ is holomorphically extensible to $\Omega$ and (4.16) holds for $0 \leq n \leq N - 1$. Let $\xi \in \Omega$ and $i \geq 2$. Then from $|e^{p\xi^p}| \leq e^{-d|\xi|^p}$ ($d > 0$) in $\Omega$, there exists $C_0$ such that $|e^{p\xi^p}\xi^k/k!| \leq C_0^k$ ($k \geq 1$). Therefore we obtain

$$
\left| e^{p\xi^p}\xi^k/k! v_N-k(\xi) \right| \leq \frac{KC_0^{N-k}C_0^k|\xi|^{N-k+1-p}}{\Gamma((N-k+1)/p)} e^{c_1|\xi|^p}.
$$

It follows from

$$
\left| \psi_i(\xi) + \cdots + \psi_i(\xi) \right| \leq C^k|\xi|^{k-p}e^{c''|\xi|^p}/\Gamma(k/p)
$$

and Lemma 4.4, by taking $c_1 \geq c''$, that

$$
\left| \psi_i(\xi) + \cdots + \psi_i(\xi) \right| \left| e^{p\xi^p}\xi^k/k! v_N-k(\xi) \right| \leq \frac{KC_0^{N-k}(C_0C_1)^k|\xi|^{N+1-p}}{\Gamma((N+1)/p)} e^{c_1|\xi|^p}.
$$

Choose $C$ so large that $\sum_{k=1}^{\infty} (C_1C_2/C)^k \leq 1/MA_1$, where $M$ is that in Lemma 4.2 and $A_1 = \sum_{k=2}^{m} |a_k|$. Then

$$
|v_N(\xi)| \leq |h(\xi)|^{-1} \sum_{i=2}^{m} |a_i| \sum_{k=1}^{N} \left| \psi_i(\xi) + \cdots + \psi_i(\xi) \right| \left| e^{p\xi^p}\xi^k/k! v_N-k(\xi) \right| \leq \frac{MA_1KC_0|\xi|^{N+1-p}}{\Gamma((N+1)/p)} e^{c_1|\xi|^p} \sum_{k=1}^{N} \left( \frac{C_0C_1}{C} \right)^k \leq \frac{KC_0^{N+1-p}}{\Gamma((N+1)/p)} e^{c_1|\xi|^p}.
$$

Hence $v_n(\xi)$ is holomorphically extensible to $\Omega$ and (4.16) holds. \qed

Proposition 4.6. Assume $\sum_{k=1}^{m} a_k \neq 0$. Then there exists a solution $v(\xi)$ of (4.7) that is holomorphic in $\Omega = S^*(\theta_0, \delta) \cup \{ 0 < |\xi| < r_1 \}$ for small $\delta,r_1 > 0$ such that

$$
|v(\xi)| \leq K_0|\xi|^{1-p}e^{c_0|\xi|^p} \quad (\xi \in \Omega)
$$

for some constants $K_0$ and $c_0$. Moreover, $\xi^{-p}v(\xi)$ is holomorphic in $\{ |\xi| < r_1 \}$, that is, $v(\xi)$ has an expansion

$$
v(\xi) = \sum_{\ell=1}^{+\infty} \xi^\ell
$$

in $\{ 0 < |\xi| < r_1 \}$. 

Proof. Let $v(\xi) = \sum_{n=0}^{+\infty} v_n(\xi)$. By Proposition 4.5 there exist positive constants $K_0$ and $c_0$ such that for $\xi \in \Omega$

$$\sum_{n=0}^{+\infty} |v_n(\xi)| \leq K|\xi|^{1-p} e^{c_1|\xi|^p} \sum_{n=0}^{+\infty} \frac{C^n|\xi|^n}{\Gamma((n+1)/p)} \leq K_0|\xi|^{1-p} e^{c_0|\xi|^p}.$$  

Hence $v(\xi)$ is holomorphic in $\Omega$. It follows from Proposition 4.3 that

$$v(\xi) = \sum_{n=0}^{+\infty} \left( \sum_{\ell=n+1}^{+\infty} v_{n,\ell} \xi^{\ell-p} \right) = \sum_{\ell=1}^{+\infty} \left( \sum_{n=0}^{\ell-1} v_{n,\ell} \right) \xi^{\ell-p} = \sum_{\ell=1}^{+\infty} v_\ell \xi^{\ell-p}.$$  

