Scattering Problem for the Supercritical Nonlinear Schrödinger Equation in 1d

By

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Abstract. We consider the one dimensional nonlinear Schrödinger equation $iu_t + u_{xx}/2 = f(u)$, $x \in \mathbf{R}$, $t > 0$, $u(0, x) = u_0(x)$, $x \in \mathbf{R}$, with a super critical nonlinearity $f(u) = \sum_{j \neq 0} f_j(u)$, and $f_j(u)$ are such that $f_j(u) = \lambda_j |u|^{\sigma_j - j} u^j$, where $\lambda_j \in \mathbf{C}$, $\sigma_j > 3$. We prove the existence of the scattering operator in the weighted Sobolev spaces.

Key Words and Phrases. Scattering operator, Asymptotic behavior in time, Nonlinear Schrödinger, Power nonlinearity.

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1. Introduction

We consider the one dimensional nonlinear Schrödinger equation

$$(1) \quad \begin{cases} iu_t + \frac{1}{2}u_{xx} = f(u), & x \in \mathbf{R}, t > T, \\ u(T, x) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

with a super critical nonlinearity $f(u) = \sum_{j \neq 0} f_j(u)$, and $f_j(u)$ are such that

$$f_j(u) = \lambda_j |u|^{\sigma_j - j} u^j,$$

where $\lambda_j \in \mathbf{C}$, $\sigma = \min \sigma_j > 3$ (really, we use the following properties $f_j \in \mathbf{C}^2$ and $f_j(re^{i\theta}u) = r^{\sigma_j} e^{ij\theta} f_j(u)$ for all $r > 0$, $\theta \in \mathbf{R}$). Here and below by $\sum_{j \neq 0}$ we mean a finite sum over $j \neq 0$.

When $j = 1$, global existence, asymptotic behavior, blow up of solutions for (1) were well studied in general space dimension by many authors (see, a text book [2]) due to the gauge invariant property of the nonlinearity. On the other hand, in the case of $j \neq 1$, equation (1) was not studied well. Global in time existence of small solutions for (1) including $j = 0$ in general space dimension n was shown in [18] for $\sigma > \sigma_s(n)$, where $\sigma_s(n) = (n + 2 + \sqrt{4 + 12n + n^2})/(2n)$ is known as the Strauss exponent [17]. The optimal time decay estimate

$$\|u(t)\|_{\mathbf{L}^{\sigma+1}} \leq C|t|^{-(n/2)(1-2/(\sigma+1))}$$

for small solutions was also obtained in [18] by the method of the $\mathbf{L}^{\sigma+1} - \mathbf{L}^{(\sigma+1)/\sigma}$ —time decay estimate for the free Schrödinger evolution group. From the previous works for the gauge invariant nonlinearity, it is known that the critical exponent is $1 + 2/n$ which is called the short range exponent for the scattering problem ([7], [2]). Therefore it is conjectured that $1 + 2/n$ is the critical case for (1). Indeed the asymptotic profile of solutions for (1) with the nonlinearity u^3 in one space dimension is different from that of linear problem. This result was shown in [9]. As far as we know, global existence and scattering of solutions were not obtained for case $1 + 2/n < \sigma \leq \sigma_s(n)$. Our purpose in the present paper is to fill this gap when $n = 1$. Our analysis in the present paper requires some regularity condition on the nonlinearity and does not work well for the nonlinearity of the form $f_0(u) = \lambda_0|u|^\sigma$. Hence the problem is still open for higher space dimensions and the case of the nonlinearity $|u|^\sigma$ with $\sigma \leq \sigma_s(n)$. On the other hand, the nonlinear Klein-Gordon equation which is considered as the relativistic version of (1) including $j = 0$ was studied in [5], [8] and the similar problem was solved for $1 \leq n \leq 3$.

In the middle of 80's, the vector field and the normal form methods were introduced by [13] and [15] to study the large time asymptotics of solutions to quadratic nonlinear Klein-Gordon equations in three space dimensions. Now these methods are very important tools in the field of nonlinear dispersive equations. The algebraic normal form method by [14], [19] was used for the study of the quadratic nonlinear Klein-Gordon equations in two space dimensions. Also the method of [15] was applied to quadratic nonlinear Schrödinger equations in two space dimensions for the nonlinearity $\bar{u}^2$, see [3]. These methods do work well for the quadratic or cubic cases but do not work well for the physically important case $|u|u$. Our method stated below is more general than these methods since it can be used for scattering problem with another types of fractional nonlinearities. Our method depends on the equation for the new dependent variable $v(t) = \mathcal{FM}\mathcal{U}(-t)u(t)$ which is written as (4) below. This equation is useful to clarify the difference between the oscillating factor of linear solutions and that of nonlinear terms denoted by E . From this equation we find that the derivative of the oscillating factor gains a time growth but integrating it by parts yields a better time decay. Our weighted homogeneous Sobolev space (which is not necessarily $\mathbf{L}^2$) is chosen carefully on the balance of the time growth by the derivative and time decay via the integration by parts. As a by-product, we find that our function space is close to the invariant space in the short range exponent. This is the main difference between our previous work [9] and the present paper.

If we restrict our attention to the life span of local solutions, we have an application to a system of Schrödinger equation such that

$$i\partial_t v_1 + \frac{1}{2m_1} \partial_x^2 v_1 = \lambda \bar{v}_1 v_2,$$

$$i\partial_t v_2 + \frac{1}{2m_2} \partial_x^2 v_2 = \mu v_1^2,$$

where $\lambda, \mu \in \mathbf{C} \setminus \{0\}$, the mass of particles m_1, m_2 are positive and satisfy a non resonance mass condition

$$2m_1 \neq m_2, \quad m_1 \neq m_2.$$

This system in higher space dimensions was studied in [10], [11] (see also [4]) for the resonance case $2m_1 = m_2$, which is easier since the operator $J_{m_j} = x + it\partial_x/m_j$ works well.

To state our result, we introduce the usual Lebesgue space $\mathbf{L}^p$ with the norm $\|\phi\|_{\mathbf{L}^p} = (\int_{\mathbf{R}} |\phi(x)|^p dx)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}\{|\phi(x)|; x \in \mathbf{R}\}$. The weighted Sobolev space is defined by

$$\mathbf{H}_p^{m,k} = \{\phi \in \mathbf{L}^2 : \|\langle x \rangle^k \langle i\partial_x \rangle^m \phi\|_{\mathbf{L}^p} < \infty\},$$

where $m, k \in \mathbf{R}^+$, $1 \leq p \leq \infty$, $\langle x \rangle = \sqrt{1 + |x|^2}$. For simplicity we write $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$. Homogeneous Sobolev space $\dot{\mathbf{H}}^{\alpha,\beta}$ is defined by

$$\dot{\mathbf{H}}^{\alpha,\beta} = \{\phi \in \mathbf{S}' ; \|\phi\|_{\dot{\mathbf{H}}^{\alpha,\beta}} = \| |x|^\beta (-\partial_x^2)^{\alpha/2} \phi \|_{\mathbf{L}^2} < \infty\}.$$

We introduce a closed ball $(\mathbf{X})_{\leq \rho} = \{\phi \in \mathbf{X}; \|\phi\|_{\mathbf{X}} \leq \rho\}$. Let $\mathbf{C}(\mathbf{I}; \mathbf{X})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{X}$. Different positive constants might be denoted by the same letter C .

We consider the Cauchy problem with initial data at $T = 1$. Denote by $\mathcal{U}(t) = e^{it\partial_x^2/2}$ the free Schrödinger evolution group.

Theorem 1.1. *Assume that $\sigma > 3$ and the initial data u_0 are such that $\mathcal{U}(-1)u_0 \in \dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}$, $0 \leq \gamma < 1/2 < \alpha < 1$. Then there exists $\varepsilon_0 > 0$ such that (1) has a unique global solution*

$$\mathcal{U}(-t)u \in \mathbf{C}([1, \infty); \dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\alpha})$$

satisfying the estimate

$$(2) \quad \|\mathcal{U}(-t)u(t)\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\alpha}} \leq Ct^{\max\{0, \alpha+5/2-\sigma\}/2}$$

for all $t > 1$, and for any u_0 such that $\|\mathcal{U}(-1)u_0\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} = \varepsilon \leq \varepsilon_0$. Moreover for any global solution u , there exists a unique scattering state $u_+ \in \dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}$, with $0 \leq \gamma < 1/2 < \beta < \sigma - 5/2$, such that

$$\|\mathcal{U}(-t)u(t) - u_+\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} \rightarrow 0$$

as $t \rightarrow \infty$.

