Solvability of Nonlinear Schrödinger Equations with Some Critical Singular Potential via Generalized Hardy-Rellich Inequalities

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Abstract. Nonlinear Schrödinger equations with inverse-square potentials (NLS) are considered. Since the potential \( |x|^{-2} \) is quite singular, the scaling argument does not work well. In view of the selfadjointness of \( P_a := -\Delta + a|x|^{-2} \), \( a = a(N) := -(N - 2)^2/4 \) seems to be the threshold of the unique solvability. In fact, if \( a > a(N) \), then the unique solvability for (NLS) is proved by the energy methods established by Okazawa-Suzuki-Yokota [12]. On the other hand, if \( a < a(N) \), then \( P_a \) is not nonnegative in \( L^2(\mathbb{R}^N) \) and has a lot of selfadjoint extensions. Here \( P_a(N) \) is nonnegative and selfadjoint in \( L^2(\mathbb{R}^N) \) in the sense of form-sum. But the energy space \( D((1 + P_a(N))^{1/2}) \) does not coincide with \( H^1(\mathbb{R}^N) \). Thus we identify the energy space by applying generalized Hardy-Rellich inequalities. By virtue of the identification we can apply the energy methods and conclude the global solvability for (NLS) with \( a = a(N) \), the critical coefficient. Moreover, the uniqueness can be shown by using the Strichartz estimates for \( e^{-itP_a(N)} \) which is also proved.

Key Words and Phrases. Nonlinear Schrödinger equation, Inverse-square potential, Hardy-Rellich inequality, Spherical harmonics decomposition, Fractional Sobolev spaces, Energy methods, Strichartz estimates.

2010 Mathematics Subject Classification Numbers. Primary 35Q55, 35Q40; Secondary 81Q15.

1. Introduction and main results

In this paper we are interested in the following Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials of the critical coefficient:

\[
\begin{cases}
  i \frac{\partial u}{\partial t} = \left(-\Delta - \frac{(N - 2)^2}{4|x|^2}\right)u + \lambda |u|^{p-1}u & \text{in } \mathbb{R} \times \mathbb{R}^N, \\
  u(0, x) = u_0(x) & \text{in } \mathbb{R}^N,
\end{cases}
\]

(1.1)

\( (\text{CP}) \)

where \( i = \sqrt{-1}, \ N \geq 3, \ \lambda \in \mathbb{R} \) and \( p \geq 1 \).

First we consider the linear operator in \( L^2(\mathbb{R}^N) (N \geq 2) \):

\[
P_a = -\Delta + \frac{a}{|x|^2}, \quad a \geq -\frac{(N - 2)^2}{4}.
\]
It is well-known that $P_a$ is essentially selfadjoint on $C_0^{\infty}(\mathbb{R}^N\setminus\{0\})$ if $a \geq -N(N-4)/4$ (see e.g. Reed-Simon [15, Theorem X.11]). Moreover, $P_a$ is selfadjoint in $L^2(\mathbb{R}^N)$ if $a > -N(N-4)/4$ (see e.g. Okazawa [9, Theorem 6.8]) in the sense of operator-sum. In particular, $|x|^{-3}$ is $(-\Delta)$-bounded in $L^2(\mathbb{R}^N)$ if $N \geq 5$. This is a consequence of the usual Rellich inequality

\begin{equation}
(1.2) \quad \| |x|^{-2}u \|_{L^2(\mathbb{R}^N)} \leq \frac{4}{N(N-4)} \| Au \|_{L^2(\mathbb{R}^N)} \quad \forall f \in \dot{H}^2(\mathbb{R}^N), \ N \geq 5.
\end{equation}

On the other hand, $P_a$ is nonnegative and selfadjoint in $L^2(\mathbb{R}^N)$ if $a \geq -(N-2)^2/4$ in the sense of form-sum (Friedrichs extension); note that it should be written $\Delta + a|x|^{-2}$ replaced with $-\Delta + a|x|^{-2}$ (see e.g. Kato [6, Section VI.4.3]). The nonnegativity and selfadjointness of $P_a$ ($a \geq -(N-2)^2/4$) is followed by the usual Hardy inequality

\begin{equation}
(1.3) \quad \| |x|^{-1}u \|_{L^2(\mathbb{R}^N)} \leq \frac{2}{N-2} \| Vu \|_{L^2(\mathbb{R}^N)} \quad \forall f \in \dot{H}^1(\mathbb{R}^N), \ N \geq 3,
\end{equation}

where the constant $2/(N-2)$ is optimal (see e.g. Okazawa [10, Section 5.3]). The operator $P_a$ is also considered in $H^{-1}(\mathbb{R}^N)$, a wider space than $L^2(\mathbb{R}^N)$. Okazawa-Suzuki-Yokota [11] showed that if $N \geq 3$ and $a > -(N-2)^2/4$, then $P_a$ is nonnegative and selfadjoint in $H^{-1}(\mathbb{R}^N)$. This is the consequence of the norm-equivalence

\begin{equation}
(1.4) \quad c_1(a)\|(-\Delta)^{1/2}f\|_{L^2(\mathbb{R}^N)} \leq \| P_a^{1/2}f \|_{L^2(\mathbb{R}^N)} \leq c_2(a)\|(-\Delta)^{1/2}f\|_{L^2(\mathbb{R}^N)}
\end{equation}

\forall f \in \dot{H}^1(\mathbb{R}^N),

where

\[ c_1(a) := 1 - \frac{4a_-}{(N-2)^2}, \quad c_2(a) := 1 + \frac{4a_+}{(N-2)^2} \]

($a_+ = \max\{0, a\}, \ a_- = \max\{0, -a\}$).

In view of the selfadjointness of $P_a$ in $H^{-1}(\mathbb{R}^N)$, we consider the following Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials:

\begin{equation}
(\text{NLS})_a \quad \begin{cases}
  i \frac{\partial u}{\partial t} = \left( -\Delta + \frac{a}{|x|^2} \right) u + f(u) \quad \text{in} \ R \times \mathbb{R}^N, \\
  u(0, x) = u_0(x) \quad \text{in} \ \mathbb{R}^N,
\end{cases}
\end{equation}

where $a \geq -(N-2)^2/4$. Needless to say, (CP) is the special case $a = -(N-2)^2/4$ of (NLS)$_a$. If $a > -(N-2)^2/4$ and $f(u) := \lambda |u|^{p-1}u$, then Okazawa-Suzuki-Yokota [12] proved the wellposedness. More precisely, there
exists a unique global weak solution $u \in C(R; H^1(R^N)) \cap C^1(R; H^{-1}(R^N))$ to (NLS)$_0$ under the condition

$$1 \leq p < \left\{ \begin{array}{ll}
(N+2)/(N-2), & \lambda > 0, \\
1+4/N, & \lambda < 0.
\end{array} \right.$$  

But they did not deal with the critical case

$$a = a(N) := -\frac{(N-2)^2}{4}.$$  

The reason of the exception to the case (1.5) is that the norm-equivalence (1.4) seems to be broken down; $c_1(a(N)) = 0$. In other words, $D(P^{1/2}_{a(N)}) \neq H^1(R^N)$ (see also Table 1). Here we see from (1.4) that $H^1(R^N) \subset D(P^{1/2}_{a(N)})$ ($c_2(a(N)) = 1$). If we find a suitable space $X$ such that $D((1 + P_{a(N)})^{1/2}) \subset X \subset L^2(R^N)$, we can solve (CP) by the energy methods. Now let $\Omega$ be a bounded open set with smooth boundary and $0 \in \Omega$. Then Vazquez-Zuazua [19, Corollary 2.3] proved the Hardy-Poincaré inequality:

$$\|u\|^2_{H^s(\Omega)} \leq C(\Omega, s) \int_{\Omega} \left( |\nabla u|^2 - \frac{(N-2)^2}{4} \left| \frac{u}{|x|} \right|^2 \right) dx$$

\forall u \in H^1_0(\Omega), \forall s \in [0, 1).$$

Thus we see that $D((1 + P_{a(N)})^{1/2}) \subset H^s(\Omega)$ for all $s \in [0, 1)$. Now we turn our eyes to the case $\Omega = R^N$, an unbounded domain. The Hardy-Poincaré inequality (1.6) does not hold in $R^N$. So we attempt to construct other type inequalities in $R^N$ and analyze the operator $P_{a(N)}$. In fact, we show the following two inequalities:

$$\left\| |x|^{-s}u \right\|_{L^2(R^N)} \leq C_s \left\| P^{s/2}_{a(N)} u \right\|_{L^2(R^N)},$$

$$\left\| (-\Delta)^{s/2} u \right\|_{L^2(R^N)} \leq C'_s \left\| P^{s/2}_{a(N)} u \right\|_{L^2(R^N)},$$

where $N \geq 3$, $0 \leq s < 1$ and $C_s$, $C'_s$ are explicitly denoted by

$$C_s = \frac{\Gamma((1-s)/2)}{2^s \Gamma((1+s)/2)}, \quad C'_s = \frac{\Gamma((N+2s)/4)\Gamma((1-s)/2)}{\Gamma((N-2s)/4)\Gamma((1+s)/2)}$$


| $a > a(N)$ | $N \geq 3$ | $H^1(R^N)$ | $N = 2$ | $H^1(R^2) \cap D(|x|^{-1})$ |
|---|---|---|---|---|
| $a = a(N)$ | wider than $H^1(R^N)$ | $H^1(R^2)$ |

Table 1: Representation of $D((1 + P_{a})^{1/2})$
Comparing (1.2) and (1.3) with (1.7), we see that the inequality (1.7) generalizes the Hardy and Rellich inequalities. So (1.7) and (1.8) may be called generalized Hardy-Rellich inequalities. Applying these inequalities and the fractional Sobolev embeddings we can show the existence of global weak solutions to (CP) via the energy methods (see Theorem 4.1 for details).

This paper is divided into four sections. In Section 2, we prepare for the tools of analyzing $P_{a}^{1/2}$. In Section 3, we prove the generalized Hardy-Rellich inequalities (1.7) and (1.8). We apply the generalized Hardy-Rellich inequalities to solving nonlinear Schrödinger equations with the critical inverse-square potential as like (CP) in Section 4.

2. Preliminaries

2.1. Gamma function

First we collect the properties of Gamma functions. Here Gamma function is defined

$$
\Gamma(z) := \int_{0}^{\infty} t^{z-1} e^{-t} dt, \quad \Re z > 0.
$$

It follows from the definition that $\Gamma(z) = \Gamma(z)$ and $\Gamma(z + 1) = z\Gamma(z)$ for all $z \in \mathbb{C}$ with $\Re z > 0$. Gamma function is also written by the Euler limit formula

$$
(2.1) \quad \Gamma(z) = \lim_{k \to \infty} \frac{k! k^{z}}{z(z+1) \ldots (z+k)} = \frac{1}{z} \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k} \right)^{z} \left( 1 + \frac{z}{k} \right)^{-1}, \quad \Re z > 0.
$$

The Euler limit formula (2.1) implies that

$$
(2.2) \quad |\Gamma(x + iy)| = \Gamma(x) \prod_{k=0}^{\infty} \left( 1 + \frac{y^{2}}{(x+k)^{2}} \right)^{-1/2}, \quad \forall x > 0, \forall y \in \mathbb{R}.
$$

Now we consider the psi function (or digamma function) as

$$
(2.3) \quad \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = \frac{d}{dx} \log \Gamma(x), \quad x > 0.
$$

Then (2.1) implies that

$$
(2.4) \quad \psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n}}, \quad \forall x > 0, \forall n \in \mathbb{N}
$$

and hence $(-1)^{n+1} \psi^{(n)}(x) > 0$ for all $x > 0$.

The next two lemmas are applied to Proposition 3.3 and Proposition 3.4.
Lemma 2.1. Assume \( a < b \). Then \( \Gamma(a + x)\Gamma(b + x)^{-1} \) is decreasing for \( x > -a \).

Proof. Denote \( \gamma(x) := \log[\Gamma(a + x)\Gamma(b + x)^{-1}] \). It suffices to show that \( \gamma'(x) < 0 \) for \( x > -a \). Now we have \( \gamma'(x) = \psi(x + a) - \psi(x + b) \). It follows from the mean-value theorem that \( \psi(x + a) - \psi(x + b) = (a - b)\psi'(\xi) < 0 \) \( (x + a < \xi < x + b) \). Thus we conclude \( \gamma'(x) < 0 \) for \( x > -a \). \hfill \Box

Lemma 2.2. Let \( \alpha, \beta \in C([0, \infty); \mathbb{R}) \cap C^1((0, \infty); \mathbb{R}) \) and \( c > 0 \) such that

\[
\alpha(x) - c > \beta(x) > 0 \quad \forall x \geq 0,
\]
\[
0 \leq \alpha'(x) \leq \beta'(x) \quad \forall x > 0.
\]

Define the function \( \gamma \) as

\[
\gamma(x) := \frac{\Gamma(\alpha(x))\Gamma(\beta(x))}{\Gamma(\alpha(x) - c)\Gamma(\beta(x) + c)}.
\]

Then \( \gamma \) is decreasing in \( x \geq 0 \). Hence \( \gamma(x) \leq \gamma(0) \) for \( x \geq 0 \).