We have

$$\left( a_1 + \sum_{i=2}^{m} a_i e^{\theta_i \xi^p} \right) v(\xi) + \sum_{i=2}^{m} a_i \left( \sum_{k=1}^{\infty} \psi_k(\xi) *_p \cdots *_p \psi_k(\xi) *_p \left( e^{\theta_k \xi^p} \frac{\xi^{pk}}{k!} v(\xi) \right) \right)$$

$$= h(\xi) \sum_{n=0}^{+\infty} v_n(\xi)$$

$$+ \sum_{i=2}^{m} a_i \left( \sum_{k=1}^{\infty} \psi_k(\xi) *_p \cdots *_p \psi_k(\xi) *_p \left( \sum_{n=k}^{+\infty} e^{\theta_k \xi^p} \frac{\xi^{pk}}{k!} v_{n-k}(\xi) \right) \right)$$

$$= \sum_{n=0}^{+\infty} h(\xi) v_n(\xi)$$

$$+ \sum_{i=2}^{m} a_i \left( \sum_{k=1}^{\infty} \psi_k(\xi) *_p \cdots *_p \psi_k(\xi) *_p \left( \sum_{n=k}^{+\infty} e^{\theta_k \xi^p} \frac{\xi^{pk}}{k!} v_{n-k}(\xi) \right) \right)$$

$$= \sum_{n=0}^{+\infty} h(\xi) v_n(\xi)$$

$$+ \sum_{n=1}^{+\infty} \left( \sum_{i=2}^{m} a_i \left( \sum_{k=1}^{n} \psi_k(\xi) *_p \cdots *_p \psi_k(\xi) *_p \left( e^{\theta_k \xi^p} \frac{\xi^{pk}}{k!} v_{n-k}(\xi) \right) \right) \right)$$

$$= \left( a_1 + \sum_{i=2}^{m} a_i e^{\theta_i \xi^p} \right) v_0(\xi) = \hat{f}(\xi).$$  

More generally we have for $\hat{f}(\xi) \in \Exp(p, S^*(\theta_0, \delta))$. 

\[ \square \]
Proposition 4.7. Assume that $\sum_{i=1}^{m} a_i \neq 0$ and $\hat{f}(\zeta) \in \text{Exp}(p, S^*(\theta_0, \delta))$ in (4.7) satisfies $|\hat{f}(\zeta)| \leq A|\zeta|^{c-p}$ ($c > 0$) in $S^*(\theta_0, \delta) \cap \{ |\zeta| < r \}$. Then there exists a unique solution $v(\zeta) \in \text{Exp}(p, S^*(\theta_0, \delta_1))$ of (4.7) with $|v(\zeta)| \leq C|\zeta|^{c-p}$ in $S^*(\theta_0, \delta_1) \cap \{ |\zeta| < r_1 \}$ for some $0 < \delta_1 < \delta$ and $0 < r_1 < r$.

Proof. The proof of the existence is almost same as those of Propositions 4.5 and 4.6. We show the uniqueness. Let $\hat{f} = 0$. We show $|v(\zeta)| \leq A_0 C_0^n |\zeta|^{n+\varepsilon-p}/\Gamma((n+\varepsilon)/p)$ in $S^*(\theta_0, \delta) \cap \{ 0 < |\zeta| < r_1 \}$ for any $n$. It holds for $n = 0$. Assume for $0 \leq n \leq N - 1$ the estimate holds. Then

$$|v(\zeta)| \leq \frac{A_0 A_1 C_0^{N-1}}{|b(\xi)|} \sum_{k=1}^{\infty} C_1^{k+1} \frac{|\zeta|^{N-1+k+\varepsilon-p}}{\Gamma((N-1+k+\varepsilon)/p)}$$

and

$$|v(\zeta)| \leq A_0 A_1 C_0^{N-1} \frac{|\zeta|^{N+\varepsilon-p}}{\Gamma((N+\varepsilon)/p)} \sum_{k=1}^{\infty} C_1^{k+1} \frac{\Gamma((N+\varepsilon)/p)|\zeta|^{k-1}}{\Gamma((N-1+k+\varepsilon)/p)}$$

$$\leq A_0 C_0^{N} \frac{|\zeta|^{N+\varepsilon-p}}{\Gamma((N+\varepsilon)/p)}.$$ 

Thus we have the estimate for $n = N$. By taking a limit as $n \to \infty$, $v(\zeta) = 0$. \qed

Let us proceed to the case when

$$\sum_{i=1}^{m} a_i = 0 \quad \text{and} \quad \sum_{\{i : p_i = p \}} a_i b_{i,p_i} \neq 0 \quad (4.19)$$

and

$$f(0) = f'(0) = \cdots = f^{(p)}(0) = 0 \quad (4.20)$$

hold.

Proposition 4.8. Assume (4.19) and (4.20). Then $\{v_n(\zeta)\}_{n=0}^{\infty}$ are holomorphic in $\omega = \{ 0 < |\zeta| < r_1 \}$ for some $r_1 > 0$ and there exist constant $K$ and $C$ such that

$$|v_n(\zeta)| \leq \frac{KC^n |\zeta|^{n+1-p}}{\Gamma((n+1)/p)} \quad (4.21).$$
Moreover, \(v(\zeta) = \sum_{n=0}^{\infty} v_n(\zeta)\) is a solution of (4.7) in \(\{0 < |\zeta| < r_1\}\) and \(\zeta^{p-1}v(\zeta)\) is holomorphic at \(\zeta = 0\),

\begin{equation}
\tag{4.22}
v(\zeta) = \sum_{\ell=1}^{+\infty} v^*_{\ell} \zeta^{\ell-p}.
\end{equation}