Remark 1.1. If we take α satisfying $1/2 < \alpha < \sigma - 5/2$, then $\beta = \alpha$ is permitted in Theorem 1.1. Hence there exists a scattering state in $\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\alpha}$. In the case of $1/2 < \alpha < \sigma - 5/2$, estimate (2) gives us $\|\mathcal{U}(-t)u(t)\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\alpha}} \leq C$ from which the sharp time decay estimate $\|u(t)\|_{\mathbf{L}^\infty} \leq Ct^{-1/2}$ follows.

Remark 1.2. Define $u_\lambda(t) = \lambda^{2/(\sigma-1)}u(\lambda^2 t, \lambda x)$. Note that $u_\lambda(t)$ satisfies (1) with the initial data $u_{0\lambda}(x) = \lambda^{(\sigma-1)/2}u_0(\lambda x)$. We have

$$\|u_{0\lambda}\|_{\dot{\mathbf{H}}^{0,(\sigma-2)/2}} = \|u_0\|_{\dot{\mathbf{H}}^{0,(\sigma-2)/2}},$$

which implies that $\dot{\mathbf{H}}^{0,(\sigma-2)/2}$ is the so-called invariant space for problem (1). When $\sigma > 3$ is close to 3, then the function space $\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\alpha}$, $\gamma < 1/2 < \alpha < \sigma - 5/2$, in Theorem 1.1 is close to the invariant space.

Our second result concerns the Cauchy problem with initial data at $T = 0$.

Theorem 1.2. *Assume that $\sigma > 3$, the initial data $u_0 = u(0, x) \in \mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha}$, $1/2 < \alpha < \delta < 1$. Then there exists $\varepsilon_0 > 0$ such that (1) has a unique global solution*

$$\mathcal{U}(-t)u \in \mathbf{C}([0, \infty); \mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha})$$

satisfying the estimate

$$\|\mathcal{U}(-t)u(t)\|_{\mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha}} \leq C \langle t \rangle^{\max\{0, \alpha+5/2-\sigma\}/2}$$

for all $t > 0$, for any u_0 such that $\|u_0\|_{\mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha}} = \varepsilon \leq \varepsilon_0$. Moreover for any global solution u , there exists a unique scattering state $u_+ \in \mathbf{H}^\delta \cap \mathbf{H}^{0,\beta}$, with $1/2 < \beta < \sigma - 5/2$, such that

$$\|\mathcal{U}(-t)u(t) - u_+\|_{\mathbf{H}^\delta \cap \mathbf{H}^{0,\beta}} \rightarrow 0$$

as $t \rightarrow \infty$.

Remark 1.3. By Theorem 1.2, the inverse wave operator

$$W_+^{-1} : u_0 \in (\mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha})_{\leq \varepsilon} \mapsto u_+ \in (\mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha})_{\leq \sqrt{\varepsilon}}$$

is defined. The proofs of Theorems 1.1 and 1.2 work for the final value problem. Therefore we find that there exists the wave operator

$$W_- : u_- \in (\mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha})_{\leq \rho} \mapsto u_0 \in (\mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha})_{\leq \sqrt{\rho}}.$$

We take $\varepsilon = \sqrt{\rho}$, then the scattering operator

$$S_{-,+} = W_+^{-1} W_- : u_- \in (\mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha})_{\leq \rho} \mapsto u_+ \in (\mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha})_{\leq \rho^{1/4}}$$

is defined.

The factorization formula from paper [12] for the free Schrödinger evolution group is presented by

$$(3) \quad \mathcal{U}(t) = e^{i\partial_x^2/2} = M(t)\mathcal{D}(t)\mathcal{V}(t)\mathcal{F},$$

where $M(t, x) = e^{i|x|^2/(2t)}$, the dilation operator $(\mathcal{D}(t)\phi)(x) = (it)^{-1/2}\phi(x/t)$, and the operator $\mathcal{V}(t) = \mathcal{F}M(t)\mathcal{F}^{-1}$. Here $\mathcal{F}\phi = (2\pi)^{-1/2} \int_{\mathbf{R}} e^{-ix\xi} \phi(x) dx$ denotes the Fourier transform of the function ϕ , and $\mathcal{F}^{-1}$ is the inverse Fourier transformation defined by $\mathcal{F}^{-1}\phi = (2\pi)^{-1/2} \int_{\mathbf{R}} e^{ix\xi} \phi(\xi) d\xi$. Formula (3) is useful for studying the large time asymptotic behavior of solutions of the nonlinear Schrödinger equations. We also have

$$\mathcal{F}\mathcal{U}(-t) = i\mathcal{V}(-t)\bar{E}(t)\mathcal{D}\left(\frac{1}{t}\right),$$

where $E(t, \xi) = e^{it|\xi|^2/2}$. Here we have used the commutation identity $\mathcal{D}(1/t)M(t) = E(t)\mathcal{D}(1/t)$. Using the new dependent variables $\varphi(t) = \mathcal{F}\mathcal{U}(-t)u(t)$ and

$$v(t) = \mathcal{F}M\mathcal{U}(-t)u(t) = \mathcal{V}(t)\varphi(t) = i\bar{E}(t)\mathcal{D}\left(\frac{1}{t}\right)u(t),$$

we represent the solution as

$$u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\varphi(t) = M(t)\mathcal{D}(t)v(t).$$

In the same way as in paper [12] we multiply both sides of (1) by $\mathcal{F}M\mathcal{U}(-t)$ to get

$$(4) \quad \mathcal{L}v = \sum_{j \neq 0} i^{(1-j)/2} t^{(1-\sigma_j)/2} E^{j-1} f_j(v),$$

where $\mathcal{L} = i\partial_t + \partial_\xi^2/(2t^2)$, since

$$\mathcal{F}M\mathcal{U}(-t) \left(i\partial_t + \frac{1}{2} \partial_x^2 \right) = \left(i\partial_t + \frac{1}{2t^2} \partial_\xi^2 \right) \mathcal{F}M\mathcal{U}(-t),$$

and

$$\begin{aligned} \mathcal{F}M\mathcal{U}(-t)f_j(u) &= i\bar{E}(t)\mathcal{D}\left(\frac{1}{t}\right)f_j(u) \\ &= i\lambda_j\bar{E}(t)\mathcal{D}\left(\frac{1}{t}\right)|u|^{\sigma_j-j}u^j \\ &= i^{(1-j)/2}\lambda_j\bar{E}^{1-j}t^{(1-\sigma_j)/2} \left| i\bar{E}\mathcal{D}\left(\frac{1}{t}\right)u \right|^{\sigma_j-j} \left(i\bar{E}\mathcal{D}\left(\frac{1}{t}\right)u \right)^j \\ &= i^{(1-j)/2}t^{(1-\sigma_j)/2}E^{j-1}\lambda_j|v|^{\sigma_j-j}v^j = i^{(1-j)/2}t^{(1-\sigma_j)/2}E^{j-1}f_j(v). \end{aligned}$$

When we differentiate equation (4), then we encounter an undesirable time growth which appears from E^{j-1} , for $j \neq 1$. Therefore we use the method originated by [6] and developed in [9].

We organize our paper as follows. In Section 2 we prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. In Section 4 we state some estimates for the fractional derivatives from paper [9].