Proof. Since \( \gamma(x) > 0 \) for \( x \geq 0 \), it suffices to prove that \( \log \gamma(x) \) is decreasing in \( x \geq 0 \). By definition of \( \psi = (\log \Gamma)' \) we have

\[
(\log \gamma(x))' = \psi(\alpha(x))\alpha'(x) + \psi(\beta(x))\beta'(x) - \psi(\alpha(x) - c)\alpha'(x) - \psi(\beta(x) + c)\beta'(x).
\]

We see from (2.4) that \( \psi''(z) < 0 \) for \( z > 0 \). Thus \(-\psi(z)\) is convex in \( z > 0 \). Hence we calculate

\[
(\log \gamma(x))' \leq \psi(\alpha(x))\alpha'(x) + \psi(\beta(x))\beta'(x) - [(1 - t)\psi(\alpha(x)) + t\psi(\beta(x))] \alpha'(x)
\]
\[
- [t\psi(\alpha(x)) + (1 - t)\psi(\beta(x))] \beta'(x)
\]
\[
= t[\psi(\alpha(x)) - \psi(\beta(x))] [\alpha'(x) - \beta'(x)].
\]

Here \( t := c/(\alpha(x) - \beta(x)) \in (0, 1) \). Since \( \psi'(z) > 0 \) for \( z > 0 \), the function \( \psi \) is increasing. Thus \( \psi(\alpha(x)) - \psi(\beta(x)) \geq 0 \). Moreover, \( \alpha'(x) - \beta'(x) \leq 0 \) by assumption. Therefore we conclude that \( (\log \gamma(x))' \leq 0 \) for \( x > 0 \). \hfill \Box

2.2. Spherical harmonics decomposition

Next we consider the spherical harmonics decomposition; see [17, Chapter IV] for details. A function \( Q : \mathbb{R}^N \rightarrow \mathbb{C} \) is said to be \( \ell \)-th solid harmonic if \( Q \) is harmonic (i.e., \( \Delta Q = 0 \)) and a homogeneous polynomial of degree \( \ell \) (i.e., \( Q(x) = |x|^{\ell}Q(|x|) \)). Let \( \mathcal{H}_\ell \) be a family of \( \ell \)-th solid harmonic functions. Then \( \mathcal{H}_\ell \) is a finite dimensional vector space. Moreover, \( \mathcal{H}_\ell \) has an orthogonal
normal system \( \{ Y_{\ell,k} \} \):

\[
\mathcal{A}_\ell = \text{Span}\{ Y_{\ell,k} \}, \quad \int_{S^{N-1}} Y_{\ell,k_1}(x') \overline{Y_{\ell,k_2}(x')} dS(x') = \begin{cases} 1 & k_1 = k_2, \\ 0 & k_1 \neq k_2. \end{cases}
\]

Let \( Q \in \mathcal{A}_\ell \) and \( \tilde{f} \in L^2(0, \infty) \). Then

\[
f(x) := |x|^{-\ell-(N-1)/2} \tilde{f}(|x|) Q(x) \in L^2(\mathbb{R}^N).
\]

In fact, we can calculate as follows:

\[
(2.6) \quad \int_{\mathbb{R}^N} |f(x)|^2 dx = \left( \int_{|x|=1} |Q(x')|^2 dS \right) \left( \int_0^\infty |\tilde{f}(r)|^2 dr \right) < \infty.
\]

Thus we define the subspace of \( L^2(\mathbb{R}^N) \) as

\[
L^2_{=\ell}(\mathbb{R}^N) := \left\{ \sum_k |x|^{-\ell-(N-1)/2} \tilde{f}_k(|x|) Y_{\ell,k}(x) ; \tilde{f}_k \in L^2(0, \infty), Y_{\ell,k} \in \mathcal{A}_\ell \right\}.
\]

In particular, \( L^2_{=0}(\mathbb{R}^N) = L^2_{\text{rad}}(\mathbb{R}^N) \), the family of radially symmetric functions. Also we define

\[
L^2_{\geq \ell}(\mathbb{R}^N) := \bigoplus_{\ell \geq d} L^2_{=\ell}(\mathbb{R}^N).
\]

Now we have the following (see e.g. [17, Lemma IV.2.18]).

**Proposition 2.3.** Let \( \ell, \ell_1, \ell_2 \) be nonnegative integers. Then one has

(i) \( L^2_{=\ell}(\mathbb{R}^N) \) is a closed subspace of \( L^2(\mathbb{R}^N) \);

(ii) \( L^2_{=\ell_1}(\mathbb{R}^N) \perp L^2_{=\ell_2}(\mathbb{R}^N) \) if \( \ell_1 \neq \ell_2 \), i.e.,

\[
\int_{\mathbb{R}^N} f_1(x) \overline{f_2(x)} dx = 0 \quad \forall f_1 \in L^2_{=\ell_1}(\mathbb{R}^N), \forall f_2 \in L^2_{=\ell_2}(\mathbb{R}^N);
\]

(iii) \( L^2(\mathbb{R}^N) = \bigoplus_{\ell=0}^{\infty} L^2_{=\ell}(\mathbb{R}^N) \), i.e., for every \( f \in L^2(\mathbb{R}^N) \) there uniquely exists \( \{ f_\ell \} \subset L^2(\mathbb{R}^N) \) such that \( f_\ell \in L^2_{=\ell}(\mathbb{R}^N) \) for \( \ell \in \mathbb{N} \cup \{0\} \) and

\[
f = \sum_{\ell=0}^{\infty} f_\ell \quad (\text{spherical harmonics decomposition}).
\]

Next we consider the representation of \(-\Delta\) and \( P_a \) in \( L^2_{=\ell}(\mathbb{R}^N) \). First we remark that

\[
(-\Delta)f(|x|) = -\frac{\partial^2 f}{\partial r^2}(|x|) - \frac{N-1}{|x|} \frac{\partial f}{\partial r}(|x|), \quad r = |x|.
\]

Now let \( \tilde{f}_0 \in L^2(0, \infty) \) and \( Y \in \mathcal{A}_\ell \). Put \( \tilde{f}(r) := \tilde{f}_0(r) r^{-(N-1)/2} \). Then (2.6) implies that \( f(x) := \tilde{f}(|x|) Y(x/|x|) \) belongs to \( L^2(\mathbb{R}^N) \). We compute
\(-\Delta [\tilde{f}(|x|) Y(x/|x|)]\) as follows:

\begin{equation}
-\Delta [\tilde{f}(|x|) Y(x/|x|)] = - \left( \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} \right) [\tilde{f}(|x|)|x|^{-\ell}] \times Y(x) \\
- \frac{2}{r} \frac{\partial}{\partial r} [\tilde{f}(|x|)|x|^{-\ell}] \times (x \cdot \nabla Y(x)),
\end{equation}

where \(r = |x|\). Note that

\[
\hat{\partial}_r [\tilde{f}(|x|)|x|^{-\ell}] = r^{-\ell} \left[ \frac{\partial^2}{\partial r^2} (\hat{f})(r) - \frac{\ell}{r} \hat{f}(r) \right],
\]

\[
\hat{\partial}_r^2 [\tilde{f}(|x|)|x|^{-\ell}] = r^{-\ell} \left[ \frac{\partial^2}{\partial r^2} (\hat{f})(r) - \frac{2\ell}{r} \frac{\partial}{\partial r} (\hat{f})(r) + \frac{\ell^2 + \ell}{r^2} \hat{f}(r) \right].
\]

Since \(Y \in \mathcal{D}_\ell\), we have \(x \cdot \nabla Y(x) = \ell Y(x)\). Combining these into (2.7) we obtain

\begin{equation}
-\Delta [f(|x|) Y(x/|x|)] = \frac{Y(x)}{|x|^{\ell}} \left[ - \frac{\partial^2}{\partial r^2} (\hat{f})(r) - \frac{N-1}{r} \frac{\partial}{\partial r} (\hat{f})(r) + \frac{\ell(N-2+\ell)}{r^2} \hat{f}(r) \right]
\end{equation}

\[
= (A_{\mu(\ell)} \tilde{f})(x) Y(x/|x|).
\]

Here \(\lambda = (N-2)/2\) and

\begin{equation}
A_\nu \tilde{f} := -\hat{\partial}_r^2 \tilde{f} - \frac{N-1}{r} \hat{\partial}_r \tilde{f} + \frac{\nu^2 - \lambda^2}{r^2} \tilde{f},
\end{equation}

\begin{equation}
\mu(\ell) := \mu_\ell = \lambda + \ell.
\end{equation}

Therefore we conclude that

\begin{equation}
-\Delta f = A_{\mu(\ell)} f \quad \forall f \in L^2_{\nu,\ell}(\mathbb{R}^N).
\end{equation}

Moreover, since \(P_a = -\Delta + a|x|^{-2}\), we have

\[
P_a f = -\hat{\partial}_r^2 f - \frac{N-1}{r} \hat{\partial}_r f + \frac{(\mu(\ell)^2 + a) - \lambda^2}{r^2} f \quad \forall f \in L^2_{\nu,\ell}(\mathbb{R}^N).
\]

Now we set

\begin{equation}
v(\ell) := v_\ell = [(\lambda + \ell)^2 + a]^{1/2}.
\end{equation}

Then we obtain

\begin{equation}
P_a f = (-\Delta + a|x|^{-2}) f = A_{v(\ell)} f \quad \forall f \in L^2_{\nu,\ell}(\mathbb{R}^N).
\end{equation}
The equalities (2.11) and (2.13) are the fundamental ideas to prove the generalized Hardy-Rellich inequalities.

### 2.3. The Mellin transform and fractional power of $A_v$

First of the section we introduce the Mellin transform.

**Definition 2.1.** Let $f$ be a complex-valued measurable function such that $x^{\gamma-1}f(x) \in L^1(0, \infty; C)$ for some $\gamma \in \mathbb{R}$. Then the Mellin transform of $f$ is defined as

$$\mathcal{M}[f](z) := \int_0^\infty r^{z-1}f(r) \, dr.$$ 

Note that $\mathcal{M}[r^af(r)](z) = \mathcal{M}[f(r)](z+a)$ for $a \in \mathbb{R}$. In general, let $f \in L^2(\mathbb{R}^N)$ ($N \geq 2$). By virtue of Lemma 2.5 (Step 1) the Mellin transform of $f$ is defined as

$$\mathcal{M}[f(x)](z) := \int_0^\infty r^{z-1}f\left(\frac{rX}{|x|}\right) \, dr.$$ 

Now we compute the Mellin transform of $A_vf$ ($v \geq 0$) for $f \in L^2(\mathbb{R}^N)$. First we note that

(2.14) $$\mathcal{M}[\partial_v f](z) = (1-z)\mathcal{M}[f](z-1).$$

In fact, let $z \in \mathbb{C}$. Then

$$\mathcal{M}[\partial_v f](z) = \int_0^\infty f_v(r)r^{z-1} \, dr = \int_0^{\infty} f(r)r^{z-1}|_{r=0}^{\infty} - \int_0^\infty f(r)(z-1)r^{z-2} \, dr = (1-z)\int_0^\infty f(r)r^{z-2} \, dr = (1-z)\mathcal{M}[f](z-1).$$

This is nothing but (2.14). Now we compute the Mellin transform of $A_vf$.

**Lemma 2.4.** Let $v \geq 0$ and $f \in L^2(\mathbb{R}^N)$. Then

(2.15) $$\mathcal{M}[A_v f](z) = -(z+\lambda-2+v)(z-\lambda-2-v)\mathcal{M}[f](z-2).$$

**Proof.** Applying (2.14) to $A_vf$ we obtain

$$\mathcal{M}[A_v f](z) = -(1-z)(2-z)\mathcal{M}[f](z) - (N-1)(2-z)\mathcal{M}[f](z) + (v^2-\lambda^2)\mathcal{M}[|x|^{-2}f](z) = -(z+\lambda-2+v)(z-\lambda-2-v)\mathcal{M}[f](z-2).$$

This is nothing but (2.15).
The linear operator \( A_v \) has another representation. To carry out this we consider the Hankel transform of order \( v \) of \( f \):

\[
(\mathcal{H}_v f)(\xi) := \int_0^\infty (r|\xi|)^{-(N-2)/2} J_v(r|\xi|) f\left(\frac{r^2}{|\xi|^2}\right) r^{N-1} dr,
\]

where \( J_v \) is Bessel functions of the first kind of order \( v \). Applying the Mellin transform to \( \mathcal{H}_v f \) we have

\[
\mathcal{M}[\mathcal{H}_v f](z) = 2^{z-\lambda-1} \frac{\Gamma((z-\lambda+v)/2)}{\Gamma(1-(z-\lambda-v)/2)} \mathcal{M}[f](N-z)
\]

(see [14, Section 2.3]). This ensures that \( \mathcal{H}_v^2 = 1 \) and hence \( \mathcal{H}_v \) is a unitary operator on \( L^2(\mathbb{R}^N) \). Moreover, for \( v \geq 0 \)

\[
A_v = \mathcal{H}_v |\xi|^2 \mathcal{H}_v.
\]

To verify (2.18) first we compute the Mellin transform of \( \mathcal{H}_v |\xi|^{\sigma} \mathcal{H}_v f \) \((\sigma \geq 0)\) as follows:

\[
\mathcal{M}[\mathcal{H}_v |\xi|^{\sigma} \mathcal{H}_v f](z) = 2^{z-\lambda-1} \frac{\Gamma((z-\lambda+v)/2)}{\Gamma(1-(z-\lambda-v)/2)} \mathcal{M}[f](N-z+\sigma)
\]

\[
= 2^\sigma \frac{\Gamma((z-\lambda+v)/2)}{\Gamma(1-(z-\lambda-v)/2)} \mathcal{M}[f](z-\sigma).
\]

Putting \( \sigma = 2 \) in (2.19) we find that \( \mathcal{M}[\mathcal{H}_v |\xi|^2 \mathcal{H}_v f] \) coincides with (2.15). Therefore (2.19) and \( A_v^{\sigma/2} = \mathcal{H}_v |\xi|^{\sigma} \mathcal{H}_v \) \((v \geq 0, \sigma \geq 0)\) imply that

\[
\mathcal{M}[A_v^{\sigma/2} f](z) = 2^\sigma \frac{\Gamma((z-\lambda+v)/2)}{\Gamma(1-(z-\lambda-v)/2)} \mathcal{M}[f](z-\sigma).
\]

