Global Existence of Solutions to a Parabolic-Elliptic System of Drift-Diffusion Type in $\mathbf{R}^2$

By

Toshitaka NAGAI and Takayoshi OGAWA

(Hiroshima University and Tohoku University, Japan)

Abstract. We consider the Cauchy problem of a parabolic-elliptic system in $\mathbf{R}^2$, which appears in various fields in physics and biology. Under a mild restriction on the initial data, we show the global existence of nonnegative solutions to the Cauchy problem for both the subcritical and the critical cases, that is, the total mass of the solution is less than or equal to 8π .

Key Words and Phrases. Interaction particle, Gravitational model, Chemotaxis model, Global existence, Subcritical case, Two dimensions.

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1. Introduction

We consider the following Cauchy problem of the parabolic-elliptic system in two space dimensions:

$$(1.1) \quad \begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbf{R}^2, \\ -\Delta \psi = u, & t > 0, x \in \mathbf{R}^2, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^2, \end{cases}$$

where the unknown function $u = u(t, x) : \mathbf{R}_+ \times \mathbf{R}^2 \rightarrow \mathbf{R}$ denotes the density of the particles and $\psi = \psi(t, x) : \mathbf{R}_+ \times \mathbf{R}^2 \rightarrow \mathbf{R}$ denotes the potential induced by the particle density and $u_0 = u_0(x)$ is a nonnegative initial data. This system appears in various fields of physics and biology. One of the important model described by (1.1) is the movement of the gravitational particles in astronomy. The above system is also connected to a simplest equation of the semiconductor device simulation (cf. Mock [41]) and a model of random walk Patlak [55]. The semi-conductor device model has a stabler sign of the nonlinearity that makes the system admits the large data global solutions.

We also mention that the semi-linear type of the equation is known as a model of chemotaxis called as the Keller-Segel system. Namely

$$(1.2) \quad \begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbf{R}^n, \\ \kappa \partial_t \psi - \Delta \psi + \lambda \psi = u, & t > 0, x \in \mathbf{R}^n, \end{cases}$$

where $\kappa \geq 0$, $\lambda > 0$, and $u \geq 0$ denotes the density of microorganisms and ψ the concentration of a chemoattractant secreted by themselves. The system (1.1) is reduced from the original parabolic system (1.2) with $\kappa > 0$ due to Keller-Segel [31] (see also [18]) under the condition that the chemical attraction is governed under large diffusive coefficient. It has been studied in detail for the behavior of the system by many authors Biler [4], Biler-Nadzieja [9], Gajewski-Zacharias [19], Herrero-Velázquez [26], Horstmann-Wang [28], Jäger-Luckhaus [29], Nagai [42], Nagai-Senba-Yoshida [48], Senba-Suzuki [58] and reference therein. Related results for chemotaxis models, see [25, 43, 47, 59], and for models of self-interacting particles, see [4, 6, 7, 13, 35, 52, 53, 64]. Many of these results for chemotaxis models also can be found in Horstmann [27] and Suzuki [61].

In this paper, we consider the global existence of the solution to (1.1) under some proper condition on the initial data. Before going to the global existence, we mention the time local existence and uniqueness. It is indeed strongly related to the results on the Navier-Stokes system in $\mathbf{R}^2$. The existence of the time local solution is essentially obtained by the similar argument of the case for the vorticity system of two dimensional Navier-Stokes equation:

$$(1.3) \quad \begin{cases} \partial_t \omega - \Delta \omega + \nabla \cdot (u\omega) = 0, & t > 0, x \in \mathbf{R}^2, \\ -\Delta u = \nabla^\perp \omega, & t > 0, x \in \mathbf{R}^2, \\ \omega(0, x) = \text{rot } u_0(x), & x \in \mathbf{R}^2, \end{cases}$$

where $u = (u_1(t, x), u_2(t, x))$ denotes the velocity fields for the incompressible flow and $\omega = \omega(t, x)$ is the vorticity of the fluid given by $\omega = \text{rot } u = \partial_1 u_2 - \partial_2 u_1$. The existence and uniqueness of the time local solution is given by Giga-Miyakawa-Osada [23], Giga-Miyakawa [22], Giga-Kambe [21], Kato [30], Ben-Arti [1], Brezis [15] (see also [20]). Following their arguments, we may obtain the time local existence of the solution in some proper space.