2. Proof of Theorem 1.1

In this section we consider the linearized version of (4), which is written as

$$(5) \quad \begin{cases} \mathcal{L}v^{(n)} = \sum_{j \neq 0} i^{(1-j)/2} t^{(1-\sigma_j)/2} E^{j-1} f_j(v^{(n-1)}), \\ v^{(n)}(1, x) = i^{1/2} e^{-i|x|^2/2} u_0(x), \end{cases}$$

where $v(t) = \mathcal{F}M(t)\mathcal{U}(-t)u(t)$ and $\mathcal{L}v^{(0)} = 0$, in the Banach space $\mathbf{X}$ with the following norm

$$\|v\|_{\mathbf{X}} = \sup_{t > 1} (\|\partial|^\gamma v\|_{\mathbf{L}^2} + \|\partial|^\beta v\|_{\mathbf{L}^2} + t^{-A} \|\partial|^\alpha v\|_{\mathbf{L}^2}),$$

where $A = \max\{0, \alpha + 5/2 - \sigma\}/2$, $0 \leq \gamma < 1/2 < \beta < \sigma - 5/2$, $1/2 < \alpha < 1$, and the fractional derivative $|\partial|^\beta = (-\partial_x^2)^{\beta/2}$. By the Sobolev embedding theorem we have

$$(6) \quad \|v\|_{\mathbf{L}^\infty} \leq C \|\partial|^\gamma v\|_{\mathbf{L}^2} + C \|\partial|^\beta v\|_{\mathbf{L}^2} \leq C \|v\|_{\mathbf{X}}.$$

We introduce the regularizing operator

$$\mathcal{K}\phi = \sqrt{t} \int_{\mathbf{R}} K(\xi\sqrt{t}, \eta\sqrt{t}) \phi(\xi - \eta) d\eta$$

with a kernel $K(x, y) = \pi^{-1/2} \langle x \rangle e^{-\langle x \rangle^2 y^2}$ to treat the problem in lower order Sobolev spaces and write (5) as

$$(7) \quad \mathcal{L}v^{(n)} = \sum_{j \neq 0, 1} i^{(1-j)/2} t^{(1-\sigma_j)/2} E^{j-1} f_j(\mathcal{K}v^{(n-1)}) + R_1,$$

where the remainder term

$$R_1 = \sum_{j \neq 0, 1} i^{(1-j)/2} t^{(1-\sigma_j)/2} E^{j-1} (f_j(v^{(n-1)}) - f_j(\mathcal{K}v^{(n-1)})) + t^{(1-\sigma_1)/2} f_1(v^{(n-1)})$$

and

$$v^{(n-1)} \in (\mathbf{X})_{\leq \varepsilon} = \{v \in \mathbf{X}; \|v\|_{\mathbf{X}} \leq \varepsilon\}.$$

For simplicity we denote $v = v^{(n)}$, $w = v^{(n-1)}$. To eliminate the first summand in the right-hand side of (7) we use the identity

$$(8) \quad \mathcal{L}(t^{\mu_j} E^{j-1} \phi) = \frac{i}{A_j} t^{\mu_j-1} E^{j-1} \phi + i t^{\mu_j-1} (j-1) E^{j-1} \xi \partial_\xi \phi + t^{\mu_j} E^{j-1} \mathcal{L} \phi,$$

since $\mathcal{L}(t^{\mu_j} E^{j-1}) = i A_j^{-1} t^{\mu_j-1} E^{j-1}$, where

$$A_j = \left(\left(\mu_j + \frac{j-1}{2} \right) + \frac{it}{2} j(j-1) \xi^2 \right)^{-1}$$

for $j \neq 0, 1$. When $(j-1)j \neq 0$, the coefficient A_j compensates the time growth, which comes from the derivatives of E^{j-1} . So identity (8) is useful for $j \neq 0, 1$. However we note that the exponent E^{j-1} is absent in equation (7) for the case of $j = 1$. In view of (8), we get from (7)

$$(9) \quad \begin{aligned} & \mathcal{L} \left(v + \sum_{j \neq 0, 1} t^{\mu_j} E^{j-1} \phi_j \right) \\ &= i \sum_{j \neq 0, 1} t^{\mu_j-1} E^{j-1} \left(\frac{1}{A_j} \phi_j - i^{(3-j)/2} t^{(3-\sigma_j)/2 - \mu_j} f_j(\mathcal{K}w) \right) \\ &+ i \sum_{j \neq 0, 1} (j-1) t^{\mu_j-1} E^{j-1} \xi \partial_\xi \phi_j + \sum_{j \neq 0, 1} t^{\mu_j} E^{j-1} \mathcal{L} \phi_j + R_1. \end{aligned}$$

We now choose $\mu_j = (3 - \sigma_j)/2$ and

$$\phi_j = i^{(3-j)/2} A_j f_j(\mathcal{K}w) = i^{(3-j)/2} A_j \lambda_j |\mathcal{K}w|^{\sigma_j-j} (\mathcal{K}w)^j$$

to eliminate the first summand in the right-hand side of (9). By the identity

$$\mathcal{L}(\psi \chi) = \chi \mathcal{L} \psi + t^{-2} \psi_\xi \chi_\xi + \psi \mathcal{L} \chi,$$

we obtain

$$(10) \quad \begin{aligned} \mathcal{L} \phi_j &= i^{(3-j)/2} \mathcal{L}(A_j f_j(\tilde{w})) \\ &= i^{(3-j)/2} f_j(\tilde{w}) \mathcal{L} A_j + i^{(3-j)/2} t^{-2} (\partial_\xi A_j) \partial_\xi f_j(\tilde{w}) + i^{(3-j)/2} A_j \mathcal{L} f_j(\tilde{w}) \end{aligned}$$

with $\tilde{w} = \mathcal{K}w$. Therefore by virtue of (10), we get from (9)

$$(11) \quad \begin{aligned} & \mathcal{L} \left(v + \sum_{j \neq 0, 1} i^{(3-j)/2} t^{(3-\sigma_j)/2} E^{j-1} A_j f_j(\tilde{w}) \right) \\ &= \sum_{j \neq 0, 1} i^{(3-j)/2} t^{(1-\sigma_j)/2} E^{j-1} ((j-1) i \xi \partial_\xi A_j + t \mathcal{L} A_j) f_j(\tilde{w}) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j \neq 0, 1} i^{(3-j)/2} t^{(1-\sigma_j)/2} E^{j-1} ((j-1)i\zeta A_j + t^{-1}\partial_\xi A_j) \partial_\xi f_j(\tilde{w}) \\
& + \sum_{j \neq 0, 1} i^{(3-j)/2} t^{(3-\sigma_j)/2} E^{j-1} A_j \mathcal{L}f_j(\tilde{w}) + R_1.
\end{aligned}$$

We denote

$$\begin{aligned}
R_0 &= \sum_{j \neq 0, 1} i^{(3-j)/2} t^{(3-\sigma_j)/2} E^{j-1} A_j f_j(\tilde{w}), & R_2 &= \sum_{j \neq 0, 1} t^{(1-\sigma_j)/2} E^{j-1} a_j f_j(\tilde{w}), \\
R_3 &= \sum_{j \neq 0, 1} t^{(1-\sigma_j)/2} E^{j-1} b_j \partial_\xi f_j(\tilde{w}), & R_4 &= \sum_{j \neq 0, 1} i^{(3-j)/2} t^{(3-\sigma_j)/2} E^{j-1} A_j \mathcal{L}f_j(\tilde{w}),
\end{aligned}$$

with $a_j = i^{(3-j)/2} ((j-1)i\zeta \partial_\xi A_j + t \mathcal{L}A_j)$ and $b_j = i^{(3-j)/2} ((j-1)i\zeta A_j + t^{-1}\partial_\xi A_j)$. Then we obtain our target equation

$$(12) \quad \mathcal{L}(v + R_0) = \sum_{l=1}^4 R_l.$$

We apply derivative $|\partial|^\alpha$ to equation (12)

$$(13) \quad \mathcal{L}(|\partial|^\alpha v + |\partial|^\alpha R_0) = \sum_{l=1}^3 |\partial|^\alpha R_l.$$