Next we consider the linear operators in \( L^2(\mathbb{R}^N) \) defined by the Mellin transform. Let \( B \) be a linear operator in \( L^2(\mathbb{R}^N) \) defined as

\[
\mathcal{M}[B f](z) = \Phi(z) \cdot \mathcal{M}[f](z-\zeta),
\]

where \( \zeta \in \mathbb{C} \). Then we have

\[
\mathcal{M}[B^{-1} f](z) = \Phi(z+\zeta)^{-1} \mathcal{M}[f](z+\zeta), \quad \mathcal{M}[B^{-1} f](z-\zeta) = \Phi(z)^{-1} \mathcal{M}[f](z).
\]

The case \( \zeta = 0 \) yields the \( L^2 \)-boundedness of the operator \( B \):

**Lemma 2.5.** Assume that the linear operator \( B \) in \( L^2(\mathbb{R}^N) \) satisfies

\[
\mathcal{M}[B f](z) = F(z) \cdot \mathcal{M}[f](z), \quad C_F := \sup_y |F\left(\frac{N}{2} + iy\right)| < +\infty.
\]
Then $B$ is bounded. Moreover,
\begin{equation}
\|Bf\|_{L^2(\mathbb{R}^N)} \leq C_F \|f\|_{L^2(\mathbb{R}^N)} \quad \forall f \in L^2(\mathbb{R}^N).
\end{equation}

**Proof.** Step 1. First we confirm that
\begin{equation}
\int_0^\infty f(s)g(s)s^{N-1} \, ds = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{M}[f]\left(\frac{N}{2} + iy\right) \mathcal{M}[g]\left(\frac{N}{2} + iy\right) \, dy.
\end{equation}
By definition of the Mellin transform we have
\begin{equation}
\mathcal{M}[f](z) = \int_{-\infty}^\infty f(e^r)e^{-rz} \, dr.
\end{equation}
Putting $z = (N/2) + iy$ we see that
\begin{equation}
\mathcal{M}[f]\left((N/2) + iy\right) = \int_{-\infty}^\infty f(e^r)e^{Njr/2}e^{iyr} \, dr = \sqrt{2\pi} \mathcal{F}_r[f(e^r)e^{Njr/2}](y),
\end{equation}
where $\mathcal{F}_r$ is usual Fourier transform:
\begin{equation}
\mathcal{F}_r[f_0(r)](\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f_0(r)e^{-i\rho r} \, dr.
\end{equation}
The Plancherel theorem for Fourier transform $\mathcal{F}_r$ implies that
\begin{align*}
\int_{-\infty}^\infty \mathcal{M}[f]\left(\frac{N}{2} + iy\right) \mathcal{M}[g]\left(\frac{N}{2} + iy\right) \, dy \\
= 2\pi \int_{-\infty}^\infty \mathcal{F}_r[f(e^r)e^{Njr/2}](y) \mathcal{F}_r[g(e^r)e^{Njr/2}](y) \, dy \\
= 2\pi \int_{-\infty}^\infty f(e^r)g(e^r)e^{Njr} \, dr = 2\pi \int_0^\infty f(s)g(s)s^{N-1} \, ds.
\end{align*}
Hence (2.23) is verified.

Step 2. Replacing $f$ and $g$ into $Bf$ of (2.23) and seeing (2.21) we obtain
\begin{align*}
\|Bf\|_{L^2(\mathbb{R}^N)}^2 &= \frac{1}{2\pi} \int_{S^{N-1}} \int_{-\infty}^\infty \left| \mathcal{M}[Bf](sx') \left(\frac{N}{2} + iy\right) \right|^2 \, dy \, ds(x') \\
&= \int_{S^{N-1}} \frac{1}{2\pi} \int_{-\infty}^\infty \left| F\left(\frac{N}{2} + iy\right) \mathcal{M}[f](sx') \left(\frac{N}{2} + iy\right) \right|^2 \, dy \, ds(x').
\end{align*}
\[
\left( \sup_y \left| F\left(\frac{N}{2} + iy\right) \right|^2 \right) \int_{S^{N-1}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}(f(x')) \left| \frac{N}{2} + iy \right|^2 dydS(x') = C_F^2 \int_{S^{N-1}} \int_0^{\infty} |f(x')|^2 drdS(x') = C_F^2 \| f \|^2_{L^2(R^N)}.
\]
This ensures (2.22). 

3. Proofs of generalized Hardy-Rellich inequalities

In this section we prove the following two types of generalized Hardy-Rellich inequalities.

**Theorem 3.1.** Let \( N \geq 2 \) and \( s \in [0, 1) \). Then

\[
\| |x|^{-\sigma} f \|_{L^2(R^N)} \leq \frac{\Gamma((1-s)/2)}{2\Gamma((1+s)/2)} \| P^{s/2}_{a(N)} f \|_{L^2(R^N)} \quad \forall f \in D(P^{s/2}_{a(N)}).
\]

**Theorem 3.2.** Let \( N \geq 2 \) and \( s \in [0, 1) \). Then

\[
\| (-A)^{s/2} f \|_{L^2(R^N)} \leq \frac{\Gamma((N+2s)/4)\Gamma((1-s)/2)}{\Gamma((N-2s)/4)\Gamma((1+s)/2)} \| P^{s/2}_{a(N)} f \|_{L^2(R^N)} \quad \forall f \in D(P^{s/2}_{a(N)}).
\]

First we consider the relation between \( |x|^{-\sigma} \) and \( P^{s/2}_{a(N)} \) and prove Theorem 3.1.

**Proposition 3.3.** Let \( N \geq 2 \), \( a \geq -(N-2)^2/4 \) and \( \sigma \geq 0 \). Assume that

\[
v(0) = \left[ \left( \frac{N-2}{2} \right)^2 + a \right]^{1/2} > \sigma - 1.
\]
Then \( |x|^{-\sigma} \) is relatively bounded with respect to \( P_a \) in \( L^2(R^N) \). Moreover,

\[
\| |x|^{-\sigma} f \|_{L^2(R^N)} \leq \frac{\Gamma((1-\sigma + v_0)/2)}{2^\sigma \Gamma((1+\sigma + v_0)/2)} \| P_a^{\sigma/2} f \|_{L^2(R^N)} \quad \forall f \in D(P_a^{\sigma/2}).
\]

**Proof.** Step 1. First we consider the operator \( |x|^{-\sigma} A_v^{-\sigma/2} \) in \( L^2(R^N) \). The Mellin transform of \( |x|^{-\sigma} A_v^{-\sigma/2} \) is as follows:

\[
\mathcal{M}[|x|^{-\sigma} A_v^{-\sigma/2} f](z) = \mathcal{M}[A_v^{-\sigma/2} f](z-\sigma) = F(z),
\]
where

\[
F(z) := \frac{\Gamma((z-\sigma - \lambda + v)/2)\Gamma(1 - (z-\sigma - \lambda + v)/2)}{2^\sigma \Gamma((z-\lambda + v)/2)\Gamma(1 - (z-\lambda + v)/2)}.
\]
To apply Lemma 2.5 we evaluate \( |F((N/2) + iy)| \). It follows from (2.2) and \( \sigma \geq 0 \) that
\[ |F((N/2) + iy)| = \frac{|\Gamma((1 - \sigma + v + iy)/2)|}{2^\sigma |\Gamma((1 + \sigma + v - iy)/2)|} \]

\[ = \frac{\Gamma((1 - \sigma + v)/2)}{2^\sigma \Gamma((1 + \sigma + v)/2)} \prod_{k=0}^{\infty} \left( 1 + \frac{y^2}{(1 - \sigma + v + 2k)^2} \right)^{-1/2} \times \left( 1 + \frac{y^2}{(1 + \sigma + v + 2k)^2} \right)^{1/2} \leq \frac{\Gamma((1 - \sigma + v)/2)}{2^\sigma \Gamma((1 + \sigma + v)/2)} \quad \forall y \in \mathbb{R}. \]

Thus we obtain

\[ \|x\|^{-\alpha} A_{\alpha}^{\alpha/2} f \rVert_{L^2(\mathbb{R}^N)} \leq \frac{\Gamma((1 - \sigma + v)/2)}{2^\sigma \Gamma((1 + \sigma + v)/2)} \|f\|_{L^2(\mathbb{R}^N)} \quad \forall f \in L^2(\mathbb{R}^N). \]

**Step 2.** To confirm (3.4) we use spherical harmonics decomposition. Put \( v = v(\ell) \) [see (2.12)]. It follows from Lemma 2.1 with regarding \( a = (1 - \sigma)/2, \ b = (1 + \sigma)/2 \) and \( x = v(\ell)/2 \) that

\[ \sup_{\ell} \frac{\Gamma((1 - \sigma + v(\ell))/2)}{2^\sigma \Gamma((1 + \sigma + v(\ell))/2)} = \frac{\Gamma((1 - \sigma + v(0))/2)}{2^\sigma \Gamma((1 + \sigma + v(0))/2)}. \]

Fix \( f \in L^2(\mathbb{R}^N) \). Set \( f_\ell \) be a \( \ell \)-th spherical harmonics part of \( f \) (see Proposition 2.3 (iii)). Then \( A_{\alpha}^{\alpha/2} f_\ell = P_{\alpha}^{\alpha/2} f_\ell \in L^2_{\alpha}(\mathbb{R}^N) \). It follows from (3.5) with \( f = A_{\alpha}^{\alpha/2} f_\ell \) that

\[ \|x\|^{-\alpha} f_\ell \rVert_{L^2(\mathbb{R}^N)} \leq \frac{\Gamma((1 - \sigma + v(0))/2)}{2^\sigma \Gamma((1 + \sigma + v(0))/2)} \|A_{\alpha}^{\alpha/2} f_\ell\|_{L^2(\mathbb{R}^N)} = \frac{\Gamma((1 - \sigma + v(0))/2)}{2^\sigma \Gamma((1 + \sigma + v(0))/2)} \|P_{\alpha}^{\alpha/2} f_\ell\|_{L^2(\mathbb{R}^N)}. \]

Summarizing over \( \ell \) we conclude that

\[ \|x\|^{-\alpha} f \rVert^2_{L^2(\mathbb{R}^N)} = \sum_{\ell=0}^{\infty} \|x\|^{-\alpha} f_\ell \rVert^2_{L^2(\mathbb{R}^N)} \leq \left( \frac{\Gamma((1 - \sigma + v_0)/2)}{2^\sigma \Gamma((1 + \sigma + v_0)/2)} \right)^2 \sum_{\ell=0}^{\infty} \|P_{\alpha}^{\alpha/2} f_\ell\|_{L^2(\mathbb{R}^N)} \rVert^2_{L^2(\mathbb{R}^N)} = \left( \frac{\Gamma((1 - \sigma + v(0))/2)}{2^\sigma \Gamma((1 + \sigma + v(0))/2)} \right)^2 \|P_{\alpha}^{\alpha/2} f\|_{L^2(\mathbb{R}^N)} \rVert^2_{L^2(\mathbb{R}^N)}.
\]

Therefore (3.4) is confirmed.
Proof of Theorem 3.1. Apply Proposition 3.3 with \( a := -(N - 2)^2/4 \) and \( \sigma := s \). Note that \( v(0) = 0 \) if \( a = -(N - 2)^2/4 \). Thus the condition (3.3) implies \( s < 1 \). 

Remark 3.1. Set \( a = 0 \) in Proposition 3.3. Then we have

\[
\int \langle |x|^{-s} f \rangle_{L^2(E^2)} \leq \frac{\Gamma((N - 2s)/4)}{2^s \Gamma((N + 2s)/4)} \|(-\Delta)^{s/2} f\|_{L^2(E^2)},
\]

where \( N \geq 2 \) and \( s < N/2 \). The inequality (3.7) implies the usual Hardy inequality (1.3) for \( N \geq 3 \) (\( s = 1 \)) and the usual Rellich inequality (1.2) for \( N \geq 5 \) (\( s = 2 \)). Moreover, setting \( s = 1/2 \) and \( N = 3 \), (3.7) implies that

\[
\langle |x|^{-1} f, f \rangle_{L^2(E^2)} \leq \frac{\pi}{2} \langle (-\Delta)^{1/2} f, f \rangle_{L^2(E^2)} \quad \forall f \in H^{1/2}(E^3).
\]

This is related to the essential selfadjointness of \(-\Delta + \epsilon_0 |x|^{-1}\) on \( C_0^\infty(E^3) \) (see Kato [6, Remark V-5.12]; the coefficient \( \pi/2 \) is specified). Furthermore, Beckner [1] first showed (3.7); later Yafaev [20] proved not only (3.7) but also the constant \( 2^{-4} \Gamma((N - 2)/2) \Gamma((N + 2)/2)^{-1} \) being optimal. On the one hand, Lieb [8] showed (3.7) by applying the classical weighted Hardy-Littlewood-Sobolev type inequality. In connection with these results the inequality (3.4), especially (3.1) seems to be optimal.

In a similar way to Proposition 3.3 we can prove the equivalence of \( D(P_{\sigma/2}^a) \) and \( \tilde{H}^\sigma(E^N) \).

Proposition 3.4. Let \( N \geq 2 \), \( a \geq -(N - 2)^2/4 \) and \( \sigma \geq 0 \).