**Proof.** The proof is similar to that of Proposition 4.5. We note that \(\zeta^p/h(\zeta)\) is holomorphic at \(\zeta = 0\), \(f(\zeta) = \sum_{n=p+1}^{\infty} f_n \zeta^{n-p}/\Gamma(n/p)\) and \(|f(\zeta)| \leq F|\zeta|^{1-p}\). We have \(|v_0(\zeta)| \leq MF|\zeta|^{1-p}\). We assume that \(v_n(\zeta)\) is holomorphic in \(\omega\) and (4.21) holds for \(0 \leq n \leq N - 1\). Let \(\zeta \in \omega\) and \(i \geq 2\). Then we have 

\[
|e^{ph_i} \zeta^p / k! v_{N-k}(\zeta)| \leq \frac{KC^{N-k} C_0^k |\zeta|^{N-k+1}}{\Gamma((N-k+1)/p)}
\]

\[
= \frac{KC^{N-k} C_0^k ((N-k+1)/p)|\zeta|^{N-k+1}}{\Gamma((N-k+1)/p+1)}.
\]

From

\[
\left| \frac{1}{h(\zeta)} \prod_{p=1}^{k} v_{i}(\zeta) \prod_{p=1}^{k} e^{ph_i} \zeta^p / k! v_{N-k}(\zeta) \right| \leq C_1^k |\zeta|^{k-p} / \Gamma(k/p)
\]

and Lemma 4.4,

\[
\left| \frac{1}{h(\zeta)} \prod_{p=1}^{k} v_{i}(\zeta) \prod_{p=1}^{k} e^{ph_i} \zeta^p / k! v_{N-k}(\zeta) \right| \leq \frac{KC^{N-k} (C_0 C_1)^k C_2 ((N-k+1)/p)|\zeta|^{N+1-p}}{\Gamma((N+1)/p+1)} \leq \frac{KC^{N-k} (C_0 C_1)^k C_2 |\zeta|^{N+1-p}}{\Gamma((N+1)/p)}.
\]

It follows from the same way as the proof of Propositions 4.5 and 4.6 that (4.21) holds for \(n = N\) and \(v(\zeta) = \sum_{n=0}^{\infty} v_n(\zeta)\) converges in \(\omega\). It is a solution of (4.7). It follows from Proposition 4.3 that

\[
v(\zeta) = \sum_{n=0}^{\infty} \left( \sum_{\ell=1}^{+\infty} v_{n,\ell} \zeta^{\ell-p} \right) = \sum_{\ell=1}^{+\infty} \left( \sum_{n=0}^{+\infty} v_{n,\ell} \right) \zeta^{\ell-p} = \sum_{\ell=1}^{+\infty} v^*_{\ell} \zeta^{\ell-p}.
\]

We have a local solution \(v(\zeta) = \sum_{n=0}^{\infty} v_n(\zeta)\) in \(\omega = \{0 < |\zeta| < r_1\}\) by Proposition 4.8. It will be shown that \(v(\zeta)\) can be analytically continued to a sector \(S^*(\theta_0, \delta)\) with its bound for small \(\delta > 0\). We transform (4.7) in order to reduce our problem to the case \(\sum_{i=1}^{m} a_i \neq 0\). Let \(\omega = \{0 < |\zeta| < r_0\}\) and
\( \Psi(\xi) \in \text{Exp}(p, S^*(\theta_0, \delta)) \) and \( v(\xi) \in \mathcal{C}(S^*(\theta_0, \delta) \cap \omega) \) with \( |\Psi(\xi)|, |v(\xi)| \leq A|\xi|^{r-p} \) \((\epsilon > 0)\) for \( \xi \in S^*(\theta_0, \delta) \cap \omega \). Let \( a = |a|e^{i\theta_0} \in S^*(\theta_0, \delta) \cap \omega \) and

\[
(\Psi * p v)(\xi) = \left( \int_0^a + \int_\delta^\infty \right) \Psi((\xi^p - \eta^p)^{1/p})v(\eta)d\eta^p = I + II,
\]

where

\[
I = \int_0^a \Psi((\xi^p - \eta^p)^{1/p})v(\eta)d\eta^p = \int_0^a \Psi((\tau^p - \rho^p + \alpha^p)^{1/p})v(\rho)d\rho^p|_{\tau=(\xi^p-\alpha^p)^{1/p}},
\]

\[
II = \int_0^\tau \Psi((\tau^p - \rho^p)^{1/p})v((\rho^p + \alpha^p)^{1/p})d\rho^p|_{\tau=(\xi^p-\alpha^p)^{1/p}}.
\]

**Lemma 4.9.** Suppose \( |\Psi(\xi)| \leq K|\xi|^{r-p}e^{c|\xi|^s}(s > 0) \) for \( \xi \in S^*(\theta_0, \delta) \) and let

\[
g(\tau) = \int_0^a \Psi((\tau^p - \rho^p + \alpha^p)^{1/p})v(\rho)d\rho^p.
\]

Then there exists a constant \( C > 0 \) such that if \( s \geq p \),

\[
|g(\tau)| \leq AKC^{s-p+1}(1 + |\tau|)^{s-p}e^{c\tau^p} \quad (\tau \in S^*(\theta_0, \delta))
\]

and if \( s < p \),

\[
|g(\tau)| \leq AKC^{p-s+1}|\tau|^{s-p}e^{c\tau^p} \quad (\tau \in S^*(\theta_0, \delta)).
\]