Next we estimate the terms $|\partial|^\alpha R_j$ in equation (13). By the second estimate of Lemma 4.2 it follows that

$$\begin{aligned}
\| |\partial|^\alpha (E^{j-1} \psi) \|_{\mathbf{L}^2} &\leq \| E^{j-1} |\partial|^\alpha \psi \|_{\mathbf{L}^2} + \| [|\partial|^\alpha, E^{j-1}] \psi \|_{\mathbf{L}^2} \\
&\leq C \| |\partial|^\alpha \psi \|_{\mathbf{L}^2} + C t^{\alpha/2} \| B^\alpha \psi \|_{\mathbf{L}^2},
\end{aligned}$$

where $B = \langle \xi \sqrt{t} \rangle$. Also since $A_j = O(B^{-2})$ we have

$$\begin{aligned}
\| [|\partial|^\alpha, A_j] \psi \|_{\mathbf{L}^2} &= C \left\| \int_{\mathbf{R}} (\tilde{A}_j - A_j) \frac{\tilde{\psi} dy}{|y|^{1+\alpha}} \right\|_{\mathbf{L}^2} \\
&= C t \left\| \int_{\mathbf{R}} \tilde{A}_j A_j (2\xi - y) y \frac{\tilde{\psi} dy}{|y|^{\alpha+1}} \right\|_{\mathbf{L}^2} \\
&\leq C t^{1/2} \| B^{-1} \psi \|_{\mathbf{L}^2} \int_{\mathbf{R}} \frac{dy}{\langle y \sqrt{t} \rangle |y|^\alpha} \leq C t^{\alpha/2} \| B^{-1} \psi \|_{\mathbf{L}^2},
\end{aligned}$$

where $\tilde{A}_j = A_j(\cdot - y)$, $\tilde{\psi} = \psi(\cdot - y)$. Hence we find

$$(14) \quad \| |\partial|^\alpha (E^{j-1} A_j \psi) \|_{\mathbf{L}^2} \leq C \| |\partial|^\alpha \psi \|_{\mathbf{L}^2} + C t^{\alpha/2} \| B^{-1} \psi \|_{\mathbf{L}^2}.$$

Also by Lemma 4.4 and (6) we obtain

$$\| |\partial|^\alpha f_j(\tilde{w}) \|_{\mathbf{L}^2} \leq C \|\tilde{w}\|_{\mathbf{L}^\infty}^{\sigma_j-1} \| |\partial|^\alpha \tilde{w} \|_{\mathbf{L}^2} \leq C \varepsilon^{\sigma_j} t^A$$

and

$$\| B^{-1} f_j(\tilde{w}) \|_{\mathbf{L}^2} \leq C \|\tilde{w}\|_{\mathbf{L}^\infty}^{\sigma_j} \| B^{-1} \|_{\mathbf{L}^2} \leq C \varepsilon^{\sigma_j} t^{-1/4}.$$

Hence by (14) we get

$$\begin{aligned} (15) \quad \| |\partial|^\alpha R_0 \|_{\mathbf{L}^2} &\leq C \sum_{j \neq 0,1} t^{(3-\sigma_j)/2} \| |\partial|^\alpha (E^{j-1} A_j f_j(\tilde{w})) \|_{\mathbf{L}^2} \\ &\leq C \sum_{j \neq 0,1} t^{(3-\sigma_j)/2} \| |\partial|^\alpha f_j(\tilde{w}) \|_{\mathbf{L}^2} + C \sum_{j \neq 0,1} t^{(3-\sigma_j+\alpha)/2} \| B^{-1} f_j(\tilde{w}) \|_{\mathbf{L}^2} \\ &\leq C \sum_{j \neq 0,1} \varepsilon^{\sigma_j} t^{(3-\sigma_j)/2+A} + C \sum_{j \neq 0,1} \varepsilon^{\sigma_j} t^{(3-\sigma_j+\alpha)/2-1/4} \leq C \varepsilon^\sigma t^A. \end{aligned}$$

In the same manner since $a_j = O(B^{-2})$ we find

$$\begin{aligned} (16) \quad \| |\partial|^\alpha R_2 \|_{\mathbf{L}^2} &\leq \sum_{j \neq 0,1} t^{(1-\sigma_j)/2} \| |\partial|^\alpha (E^{j-1} a_j f_j(\tilde{w})) \|_{\mathbf{L}^2} \\ &\leq C \sum_{j \neq 0,1} \varepsilon^{\sigma_j} t^{(1-\sigma_j)/2+A} + C \sum_{j \neq 0,1} \varepsilon^{\sigma_j} t^{(1-\sigma_j+\alpha)/2-1/4} \\ &\leq C \varepsilon^\sigma t^{(1-\sigma)/2+A} + C \varepsilon^\sigma t^{(\alpha-\sigma+1/2)/2}. \end{aligned}$$

Next by the second estimate of Lemma 4.2 we have

$$\begin{aligned} \| |\partial|^\alpha R_1 \|_{\mathbf{L}^2} &\leq C \sum_{j \neq 0,1} t^{(1-\sigma_j)/2} \| |\partial|^\alpha (f_j(w) - f_j(\tilde{w})) \|_{\mathbf{L}^2} + C t^{(1-\sigma_1)/2} \| |\partial|^\alpha f_1(w) \|_{\mathbf{L}^2} \\ &\quad + C \sum_{j \neq 0,1} t^{(1-\sigma_j+\alpha)/2} \| B^\alpha (f_j(w) - f_j(\tilde{w})) \|_{\mathbf{L}^2}. \end{aligned}$$

Denote $W = w - \tilde{w}$, since $K_0 = \int_{\mathbf{R}} K(\xi \sqrt{t}, y) dy = 1$, then by Lemma 4.3 we find

$$\| |\partial|^\alpha W \|_{\mathbf{L}^2} + t^{\alpha/2} \| B^\alpha W \|_{\mathbf{L}^2} \leq C \| |\partial|^\alpha w \|_{\mathbf{L}^2} \leq C \varepsilon t^A,$$

and also

$$\| |\partial|^\alpha \tilde{w} \|_{\mathbf{L}^2} \leq \| |\partial|^\alpha W \|_{\mathbf{L}^2} + \| |\partial|^\alpha w \|_{\mathbf{L}^2} \leq C \varepsilon t^A.$$

Then by Lemma 4.4 we obtain

$$\| |\partial|^\alpha (f_j(w) - f_j(\tilde{w})) \|_{\mathbf{L}^2} \leq C \|w\|_{\mathbf{L}^\infty}^{\sigma_j-1} \| |\partial|^\alpha w \|_{\mathbf{L}^2} \leq C \varepsilon^\sigma t^A$$

and

$$\begin{aligned} t^{\alpha/2} \|B^\alpha(f_j(w) - f_j(\tilde{w}))\|_{\mathbf{L}^2} &\leq Ct^{\alpha/2} (\|w\|_{\mathbf{L}^\infty}^{\sigma_j-1} + \|\tilde{w}\|_{\mathbf{L}^\infty}^{\sigma_j-1}) \|B^\alpha W\|_{\mathbf{L}^2} \\ &\leq C\varepsilon^\sigma t^A. \end{aligned}$$

Therefore we find

$$(17) \quad \|\partial|^\alpha R_1\|_{\mathbf{L}^2} \leq C\varepsilon^\sigma t^{(1-\sigma)/2+A}.$$

Next we estimate the term

$$\begin{aligned} |\partial|^\alpha R_3 &= \sum_{j \neq 0,1} t^{(1-\sigma_j)/2} |\partial|^\alpha (E^{j-1} b_j \partial_\xi f_j(\tilde{w})) \\ &= \sum_{j \neq 0,1} t^{(1-\sigma_j)/2} |\partial|^\alpha (E^{j-1} b_j (\partial_{\tilde{w}} f_j(\tilde{w}) \partial_\xi \tilde{w} + \partial_{\tilde{w}} f_j(\tilde{w}) \overline{\partial_\xi \tilde{w}})). \end{aligned}$$