(i) Assume that (3.3) and \( a \leq 0 \). Then \((-\Delta)^{\sigma/2}\) is relatively bounded with respect to \( P_{\sigma/2}^a \) in \( L^2(E^N) \). Moreover, for \( f \in D(P_{\sigma/2}^a) \)

\[
\|(-\Delta)^{\sigma/2} f\|_{L^2(E^N)} \leq \frac{\Gamma((N + 2\sigma)/4) \Gamma((1 - \sigma + v(0))/2)}{\Gamma((N - 2\sigma)/4) \Gamma((1 + \sigma + v(0))/2)} \|P_{\sigma/2}^a f\|_{L^2(E^N)};
\]

(ii) Assume that \( a \geq 0 \) and \( N > 2\sigma \). Then \( P_{\sigma/2}^a \) is relatively bounded with respect to \((-\Delta)^{\sigma/2}\) in \( L^2(E^N) \). Moreover, for \( f \in \tilde{H}^\sigma(E^N) \)

\[
\|P_{\sigma/2}^a f\|_{L^2(E^N)} \leq \frac{\Gamma((1 + \sigma + v(0))/2) \Gamma((N - 2\sigma)/4)}{\Gamma((1 - \sigma + v(0))/2) \Gamma((N + 2\sigma)/4)} \|(-\Delta)^{\sigma/2} f\|_{L^2(E^N)}.
\]

Proof. Step 1. Let \( \mu > v \geq 0 \) and \( \sigma \geq 0 \). First we consider the operator \( A_{\mu}^{\sigma/2} A_{\nu}^{-\sigma/2} \). The Mellin transform of \( A_{\mu}^{\sigma/2} A_{\nu}^{-\sigma/2} f \) is as follows:

\[
\mathcal{M}[A_{\mu}^{\sigma/2} A_{\nu}^{-\sigma/2} f](z) = 2^\sigma \frac{\Gamma((z - \lambda + \mu)/2) \Gamma((1 - (z - \sigma - \lambda - \mu)/2)}{\Gamma((z - \sigma - \lambda + \mu)/2) \Gamma(1 - (z - \lambda - \mu)/2)} \mathcal{M}[A_{\nu}^{-\sigma/2} f](z - \sigma) = F(z) \mathcal{M}[f](z),
\]
where
\[
F(z) := \frac{\Gamma((z - \lambda + \mu)/2)\Gamma(1 - (z - \sigma - \lambda - \mu)/2)}{\Gamma((z - \sigma - \lambda + \mu)/2)\Gamma(1 - (z - \lambda - \mu)/2)} \\
\times \frac{\Gamma((z - \sigma - \lambda + v)/2)\Gamma(1 - (z - \lambda - v)/2)}{\Gamma((z - \lambda + v)/2)\Gamma(1 - (z - \sigma - \lambda - v)/2)}.
\]

We evaluate \(|F((N/2) + iy)|\) to apply Lemma 2.5. It follows from (2.2) that
\[
|F((N/2) + iy)| = \left| \frac{\Gamma((1 + \sigma + \mu + iy)/2)}{\Gamma((1 - \sigma + \mu + iy)/2)} \times \frac{\Gamma((1 - \sigma + v + iy)/2)}{\Gamma((1 + \sigma + v - iy)/2)} \right| \times \left[ \prod_{k=0}^{\infty} \left(1 + y^2/(1 + \sigma + v + 2k)^2\right) \right]^{1/2}
\]
\[
= \frac{\Gamma((1 + \sigma + \mu)/2)\Gamma((1 - \sigma + v)/2)}{\Gamma((1 - \sigma + \mu)/2)\Gamma((1 + \sigma + v)/2)} \left[ \prod_{k=0}^{\infty} R_k(y) \right]^{1/2}.
\]
Now we set \(M_k := 1 + \mu + 2k\) and \(N_k := 1 + v + 2k\). Note that \(\mu > v\) implies that \(M_k > N_k\). Since \(\varphi(t) := t/(t^2 - \sigma^2)^2\) is decreasing in \(t > \sigma\) and \(M_k > N_k\), we obtain
\[
R_k(y) = \frac{(1 + y^2/(N_k + \sigma)^2)(1 + y^2/(M_k - \sigma)^2)}{(1 + y^2/(N_k - \sigma)^2)(1 + y^2/(M_k + \sigma)^2)}
\]
\[
= 1 - y^2 \frac{4\sigma|N_k/(N_k^2 - \sigma^2) - M_k/(M_k^2 - \sigma^2)|}{(1 + y^2/(N_k - \sigma)^2)(1 + y^2/(M_k + \sigma)^2)}
\]
\[
- y^4 \frac{4\sigma(M_k - N_k)(N_k M_k - \sigma^2)}{(M_k^2 - \sigma^2)^2(N_k^2 - \sigma^2)^2(1 + y^2/(N_k - \sigma)^2)(1 + y^2/(M_k + \sigma)^2)}
\]
\[
\leq 1 \quad \forall y \in \mathbb{R}.
\]
Therefore we conclude that \(\mu > v\) ensures
\[
(3.10) \quad \left| F\left(\frac{N}{2} + iy\right) \right| \leq \frac{\Gamma((1 + \sigma + \mu)/2)\Gamma((1 - \sigma + v)/2)}{\Gamma((1 - \sigma + \mu)/2)\Gamma((1 + \sigma + v)/2)} =: C(\mu, v) \quad \forall y \in \mathbb{R}.
\]

Step 2. We prove (i). Put \(\mu = \mu(\ell)\) [see (2.10)] and \(v = v(\ell)\) [see (2.12)] for \(\ell \in \mathbb{N}\cup \{0\}\). Since \(a < 0\), \(\mu = \mu(\ell) > v(\ell) = v\) is verified. Now we
evaluate

\begin{equation}
(3.11) \quad C_\sigma := \sup_{\ell \geq 0} C(\mu(\ell), v(\ell)).
\end{equation}

Now we set

\[
x(x) = \frac{x}{2} + \frac{N - 2}{4} + \frac{1 + \sigma}{2}, \quad \beta(x) = \sqrt{\left(\frac{x}{2} + \frac{N - 2}{4}\right)^2 + \frac{a + 1 - \sigma}{4}}
\]

and \(c := \sigma\) in Lemma 2.2. Note that \(\gamma(\ell) = C(\mu(\ell), v(\ell))\) for \(\ell \in \mathbb{N} \cup \{0\}\). Therefore Lemma 2.2 ensures that

\[
\gamma(\ell) \leq \gamma(0) = \frac{\Gamma((1 + \sigma + n/2)/2)\Gamma((1 - \sigma + n(0))/2)}{\Gamma((1 - \sigma + n(0))/2)\Gamma((1 + \sigma + n(0))/2)},
\]

where \(\mu(0) = n = (N - 2)/2\) and \(v(0) = \sqrt{a + [(N - 2)/2]^2}\). In a way similar to (3.6) we obtain (3.8).

**Step 3.** We prove (ii). Put \(\mu = v(\ell)\) [see (2.12)] and \(v = \mu(\ell)\) [see (2.10)] for \(\ell \in \mathbb{N} \cup \{0\}\). Since \(a > 0\), \(\mu = \mu(\ell) > v(\ell) = v\) is verified. Now we evaluate (3.11). In a way similar to Step 2 we see that

\[
\gamma(\ell) \leq \gamma(0) = \frac{\Gamma((1 + \sigma + n(0))/2)\Gamma((1 - \sigma + n(0))/2)}{\Gamma((1 - \sigma + n(0))/2)\Gamma((1 + \sigma + n(0))/2)},
\]

where \(\gamma\) is defined in (2.5) with

\[
x(x) := \sqrt{\left(\frac{x}{2} + \frac{N - 2}{4}\right)^2 + \frac{a + 1 + \sigma}{4}}, \quad \beta(x) := \frac{x}{2} + \frac{N - 2}{4} + \frac{1 - \sigma}{2}
\]

and \(c := \sigma\). Therefore we obtain (3.9) in a way similar to (3.6).

**Proof of Theorem 3.2.** Apply Proposition 3.4 (i) with \(a = -(N - 2)^2/4\) and \(\sigma = s\). Note that \(v(0) = 0\) if \(a = -(N - 2)^2/4\). Thus the condition (3.3) implies \(s < 1\).

**Remark 3.2.** First let \(a \in \left[-(N - 2)^2/4, 0\right]\). Then (1.4) implies that

\[
\|P_a^{1/2}u\|_{L^2(R^N)} \leq \|(-A)^{1/2}u\|_{L^2(R^N)} \quad \forall u \in \hat{H}^1(R^N).
\]

The Heinz-Kato inequality [5, Theorem 2] yields

\[
\|P_a^{1/2}u\|_{L^2(R^N)} \leq \|(-A)^{s/2}u\|_{L^2(R^N)} \quad \forall u \in \hat{H}^s(R^N), \forall s \in [0, 1].
\]

Thus we see from (3.8) that for \(u \in \hat{H}^s(R^N)\) with \(s \in [0, 1]\)

\[
\|P_a^{1/2}u\|_{L^2(R^N)} \leq \|(-A)^{s/2}u\|_{L^2(R^N)} \leq C\|P_a^{s/2}u\|_{L^2(R^N)}.
\]

This is nothing but the norm equivalence \(D(P_a^{1/2}) = \hat{H}^s(R^N)\) \((0 \leq s < 1)\); note that \(a = -(N - 2)^2/4\) is included. On the other hand, if \(a \geq 0\), then for
4. Applications to nonlinear Schrödinger equations

First it follows from Theorem 3.2 that $D((1 + P_{a(N)})^{1/2})$ is embedded into $H^s(R^N)$ $(0 \leq s < 1$ and $N \geq 3)$:

$$C_s^{-1} \|(-\Delta)^{s/2} f\|_{L^2(R^N)} \leq \|P_{a(N)}^{s/2} f\|_{L^2(R^N)} \leq C \|f\|_{L^2(R^N)} \|P_{a(N)}^{1/2} f\|_{L^2(R^N)} \leq C' \|(1 + P_{a(N)})^{1/2} f\|_{L^2(R^N)}.$$ 

Now we apply the inequalities (3.2) and (3.1) to the Cauchy problem for nonlinear Schrödinger equations with the inverse-square potential of the critical coefficients. To simplify we set

$$X^1(R^N) := D((1 + P_{a(N)})^{1/2}), \quad X^{-1}(R^N) := X^1(R^N)^* = D((1 + P_{a(N)})^{-1/2}).$$

Here we have the continuous inclusions:

$$H^1(R^N) \subset X^1(R^N) \subset H^s(R^N) \quad \forall s \in [0, 1).$$

$X^1(R^N)$ is also regarded as the closure of $H^1(R^N)$ in the norm

$$\|u\|_{X^1(R^N)} := \left( \int_{R^N} \left[ |u|^2 + |\nabla u|^2 - \frac{(N - 2)^2}{4|x|^2} |u|^2 \right] dx \right)^{1/2} = \|(1 + P_{a(N)})^{1/2} f\|_{L^2(R^N)}.$$

Now we consider the Cauchy problem for nonlinear Schrödinger equations with inverse-square potential of critical coefficient:

\[ \begin{cases} 
   i \frac{\partial u}{\partial t} = -\Delta u - \frac{(N - 2)^2}{4|x|^2} u + \frac{b}{|x|^r} |u|^{p-1} u & \text{in } R \times R^N, \\
   u(0, x) = u_0(x) & \text{in } R^N,
\end{cases} \]

where $i = \sqrt{-1}$, $b \in R$, $p \geq 1$ and $r \geq 0$ such that

$$1 \leq p < \begin{cases} 
   (N + 2 - 2r)/(N - 2) & \text{if } b > 0, \\
   1 + (4 - 2r)/N & \text{if } b < 0.
\end{cases} \tag{4.1}$$

Note that (CP) is a special case $r = 0$ of (NLS)$_{a(N)}$. 

$u \in \dot{H}^s(R^N)$ with $s \in [0, 1)$

$$\|(-\Delta)^{s/2} u\|_{L^2(R^N)} \leq \|P_{a(N)}^{s/2} u\|_{L^2(R^N)} \leq \tilde{C}_s \|(-\Delta)^{s/2} u\|_{L^2(R^N)}.$$

Hence we conclude that $D(P_{a(N)}^{s/2}) = \dot{H}^s(R^N)$ $(0 \leq s < 1)$ for $a \geq -(N - 2)^2/4$. Finally, we remark again that the equivalence $D(P_{a(N)}^{1/2}) = H^1(R^N)$ is broken down when $a = -(N - 2)^2/4$ [see (1.4)].
Theorem 4.1. Assume (4.1). Then for every \( u_0 \in X^1(\mathbb{R}^N) \) there exists a unique weak solution \( u \in C(\mathbb{R}; X^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; X^1(\mathbb{R}^N)) \) to \((\text{NLS})_{a(N)}\). Moreover, \( u \) satisfies
\[
\|u(t)\|_{L^2(\mathbb{R}^N)} = \|u_0\|_{L^2(\mathbb{R}^N)}; \quad E(u(t)) = E(u_0) \quad \forall t \in \mathbb{R},
\]
where the energy functional \( E \) is defined by
\[
E(\varphi) := \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla \varphi|^2 - \frac{(N-2)^2}{4|x|^2} |\varphi|^2 \right) dx + \frac{b}{p+1} \int_{\mathbb{R}^N} \frac{|\varphi|^{p+1}}{|x|^r} dx, \quad \varphi \in X^1(\mathbb{R}^N).
\]

Theorem 4.1 is proved by energy methods established by Okazawa-Suzuki-Yokota [12] (see Section 4.2). But we need to consider \((\text{NLS})_{a(N)}\) in \( X^1(\mathbb{R}^N) = D((1 + P_{a(N)})^{1/2}) = X_S \) (\( S = P_{a(N)} \)). Thus we modify the composition mappings for applying the energy methods (see Figure 1).

4.1. Fractional Sobolev spaces and some interpolation spaces

In this section we set \( \Omega \subset \mathbb{R}^N \). \( L^p(\Omega) \) (\( 1 \leq p \leq \infty \)) is the usual Lebesgue space with the usual norm. \( p' = p/(p-1) \) is the Hölder conjugate of \( p \) for \( 1 \leq p < \infty \). Note that \( [L^p(\Omega')]' \), the dual of \( L^p(\Omega) \) is equal to \( L^{p'}(\Omega) \) for \( 1 \leq p < \infty \).