**Proof.** Suppose \( s \geq p \). Then

\[
|g(\tau)| \leq \left| \int_0^a \Psi((\tau^p - \rho^p + \alpha^p)^{1/p})v(\rho)d\rho^p \right|
\]

\[
\leq AKC_0e^{c\tau^p} \int_0^{|\alpha|} (|\tau|^p + r^p + |a|^p)^{s/p-1}r^{s-p} dr^p
\]

\[
\leq AKC^{s-p+1}(1 + |\tau|)^{s-p}e^{c\tau^p}.
\]

Suppose \( 0 < s < p \). For \( a = |a|e^{i\theta_0}, \rho = |\rho|e^{i\theta_0} \) \((0 < |\rho| \leq |a|)\) and \( \tau \in S^*(\theta_0, \delta) \)

\( C_1|\tau^p - \rho^p + \alpha^p|^{1/p} \geq |\tau| \) holds for some \( C_1 > 0 \). Then

\[
|g(\tau)| \leq \left| \int_0^a \Psi((\tau^p - \rho^p + \alpha^p)^{1/p})v(\rho)d\rho^p \right|
\]

\[
\leq AKC_0C_1^{p-s}|\tau|^{s-p}e^{c\tau^p} \int_0^{|\alpha|} r^{s-p} dr^p \leq AKC^{p-s+1}|\tau|^{s-p}e^{c\tau^p}. \quad \square
\]
We have the following relations

\[(\Psi * p v)(\xi) = g(\tau)|_{\tau=(\xi^p - a\tau)^{1/p}} + V(\tau)|_{\tau=(\xi^p - a\tau)^{1/p}},\]

\[V(\tau) = \int_{0}^{\tau} \Psi((\tau^p - p^p)^{1/p})v((\rho^p + a^p)^{1/p})d\rho^p.\]

**Proposition 4.10.** Assume (4.19) and (4.20). Then there exists a solution \(v(\xi)\) of (4.7) that is holomorphic in \(\Omega = S^{\ast}(\theta_0, \delta) \cup \{0 < |\xi| < r_1\}\) for small \(\delta, r_1 > 0\) such that

\[|v(\xi)| \leq K_0|\xi|^{-p}e^{c_0|\xi|^p} \quad (\xi \in \Omega)\]

for some positive constants \(K_0\) and \(c_0\). Further \(\xi^{p-1}v(\xi)\) is holomorphic in \(\{|\xi| < r_1\}\), that is, \(v(\xi)\) has an asymptotic expansion

\[v(\xi) = \sum_{\ell=1}^{\infty} v_{\xi}^{\ell} |\xi|^{-p}\]

in \(\{0 < |\xi| < r_1\}\).

**Proof.** Let \(v(\xi)\) be a local solution of (4.7) in \(\{0 < |\xi| < r_1\}\). Choose \(a = |a|e^{i\theta_0} \quad (0 < |a| < r_1)\). Then \(h(a) \neq 0\) from Lemma 4.2. Put \(w(\tau) = v((\tau^p + a^p)^{1/p})\) and

\[\Psi_{i,\ell}(\xi) = (a, \psi^{\ast} \cdots \psi^{\ast}) \xi^{\ell}/\ell!.\]

Then \(|\Psi_{i,\ell}(\xi)| \leq C_1 |\xi|^{-p}e^{c_0|\xi|^p}/\Gamma(\ell/p)\ell!\), \(w(\tau)|_{\tau=(\xi^p - a\tau)^{1/p}} = v(\xi)\) and

\[(\Psi_{i,\ell} * p (\rho^{\ell/p} e^{ib\rho^{1/p}}v(\rho)))|_{\tau=(\xi^p - a\tau)^{1/p}} = g_{i,\ell}(\tau) + (\Psi_{i,\ell} * p ((\rho^p + a^p)^{\ell/p}e^{ib(\rho^p + a^p)^{1/p}}v((\rho^p + a^p)^{1/p}))(\tau)|_{\tau=(\xi^p - a\tau)^{1/p}}\]

where

\[g_{i,\ell}(\tau) = \int_{0}^{\tau} \Psi_{i,\ell}((\tau^p - p^p)^{1/p})(\rho^{p/\ell} e^{ib\rho^{1/p}}v(\rho))d\rho^p.\]

Put \(H(\tau) = \sum_{\ell=1}^{m} a_ie^{ib(\tau^p + a^p)}\). Then from (4.7) we get a convolution equation

\[H(\tau)w(\tau) + \sum_{\ell=2}^{m} \sum_{i=1}^{\infty} \Psi_{i,\ell} * p ((\rho^p + a^p)^{\ell/p}e^{ib(\rho^p + a^p)}w(\rho)) = F(\tau)\]
of \( w(\tau) \), where

\[
F(\tau) = \hat{f} ((\tau^p + a^p)^{1/p}) - \sum_{i=2}^{m} \sum_{\ell=1}^{\infty} g_{i,\ell}(\tau).
\]