We have the identity

$$\begin{aligned} t^{-1/2} B^{-1} \partial_\xi \tilde{w} &= \frac{1}{\sqrt{\pi} \langle \xi \sqrt{t} \rangle} \partial_\xi \langle \xi \sqrt{t} \rangle \int_{\mathbf{R}} e^{-\langle \xi \sqrt{t} \rangle^2 ((\xi - \eta) \sqrt{t})^2} w(t, \eta) d\eta \\ &= \mathcal{K}_1 w, \end{aligned}$$

where we denote the operator

$$\mathcal{K}_1 \phi = \sqrt{t} \int_{\mathbf{R}} K_1(\xi \sqrt{t}, \eta \sqrt{t}) \phi(\xi - \eta) d\eta,$$

with kernel

$$\begin{aligned} K_1(x, y) &= \langle x \rangle^{-1} (\partial_x + \partial_y) K(x, y) \\ &= \frac{1}{\sqrt{\pi}} (x \langle x \rangle^{-2} (1 - 2 \langle x \rangle^2 y^2) - 2 \langle x \rangle^2 y) e^{-\langle x \rangle^2 y^2}, \end{aligned}$$

which satisfies the conditions of Lemmas 4.1 and 4.3. Also we have

$$K_{1,0} = \int_{\mathbf{R}} K_1(x, y) dy = 0.$$

Note that $b_j = i^{(3-j)/2} ((j-1)i\xi A_j + t^{-1} \partial_\xi A_j) = O(t^{-1/2} B^{-1})$. Then by Lemma 4.2 we get

$$\begin{aligned} \|\partial|^\alpha (E^{j-1} b_j \partial_{\tilde{w}} f_j(\tilde{w}) \partial_\xi \tilde{w})\|_{\mathbf{L}^2} &= t^{1/2} \|\partial|^\alpha (E^{j-1} b_j B \partial_{\tilde{w}} f_j(\tilde{w}) \mathcal{K}_1 w)\|_{\mathbf{L}^2} \\ &\leq \|\partial|^\alpha (\partial_{\tilde{w}} f_j(\tilde{w})) (\mathcal{K}_1 - K_{1,0}) w\|_{\mathbf{L}^2} \\ &\quad + Ct^{\alpha/2} \|\tilde{w}\|_{\mathbf{L}^\infty}^{\sigma_j-1} \|B^\alpha (\mathcal{K}_1 - K_{1,0}) w\|_{\mathbf{L}^2} \\ &\leq C\varepsilon^\sigma t^A + C\varepsilon^\sigma t^{\alpha/2-1/4}. \end{aligned}$$

Hence

$$(18) \quad \begin{aligned} \|\partial|^\alpha R_3\|_{\mathbf{L}^2} &\leq \sum_{j \neq 0,1} t^{(1-\sigma_j)/2} \|\partial|^\alpha (E^{j-1} b_j (\partial_{\tilde{w}} f_j(\tilde{w}) \partial_\xi \tilde{w} + \partial_{\tilde{w}} f_j(\tilde{w}) \overline{\partial_\xi \tilde{w}}))\|_{\mathbf{L}^2} \\ &\leq C\varepsilon^\sigma t^{(1-\sigma)/2+A} + C\varepsilon^\sigma t^{(\alpha-\sigma+1/2)/2}. \end{aligned}$$

Finally we estimate the term $|\partial|^\alpha R_4$. We have

$$\begin{aligned} \mathcal{L}f_j(\tilde{w}) &= (\partial_{\tilde{w}} f_j(\tilde{w})) \mathcal{L}\tilde{w} - (\partial_{\tilde{w}} f_j(\tilde{w})) \overline{\mathcal{L}\tilde{w}} \\ &\quad + \frac{1}{t^2} (\partial_{\tilde{w}} f_j(\tilde{w})) (\partial_\xi^2 \tilde{w}) \\ &\quad + \frac{1}{2t^2} (\partial_{\tilde{w}}^2 f_j(\tilde{w})) (\partial_\xi \tilde{w})^2 + \frac{1}{2t^2} (\partial_{\tilde{w}}^2 f_j(\tilde{w})) (\overline{\partial_\xi \tilde{w}})^2 \\ &\quad + \frac{1}{t^2} (\partial_{\tilde{w}} \partial_{\tilde{w}} f_j(\tilde{w})) |\partial_\xi \tilde{w}|^2. \end{aligned}$$

We find

$$\begin{aligned} (19) \quad &\| |\partial|^\alpha (E^{j-1} A_j (\partial_{\tilde{w}}^2 f_j(\tilde{w})) (\partial_\xi \tilde{w})^2) \|_{\mathbf{L}^2} \\ &\leq Ct \| |\partial|^\alpha (E^{j-1} A_j B^2 (\partial_{\tilde{w}}^2 f_j(\tilde{w})) (\mathcal{K}_1 w)^2) \|_{\mathbf{L}^2} \\ &\leq Ct \| |\partial|^\alpha ((\partial_{\tilde{w}}^2 f_j(\tilde{w})) ((\mathcal{K}_1 - K_{1,0}) w)^2) \|_{\mathbf{L}^2} \\ &\quad + Ct^{1+\alpha/2} \|\tilde{w}\|_{\mathbf{L}^\infty}^{\sigma_j-2} \|\mathcal{K}_1 w\|_{\mathbf{L}^\infty} \|B^\alpha (\mathcal{K}_1 - K_{1,0}) w\|_{\mathbf{L}^2} \\ &\leq C\varepsilon^\sigma t^{1+A} + C\varepsilon^\sigma t^{\alpha/2+3/4}. \end{aligned}$$

Similarly,

$$\begin{aligned} (20) \quad &\| |\partial|^\alpha (E^{j-1} A_j (\partial_{\tilde{w}}^2 f_j(\tilde{w})) (\partial_\xi \tilde{w})^2) \|_{\mathbf{L}^2} \\ &\quad + \| |\partial|^\alpha (E^{j-1} A_j (\partial_{\tilde{w}} \partial_{\tilde{w}} f_j(\tilde{w})) |\partial_\xi \tilde{w}|^2) \|_{\mathbf{L}^2} \\ &\leq C\varepsilon^\sigma t^{1+A} + C\varepsilon^\sigma t^{\alpha/2+3/4}. \end{aligned}$$

In the same manner as in the definition of $\mathcal{K}_1$ we write

$$\begin{aligned} t^{-1} B^{-2} \partial_\xi^2 \tilde{w} &= \frac{1}{\sqrt{\pi t} \langle \xi \sqrt{t} \rangle^2} \partial_\xi^2 \langle \xi \sqrt{t} \rangle \int_{\mathbf{R}} e^{-\langle \xi \sqrt{t} \rangle^2 ((\xi - \eta) \sqrt{t})^2} w(t, \eta) d\eta \\ &= \mathcal{K}_2 w, \end{aligned}$$

where we denote the operator

$$\mathcal{K}_2 \phi = \sqrt{t} \int_{\mathbf{R}} K_2(\xi \sqrt{t}, \eta \sqrt{t}) \phi(\xi - \eta) d\eta,$$

with kernel

$$K_2(x, y) = \langle x \rangle^{-2} (\partial_x + \partial_y)^2 K(x, y),$$

which satisfies the conditions of Lemmas 4.1 and 4.3. Also we have

$$K_{2,0} = \int_{\mathbf{R}} K_2(x, y) dy = 0.$$

Therefore as above we get

$$\begin{aligned} (21) \quad & \| |\partial|^\alpha (E^{j-1} A_j (\partial_{\tilde{w}} f_j(\tilde{w})) (\overline{\partial_{\tilde{\xi}}^2 \tilde{w}})) \|_{\mathbf{L}^2} \\ & \leq Ct \| |\partial|^\alpha (E^{j-1} A_j B^2 (\partial_{\tilde{w}} f_j(\tilde{w})) \overline{\mathcal{K}_2 w}) \|_{\mathbf{L}^2} \\ & \leq Ct \| |\partial|^\alpha ((\partial_{\tilde{w}} f_j(\tilde{w})) \overline{(\mathcal{K}_2 - K_{2,0}) w}) \|_{\mathbf{L}^2} \\ & \quad + Ct^{1+\alpha/2} \|\tilde{w}\|_{\mathbf{L}^\infty}^{\sigma_j-1} \|B^\alpha (\mathcal{K}_2 - K_{2,0}) w\|_{\mathbf{L}^2} \\ & \leq C\varepsilon^\sigma t^{1+A} + C\varepsilon^\sigma t^{\alpha/2+3/4}. \end{aligned}$$