**Lorentz spaces** The definition is fully written in Bergh-Löfström [2] and Triebel [18]. Let \( f \in L^1_{\text{loc}}(\Omega) \). Then the distribution function of \( f \) is defined as
\[
m(f; s) := |\{ x \in \mathbb{R}^N; |f(x)| > s \}|.
\]
Since \( m(f; s) \) is decreasing in \( s \), we can consider the decreasing rearrangement of \( f \) as
\[
f^*(t) := \inf \{ s; m(f, s) \leq t \}.
\]
Now Lorentz spaces $L^{p,q}(\Omega)$ ($p \in (1, \infty)$ and $q \in [1, \infty]$) is specified the family of $f$ such that the following (quasi-)norm $|f|_{L^{p,q}(\Omega)}$ is finite:

$$|f|_{L^{p,q}(\Omega)} := \left\{ \begin{array}{ll} \int_0^\infty [t^{1/p} f^*(t)]^q t^{-1} \, dt & 1 \leq q < \infty, \\
\sup_{t>0} t^{1/p} f^*(t) & q = \infty. \end{array} \right.$$ 

Note that $|f|_{L^{p,q}(\Omega)} = \|f\|_{L^p(\Omega)}$ and there exists an equivalent norm $\|\cdot\|_{L^{p,q}(\Omega)}$ of $|\cdot|_{L^{p,q}(\Omega)}$ for $p \in (1, \infty)$ and $q \in [1, \infty]$. Also Lorentz spaces $L^{p,q}(\Omega)$ is characterized in the real interpolation between the usual Lebesgue spaces:

$$L^{p,q}(\Omega) = (L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,q}, \quad \frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1.$$ 

Typical example belonging to Lorentz spaces is $|x|^{-1} \in L^{N,\infty}(\mathbb{R}^N)$ ($N \geq 2$). Note that $L^{p,p}(\Omega) = L^p(\Omega)$ and $L^{p,1}(\Omega) \subset L^{p,q_1}(\Omega) \subset L^{p,q_2}(\Omega) \subset L^{p,\infty}(\Omega)$ for $1 \leq q_1 \leq q_2 \leq \infty$. Now we use the following Hölder inequality in Lorentz spaces (see, e.g. O’Neil [13]).

**Lemma 4.2.** Let $f \in L^{p_1,q_1}(\Omega)$ and $g \in L^{p_2,q_2}(\Omega)$. Assume that $p \in (1, \infty)$, $q \in [1, \infty]$ satisfy

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$ 

Then $fg \in L^{p,q}(\Omega)$. Moreover, the Hölder inequality in Lorentz space is fulfilled:

$$\|fg\|_{L^{p,q}(\Omega)} \leq 3p'\|f\|_{L^{p_1,q_1}(\Omega)}\|g\|_{L^{p_2,q_2}(\Omega)}.$$ 

In connection with Lorentz spaces, we can consider the real interpolation between the space-time (vector-valued) Lebesgue spaces (see [18, Theorem 1.18.4]):

$$(L^{p_0}(I;L^{p_0}(\mathbb{R}^N)), L^{p_1}(I;L^{p_1}(\mathbb{R}^N)))_{\theta,p} = L^p(I;L^{q,p}(\mathbb{R}^N)),$$

where $p_0, p_1 \in (1, \infty)$, $q_0, q_1 \in [1, \infty]$, $\theta \in (0,1)$ and

$$\frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1}{q_0} + \frac{\theta}{q_1}.$$ 

We apply these real interpolation spaces to the Strichartz estimates for $e^{itA}$ in $L^2(\mathbb{R}^2)$. Here the usual Strichartz estimates for $e^{itA}$ in $L^2(\mathbb{R}^2)$ is

$$\|e^{itA}\varphi\|_{L^1(\mathbb{R};L^2(\mathbb{R}^2))} \leq C_{\tau}\|\varphi\|_{L^2(\mathbb{R}^2)}, \quad \varphi \in L^2(\mathbb{R}^2),$$

where $(\tau, \rho)$ is the Schrödinger admissible pair in dimension two:

$$\frac{1}{\tau} + \frac{1}{\rho} = \frac{1}{2}, \quad \tau, \rho \in [2, \infty], \quad (\tau, \rho) \neq (2, \infty).$$
Lemma 4.3. Let \((\tau, \rho)\) be a Schrödinger admissible pair of 2-dimensional. Then
\[
\|e^{itA}\varphi\|_{L^{s_0}(\mathbb{R}; L^{m}(\mathbb{R}^N))} \leq C \|\varphi\|_{L^{q_0}(\mathbb{R}^N)}, \quad \varphi \in L^2(\mathbb{R}^2),
\]

Proof. First note that if a bounded linear operator \(F\) maps from \(X\) to \(E_0\) and from \(X\) to \((E_0, E_1)_{\theta, q}\) and the mapping is also bounded (see, e.g. [2, Theorem 3.1.2]). Now we put \(F = e^{itA}, X = L^2(\mathbb{R}^2), E_j = L^{j}(\mathbb{R}; L^{\rho_j}(\mathbb{R}^2))\) \((j = 0, 1)\).

Since \((\tau_j, \rho_j)\) \((j = 0, 1)\) are the Schrödinger admissible pair, so is \((\tau_0, \rho_0)\), where
\[
\frac{1}{\tau_0} = 1 - \theta + \frac{\theta}{\tau_1}, \quad \frac{1}{\rho_0} = 1 - \theta + \frac{\theta}{\rho_1}, \quad 0 < \theta < 1.
\]
Conversely, for any Schrödinger admissible pair \((\tau, \rho)\), there exist \(\theta \in (0, 1)\) and Schrödinger admissible pairs \((\tau_j, \rho_j)\) \((j = 0, 1)\) such that \((\tau, \rho) = (\tau_0, \rho_0)\). It follows from (4.3) that
\[
(L^{\tau_0}(\mathbb{R}; L^{\rho_0}(\mathbb{R}^2)), L^{\tau_1}(\mathbb{R}; L^{\rho_1}(\mathbb{R}^N)))_{\theta, t} = L^{\tau}(\mathbb{R}; L^{\rho}(\mathbb{R}^2)).
\]
Thus \(F = e^{itA}\) maps from \(L^2(\mathbb{R}^2)\) to \(L^{\tau}(\mathbb{R}; L^{\rho}(\mathbb{R}^2))\) and the mapping is also bounded.

Strichartz estimates is also considered in higher dimension (see e.g. [7]). In fact,
\[
(4.4) \quad \|e^{itA}\varphi\|_{L^{\tau_0}(\mathbb{R}; L^{m}(\mathbb{R}^N))} \leq \tilde{C} \|\varphi\|_{L^{q_0}(\mathbb{R}^N)},
\]
\[
\left\| \int_0^t e^{i(t-s)A} F(s) ds \right\|_{L^{s_0}(\mathbb{R}; L^{m}(\mathbb{R}^N))} \leq \tilde{C} \|F\|_{L^{s_0}(\mathbb{R}; L^{s_0}(\mathbb{R}^N))},
\]
where \((\tau_j, \rho_j)\) \((j = 0, 1, 2)\) are the \((N\text{-dimensional)}\) Schrödinger admissible pairs:
\[
(4.5) \quad \frac{2}{\tau} + \frac{N}{\rho} = \frac{N}{2}, \quad \tau, \rho \in [2, \infty], \quad (\tau, \rho) \neq (2, \infty).
\]
Moreover, Keel-Tao [7] proved
\[
(4.6) \quad \left\| \int_0^t e^{i(t-s)A} F(s) ds \right\|_{L^{s_0}(\mathbb{R}; L^{m}(\mathbb{R}^N))} \leq \tilde{C} \|F\|_{L^{s_0}(\mathbb{R}; L^{s_0}(\mathbb{R}^N))},
\]

**Fractional Sobolev space** If \(\Omega = \mathbb{R}^N\), then the fractional Sobolev spaces \(H^s(\mathbb{R}^N)\) \((s \in \mathbb{R})\) and the homogeneous Sobolev spaces \(\dot{H}^s(\mathbb{R}^N)\) are characterized as follows:
\[ H^s(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N); (1 + |\xi|^2)^{s/2} \hat{f}(\xi) \in L^2(\mathbb{R}^N) \}, \]

\[ \hat{H}^s(\mathbb{R}^N) = \{ f \in \mathcal{S}'(\mathbb{R}^N) \setminus P(\mathbb{R}^N); |\xi|^s \hat{f}(\xi) \in L^2(\mathbb{R}^N) \}, \]

where \( \hat{f} \) is the Fourier transform of \( f \) in \( N \)-dimensional and \( P(\mathbb{R}^N) \) is the family of polynomials. Moreover,

\[
\|u\|_{H^s(\mathbb{R}^N)} = \|(1 - \Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)} = \|(1 + |\xi|^2)^{s/2} \hat{u}\|_{L^2(\mathbb{R}^N)} \quad \forall u \in H^s(\mathbb{R}^N),
\]

\[
\|u\|_{\hat{H}^s(\mathbb{R}^N)} = \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)} = \| |\xi|^s \hat{u}\|_{L^2(\mathbb{R}^N)} \quad \forall u \in \hat{H}^s(\mathbb{R}^N).
\]

Note that \([H^s(\mathbb{R}^N)]^* = H^{-s}(\mathbb{R}^N)\) for \( s \in \mathbb{R} \). See [18, Chapter 4] for definition of \( H^s(\Omega) \) (\( s \geq 0 \)). Fractional Sobolev spaces can be also characterized in the real interpolation (see [18, Theorem 1 in Section 4.3.1]). In fact, let \( s \in (0, 1) \). Then

\[ H^s(\Omega) = (L^2(\Omega), H^1(\Omega))_{s, 2}. \]

Sobolev embeddings in the fractional Sobolev spaces are available (see Remark 3 in Section 2.8.1 and Remark 1 in Section 4.6.1 as in [18] for details).

**Lemma 4.4.** Let \( s \in (0, 1] \). Assume that \( 2 \leq p \leq 2N/(N - 2s) \). Then

\[ \|\varphi\|_{H^s(\Omega)} \leq C_{s, p, N}\|\varphi\|_{H^s(\Omega)}, \quad \varphi \in H^s(\Omega). \]

Rellich’s compactness lemma in fractional Sobolev spaces is also applied in later.

**Lemma 4.5.** Let \( \Omega \subset \mathbb{R}^N \) be an open and bounded set with smooth boundary. Assume that \( 0 < s \leq 1 \) and \( 2 \leq p < 2N/(N - 2s) \). Then the embedding \( H^s(\Omega) \subset L^p(\Omega) \) is compact.

**Proof.** First the case \( s = 1 \) is well-known. Thus we only consider the case \( 0 < s < 1 \). Since \( H^s(\Omega) = (L^2(\Omega), H^1(\Omega))_{s, 2} \), \( H^s(\Omega) \subset L^2(\Omega) \) is compact (see e.g. [18, Theorem 1 in Section 1.16.4]).

Next we show that \( H^s(\Omega) \subset L^p(\Omega) \) (\( 2 < p < 2N/(N - 2s) \)) is compact. Let \( \{u_n\}_n \subset H^s(\Omega) \) be a bounded sequence and \( M := \sup_n\|u_n\|_{H^s(\Omega)} < \infty \). Then the compactness of \( H^s(\Omega) \subset L^2(\Omega) \) implies that there exist \( u \in L^2(\Omega) \) and a subsequence \( \{u_n(j)\}_j \subset \{n\}_n \) such that \( u_n \to u \) (\( n \to \infty \)) strongly in \( L^2(\Omega) \). Hölder inequality and Sobolev embedding (4.7) yield that

\[
\|u_n(j) - u_{n(j')}\|_{L^p(\Omega)} \leq \|u_n(j) - u_{n(j')}\|_{L^2(\Omega)}^{1-\theta} \|u_{n(j)} - u_{n(j')}\|_{L^2(\Omega)}^{\theta} \leq (2C_{s, p, N}M)^{\theta}\|u_n(j) - u_{n(j')}\|_{L^2(\Omega)} \to 0 \quad (j, j' \to \infty),
\]

}\[ \text{in } L^p(\Omega). \]
where \( \theta = N(p - 2)/(2ps) \in (0, 1) \). Hence we see from the continuous embedding \( L^p(\Omega) \subset L^2(\Omega) \) that \( u_n(j) \to u \) strongly in \( L^p(\Omega) \). Therefore \( H^s(\Omega) \subset L^p(\Omega) \) \( (2 \leq p < 2N/(N - 2s)) \) is compact.