By Lemma 4.9 \( F(\tau) \in \text{Exp}(p, S^*(\theta_0, \delta')) \) and \( |F(\tau)| \leq C' |\tau|^{1-p} \) in \( S^*(\theta_0, \delta') \cap \{ 0 < |\tau| < r' \} \) for small \( \delta', r' > 0 \). Note \( H(0) = h(a) \neq 0 \). We can show that there exists a unique solution \( w(\tau) \) of (4.31) such that \( w(\tau) \in \text{Exp}(p, S^*(\theta_0, \delta'')) \) with \( |w(\tau)| \leq C |\tau|^{1-p} \) in \( S^*(\theta_0, \delta'') \cap \{ 0 < |\tau| < r'' \} \) \( (0 < \delta'' < \delta', 0 < r'' < r') \), whose proof is almost same as those of Propositions 4.5, 4.6 and 4.7. By the uniqueness \( w(\tau)|_{\tau=(\epsilon_p-a^p)/p} = v(\zeta) \) holds in a small sector with vertex \( \zeta = a \). Hence \( v(\zeta) \) has a holomorphic extension to \( S^*(\theta_0, \delta) \) such that \( v(\zeta) \in \text{Exp}(p, S^*(\theta_0, \delta)) \) for small \( \delta > 0 \).

**Proof of Theorem 2.3.** It follows from Propositions 4.6 and 4.10 that there exists a solution \( v(\zeta) \) of (4.7) with the desired estimate. Define

\[
u(z) = \int_{0}^{\infty} \exp \left( - \left( \frac{\zeta}{z} \right)^p \right) v(\zeta) d\zeta.
\]

Then \( u(z) \in \mathcal{C}(S_{0}(\theta_0, \pi/(2p) + \delta)) \) and by (4.18) and (4.29) we obtain

\[
u(z) \sim_p \sum_{n=1}^{\infty} v_n \Gamma \left( \frac{n}{p} \right) z^n.
\]

Since \( v(\zeta) \) is a solution of (4.7), \( u(z) \) is a solution of (2.1). The uniqueness of (2.1) in \( C[[z]] \) implies \( \tilde{u}(z) = \sum_{n=1}^{\infty} v_n \Gamma \left( \frac{n}{p} \right) z^n \). Hence \( u(z) \sim_p \tilde{u}(z) \). Consequently \( \tilde{u}(z) \) is \( p \)-Borel summable in the direction \( \theta_0 \).

Let us give a simple example,

\[
\begin{aligned}
& a_1 u(z) + a_2 u \left( \frac{z}{\sqrt[1-b_2 z^p]} \right) + a_3 u \left( \frac{z}{\sqrt[1-b_3 z^p]} \right) = f(z),
\end{aligned}
\]

where \( a_1 a_2 a_3 b_2 b_3 \neq 0 \). We have, by setting \( h(\zeta) = a_1 + a_2 e^{b_2 \zeta^p} + a_3 e^{b_3 \zeta^p} \),

\[
u(z) = \int_{0}^{\infty} \exp \left( - \left( \frac{\zeta}{z} \right)^p \right) f(\zeta) d\zeta.
\]

Assume \( \Re b_2 < 0 \) and \( \Re b_3 < 0 \). We can choose \( \theta_0 \) and \( \delta > 0 \) such that \( h(\zeta) \neq 0 \) in the sector \( S^*(\theta_0, \delta) \) and \( |h(\zeta)|^{-1} \leq M_1 \) in \( S^*(\theta_0, \delta) \cap \{ \zeta : |\zeta| \geq 1 \} \) for some \( M_1 > 0 \). Set \( c = e^{i\theta}(0, \infty) \). Suppose \( a_1 + a_2 + a_3 \neq 0 \). Then \( h(0) \neq 0 \) and \( |h(\zeta)|^{-1} \leq M \) on \( c \). Suppose \( a_1 + a_2 + a_3 = 0 \) and \( a_2 b_2 + a_3 b_3 \neq 0 \). Then \( h(\zeta) = O(\zeta^p) \). If \( f(0) = f'(0) = \cdots = f^{(p)}(0) = 0 \), then \( \hat{f}(\zeta) = O(\zeta) \) and
\( \hat{f}(\zeta) d\zeta / h(\zeta) = O(1) d\zeta, \) hence the integral near \( \zeta = 0 \) converges. Therefore we can integrate on \( C. \)

5. Remarks on the case \( \sum_{i=1}^{m} a_i = 0 \)

We assume \( \sum_{i=1}^{m} a_i = 0 \) and Condition \( \tilde{B} \) in this section.