Let us define the solution of the two dimensional Poisson equation in (1.1) as

$$(1.4) \quad \psi(t, x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \log \frac{1}{|x - y|} u(t, y) dy.$$

We denote by $L^p(\mathbf{R}^2)$ the usual Lebesgue space with the norm $\|\cdot\|_p$ for $1 \leq p \leq \infty$ and by $W^{s,p}(\mathbf{R}^2)$ the Sobolev space with $s > 0$ and $1 \leq p \leq \infty$. For the following proposition on time local solutions of (1.1), see [45].

Proposition 1.1 (local existence and uniqueness). *Let $u_0 \in L^1(\mathbf{R}^2)$ satisfy $u_0 \log(1 + |x|) \in L^1(\mathbf{R}^2)$. Then there exists $T > 0$ and a unique mild solution (u, ψ) of (1.1) such that*

(1) u is in $C([0, T]; L^1(\mathbf{R}^2)) \cap C((0, T); L^{4/3}(\mathbf{R}^2))$ under the conditions

$$\sup_{0 < t < T} t^{1/4} \|u(t)\|_{4/3} < \infty, \quad \sup_{0 < t < T} \|u(t) \log(1 + |x|)\|_1 < \infty,$$

(2) (u, ψ) , where ψ is defined by (1.4), satisfies the following integral equation

$$u(t) = e^{tA}u_0 - \int_0^t \nabla \cdot e^{(t-s)A}(u(s)\nabla\psi(s))ds, \quad 0 < t < T,$$

where e^{tA} is the heat semigroup defined by

$$(e^{tA}f)(x) = \int_{\mathbf{R}^2} \frac{1}{4\pi t} e^{-|x-y|^2/(4t)} f(y) dy.$$

Moreover, (u, ψ) satisfies the following:

- (1) Both u and ψ are smooth and (u, ψ) is a classical solution of (1.1).
- (2) If $u_0(x) \geq 0$, $u_0 \not\equiv 0$ then $u(t, x) > 0$ for all $t \in (0, T)$ and $x \in \mathbf{R}^2$.
- (3) The total mass is conserved, namely $\int_{\mathbf{R}^2} u(t) dx = \int_{\mathbf{R}^2} u_0 dx$ for $0 < t < T$.

Remark. For (u, ψ) in Proposition 1.1, $\psi(t)$ is well-defined because of $u(t) \log(1 + |x|) \in L^1(\mathbf{R}^2)$, and u satisfies $\partial_t^k \partial_x^\ell u \in L^p(\mathbf{R}^2)$ for all $k \geq 0$, $\ell \geq 0$ and all $1 < p \leq \infty$ (see [45]).

The existence of the time local solution is also discussed in Kurokiba-Ogawa [34] in the weighted L^2 space such as

$$\int_{\mathbf{R}^2} |x|^{2s} |u(t)|^2 dx < \infty$$

for $s > 1$, that is included in $L^1(\mathbf{R}^2)$ and [36] for $s = 1$. Note that the uniqueness up to the constant for the Poisson equation is rather subtle (cf. Benilan-Brezis-Crandall [2]). Under this weight condition, the solution of the Poisson equation is determined uniquely up to constant.

We now concentrate our attention on the global existence of the solution. For the positive large data, it is known that the solution does not exist globally in time (see, [4, 43, 58, 59, 61]). On the other hand, the global existence of the solution is obtained for the bounded domain $\Omega \subset \mathbf{R}^2$ under the smallness assumption on the initial mass

$$\int_{\Omega} u_0(x) dx < 8\pi$$

when the solution is nonnegative. The argument heavily depends on the use of the free energy functional

$$(1.5) \quad F[u](t) \equiv \int_{\Omega} u(t) \log u(t) dx - \frac{1}{2} \|\nabla \psi(t)\|_2^2.$$

It is used that the free energy functional satisfies the following estimate

$$F[u](t) + \int_0^t \int_{\Omega} u |\nabla \log u(t) - \nabla \psi(t)|^2 dx dt \leq F[u_0]$$

so that the free energy is monotone decreasing in time and if the data is small then one may obtain the a priori estimate for the higher regularity functional of the solution such as $u(t) \in L^\infty(0, T; L \log L(\Omega))$. The standard parabolic regularity argument gives the boundedness of the solution u and hence the global existence for (1.1) for bounded domain with 0-Neumann boundary condition is obtained.