### 4.2. Preliminaries for nonlinear Schrödinger equations

Let \( S \) be a nonnegative selfadjoint operator in a complex Hilbert space \( X \).
Put \( X_S := D((1 + S)^{1/2}) \). Then we have the usual triplet: \( X_S \subset X = X^* \subset X_S^* \) (* denotes conjugate space). Under this setting \( S \) can be extended a nonnegative selfadjoint operator in \( X_S^* \) with domain \( X_S \). Now we consider

\[
\begin{aligned}
&\begin{cases} 
  i \frac{du}{dt} = Su + g(u), \\
u(0) = u_0, 
\end{cases} \\
\text{(ACP)}
\end{aligned}
\]

where \( i = \sqrt{-1} \), \( g : X_S \to X_S^* \) is a nonlinear operator satisfying

\**G1** Existence of energy functional: there exists \( G \in C^1(X_S; \mathbb{R}) \) such that \( G' = g \), that is, given \( u \in X_S \), for every \( \varepsilon > 0 \) there exists \( \delta = \delta(u, \varepsilon) > 0 \) such that

\[
|G(u + v) - G(u) - \Re \langle g(u), v \rangle_{X_S^*, X_S}| \leq \varepsilon \|v\|_{X_S} \quad \forall v \in X_S \text{ with } \|v\|_{X_S} < \delta;
\]

\**G2** Local Lipschitz continuity: for all \( M > 0 \) there exists \( C(M) > 0 \) such that

\[
\|g(u) - g(v)\|_{X_S^*} \leq C(M)\|u - v\|_{X_S} \quad \forall u, v \in X_S \text{ with } \|u\|_{X_S}, \|v\|_{X_S} \leq M;
\]

\**G3** Hölder-like continuity of energy functional: given \( M > 0 \), for all \( \delta > 0 \) there exists a constant \( C_\delta(M) > 0 \) such that

\[
|G(u) - G(v)| \leq \delta + C_\delta(M)\|u - v\|_X \quad \forall u, v \in X_S \text{ with } \|u\|_{X_S}, \|v\|_{X_S} \leq M;
\]

\**G4** Gauge type condition for the conservation of charge:

\[
\text{Im} \langle g(u), u \rangle_{X_S^*, X_S} = 0 \quad \forall u \in X_S;
\]

\**G5** Closedness type condition: given a bounded open interval \( I \subset \mathbb{R} \), let \( \{w_n\}_n \) be any bounded sequence in \( L^\infty(I; X_S^*) \) such that

\[
\begin{aligned}
&\begin{cases} 
  w_n(t) \to w(t) \quad (n \to \infty) \quad \text{weakly in } X_S \text{ a.a. } t \in I, \\
g(w_n) \to f \quad (n \to \infty) \quad \text{weakly}^* \text{ in } L^\infty(I; X_S^*). 
\end{cases} \\
\end{aligned}
\]

Then

\[
\text{Im} \int_I \langle f(t), w(t) \rangle_{X_S^*, X_S} dt = \lim_{n \to \infty} \text{Im} \int_I \langle g(w_n(t)), w_n(t) \rangle_{X_S^*, X_S} dt.
\]
Here $f = g(w)$ is guaranteed if

$$w_n(t) \to w(t) \quad (n \to \infty) \quad \text{strongly in } X \text{ a.a. } t \in I;$$

\[(G6) \quad \text{Lower boundedness of the energy:} \quad \text{there exist } \varepsilon \in (0,1] \text{ and } C_0(\cdot) \geq 0 \text{ such that} \]

$$G(u) \geq -\frac{1-\varepsilon}{2}\|u\|_{X_S}^2 - C_0(\|u\|_X) \quad \forall u \in X_S.$$ 

Here we call $u$ a \textit{local weak solution} to (ACP) on $I(\geq 0)$ if $u$ belongs to $L^\infty(I;X_S) \cap W^{1,\infty}(I;X_S^*)$ and $u$ satisfies (ACP) in $L^\infty(I;X_S^*)$. In these settings Okazawa-Suzuki-Yokota [12, Theorem 2.4] established the energy methods for (ACP) as follows:

\textbf{Theorem 4.6.} Assume that $g$ satisfies (G1)–(G6) and the uniqueness of local weak solution to (ACP). Then for every $u_0 \in X_S$ there exists a global weak solution $u \in C(R;X_S) \cap C^1(R;X_S^*)$ to (ACP) and the conservation laws for the charge and energy hold:

$$\|u(t)\|_X = \|u_0\|_X, \quad E(u(t)) = E(u_0) \quad \forall t \in R,$$

where $E(\cdot)$ is the energy given by

$$E(\phi) := \frac{1}{2}\|S^{1/2}\phi\|_X^2 + G(\phi), \quad \phi \in X_S.$$ 

To verify (G5) the following lemma is useful; see [12, Lemma 5.3].

\textbf{Lemma 4.7.} Let $g : X_S \to X_S^*$. For any sequence $\{u_n\}_n \subset X_S$ assume that

\begin{equation} (4.11) \quad \begin{cases} u_n \to u \quad (n \to \infty) \quad \text{weakly in } X_S, \\ g(u_n) \to f \quad (n \to \infty) \quad \text{weakly in } X_S^* \Rightarrow f = g(u). \end{cases} \end{equation} 

Then (G4) implies (G5).

Now we note that $S := P_{a(N)} = -\Delta - ((N - 2)^2/4)|x|^{-2}$ is nonnegative and selfadjoint in $L^2(R^N)$ (in the sense of form-sum) and $X^{-1}(R^N)$. In particular the case $X^{-1}(R^N) = X_S^*$, the nonnegativity and the selfadjointness of $P_{a(N)}$ is a consequence of [12, Theorem 3.1 and Remark 2].

Next we prepare the Strichartz estimates for $e^{-itP_{a(N)}}$ to prove the uniqueness of local weak solutions to (NLS)$_{a(N)}$. 


**Proposition 4.8.** Let $N \geq 3$ and $(p_j, q_j)$ $(j = 0, 1, 2)$ be $(N$-dimensional) Schrödinger admissible pairs except the endpoint $(2, 2N/(N - 2))$. Then

\begin{equation}
\|e^{-itP_{\omega}(N)}\|_{L^{p_0}(\mathbb{R}; L^{q_0}(\mathbb{R}^N))} \leq C_{p_0}\|\phi\|_{L^2(\mathbb{R}^N)},
\end{equation}

\begin{equation}
\left\| \int_0^t e^{-it\xi P_{\omega}(N)} \Phi(s)ds \right\|_{L^{p_2}(\mathbb{R}; L^{q_2}(\mathbb{R}^N))} \leq C_{p_1,p_2}\|\Phi\|_{L^{p_1}(\mathbb{R}; L^{q_1}(\mathbb{R}^N))}.
\end{equation}

**Proof.** Step 1. First we consider $u_0 \in L^2_{\text{rad}}(\mathbb{R}^N) = L^2_{\text{rad}}(\mathbb{R}^N)$ to prove (4.12). Note that

\begin{equation}
\|u_0\|_{L^2(\mathbb{R}^N)}^2 = \int_0^\infty \omega_N r^{N-1}|u_0(r)|^2 dr = \frac{\omega_N}{2\pi} \int_0^\infty 2\pi r |r^{(N-2)/2}u_0(r)|^2 dr
\end{equation}

\begin{equation}
= \frac{\omega_N}{2\pi} \|r^{(N-2)/2}u_0(r)\|_{L^2(\mathbb{R}^N)}^2 \quad \forall u_0 \in L^2_{\text{rad}}(\mathbb{R}^N),
\end{equation}

where $\omega_N$ is a measure of $S^{N-1} = \partial B(0, 1)$. Now let $v(\cdot) \in C(\mathbb{R}; L^2(\mathbb{R}^2))$ be a solution to

\begin{equation}
\begin{cases}
\frac{i}{\partial t} \hat{v} = -\Delta v, & (t, x) \in \mathbb{R} \times \mathbb{R}^2,
\hat{v}(0) = r^{(N-2)/2}u_0(r) \in L^2(\mathbb{R}^2).
\end{cases}
\end{equation}

Since $v(0)$ is radial, $v(t)$ is also radial for every $t \in \mathbb{R}$. Next we define $u(t, r) := r^{-(N-2)/2}v(t, r)$ ($r = |x|$, $x \in \mathbb{R}^N$). Then we see from (4.14) and the charge conservation law for (4.15) that

\begin{equation}
\|u(t, \cdot)\|_{L^2(\mathbb{R}^N)}^2 = \frac{\omega_N}{2\pi} \|r^{(N-2)/2}u(t, r)\|_{L^2(\mathbb{R}^N)}^2 = \frac{\omega_N}{2\pi} \|v(t, r)\|_{L^2(\mathbb{R}^2)}^2
\end{equation}

\begin{equation}
= \frac{\omega_N}{2\pi} \|v(0, r)\|_{L^2(\mathbb{R}^N)}^2 = \frac{\omega_N}{2\pi} \|r^{(N-2)/2}u_0(r)\|_{L^2(\mathbb{R}^2)}^2
\end{equation}

\begin{equation}
= \|u_0\|_{L^2(\mathbb{R}^N)}^2.
\end{equation}

Hence $u(\cdot) \in C(\mathbb{R}; L^2(\mathbb{R}^N))$. Moreover, $u$ satisfies

\begin{equation}
\begin{cases}
\frac{i}{\partial t} \hat{u} = -\Delta u - \frac{(N-2)^2}{4|x|^2} u, & (t, x) \in \mathbb{R} \times \mathbb{R}^N,
\hat{u}(0) = u_0 \in L^2(\mathbb{R}^N).
\end{cases}
\end{equation}

Now let $(\tau, \rho)$ be a $(N$-dimensional) Schrödinger admissible pair with $\tau \leq \rho$. By definition we calculate
\[ (4.17) \quad \| e^{-iP_{a(N)}} u_0 \|_{L^1(\mathbb{R}, L^p(\mathbb{R}^N))} = \omega_N^{1/\rho} \left( \int_0^\infty |u(t, r)|^{\rho} r^{N-1} \, dr \right)^{1/\rho} \leq \left( \frac{\omega_N}{2\pi} \right)^{1/\rho} \| r^{\rho(\frac{N-2}{2})} u(t, r) \|_{L^1(\mathbb{R}, L^p(\mathbb{R}^2))} \leq \left( \frac{\omega_N}{2\pi} \right)^{1/\rho} \| r^{-\rho(\frac{N-2}{2}) - 1/\rho} v(t, r) \|_{L^1(\mathbb{R}, L^p(\mathbb{R}^2))}. \]

Now we set \( q \) so that
\[ \frac{1}{q} = \frac{1}{2} - \frac{1}{\tau} = \frac{1}{\rho} - \frac{N-2}{N\tau}. \]

Since
\[ \frac{1}{q} + \frac{1}{2} (N-2) \left( \frac{1}{2} - \frac{1}{\rho} \right) = \frac{1}{\rho}, \quad \frac{1}{q} + \frac{1}{\infty} = \frac{1}{\rho}, \]

it follows from Lemma 4.2 and \( L^{q, \tau}(\mathbb{R}^2) \subset L^{q, \rho}(\mathbb{R}^2) \) (\( \tau \leq \rho \)) that
\[ \| r^{-\rho(\frac{N-2}{2}) - 1/\rho} v \|_{L^q(\mathbb{R}^2)} \leq C' \| r^{-\rho(\frac{N-2}{2}) - 1/\rho} v \|_{L^q(\mathbb{R}^2)} \leq C' \| v \|_{L^{q, \rho}(\mathbb{R}^2)}. \]

Moreover, since \( 2/\tau + 2/q = 2/2 \), we see that \((\tau, q)\) is a \((2\text{-dimentional})\) Schrodinger admissible pair. Applying Lemmas 4.2 and 4.3 we calculate
\[ \| r^{-\rho(\frac{N-2}{2}) - 1/\rho} v \|_{L^q(\mathbb{R}^2)} \leq C' \| r^{-\rho(\frac{N-2}{2}) - 1/\rho} u_0 \|_{L^q(\mathbb{R}^2)} \leq C' \| r^{-\rho(\frac{N-2}{2}) - 1/\rho} v \|_{L^q(\mathbb{R}^2)} \leq C' \| v \|_{L^{q, \rho}(\mathbb{R}^2)}. \]

Since \((\tau, \rho)\) is \(N\)-dimensional Schrödinger admissible pair, \( \tau \leq \rho \) is rewritten as
\[ 2 + 4/N \leq \rho < 2N/(N - 2). \]

Combining (4.18) into (4.17) we obtain
\[ \| e^{-iP_{a(N)}} u_0 \|_{L^q(\mathbb{R}, L^p(\mathbb{R}^N))} \leq C' \| v \|_{L^{q, \rho}(\mathbb{R}^2)} \leq C' \| u_0 \|_{L^q(\mathbb{R}^N)}. \]

On the other hand, the selfadjointness of \( P_{a(N)} \) yields that
\[ \| e^{-iP_{a(N)}} u_0 \|_{L^q(\mathbb{R}, L^p(\mathbb{R}^N))} = \| u_0 \|_{L^q(\mathbb{R}^N)}. \]
It follows from (4.19), (4.20) and the interpolation theorem that

\( \| e^{-iP_{ad(2)}}u_0 \|_{L^r(\mathbb{R}, L^s(\mathbb{R}^N))} \leq C_1 \| u_0 \|_{L^2(\mathbb{R}^N)} \),

\( 2 \leq \rho < \frac{2N}{N-2}, \quad u_0 \in L^2_{\text{rad}}(\mathbb{R}^N). \)

**Step 2.** Next we consider \( u_0 \in L^2_{\geq 1}(\mathbb{R}^N) = L^2_{\text{rad}}(\mathbb{R}^N)^\perp \) in \( L^2(\mathbb{R}^N) \) to prove (4.12). First note that Burq, Planchon, Stalker, and Tahvildar-Zadeh [3, Theorem 1] proved the following even in the case \( a = -(N-2)^2/4 \) and \( 0 < \alpha < \nu(d) = (1/4) + (1/2)\sqrt{d^2 + (N-2)d} \):

\( \| x^{-1/2-2\alpha} P_{a(N)}^{1/4-\alpha} e^{-itP_{a(N)}} \phi \|_{L^2(\mathbb{R}, L^2(\mathbb{R}^N))} \leq C_{d, \alpha} \| \phi \|_{L^2(\mathbb{R}^N)}, \quad \phi \in L^2_{\geq d}(\mathbb{R}^N). \)