**Definition 5.1.** (1) The set of all formal series \( \bar{v}(z) \) of the form

\[
\bar{v}(z) = \sum_{n=n_0}^{\infty} v_n z^n + v^* \log z \quad (n_0 \in \mathbb{Z}, v_n, v^* \in C)
\]

is denoted by \( C_{d, \log[[z]]}. \)

(2) The set of all formal series \( \bar{v}(z) \in C_{d, \log[[z]]} \) such that \( \sum_{n=0}^{\infty} v_n z^n \in C_{\{1\}}[[z]] \) is denoted by \( C_{d, \log, \{1\}}[[z]]. \)

(3) Let \( \bar{v}(z) \in C_{d, \log, \{s\}}[[z]] \) and \( \nu = 1/s. \) If \( \sum_{n=0}^{\infty} v_n z^n \) is \( \gamma \)-Borel summable in a direction \( \theta, \) then we say that \( \bar{v}(z) \) is \( \gamma \)-Borel summable in a direction \( \theta. \)

**Lemma 5.2.** Let \( f(z) \in C[[z]]. \) Assume \( \sum_{i=1}^{m} a_i b_i \neq 0. \) Then there exists \( v(z) = \sum_{k=-p}^{-1} c_k z^k + c^* \log z \) such that \( \sum_{i=1}^{m} a_i v(\phi_i(z)) - f(z) = O(z^{p+1}). \)

**Proof.** We have

\[
\phi_i(z)^k = z^k (1 + b_i z^p + O(z^{p+1})) = z^k (1 + kb_i z^p + O(z^{p+1}))
\]

and

\[
\log \phi_i(z) = \log z + \log(1 + b_i z^p + O(z^{p+1})) = \log z + b_i z^p + O(z^{p+1}),
\]

hence

\[
\sum_{i=1}^{m} a_i v(\phi_i(z)) = \sum_{k=-p}^{-1} c_k \sum_{i=1}^{m} a_i (z^k + kb_i z^{p+k} + O(z^{p+k+1}))
\]

\[
+ c^* \sum_{i=1}^{m} a_i (\log z + b_i z^p + O(z^{p+1})).
\]

Since \( \sum_{i=1}^{m} a_i = 0, \) we try to find \( \{c_k\}_{k=-p}^{-1} \) and \( c^* \) such that

\[
\sum_{k=-p}^{-1} c_k \left( \sum_{i=1}^{m} a_i b_i \right) z^{p+k} + O(z^{p+k+1}) + c^* \left( \sum_{i=1}^{m} a_i b_i \right) z^p + O(z^{p+1})
\]

\[
= f_0 + f_1 z + \cdots + f_p z^p + O(z^{p+1}),
\]

which is written in the form
Theorems 2.1, 2.2 and 2.3.

where $g_k(z) = O(z^{k+1})$. We have $-p(\sum_{i=1}^m a_i b_i) = f_0$ and $c_{-p}$ is determined. By $(1 - p)(\sum_{i=1}^m a_i b_i) c_{-p+1} + c_{-p} g_0'(0) = f_1$, $c_{-p+1}$ is determined. Successively we can determine $\{c_k\}_{k=-p}$ and $c^*$ so that (5.2) holds.

Let us return to the equation (2.1).

**Theorem 5.3.** Let $f(z) \in C[[z]]$. Suppose $\sum_{i=1}^m a_i b_i \neq 0$. Then there exists a formal solution $\tilde{u}(z)$ to (2.1) represented by $\tilde{u}(z) = \sum_{w=-p}^\infty c_w z^n + c^* \log z \in C_{\mathbf{H}, \log}[[[z]]]$. Moreover,

1. If $f(z) \in C_{1/p}[[[z]]]$, then $\tilde{u}(z) \in C_{\mathbf{H}, \log, (1/p)}[[[z]]]$.
2. There exists a direction $\theta_0$ such that $\tilde{u}(z)$ is p-Borel summable in the direction $\theta_0$, provided $f(z)$ is p-Borel summable in the direction $\theta_0$.

**Proof.** Let $v(z)$ be that given in Lemma 5.2. Set $u(z) = v(z) + w(z)$. Then (0.1) becomes

\[
\begin{align*}
\sum_{i=1}^m a_i w(\phi_i(z)) &= g(z), \\
g(z) &= f(z) - \sum_{i=1}^m a_i v(\phi_i(z)),
\end{align*}
\]

where $g(z) \in C[[z]]$ and $g(z) = O(z^{p+1})$. The assertions follows easily from Theorems 2.1, 2.2 and 2.3.

**Example.** Consider

\[
u(\phi(z)) - u(z) = 1,
\]

which is called Abel’s functional equation studied first in [1]. We assume $\phi(z) = z(1 + bz^p + O(z^{p+1}))$, $b \neq 0$. Let $\tilde{u}(z) \in C_{\mathbf{H}, \log}[[[z]]]$ be a formal solution to Abel's equation assured in Theorem 5.3. Set $h(\xi) = e^{p\xi} - 1$. The poles of $1/h(\xi)$ are $\{\xi; \xi^p = (2\pi i/(ph)) n, n \in \mathbb{Z}\}$, which are on lines through the origin in $\xi$-space. Let $\theta_0$ be a direction such that $\lim_{|\xi| \to \infty} \exp(pb|\xi|^p e^{b\theta_0}) = 0$. Then $\tilde{u}(z)$ is p-Borel summable in this direction $\theta_0$.