Similar argument is also valid for the chemotaxis model (1.2). Indeed, the system (1.1) also has analogous property in the structure. Formally we have that the free energy functional

$$(1.6) \quad F[u](t) \equiv \int_{\mathbf{R}^2} u(t) \log u(t) dx - \frac{1}{2} \|\nabla \psi(t)\|_2^2 - \frac{\lambda}{2} \|\psi(t)\|_2^2$$

is bounded by the initial data. However the proof of the global existence for (1.1) is rather complicated since the behavior of the solution of (1.1) is different from the one in (1.2). Namely one can not expect the free energy functional holding by its original form and it is not straightforward to derive the a priori bound for the solution. The proof of the a priori estimate for u in $L \log L(\mathbf{R}^2)$ space is formally very similar to the case of (1.2) by letting $\kappa = \lambda = 0$. However, there is a significant difference of the nature of the solution of the above system (1.1) from the case of the bounded domain or the solution of (1.2) because of the behavior of ψ at $|x| \rightarrow \infty$. If we discuss the solution of the first equation $u(t)$ in $L^1(\mathbf{R}^2)$, then applying the Gauss divergence theorem to the second equation, we see that for any $R' > R > 0$,

$$\begin{aligned} \|\nabla \psi(t)\|_2^2 &\geq \int_R^{R'} \int_0^{2\pi} |\partial_r \psi(r, \omega)|^2 r d\omega dr \\ &\geq \frac{1}{2\pi} \int_R^{R'} \left(\int_0^{2\pi} (\partial_r \psi(r, \omega)) r d\omega \right)^2 \frac{1}{r} dr \\ &= \frac{1}{2\pi} \int_R^{R'} \left(\int_{B_r} u(y) dy \right)^2 \frac{1}{r} dr \geq \frac{1}{2\pi} \left(\int_{B_R} u(y) dy \right)^2 \log \frac{R'}{R} \rightarrow \infty \end{aligned}$$

as $R' \rightarrow \infty$. Hence as far as treating the positive solution for the first equation, we can not assume that $\nabla \psi(t) \in L^2(\mathbf{R}^2)$ for any time. To avoid this difficulty, one may introduce the modified form of the free energy functional

$$H[u](t) \equiv \int_{\mathbf{R}^2} u(t) \log u(t) dx - \frac{1}{2} \int_{\mathbf{R}^2} u(t) \psi(t) dx,$$

which is non-increasing in t (cf. Blanchet-Dolbeault-Perthame [14]). $H[u](t)$ is formally equivalent to the original functional $F[u](t)$ in (1.6) for the case $\lambda = 0$ in view of the Poisson equation. However the justification of the equivalence is not quite straightforward and simple cut off argument does not seem to work well.

In this paper, we employ the following modified free energy functional

$$(1.7) \quad H[u](t) \equiv \int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) dx - \frac{1}{2} \int_{\mathbf{R}^2} u(t) \psi(t) dx.$$

The modified entropy $\int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) dx$ in (1.7) has the advantage of being nonnegative and has been used to get nonnegative global solutions for (1.2) with $\kappa > 0$ by Nagai-Ogawa [46] in case of $\int_{\mathbf{R}^2} u_0 dx < 4\pi$ and by Mizoguchi [40] in case of $\int_{\mathbf{R}^2} u_0 dx < 8\pi$. The entropy $\int_{\mathbf{R}^2} u(t) \log u(t) dx$ has been used for (1.2) with $\kappa > 0$ by Calvez-Corrias [17] in case of $\int_{\mathbf{R}^2} u_0 dx < 8\pi$ and by Mizoguchi [40] in case of $\int_{\mathbf{R}^2} u_0 dx = 8\pi$.