Set \( d = 1 \) and \( \alpha = 1/4 \) in (4.22). Then

\( \| x^{-1} e^{-itP_{a(N)}} \phi \|_{L^2(\mathbb{R}, L^2(\mathbb{R}^N))} \leq C_1 \| \phi \|_{L^2(\mathbb{R}^N)} \quad \forall \phi \in L^2_{\geq 1}(\mathbb{R}^N). \)

Now let \( u_0 \in L^2_{\geq 1}(\mathbb{R}^N) \). Then \( u(t) = e^{-itP_{a(N)}}u_0 \) satisfies the following integral equations:

\( u(t) = e^{itA}u_0 + \frac{(N-2)^2}{4} \int_0^t e^{i(t-s)A} |x|^{-2} u(s) ds. \)

Applying the Strichartz estimates for \( e^{itA} \) in higher dimension [(4.4) and (4.6)] and the H"older inequality in Lorentz spaces (Lemma 4.2) we obtain

\( \| u(t) \|_{L^r(\mathbb{R}, L^s(\mathbb{R}^N))} \leq \| e^{itA}u_0 \|_{L^r(\mathbb{R}, L^s(\mathbb{R}^N))} + \left( \int_0^t e^{i(t-s)A} |x|^{-2} u(s) ds \right) \| u(t) \|_{L^r(\mathbb{R}, L^s(\mathbb{R}^N))} \)

\( \leq \tilde{C}_1 \| u_0 \|_{L^2(\mathbb{R}^N)} + \tilde{C}_2 \frac{\| u(t) \|_{L^2(\mathbb{R}^N)}}{|x|^2} \| u(t) \|_{L^2(\mathbb{R}^N)} \)

\( \leq \tilde{C}_1 \| u_0 \|_{L^2(\mathbb{R}^N)} + \tilde{C}_2 \| |x|^{-1} u(t) \|_{L^2(\mathbb{R}, L^2(\mathbb{R}^N))} \)

\( \leq (\tilde{C}_1 + \tilde{C}_2 \tilde{C}_1) \| u_0 \|_{L^2(\mathbb{R}^N)}. \)

Thus we see

\( \| e^{-itP_{a(N)}}u_0 \|_{L^r(\mathbb{R}, L^s(\mathbb{R}^N))} \leq C_2 \| u_0 \|_{L^2(\mathbb{R}^N)}, \)

\( 2 \leq \rho < \frac{2N}{N-2}, \quad u_0 \in L^2_{\geq 1}(\mathbb{R}^N). \)

**Step 3.** Now let \( u_0 \in L^2(\mathbb{R}^N) \). Then there exist \( u_1 \in L^2_{\text{rad}}(\mathbb{R}^N) \) and \( u_2 \in L^2_{\geq 1}(\mathbb{R}^N) \) such that \( u_0 = u_1 + u_2 \). Combining (4.21) and (4.23) we obtain
\[ \|e^{-itP_{a(N)}}u_0\|_{L^r(R, L^p(R^N))} \leq \|e^{-itP_{a(N)}}u_1\|_{L^r(R, L^p(R^N))} + \|e^{-itP_{a(N)}}u_2\|_{L^r(R, L^p(R^N))} \]
\[ \leq C^{(1)}_t\|u_1\|_{L^q(R^N)} + C^{(2)}_t\|u_2\|_{L^q(R^N)} \]
\[ \leq (C^{(1)}_t + C^{(2)}_t)\|u_0\|_{L^q(R^N)}. \]

This is nothing but (4.12).

**Step 4.** Next we show (4.13). Applying the duality argument to (4.12) we have

(4.24) \[ \left\| \int_{-\infty}^{\infty} e^{i\alpha_{a(N)}} F(s) ds \right\|_{L^q(R^N)} \leq C_t \|F\|_{L^r(R, L^{q'(R^N)})}. \]

Using (4.12) with \((\tau, \rho) = (p_2, q_2)\) and (4.24) with \((\tau, \rho) = (p_1, q_1)\) we obtain

\[ \left\| \int_{-\infty}^{\infty} e^{-i(t-s)P_{a(N)}} F(s) ds \right\|_{L^{q_2}(R, L^{q_2}(R^N))} \leq C_{p_2} \left\| \int_{-\infty}^{\infty} e^{i\alpha_{a(N)}} F(s) ds \right\|_{L^{q_2}(R^N)} \]
\[ \leq C_{p_2} C_{p_1} \|F\|_{L^{q_2'}(R, L^{q_2'}(R^N))}. \]

Note that \(p'_2 < 2 < p_2\). Applying the Christ-Kiselev lemma (see [4, Theorem 1.2]) we obtain

\[ \left\| \int_0^t e^{-i(t-s)P_{a(N)}} F(s) ds \right\|_{L^{q_2}(R, L^{q_2}(R^N))} \leq (1 - 2^{-1/(p_2 - 1/p'_2)})^{-1} C_{p_1} C_{p_2} \|F\|_{L^{q_2'}(R, L^{q_2'}(R^N))}. \]

This is nothing but (4.13).

\[ \boxdot \]

### 4.3. Proof of Theorem 4.1

Let \(N \geq 3\). Define the operators \(g, G\) and the indices \(s, \theta\) as

(4.25) \[ g(u) := \frac{b}{|x|^r} |u|^{p-1} u, \quad u \in X^1(R^N), \]
\[ G(u) := \frac{b}{p + 1} \int_{R^N} \frac{|u(x)|^{p+1}}{|x|^r} dx, \quad u \in X^1(R^N), \]
\[ s := \frac{N(p - 1) + 2r}{2(p + 1)} \in [0, 1], \quad \theta := \frac{2r}{N(p - 1) + 2r} \in [0, 1]. \]

Note that \(\theta = r/[s(p + 1)]\).
Lemma 4.9. Let \( s, \theta \) be as \((4.25)\). Then for every \( u_j \in H^s(R^N) \) \((j = 0, 1, 2)\)

\[
(4.26) \quad \left\| \int_{R^N} \frac{|u_0(x)|^{p-1}u_1(x)u_2(x)}{|x|^r} \, dx \right\| \leq C_s \left\| \frac{|u_0|^{p-1}}{H^s(R^N)} \right\| \left\| \frac{|u_1|}{H^s(R^N)} \right\| \left\| \frac{|u_2|}{H^s(R^N)} \right\|.
\]

Proof. First we show

\[
(4.27) \quad \int_{R^N} \frac{|\varphi|^{p+1}}{|x|^r} \, dx \leq C_s \| \varphi \|_{H^s(R^N)}^{p+1}, \quad \varphi \in H^s(R^N).
\]

By the generalized Hardy-Rellich inequality \((3.1)\) yields that \( |x|^{-s} \varphi \in L^2(R^N) \). Also fractional Sobolev inequality \((4.7)\) implies that \( \varphi \in L^{2N/(N-2\delta)}(R^N) \). Thus we have

\[
\int_{R^N} \left[ \frac{|\varphi|}{|x|^s} \right]^{r/s} \left| \varphi \right|^{(p+1)(1-r/s(p+1))} \, dx \leq \left\| \frac{|\varphi|^{p+1}}{|x|^\theta} \right\|_{L^2(R^N)} \left\| \varphi \right\|_{L^{2N/(N-2\delta)}(R^N)} \leq C(s, p, N) \| \varphi \|_{H^s(R^N)}^{p+1}.
\]

Now we show \((4.26)\). We see from Hölder inequality and \((4.27)\) that

\[
\left\| \int_{R^N} \frac{|u_0(x)|^{p-1}u_1(x)u_2(x)}{|x|^r} \, dx \right\| \leq \int_{R^N} \left[ \frac{|u_0(x)|}{|x|^{r/(p+1)}} \right]^{p-1} \frac{|u_1(x)|}{|x|^{r/(p+1)}} \frac{|u_2(x)|}{|x|^{r/(p+1)}} \, dx \leq C(s, p, N) \| u_0 \|_{H^s(R^N)} \| u_1 \|_{H^s(R^N)} \| u_2 \|_{H^s(R^N)}.
\]

This is nothing but \((4.26)\).

\[\square\]

Now we are the position to show Theorem 4.1. First we verify the conditions \((G1)-(G6)\) as in Section 4.2.

Verification of \((G1)\) By a simple calculation we have for \( u, v \in C \)

\[
|u + v|^{p+1} - |u|^{p+1} - (p + 1)|u|^{p-1} \Re (u\overline{v})| \leq p(p+1)(|u| + |v|)|v|^{p-1}|v|^2.
\]

Thus we calculate for every \( u, v \in X^1(R^N) \subset H^s(R^N) \)

\[
|G(u + v) - G(u) - \Re \langle g(u), v \rangle_{X^1(R^N)}| \leq \frac{|b|}{p+1} \int_{R^N} |x|^{-r} |u + v|^{p+1} - |u|^{p+1} - (p + 1)|u|^{p-1} \Re (u\overline{v})| \, dx \leq 2^{p-1} |b| \int_{R^N} |x|^{-r}(|u|^{p-1} + |v|^{p-1})|v|^2 \, dx.
\]
Applying (4.26) we calculate for every
\[ 2^{p-1}p|b| \int_{R^N} |x|^{-r}(|u|^{p-1} + |v|^{p-1})|v|^2 \, dx \]
\[ \leq 2^{p-1}p|b|C(s, p, N)[|u|_{H^s(R^N)}^{p-1} + |v|_{H^s(R^N)}^{p-1}]|v|^2_{H^s(R^N)} \]
\[ \leq 2^{p-1}p|b|C'(s, p, N)[|u|_{X^{1}(R^N)}^{p-1} + |v|_{X^{1}(R^N)}^{p-1}]|v|^2_{X^{1}(R^N)}. \]

Now let \( M > 0 \) and \( \varepsilon > 0 \). Then we see that
\[ |G(u + v) - G(u) - \Re \langle g(u), v \rangle_{X^{-1}(R^N), X^1(R^N)}| \]
\[ \leq |b|C''(s, p, N)(M^{p-1} + 1)||v||_{X^1(R^N)}^2 \]
\[ \forall u, v \in X^1(R^N) \text{ with } ||u||_{X^1(R^N)} \leq M, ||v||_{X^1(R^N)} \leq 1. \]

Hence by setting \( \delta = \delta(u, \varepsilon) > 0 \) as \( \delta := 1 \wedge \varepsilon / ||b|C''(s, p, N)(M^{p-1} + 1)|| \), we conclude that for \( v \in X^1(R^N) \) with \( ||v||_{X^1(R^N)} \leq \delta \)
\[ |G(u + v) - G(u) - \Re \langle g(u), v \rangle_{X^{-1}(R^N), X^1(R^N)}| \leq \varepsilon ||v||_{X^1(R^N)}. \]

This is nothing but (G1).

**Verification of (G2)** We consider \( \langle g(u) - g(v), w \rangle_{X^{-1}(R^N), X^1(R^N)} \) for \( u, v, w \in X^1(R^N) \). By a simple calculation we have
\[ ||u|^{p-1}u - |v|^{p-1}v| \leq p(|u| + |v|)^{p-1}|u - v| \quad \forall u, v \in C. \]

Applying (4.26) we calculate for every \( w \in X^1(R^N) \)
\[ |\langle g(u) - g(v), w \rangle_{X^{-1}(R^N), X^1(R^N)}| \]
\[ \leq \int_{R^N} p|b| |x|^{-r}(|u| + |v|)^{p-1}|u - v| |w| \, dx \]
\[ \leq 2^{p-1}p|b|C(s, p, N)[|u|_{H^s(R^N)}^{p-1} + |v|_{H^s(R^N)}^{p-1}]||u - v|_{H^s(R^N)} ||w||_{H^s(R^N)}. \]

Thus we see that \( g(u) - g(v) \in H^{-s}(R^N) \) and for \( u, v \in H^s(R^N) \)
\[ ||g(u) - g(v)||_{H^{-s}(R^N)} \leq 2^{p-1}p|b|C(s, p, N)[|u|_{H^s(R^N)}^{p-1} + |v|_{H^s(R^N)}^{p-1}]||u - v||_{H^s(R^N)}. \]

The embedding \( X^1(R^N) \subset H^s(R^N) \) implies that for \( u, v \in X^1(R^N) \)
\[ ||g(u) - g(v)||_{X^{-1}(R^N)} \leq |b|C''(s, p, N)[|u|_{X^1(R^N)}^{p-1} + |v|_{X^1(R^N)}^{p-1}]||u - v||_{X^1(R^N)}., \]
Hence we obtain (G2):

\[ \| g(u) - g(v) \|_{X^{-1}(\mathbb{R}^N)} \leq 2|b|C''(s, p, N)M^{p-1} \| u - v \|_{X^1(\mathbb{R}^N)} \]

\[ \forall u, v \in X^1(\mathbb{R}^N) \text{ with } \| u \|_{X^1(\mathbb{R}^N)}, \| v \|_{X^1(\mathbb{R}^N)} \leq M. \]

**Verification of (G3)** By a simple calculation we have

\[ |u|^{p+1} - |v|^{p+1} \leq (p + 1)(|u|^p + |v|^p)|u - v| \quad \forall u, v \in C. \]

Applying (4.26) we calculate for \( u, v \in X^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N) \)

\[ |G(u) - G(v)| \leq \int_{\mathbb{R}^N} |b| |x|^{-r} (|u|^p + |v|^p)|u - v|dx \]

\[ \leq |b|C(s, p, N)[\| u \|_{H^s(\mathbb{R}^N)}^p + \| v \|_{H^s(\mathbb{R}^N)}^p]\| u - v \|_{H^s(\mathbb{R}^N)}. \]

Since \( s < (1 + s)/2 < 1 \) we obtain

\[ \| u - v \|_{H^s(\mathbb{R}^N)} \leq \| u - v \|_{L^2(\mathbb{R}^N)}^{2/(1+s)}\| u - v \|_{L^2(\mathbb{R}^N)^{1/(1+s)}} \quad \forall u, v \in X^1(\mathbb{R}^N). \]