Finally, we consider the term

$$\sum_{j \neq 0, 1} i^{(3-j)/2} t^{(1-\sigma_j)/2} |\partial|^\alpha (E^{j-1} A_j ((\partial_{\tilde{w}} f_j(\tilde{w})) t \mathcal{L} \tilde{w} - (\partial_{\tilde{w}} f_j(\tilde{w})) \overline{t \mathcal{L} \tilde{w}})).$$

By virtue of equation (5) we have $\mathcal{L} v^{(n-1)} = \sum_{j \neq 0} i^{(1-j)/2} t^{(1-\sigma_j)/2} E^{j-1} f_j(v^{(n-2)})$. Therefore

$$\begin{aligned} t \mathcal{L} \tilde{w} &= \mathcal{K}_3 w + \mathcal{K} t \mathcal{L} w \\ &= \mathcal{K}_3 w + \sum_{l \neq 0} i^{(1-l)/2} t^{(3-\sigma_l)/2} \mathcal{K} (E^{l-1} f_l(v^{(n-2)})) \\ &= \mathcal{K}_3 w + \sum_{l \neq 0} i^{(1-l)/2} t^{(3-\sigma_l)/2} E^{l-1} \mathcal{K}_{4,l} f_l(v^{(n-2)}), \end{aligned}$$

where

$$\mathcal{K}_3 \phi = \sqrt{t} \int_{\mathbf{R}} K_3(\xi \sqrt{t}, \eta \sqrt{t}) \phi(\xi - \eta) d\eta$$

with kernel

$$K_3(\xi \sqrt{t}, \eta \sqrt{t}) = \frac{1}{\sqrt{\pi}} (\sqrt{t} \mathcal{L} (\sqrt{t} B e^{-t B^2 \eta^2}) + t^{-1} \partial_\eta \partial_\xi (B e^{-t B^2 \eta^2}))$$

and

$$\mathcal{K}_{4,l} \phi = \sqrt{t} \int_{\mathbf{R}} K_{4,l}(\xi \sqrt{t}, \eta \sqrt{t}) \phi(\xi - \eta) d\eta$$

with kernel

$$K_{4,l}(x, y) = \frac{\langle x \rangle}{\sqrt{\pi}} e^{-\langle x \rangle^2 y^2 + i(l-1)(y^2 - 2xy)/2},$$

which satisfy the conditions of Lemma 4.1 and Lemma 4.3.

Then as above we get

$$\begin{aligned}
 (22) \quad & \| |\partial|^\alpha (E^{j-1} A_j (\partial_{\tilde{w}} f_j(\tilde{w})) t \mathcal{L} \tilde{w}) \|_{\mathbf{L}^2} \\
 & \leq \| |\partial|^\alpha (E^{j-1} A_j (\partial_{\tilde{w}} f_j(\tilde{w})) \mathcal{H}_3 w) \|_{\mathbf{L}^2} \\
 & \quad + \sum_{l \neq 0} t^{(3-\sigma_l)/2} \| |\partial|^\alpha (E^{j+l-2} A_j (\partial_{\tilde{w}} f_j(\tilde{w})) \mathcal{H}_{4,l} f_l(v^{(n-2)})) \|_{\mathbf{L}^2} \\
 & \leq C \| |\partial|^\alpha (\partial_{\tilde{w}} f_j(\tilde{w})) \mathcal{H}_3 w \|_{\mathbf{L}^2} + C t^{\alpha/2} \| B^{-1} \partial_{\tilde{w}} f_j(\tilde{w}) \mathcal{H}_3 w \|_{\mathbf{L}^2} \\
 & \quad + \sum_{l \neq 0} \| |\partial|^\alpha ((\partial_{\tilde{w}} f_j(\tilde{w})) \mathcal{H}_{4,l} f_l(v^{(n-2)})) \|_{\mathbf{L}^2} \\
 & \quad + C \sum_{l \neq 0} t^{\alpha/2} \| B^{-1} (\partial_{\tilde{w}} f_j(\tilde{w})) \mathcal{H}_{4,l} f_l(v^{(n-2)}) \|_{\mathbf{L}^2} \leq C \varepsilon^\sigma t^A + C \varepsilon^\sigma t^{\alpha/2-1/4}.
 \end{aligned}$$

Similarly,

$$(23) \quad \| |\partial|^\alpha (E^{j-1} A_j (\partial_{\tilde{w}} f_j(\tilde{w})) \overline{t \mathcal{L} \tilde{w}}) \|_{\mathbf{L}^2} \leq C \varepsilon^\sigma t^A + C \varepsilon^\sigma t^{\alpha/2-1/4}.$$

Collecting estimates (19)–(23) we obtain

$$(24) \quad \| |\partial|^\alpha \mathbf{R}_4 \|_{\mathbf{L}^2} \leq C \varepsilon^\sigma t^{(1-\sigma)/2+A} + C \varepsilon^\sigma t^{(\alpha-\sigma+1/2)/2}.$$

Now substituting estimates (16), (17), (18) and (24) into (13) we find

$$\frac{d}{dt} \| |\partial|^\alpha (v + R_0) \|_{\mathbf{L}^2} \leq C \varepsilon^\sigma t^{(1-\sigma)/2+A} + C \varepsilon^\sigma t^{(\alpha-\sigma+1/2)/2}.$$

The integration with respect to time yields

$$\| |\partial|^\alpha (v + R_0) \|_{\mathbf{L}^2} \leq C \varepsilon + C \varepsilon^\sigma t^A$$

from which in view of (15) it follows that

$$(25) \quad \| |\partial|^\alpha v^{(n)} \|_{\mathbf{L}^2} \leq C \varepsilon + C \varepsilon^\sigma t^A.$$

In the same way as above, by (12) we find the estimates

$$(26) \quad \| |\partial|^\gamma v^{(n)} \|_{\mathbf{L}^2} + \| |\partial|^\beta v^{(n)} \|_{\mathbf{L}^2} \leq C \varepsilon + C \varepsilon^\sigma$$

and

$$\| |\partial|^\alpha (v^{(n)}(t) - v^{(n-1)}(t)) \|_{\mathbf{L}^2} \leq C \varepsilon^\sigma t^A,$$

which mean that $\{v^{(n)}\}$ is a Cauchy sequence in $\mathbf{X}$. Therefore we have a unique solution v satisfying the estimates

$$(27) \quad \begin{aligned} \|\partial^\alpha v\|_{\mathbf{L}^2} &\leq C\varepsilon + C\varepsilon^\sigma t^A, \\ \|\partial^\gamma v\|_{\mathbf{L}^2} + \|\partial^\beta v\|_{\mathbf{L}^2} &\leq C\varepsilon + C\varepsilon^\sigma \end{aligned}$$

and the integral equation associated with equation

$$(28) \quad \mathcal{L}(v + R_0) = \sum_{l=1}^4 R_l,$$

where

$$\begin{aligned} R_0 &= \sum_{j \neq 0,1} i^{(3-j)/2} t^{(3-\sigma_j)/2} E^{j-1} A_j f_j(\mathcal{H}v), & R_2 &= \sum_{j \neq 0,1} t^{(1-\sigma_j)/2} E^{j-1} a_j f_j(\mathcal{H}v), \\ R_1 &= \sum_{j \neq 0,1} i^{(1-j)/2} t^{(1-\sigma_j)/2} E^{j-1} (f_j(v) - f_j(\mathcal{H}v)) + t^{(1-\sigma_1)/2} f_1(v), \\ R_3 &= \sum_{j \neq 0,1} t^{(1-\sigma_j)/2} E^{j-1} b_j \partial_{\xi} f_j(\mathcal{H}v), & R_4 &= \sum_{j \neq 0,1} i^{(3-j)/2} t^{(3-\sigma_j)/2} E^{j-1} A_j \mathcal{L} f_j(\mathcal{H}v), \end{aligned}$$

with $a_j = i^{(3-j)/2}((j-1)i\tilde{\zeta}\partial_{\xi}A_j + t\mathcal{L}A_j)$ and $b_j = i^{(3-j)/2}((j-1)i\tilde{\zeta}A_j + t^{-1}\partial_{\xi}A_j)$. By (27) it is clear that $\|v\|_{\mathbf{X}} < \infty$. From (28) and the above estimates for R_j , it follows that

$$\|v(t) - v(s) + R_0(t) - R_0(s)\|_{\dot{\mathbf{H}}^\gamma \cap \dot{\mathbf{H}}^\beta} \leq C\varepsilon^\sigma s^{(\beta+5/2-\sigma)/2}$$

for $t > s > 1$ and $0 \leq \gamma < 1/2 < \beta < p - 3/2$. Also by (15) we have

$$\|R_0(s)\|_{\dot{\mathbf{H}}^\gamma \cap \dot{\mathbf{H}}^\beta} \leq C\varepsilon^\sigma s^{(\beta+5/2-\sigma)/2}.$$