Our result on the global existence for (1.1) now reads as follows:

Theorem 1.2 (Global existence). *Suppose that $u_0 \in L^1(\mathbf{R}^2)$ is nonnegative and satisfies*

$$\int_{\mathbf{R}^2} u_0 \log(1 + |x|) dx < \infty.$$

Then under the condition

$$(1.8) \quad \int_{\mathbf{R}^2} u_0 dx \leq 8\pi,$$

the nonnegative solution (u, ψ) to (1.1) in Proposition 1.1 exists globally in time. Moreover (u, ψ) satisfies the properties mentioned in Proposition 1.1 for all $0 < T < \infty$, and for any $0 < t_0 < T < \infty$,

$$(1.9) \quad \int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) dx - \frac{1}{2} \int_{\mathbf{R}^2} u(t) \psi(t) dx \leq C_{t_0, T}, \quad t \in [t_0, T].$$

In the subcritical case $M := \int_{\mathbf{R}^2} u_0 dx < 8\pi$, the global existence of non-negative weak solutions of (1.1) has been studied by Blanchet-Dolbeault-Perthame [14] with the additional restrictions on the data;

$$(1.10) \quad (1 + |x|^2) u_0 \in L^1(\mathbf{R}^2),$$

$$\int_{\mathbf{R}^2} u_0 \log(1 + u_0) dx < \infty.$$

Under these conditions, the negative part of the free energy is controlled by the second moment identity

$$(1.11) \quad \int_{\mathbf{R}^2} |x|^2 u(t) dx dx = \int_{\mathbf{R}^2} |x|^2 u_0 dx + 4M \left(1 - \frac{M}{8\pi}\right) t.$$

Without additional weight restrictions on the nonnegative mild solution for $|x|$ large, Nagai [45] showed that the L^p -norm ($1 \leq p \leq \infty$) of the solution is less than or equal to that of a radially symmetric nonnegative self-similar solution to (1.1) by using rearrangement techniques, which gives the global existence and decay estimates of the solution.

The critical mass for the existence of the global solutions is given by $M = 8\pi$. When the initial data have the critical mass $M = 8\pi$ there are some results known. See for instance [8, 56] for radially symmetric nonnegative solutions. Without symmetry assumptions on the solutions, Blanchet-Carrillo-Masmoudi [13] proved that the nonnegative solutions with finite second moment exist globally in time and converge to a Dirac delta function as time goes to infinity. The system (1.1) for the critical mass case admits the following stationary solutions

$$w(x; b) = \frac{8b}{(|x|^2 + b)^2}, \quad b > 0,$$

which have no finite second moment. Blanchet-Carlen-Carrillo [11] have studied the global existence and long time behavior for the nonnegative solutions whose second moment is infinite but first moment is finite, by introducing the following functional

$$H_b[u] = \int_{\mathbf{R}^2} (\sqrt{u(x)} - \sqrt{w(x; b)})^2 (w(x; b))^{-1/2} dx,$$

which comes from the critical fast diffusion equation in $\mathbf{R}^2$. See also López-Gómez-Nagai-Yamada [39] for global existence and large time behavior.

For the supercritical case $M > 8\pi$, under suitable assumption, the positive solution blows up in a finite time (see [4, 3, 9, 10, 42, 43] under the radially and non radially symmetric cases and [34] under non finite second moment case).

We emphasize that our result on the global existence of nonnegative solutions does not require a finite first moment of the solution and it holds true for all nonnegative initial data $u_0 \in L^1(\mathbf{R}^2)$ with the least weighted condition $u_0 \log(2 + |x|) \in L^1(\mathbf{R}^2)$. In order to prove Theorem 1.2, by virtue of Proposition 1.3 below (for the proof, see [45]) it suffices to prove the

following: For a time local nonnegative solution u over arbitrary time interval $[0, T)$,

$$(1.12) \quad \limsup_{t \rightarrow T} \int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) dx < \infty.$$

This is proved in Theorem 3.1 in Section 3.