The embeddings \( X^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N) \) and \( X^1(\mathbb{R}^N) \subset H^{(1+s)/2}(\mathbb{R}^N) \) yield that

\[ |G(u) - G(v)| \leq |b|C(s, p, N)M^{2+2s/(1+s)}\| u - v \|_{L^2(\mathbb{R}^N)}^{(1-s)/(1+s)} \]

\[ \forall u, v \in X^1(\mathbb{R}^N) \text{ with } \| u \|_{X^1(\mathbb{R}^N)}, \| v \|_{X^1(\mathbb{R}^N)} \leq M. \]

Hence by the Young inequality we obtain (G3):

\[ |G(u) - G(v)| \leq \delta + C_\delta(M)\| u - v \|_{L^2(\mathbb{R}^N)} \]

\[ \forall u, v \in X^1(\mathbb{R}^N) \text{ with } \| u \|_{X^1(\mathbb{R}^N)}, \| v \|_{X^1(\mathbb{R}^N)} \leq M. \]

**Verification of (G4)** By definition of \( g \) we conclude that

\[ \text{Re}\langle g(u), iu \rangle_{X^{-1}(\mathbb{R}^N), X^1(\mathbb{R}^N)} = \text{Re}\int_{\mathbb{R}^N} ib|x|^{-r}|u(x)|^{p+1}dx = 0 \quad \forall u \in X^1(\mathbb{R}^N). \]

**Verification of (G5)** We use Lemma 4.7. Let \( \{u_n\}_n \subset X^1(\mathbb{R}^N) \) satisfy

\[ \begin{cases} 
 u_n \to u \quad (n \to \infty) & \text{weakly in } X^1(\mathbb{R}^N), \\
 g(u_n) \to f \quad (n \to \infty) & \text{weakly in } X^{-1}(\mathbb{R}^N).
 \end{cases} \]

Then it suffices to show \( f = g(u) \). To end this let \( \Omega \subset \mathbb{R}^N \) be an arbitrary bounded domain with smooth boundary. The embedding \( X^1(\mathbb{R}^N) \subset H^{(1+s)/2}(\mathbb{R}^N) \) and Lemma 4.5 ensure that

\[ u_n \to u \quad (n \to \infty) \quad \text{strongly in } L^2(\Omega) \text{ and } L^{2N/(N-2s)}(\Omega); \]
note that $2N/(N-2s) < 2N/(N-s-1)$. In particular, $u_n \to u$ a.a. $x \in \mathbb{R}^N$. Moreover, $|u_n|^{p-1}u_n \to |u|^{p-1}u$ strongly in $L^{2N/p(N-2s)}(\Omega)$. We see from the boundedness of $\Omega$ that $|x|^{-r} \in L^{2N/(2r+1-s)}(\Omega)$. Note that

$$\frac{p(N-2s)}{2N} + \frac{2r + 1 - s}{2N} = \frac{N + 1 + s}{2N}.$$ 

Thus the Hölder inequality implies that $g(u_n) \to g(u)$ in $L^{2N/(N+1+s)}(\Omega)$. Now let $\varphi \in C^\infty_c(\mathbb{R}^N) \subset X^{-1}(\mathbb{R}^N)$. Choose $\Omega \subset \mathbb{R}^N$ in such a way that $\Omega \ni \text{supp } \varphi$, the support of $\varphi$. Then we have

$$|\langle g(u_n) - g(u), \varphi \rangle_{X^{-1}(\mathbb{R}^N), X^1(\mathbb{R}^N)}|$$

$$\leq \int_\Omega |g(u_n) - g(u)||\varphi|dx$$

$$\leq \|g(u_n) - g(u)\|_{L^{2N/(N+1+s)}(\Omega)} \|\varphi\|_{L^{2N/(N-1-s)}(\Omega)} \to 0 \quad (n \to \infty).$$

Thus we see that

$$g(u_n) \to g(u) \quad (n \to \infty) \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

in the sense of distribution. On the other hand, $g(u_n) \to f$ $(n \to \infty)$ weakly in $X^{-1}(\mathbb{R}^N)$ and hence in $\mathcal{D}'(\mathbb{R}^N)$. Therefore we conclude that $f = g(u)$. Thus Lemma 4.7 implies (G5).

**Verification of (G6)** The case $b \geq 0$ is clear since $G(u) \geq 0$ for all $u \in X^1(\mathbb{R}^N)$. So we consider the case $b < 0$. We see from (3.2) and the interpolation inequality that

$$\|u\|_{H^s(\mathbb{R}^N)} \leq c\|u\|_{X^1(\mathbb{R}^N)}^{s} \|u\|_{L^2(\mathbb{R}^N)}^{1-s} \quad \forall u \in X^1(\mathbb{R}^N).$$

Applying this to $G(u)$ we obtain

$$G(u) = -|b| \int_{\mathbb{R}^N} |x|^{-r} |u|^{p+1}dx$$

$$\geq -|b|C(s, p, N)\|u\|_{H^s(\mathbb{R}^N)}^{p+1}$$

$$\geq -|b|C'(s, p, N)\|u\|_{X^1(\mathbb{R}^N)}^{s(p+1)} \|u\|_{L^2(\mathbb{R}^N)}^{(1-s)(p+1)} \quad \forall u \in X^1(\mathbb{R}^N).$$

Note that $p < 1 + (4 - 2r)/N$ and $s(p+1) = [N(p-1) + 2r]/2$ imply $s(p+1) < 2$. Thus the Young inequality ensures that for any $\varepsilon \in (0, 1)$

$$G(u) \geq -\frac{1 - \varepsilon}{2} \|u\|_{X^1(\mathbb{R}^N)} - C_\varepsilon(\|u\|_{L^2(\mathbb{R}^N)}) \quad \forall u \in X^1(\mathbb{R}^N).$$

This is nothing but (G6).
Next we show the uniqueness of local weak solutions to \((\text{NLS})_{u(N)}\) by using the Strichartz estimates and applying the standard contraction arguments.

**Uniqueness of local weak solution** Let \(u, v \in L^\infty(\mathbb{R}^N)\) be local weak solutions to \((\text{NLS})_{u(N)}\) with initial values \(u(0) = u_0 = v(0)\). Then \(u - v\) satisfies
\[
(u(t) - v(t)) = -i \int_0^t e^{-i(t-s)P_{u(N)}} [b|x|^{-r} |u(s)|^{p-1} u(s) - |v(s)|^{p-1} v(s)] ds.
\]

Now we divide \(b|x|^{-r}\) into \(V_0 + V_1\) so that \(V_0 \in L^\infty(\mathbb{R}^N)\) and \(V_1 \in L^{2N/(2r+1-s)}(\mathbb{R}^N)\). Applying the Strichartz estimates for \(e^{-itP_{u(N)}}\) [see (4.13)] we obtain
\[
\|u - v\|_{L_t^r(\mathbb{R}^N)} \leq C_{\tau_0, r} \|V_0\|_{L^\infty(\mathbb{R}^N)} \|u|^{p-1} - |v|^{p-1}\|_{L_t^{r_0}(\mathbb{R}^N)} + C_{\tau_1, r} \|V_1\|_{L^{2N/(2r+1-s)}(\mathbb{R}^N)} \|u|^{p-1} - |v|^{p-1}\|_{L_t^{r_1}(\mathbb{R}^N)}.
\]

where \(\tau_0, \tau_1\) are constants so that \((\tau_0, p + 1)\) and \((\tau_1, 2N/(N - 2s))\) are the \((N\text{-dimensional})\) Schrödinger admissible pairs. To simplify we denote \(\rho_0 := p + 1, \rho_1 := 2N/(N - 2s)\) and \(\|\cdot\|_{L_t^r L^\rho} := \|\cdot\|_{L_r^r(\mathbb{R}^N)}\). It follows from the Hölder inequality and (4.31) that
\[
\|V_0\|_{L^\infty(\mathbb{R}^N)} \|u|^{p-1} - |v|^{p-1}\|_{L_t^{r_0}(\mathbb{R}^N)} + \|V_1\|_{L^{2N/(2r+1-s)}(\mathbb{R}^N)} \|u|^{p-1} - |v|^{p-1}\|_{L_t^{r_1}(\mathbb{R}^N)}
\]
\[
\leq p \|V_0\|_{L^\infty(\mathbb{R}^N)} \|u|^{p-1} - |v|^{p-1}\|_{L_t^{r_0}(\mathbb{R}^N)} + \|V_1\|_{L^{2N/(2r+1-s)}(\mathbb{R}^N)} \|u|^{p-1} - |v|^{p-1}\|_{L_t^{r_1}(\mathbb{R}^N)}.
\]

Since \(u, v \in L^\infty(\mathbb{R}^N)\), there exists \(M > 0\) such that
\[
\|u\|_{L^{\rho_0}_t L^{\rho_0}}, \|V\|_{L^{\rho_0}_t L^{\rho_0}}, \|u\|_{L^{\rho_1}_t L^{\rho_1}}, \|V\|_{L^{\rho_1}_t L^{\rho_1}} \leq M.
\]

Thus we see that
\[
\|u - v\|_{L_t^r L^\rho} \leq 2pC_{\tau_0, r} \|V_0\|_{L^\infty(\mathbb{R}^N)} M^{p-1}(2T)^{1-2/\tau_0} \|u - v\|_{L_t^{\rho_0} L^{\rho_0}}
\]
\[
+ 2pC_{\tau_1, r} \|V_0\|_{L^{2N/(2r+1-s)}(\mathbb{R}^N)} M^{p-1}(2T)^{1-2/\tau_1} \|u - v\|_{L_t^{\rho_1} L^{\rho_1}}.
\]
Putting \((\tau, \rho) = (\tau_j, \rho_j)\) \((j = 0, 1)\) and taking \(T > 0\) sufficiently small, we have
\[
||u - v||_{L^t_t L^\infty} + ||u - v||_{L^1_t L^p} \leq 0.
\]
Thus we conclude \(u = v\).

**Proof of Theorem 4.1.** Since (G1)–(G6) are fulfilled and the uniqueness of local weak solutions to \((\text{NLS})_{d(N)}\) is verified, we conclude from Theorem 4.6 that there exists a unique global weak solution \(u\) to \((\text{NLS})_{d(N)}\). \(\blacksquare\)

**Remark 4.1.** In a way similar to Theorem 4.1 we also consider the Cauchy problem for nonlinear Schrödinger equations of power-type nonlinearities:

\[
(i \partial_t + \Delta - \frac{(N - 2)^2}{4|x|^2})u + f(u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^N,
\]

where \(f\) satisfies as follows:

\[
\text{(N1)} \quad \text{There exist } p \in \left[1, \frac{N + 2}{(N - 2)}\right) \text{ and } K \geq 0 \text{ such that } |f(u) - f(v)| \leq K(1 + |u|^{p-1} + |v|^{p-1})|u - v| \quad \forall u, v \in C;
\]

\[
\text{(N2)} \quad f(0) = 0, \quad f(x) \in \mathbb{R} \quad (x > 0) \quad \text{and} \quad f(e^{it}z) = e^{it}f(z) \quad (z \in C, \quad t \in \mathbb{R});
\]

\[
\text{(N3)} \quad \text{There exist } q \in \left[1, 1 + \frac{4}{N}\right) \text{ and } L_1, L_2 \geq 0 \text{ such that } F(x) := \int_0^x f(s)ds \geq -L_1 x^2 - L_2 x^{q+1} \quad \forall x > 0.
\]

Then for every \(u_0 \in X^1(\mathbb{R}^N)\) there exists a unique global weak solution \(u \in C(\mathbb{R}; X^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; X^{-1}(\mathbb{R}^N))\) to \((\text{NLS})_{d(N)}\). Moreover, \(u\) satisfies the conservation laws (4.2) where the energy functional is defined by

\[
E(\phi) := \frac{1}{2} \int_{\mathbb{R}^N} \left(\nabla \phi(x)^2 - \frac{(N - 2)^2}{4|x|^2} |\phi(x)|^2 + 2F(|\phi|)\right) dx \quad \forall \phi \in X^1(\mathbb{R}^N).
\]

See [12, Theorem 5.1] for the ideas of proof.

**Remark 4.2.** We can also consider the Cauchy problem for Hartree equations with inverse-square potential of critical coefficient:

\[
(i \partial_t + \Delta - \frac{(N - 2)^2}{4|x|^2})u + (W * |u|^2)u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^N,
\]

where \(W\) satisfies
(W1) $W$ is a real-valued even function, that is, $W(-x) = W(x) \in \mathbb{R}$ a.a. $x \in \mathbb{R}^N$;

(W2) There exists $p \geq 1$ and $p > N/4$ such that $W \in L^\infty(\mathbb{R}^N) + L^p(\mathbb{R}^N)$;

(W3) There exists $q > N/2$ such that $W_- := -\min\{W, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$.

Then for every $u_0 \in X^1(\mathbb{R}^N)$ there exists a unique global weak solution $u \in C(\mathbb{R}; X^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; X^{-1}(\mathbb{R}^N))$ to \((HE)_{u(N)}\). Moreover, $u$ satisfies the conservation laws (4.2), where the energy functional $E$ is defined by

$$E(\varphi) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \varphi|^2 - \frac{(N-2)^2}{4|x|^2} |\varphi|^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} W(x-y)|\varphi(x)|^2 |\varphi(y)|^2 \, dx \, dy, \quad \varphi \in X^1(\mathbb{R}^N).$$

See [16, Theorem 1.3] for the ideas of proof.

Acknowledgement

The author would like to thank the referee for reading the original manuscript carefully. Especially, a lot of constructive comments helpful to make it as readable friendly as possible.

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(Ricevita la 13-an de januaro, 2014)
(Reviziita la 14-an de julio, 2014)