Therefore

$$\begin{aligned} &\|M(t)\mathcal{U}(-t)u(t) - M(s)\mathcal{U}(-s)u(s)\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} \\ &= \|\mathcal{F}M(t)\mathcal{U}(-t)u(t) - \mathcal{F}M(s)\mathcal{U}(-s)u(s)\|_{\dot{\mathbf{H}}^\gamma \cap \dot{\mathbf{H}}^\beta} \\ &= \|v(t) - v(s)\|_{\dot{\mathbf{H}}^\gamma \cap \dot{\mathbf{H}}^\beta} \leq C\varepsilon s^{(\beta+5/2-\sigma)/2} \rightarrow 0 \end{aligned}$$

for $t > s \rightarrow \infty$. Then there exists a unique scattering state $u_+ \in \dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}$ such that

$$\begin{aligned} &\|\mathcal{U}(-t)u(t) - M(-t)u_+\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} \\ &= \|M(t)\mathcal{U}(-t)u(t) - u_+\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} \leq C\varepsilon t^{(\beta+5/2-\sigma)/2}. \end{aligned}$$

Hence

$$\begin{aligned}
& \|\mathcal{U}(-t)u(t) - u_+\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} \\
& \leq \|\mathcal{U}(-t)u(t) - M(-t)u_+\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} + \|M(-t)u_+ - u_+\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} \\
& \leq C\varepsilon t^{(\beta+5/2-\sigma)/2} + \|M(-t)u_+ - u_+\|_{\dot{\mathbf{H}}^{0,\gamma} \cap \dot{\mathbf{H}}^{0,\beta}} \rightarrow 0
\end{aligned}$$

as $t \rightarrow \infty$. Theorem 1.1 is proved.

3. Proof of Theorem 1.2

Since the norm $\|\partial|^\alpha \phi\|_{\mathbf{L}^2}$ of the homogeneous Sobolev space $\dot{\mathbf{H}}^\alpha$ is equivalent to that of the homogeneous Besov space $\mathbf{B}_{2,2}^\alpha$ (see [1]), we have

$$\begin{aligned}
\|e^{itx^2}\phi\|_{\dot{\mathbf{H}}^\alpha} & \leq C\|e^{itx^2}\phi\|_{\mathbf{B}_{2,2}^\alpha} \\
& = C\left(\int_0^\infty x^{-1-2\alpha} \sup_{|y|\leq x} \|e^{it(z+y)^2}\phi(z+y) - e^{itz^2}\phi(z)\|_{\mathbf{L}_z^2}^2 dx\right)^{1/2}.
\end{aligned}$$

By a simple computation we find

$$\begin{aligned}
& |e^{it(z+y)^2}\phi(z+y) - e^{itz^2}\phi(z)| \\
& \leq |(e^{it((z+y)^2-z^2)} - 1)\phi(z+y)| + |\phi(z+y) - \phi(z)| \\
& \leq C|t(y(2z+y))|^\delta |\phi(z+y)| + |\phi(z+y) - \phi(z)|
\end{aligned}$$

for $0 \leq \delta \leq 1$. Substituting the last inequality into the above we get

$$\begin{aligned}
\|e^{itx^2}\phi\|_{\dot{\mathbf{H}}^\alpha} & \leq Ct^\delta \left(\int_0^1 x^{-1-2\alpha} \sup_{|y|\leq x} \int |y|^{2\delta} |2z+y|^{2\delta} |\phi(z+y)|^2 dz dx\right)^{1/2} \\
& \quad + C\left(\int_1^\infty x^{-1-2\alpha} \sup_{|y|\leq x} \int |\phi(z+y)|^2 dz dx\right)^{1/2} + C\|\phi\|_{\dot{\mathbf{H}}^\alpha} \\
& \leq Ct^\delta \left(\int_0^1 x^{-1-2\alpha+2\delta} \sup_{|y|\leq x} \int (|z+y|^{2\delta} + 1) |\phi(z+y)|^2 dz dx\right)^{1/2} \\
& \quad + C\|\phi\|_{\dot{\mathbf{H}}^\alpha} + C\|\phi\|_{\mathbf{L}^2} \\
& \leq C\|\phi\|_{\dot{\mathbf{H}}^\alpha} + Ct^\delta \|\phi\|_{\dot{\mathbf{H}}^{0,\delta}},
\end{aligned}$$

where $\alpha < \delta$. Hence

$$(29) \quad \begin{aligned} \|\mathcal{U}(-t)u\|_{\mathbf{H}^{0,\alpha}} &\leq \|e^{it\xi^2/2}\hat{u}\|_{\mathbf{H}^\alpha} \\ &\leq C(\|\hat{u}\|_{\mathbf{H}^\alpha} + t^\delta\|\hat{u}\|_{\mathbf{H}^{0,\delta}}) \leq C(\|u\|_{\mathbf{H}^{0,\alpha}} + t^\delta\|u\|_{\mathbf{H}^\delta}). \end{aligned}$$

As in the proof of Theorem 1.1, we easily see that if the initial data $u_0 = u(1, x) \in \mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha}$ with $1/2 < \alpha < \delta < 1$, then there exists $\varepsilon_0 > 0$ such that (1) has a unique global solution

$$\mathcal{U}(-t)u \in \mathbf{C}([1, \infty); \mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha}).$$

Indeed the sharp time decay estimate $\|u(t)\|_{\mathbf{L}^\infty} \leq Ct^{-1/2}$ obtained in Theorem 1.1 and the integral equation

$$(30) \quad u(t) = \mathcal{U}(t)u(1) - i \int_1^t \mathcal{U}(t-s)f(u)ds$$

associated with (1) yield

$$u \in \mathbf{C}([1, \infty); \mathbf{H}^\delta).$$

Let us consider equation (30) for $t \leq 1$. By (29), we see that existence time can be prolonged to $t = 0$, and

$$\mathcal{U}(-t)u \in \mathbf{C}([0, \infty); \mathbf{H}^\delta \cap \mathbf{H}^{0,\alpha}).$$

This completes the proof of Theorem 1.2.

4. Appendix

We first state three Lemmas which were shown in [9]. Define the operator

$$\mathcal{K}\phi = \sqrt{t} \int_{\mathbf{R}} K(\xi\sqrt{t}, \eta\sqrt{t})\phi(\xi - \eta)d\eta$$

for $t \geq 1$, where $K(x, y)$ is a given kernel. Denote $B = \langle \xi\sqrt{t} \rangle$.

Lemma 4.1. *Let the kernel K satisfy the estimates*

$$|K(x, y)| \leq \langle x \rangle K_1(\langle x \rangle y)$$

for $x, y \in \mathbf{R}$, and

$$|K(x, y)| \leq K_2(y),$$

for $|y| \geq |x|/2$, where $K_1, K_2 \in \mathbf{L}^1$. Then for any $\phi \in \mathbf{L}^2$ the estimate is true

$$\|\mathcal{K}\phi\|_{\mathbf{L}^2} \leq C\|\phi\|_{\mathbf{L}^2}.$$

Moreover if $K_1 \in \mathbf{L}^{1,\beta}$, with $\beta \geq 0$, then

$$\|B^{-\beta} \mathcal{K}\phi\|_{\mathbf{L}^\infty} \leq C \|B^{-\beta} \phi\|_{\mathbf{L}^\infty}$$

provided that the right-hand side is finite.