6. Borel summability-II

Let $\{\phi_i(z)\}_{i=1}^m$ ($m \geq 2$) be holomorphic functions in a neighborhood of $z = 0$ such that $\phi_i(0) = 0$, $\phi'_i(0) = 1$ and $\phi_i(z) \neq \phi_j(z)$ for $i \neq j$. It is our aim
in this section that we transform equation (0.1) to one to which we can apply theorems in the preceding sections.

Put $\Phi = \{\varphi_i(z); i = 1, \ldots, m\}$. Let for $i \neq j$

\[(6.1) \quad \varphi_i(z) - \varphi_j(z) = c_{i,j}z^{p_{i,j}+1} + O(z^{p_{i,j}+2}),\]

where $c_{i,j} \neq 0$ and $p_{i,j} \in \{1, 2, \ldots\}$ and put $p_{i,i} = +\infty$. Let

\[(6.2) \quad p_* = \min_{1 \leq i, j \leq m} \{p_{i,j}\}.\]

Let us introduce an equivalence relation in $\Phi$,

\[(6.3) \quad \varphi_i(z) \sim \varphi_j(z) \iff \varphi_i(z) - \varphi_j(z) = O(z^{p_*+2}).\]

By this relation we classify $\Phi$ into equivalent classes $\{\Phi_{\ell}\}_{\ell=1}^{m_0} \ (2 \leq m_0 \leq m)$. So $\Phi = \bigcup_{\ell=1}^{m_0} \Phi_{\ell}$, $\Phi_{\ell} \cap \Phi_{\ell'} = \emptyset$ for $\ell \neq \ell'$. 

**Definition 6.1.** If $\Phi_{\ell}$ consists of one element, that is, $\Phi_{\ell} = \{\varphi_{i_0}\}$ for some $i_0$, then we say that $\Phi_{\ell}$ is simple or $\varphi_{i_0}(z)$ is simple.

Let

\[(6.4) \quad \varphi_i(z) = z \left(1 + \sum_{j=1}^{p_*-1} b_{i,j}z^j + b_{i,p_*}z^{p_*} \right) + O(z^{p_*+2}).\]

Then

\[(6.5) \quad \varphi_i(z) - \varphi_j(z) = (b_{i,p_*} - b_{j,p_*})z^{p_*+1} + O(z^{p_*+2})\]

and $b_{i,p_*}$ is uniquely determined for $\Phi_{\ell}$. Hence we put $b^{0}(\ell) = b_{i,p_*}$ for $\varphi_i \in \Phi_{\ell}$ and $b^{0}(\ell) \neq b^{0}(\ell')$ for $\ell \neq \ell'$. The constants $\{b^{0}(\ell)\}_{\ell=1}^{m_0}$ depend on coordinates.

We have

**Lemma 6.2.** Let $\varphi_i(z) \in \Phi_{\ell}$, $\varphi_j(z) \in \Phi_{\ell'}$ ($\ell \neq \ell'$). Let $\phi(w)$ be a holomorphic function at $w = 0$ with $\phi(w) = cw + O(w^2)$ ($c \neq 0$). Then

\[(6.6) \quad \varphi_i(\phi(w)) - \varphi_j(\phi(w)) = (b^{0}(\ell) - b^{0}(\ell'))c^{p_*+1}w^{p_*+1} + O(w^{p_*+2}).\]

**Proof.** We have (6.6) from (6.5). \qed

Put $A^0 = \{b^0_{i,p_*}; 1 \leq i \leq m\}$. Then $A^0 = \{b^{0}(\ell); 1 \leq \ell \leq m_0\}$. We may assume $\varphi_i \in \Phi_1$. Let $w = \varphi_{i}(z)$ and $\phi(w) = \varphi_{i}^{-1}(w)$. Then it follows from Lemma 6.2 that

\[(6.7) \quad \begin{cases} \varphi_{i}(\phi(w)) = w, \\ \varphi_{i}(\phi(w)) = w + (b^0_{i,p_*} - b^0_{1,p_*})w^{p_*+1} + O(w^{p_*+2}) \quad (i \geq 2). \end{cases}\]
Set $A = \{b_{1,p}^0 - b_{1,p}^0; 1 \leq i \leq m\}$. Then $A = \{b^0(\ell) - b^0(1); 1 \leq \ell \leq m_0\}$. It is obvious that $A$ (resp. $\tilde{A}$) is a translation of $A^0$ (resp. $\tilde{A}^0$). Change the coordinate $z = \varphi_{i}^{-1}(w)$ as above and denote $w$ by $z$ again. We have from the above considerations, by putting $b_{\ell} = b^0(\ell) - b^0(1)$,

\[ \varphi_1(z) = z, \]
\[ \varphi_i(z) = z + O(z^{p-2}) \quad (\varphi_i \in \Phi_i), \]
\[ \varphi_{\ell}(z) = z + b_{\ell} z^{p-1} + O(z^{p-2}) \quad \text{with} \ b_{\ell} \neq 0 \quad (\varphi_{\ell} \in \Phi_{\ell}, \ \ell \geq 2). \]