Proposition 1.3. *Let T_m be the maximal existence time of the nonnegative solution u . If $T_m < \infty$, then*

$$(1.13) \quad \limsup_{t \rightarrow T_m} \int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) dx = \infty.$$

Remark. In [13, 14], under the assumptions $u_0 \log u_0, u_0 |x|^2 \in L^1(\mathbf{R}^2)$ it was proved that if $T_m < \infty$, then

$$(1.14) \quad \limsup_{t \rightarrow T_m} \int_{\mathbf{R}^2} u(t) \log u(t) dx = \infty.$$

However (1.13) and (1.14) are equivalent under the weaker assumption $u_0 \log(2 + |x|) \in L^1(\mathbf{R}^2)$ by virtue of Lemma 2.3 in Section 2.

As we have mentioned above, the usual free energy functional $F(t)$ in (1.5) is useful for the bounded domain $\Omega \subset \mathbf{R}^2$, however it is not suitable for the whole space case since the first and the second terms are not well behaved. Instead of that, to prove (1.12) we use the modified free energy functional $H(t)$ in (1.7). The key idea to show the global a priori bound is that we split the whole space into three pieces; a bounded ball (interior region), an exterior region and the annulus that connects the ball and the exterior region:

$$\mathbf{R}^2 = B_R \cup A_R \cup E_R,$$

where

$$\begin{aligned} B_R &= \{x \in \mathbf{R}^2; |x| < R\}; & \text{interior region,} \\ A_R &= \{x \in \mathbf{R}^2; R/2 \leq |x| \leq 2R\}; & \text{connecting annulus,} \\ E_R &= \{x \in \mathbf{R}^2; R < |x|\}; & \text{exterior region.} \end{aligned}$$

The difficulty is that since the equation has the non-local interaction, such a localization is not quite possible and there remains a term involving the integral on the out side of each regions. Thanks to the special situation of the two space dimensions and the preferable behavior of the integral kernel of $(-\Delta)^{-1}$, it is controllable when we consider the estimate in the exterior region E_R with sufficiently large $R > 0$. Then it enable us to derive the a priori estimate in the exterior region (see Proposition 3.2 below). The estimate for the annulus A_R follows from the estimate in E_R . Finally, we derive the interior estimate by

using the critical type Sobolev inequality due to Brezis-Merle [16]. Namely, for a bounded domain $\Omega \subset \mathbf{R}^2$, Brezis-Merle showed that for a solution v to the Poisson equation;

$$(1.15) \quad \begin{cases} -\Delta v = f, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases}$$

it holds that

$$(1.16) \quad \int_{\Omega} e^{|v(x)|} dx \leq \frac{Cd(\Omega)^2}{4\pi - \|f\|_1},$$

provided f in $L^1(\Omega)$ and $\|f\|_1 < 4\pi$, where $d(\Omega)$ is the diameter of Ω (see Lemma 2.7 below). With the aid of a priori estimates over the annulus A_R , we derive

$$\sup_{0 < t < T} \int_{B_R} u(t) dx < 8\pi$$

and then, applying the Brezis-Merle inequality (1.16) we establish the global bound for the modified entropy over B_R (see Proposition 3.16 below). This procedure makes us possible to obtain the time global a priori estimate for the modified entropy $\int_{\mathbf{R}^2} (1+u) \log(1+u) dx$.

It is also worth to mention that the two dimensional case is both L^1 critical case and the Sobolev critical case. Those critical cases can be observed by regarding the quasilinear parabolic type system with the degenerate diffusion $-\Delta u^\alpha$ instead of $-\Delta u$ in the equation (cf [12, 32, 33, 50, 51, 60, 62]). When $\alpha = 2 - 2/n$ it is the L^1 critical case and the L^1 norm of the initial data can determine the large time behavior of the solution while if $\alpha = 2 - 4/(n+2)$ then the (dual) Sobolev critical norm $L^{2n/(n+2)}$ of the data determines the large time behavior ([50, 51]). When $n = 2$, those critical exponents $\alpha = 2 - 2/n$ and $\alpha = 2 - 4/(n+2)$ coincides and the semi-linear case is the doubly critical and the critical mass 8π is the simultaneous threshold in the both critical cases. We also note that higher dimensional case for the system (1.1) is considered in [3, 37, 54, 57] and the solution blows up in a finite time in a weaker assumptions.