We denote the fractional derivative for $\alpha \in (0, 1)$ (see [16])

$$|\partial|^\alpha \phi = \mathcal{F}^{-1}(|\xi|^\alpha \hat{\phi}(\xi)) = C \int_{\mathbf{R}} (\tilde{\phi} - \phi) \frac{dy}{|y|^{1+\alpha}},$$

where we denote

$$\tilde{\phi} = \phi(\cdot - y), \quad C = -\frac{\alpha}{2\Gamma(1-\alpha) \cos(\pi\alpha/2)},$$

Γ is the Euler gamma function. We can write the Leibnitz rule $|\partial|^\alpha(\phi\psi) = \phi|\partial|^\alpha\psi + [|\partial|^\alpha, \phi]\psi$ with the commutator

$$[|\partial|^\alpha, \phi]\psi = C \int_{\mathbf{R}} (\tilde{\phi} - \phi) \tilde{\psi} \frac{dy}{|y|^{1+\alpha}}.$$

In the next lemma we find some estimates for the commutator $[|\partial|^\alpha, \phi]\psi$ for $\alpha \in (0, 1)$.

Lemma 4.2. *The inequalities are valid*

$$\| [|\partial|^\alpha, \phi]\psi \|_{\mathbf{L}^2} \leq C t^{\alpha/2} (\|B^v \phi\|_{\mathbf{L}^\infty} + t^{-1/2} \|B^v \partial \phi\|_{\mathbf{L}^\infty}) \|B^{-v} \psi\|_{\mathbf{L}^2},$$

where $0 \leq v < \alpha < 1$, and

$$\| [|\partial|^\alpha, \phi]\psi \|_{\mathbf{L}^2} \leq C t^{\alpha/2} (\|\phi\|_{\mathbf{L}^\infty} + t^{-1/2} \|B^{-1} \partial \phi\|_{\mathbf{L}^\infty}) \|B^\alpha \psi\|_{\mathbf{L}^2}$$

provided that the right-hand sides are finite.

Denote

$$K_0(\xi\sqrt{t}) = \sqrt{t} \int_{\mathbf{R}} K(\xi\sqrt{t}, \eta\sqrt{t}) d\eta = \int_{\mathbf{R}} K(\xi\sqrt{t}, \zeta) d\zeta.$$

In the next lemma we estimate the difference

$$\mathcal{K}\phi - K_0\phi = \sqrt{t} \int_{\mathbf{R}} K(\xi\sqrt{t}, \eta\sqrt{t}) (\phi(\xi - \eta) - \phi(\xi)) dy.$$

A typical choice of a kernel is $K(x, y) = \pi^{-1/2} \langle x \rangle e^{-\langle x \rangle^2 y^2}$, when applying Lemma 4.3.

Lemma 4.3. *Let the kernel $K \in \mathbf{C}^1(\mathbf{R}^2)$ satisfy the estimates*

$$|K(x, y)| \leq C \langle x \rangle \langle \langle x \rangle y \rangle^{-3}$$

and

$$|\partial_x K(x, y)| \leq C \langle y \rangle^{-3}$$

for all $x, y \in \mathbf{R}$. Then the estimate is true

$$\| |\partial|^\alpha (\mathcal{K}\phi - K_0\phi) \|_{\mathbf{L}^2} + t^{\alpha/2} \| B^\alpha (\mathcal{K}\phi - K_0\phi) \|_{\mathbf{L}^2} \leq C \| |\partial|^\alpha \phi \|_{\mathbf{L}^2}$$

for all $t \geq 1$.

We need the following lemma to consider the fractional order nonlinearities.

Lemma 4.4. *Let $0 < \alpha < 1$. Let f satisfy*

$$|f(\phi) - f(\psi)| \leq C(|\phi| + |\psi|)^{q-1} |\phi - \psi|$$

where $q > 1$. Then the inequalities

$$\| |\partial|^\alpha f(\phi) \|_{\mathbf{L}^2} \leq C \| \phi \|_{\mathbf{L}^\infty}^{q-1} \| |\partial|^\alpha \phi \|_{\mathbf{L}^2}$$

and

$$\| |\partial|^\alpha (f(\phi)\psi) \|_{\mathbf{L}^2} \leq C \| \phi \|_{\mathbf{L}^\infty}^{q-1} (\| \phi \|_{\mathbf{L}^\infty} \| |\partial|^\alpha \psi \|_{\mathbf{L}^2} + \| \psi \|_{\mathbf{L}^\infty} \| |\partial|^\alpha \phi \|_{\mathbf{L}^2})$$

are valid, provided that the right-hand sides are finite.

Proof. We have

$$\begin{aligned} \| f(\phi) \|_{\dot{\mathbf{H}}^\alpha} &\leq C \| f(\phi) \|_{\dot{\mathbf{B}}_{2,2}^\alpha} \\ &= C \left(\int_0^\infty x^{-1-2\alpha} \sup_{|y| \leq x} \| f(\phi(\cdot + y)) - f(\phi(\cdot)) \|_{\mathbf{L}^2}^2 dx \right)^{1/2} \\ &\leq C \| \phi \|_{\mathbf{L}^\infty}^{q-1} \left(\int_0^\infty x^{-1-2\alpha} \sup_{|y| \leq x} \| \phi(\cdot + y) - \phi(\cdot) \|_{\mathbf{L}^2}^2 dx \right)^{1/2} \\ &\leq C \| \phi \|_{\mathbf{L}^\infty}^{q-1} \| \phi \|_{\dot{\mathbf{B}}_{2,2}^\alpha} = C \| \phi \|_{\mathbf{L}^\infty}^{q-1} \| \phi \|_{\dot{\mathbf{H}}^\alpha}. \end{aligned}$$

In the same manner we find

$$\begin{aligned} \| f(\phi)\psi \|_{\dot{\mathbf{H}}^\alpha} &\leq C \| f(\phi)\psi \|_{\dot{\mathbf{B}}_{2,2}^\alpha} \\ &= C \left(\int_0^\infty x^{-1-2\alpha} \sup_{|y| \leq x} \| f(\phi(\cdot + y))\psi(\cdot + y) - f(\phi(\cdot))\psi(\cdot) \|_{\mathbf{L}^2}^2 dx \right)^{1/2} \\ &\leq C \| \phi \|_{\mathbf{L}^\infty}^{q-1} \| \psi \|_{\mathbf{L}^\infty} \left(\int_0^\infty x^{-1-2\alpha} \sup_{|y| \leq x} \| \phi(\cdot + y) - \phi(\cdot) \|_{\mathbf{L}^2}^2 dx \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& + C \|\phi\|_{\mathbf{L}^\infty}^q \left(\int_0^\infty x^{-1-2\alpha} \sup_{|y| \leq x} \|\psi(\cdot + y) - \psi(\cdot)\|_{\mathbf{L}^2}^2 dx \right)^{1/2} \\
& \leq C \|\phi\|_{\mathbf{L}^\infty}^{q-1} (\|\psi\|_{\mathbf{L}^\infty} \|\phi\|_{\dot{\mathbf{B}}_{2,2}^\alpha} + C \|\phi\|_{\mathbf{L}^\infty} \|\psi\|_{\dot{\mathbf{B}}_{2,2}^\alpha}) \\
& = C \|\phi\|_{\mathbf{L}^\infty}^{q-1} (\|\psi\|_{\mathbf{L}^\infty} \|\phi\|_{\dot{\mathbf{H}}^\alpha} + \|\phi\|_{\mathbf{L}^\infty} \|\psi\|_{\dot{\mathbf{H}}^\alpha}) \\
& \leq C \|\phi\|_{\mathbf{L}^\infty}^{q-1} (\|\phi\|_{\mathbf{L}^\infty} \|\partial|^\alpha \psi\|_{\mathbf{L}^2} + \|\psi\|_{\mathbf{L}^\infty} \|\partial|^\alpha \phi\|_{\mathbf{L}^2}).
\end{aligned}$$

Lemma 4.4 is proved. $\square$

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