For these $\{\varphi_i(z)\}_{i=1}^m$, if $\varphi_1(z)$ is simple, then $p = p_2 = p_3 = \cdots = p_m$ and $p = p^*$ (see (2.6)). Thus we can apply theorems in the previous sections to (0.1). In order to state theorems about (0.1) let us sum up the notations and conditions: $p = p^*$ and

\[ \varphi_i(z) = z \left( 1 + \sum_{j=1}^{p-1} b_{1,j} z^j + b_{1,p} z^p \right) + O(z^{p+2}) \quad 1 \leq i \leq m, \]
\[ A^0 = \{b_{1,p}, b_{2,p}, \ldots, b_{m,p}\}, \quad \tilde{A}^0 = \text{the convex hull of } A^0, \]

and conditions

\[ \sum_{i=1}^{m} a_i \neq 0, \]
\[ \sum_{i=1}^{m} a_i = 0, \quad \sum_{i=1}^{m} a_i b_{i,p}^0 \neq 0, \]
\[ f(0) = f'(0) = \cdots = f^{(p)} = 0. \]

Consequently we have the following theorems from Theorems 2.1, 2.2 and 2.3.

**Theorem 6.3.** (1) Suppose (6.10). Then there exists a unique formal solution $\tilde{u}(z) \in C[[z]]$ of (0.1).

(2) Suppose (6.11). Then there exists a unique formal solution $\tilde{u}(z) \in C[[z]]$ of (0.1) with $\tilde{u}(0) = 0$.

In the following Theorems 6.4 and 6.5 $\tilde{u}(z) \in C[[z]]$ is that in Theorem 6.3.

**Theorem 6.4.** If $f(z) \in C_{(1/p)}[[z]]$, then $\tilde{u}(z) \in C_{(1/p)}[[z]]$.

**Theorem 6.5.** Suppose that there exists $\varphi_0(z) \in \Phi$ such that $\varphi_0(z)$ is simple and $b_{0,p}$ is a vertex of $A^0$. Then there exists a direction $\theta_0 \in R$ such that $\tilde{u}(z)$ is $p$-Borel summable in the direction $\theta_0$, provided $f(z)$ is $p$-Borel summable in the direction $\theta_0$. 
7. Appendix

Let \( \varphi_i(z) \in C_0 \) (1 \( \leq i \leq m \)) such that \( \varphi_i(0) = 0 \) and \( \varphi_i'(0) = \lambda_i \). We do not assume (0.2). Let \( f(z) \in C_0 \) and consider (0.1). Here we assume \( \varphi_i(z) = z \), \( a_1 = 1 \) and \( \lambda_1 = 1 > |\lambda_i| \) for \( i \geq 2 \). Then we can show the existence of holomorphic solution to (0.1).

**Theorem 7.1.** Let \( p \in \{1, 2, \ldots\} \) such that \( \sum_{i=2}^{m} |a_i| \lambda_i|^p < 1 \). Suppose \( |f(z)| \leq K|z|^p \) for \( |z| \leq R \). Then there exists a unique solution \( u(z) \in C_0 \) of (0.1) with \( u(z) = O(z^p) \).

**Proof.** Let \( \{\kappa_i\}_{i=2}^{m} \) and \( \kappa \) such that \( |\lambda_i| < \kappa_i < 1 \) and \( \sum_{i=2}^{m} |a_i| \kappa_i^p < \kappa < 1 \). Take \( 0 < r < R \) such that \( |\varphi_i(z)| \leq \kappa_i |z| \) for \( |z| \leq r \). Let us construct a solution \( u(z) \) by the following iteration

\[
\begin{align*}
(7.1) & \quad u_0(z) = f(z), \\
(7.2) & \quad u_n(z) + \sum_{i=2}^{m} a_i u_{n-1}(\varphi_i(z)) = 0 \quad (n \geq 1).
\end{align*}
\]

We show for \( |z| \leq r \)

\[
|u_n(z)| \leq K\kappa^n |z|^p,
\]

which obviously holds for \( n = 0 \). Assume that (7.2) holds for \( n = N - 1 \). Then we have

\[
\sum_{i=2}^{m} |a_i| |u_{N-1}(\varphi_i(z))| \leq K\kappa^{N-1} \sum_{i=2}^{m} |a_i| |\varphi_i(z)|^p
\]

\[
\leq K\kappa^{N-1} \left( \sum_{i=2}^{m} |a_i| \kappa_i^p \right) |z|^p \leq K\kappa^N |z|^p.
\]

Thus (7.2) holds for all \( n \) and \( u(z) = \sum_{n=0}^\infty u_n(z) \) converges, \( |u(z)| \leq K(1 - \kappa)^{-1} |z|^p \) and satisfies (0.1). Let \( v(z) \) with \( |v(z)| \leq K' |z|^p \) be another solution of (0.1). Then \( w(z) = u(z) - v(z) \) satisfies \( \sum_{i=1}^{m} a_i w(\varphi_i(z)) = 0 \). We can show \( |w(z)| \leq M\kappa^n |z|^p \) for all \( n \) by the similar way to (7.2). Thus \( w(z) = 0 \).

\[ \square \]

References


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