In the paper, we use a universal constant C to denote a various constant, and C_η when C depends on η . $c(*, \dots, *)$ denotes a constant depending on the quantities appearing in parentheses such that $c(*, \dots, *)$ increases with the quantities. Especially, $c(u_0, *, \dots, *)$ denotes a constant depending on $\int_{\mathbf{R}^2} (2+u_0) \log(2+u_0) dx$ and the other quantities appearing in parentheses such that $c(u_0, *, \dots, *)$ increases with these quantities. The Lebesgue measure is written by $|A|$, where A is a measurable set in $\mathbf{R}^n$. $B_r(x)$ stands for the Euclidean open ball of radius r centered at x and $B_r = B_r(0)$.

This paper is organized as follows. In Section 2 we give some function inequalities such as the Sobolev inequalities, the BMO estimate, the Hardy-Littlewood-Sobolev inequality and the Brezis-Merle inequality. In Section 3 we establish the a priori estimate of the modified entropy $\int_{\mathbf{R}^2} (1+u) \log(1+u) dx$ for the nonnegative solution (see Theorem 3.1), which implies the global existence of the solution. Subsections 3.1 and 3.2 are devoted to the a priori estimates in the exterior region, Subsection 3.3 in the annulus and Subsection 3.4 in the interior region.

2. Preliminary

Before proving our main theorem, we prepare some function inequalities.

Lemma 2.1. (1) *Let $1 \leq q < \infty$. There exists a positive constant C depending only on q such that for any $g \in L^1(\mathbf{R}^2)$ with $|\nabla g| \in L^2(\mathbf{R}^2)$,*

$$(2.1) \quad \|g\|_q \leq C \|g\|_1^{1-\lambda} \|\nabla g\|_2^\lambda, \quad \lambda = 1 - \frac{1}{q}.$$

(2) *There exists a positive constant C such that for any $g \in L^2(\mathbf{R}^2)$ with $|\nabla g| \in L^1(\mathbf{R}^2)$,*

$$(2.2) \quad \|g\|_2 \leq \frac{1}{\sqrt{2}} \|\nabla g\|_1.$$

For this lemma, see [24, 49] for example.

Lemma 2.2. *For any $f \in W^{2,2}(\mathbf{R}^2)$ and nonnegative cut-off function $\phi \in C_0^\infty(\mathbf{R}^2)$, we have the following estimates:*

$$(1) \quad \int_{\mathbf{R}^2} |f|^2 \phi \, dx \leq 2 \left(\int_{\{|f|>1\} \cap \text{supp}[\phi]} |f| \, dx \right) \left(\int_{|f|>1} \frac{|\nabla f|^2}{1+|f|} \phi \, dx \right) \\ + \left(\int_{\mathbf{R}^2} |f| |\nabla \phi^{1/2}| \, dx \right)^2 + 4 \left(\int_{\mathbf{R}^2} |f| \phi \, dx \right).$$

$$(2) \quad \int_{\mathbf{R}^2} |f|^2 \phi \, dx \leq \varepsilon \left(\int_{\text{supp}[\phi]} (1+|f|) \log(1+|f|) \, dx \right) \left(\int_{\mathbf{R}^2} \frac{|\nabla f|^2}{1+|f|} \phi \, dx \right) \\ + 4 \left(\int_{\mathbf{R}^2} |f \nabla \phi^{1/2}| \, dx \right)^2 + C(\varepsilon) \left(\int_{\mathbf{R}^2} f \phi \, dx \right),$$

where ε is any positive number and $C(\varepsilon) = O(e^{\varepsilon^{-1}})$ as $\varepsilon \rightarrow 0$.

(3)

$$\begin{aligned} \int_{\mathbf{R}^2} |f|^3 \phi \, dx &\leq \varepsilon \left(\int_{\text{supp}[\phi]} (1 + |f|) \log(1 + |f|) dx \right) \left(\int_{\mathbf{R}^2} |\nabla f|^2 \phi \, dx \right) \\ &\quad + C \left(\int_{\mathbf{R}^2} |f|^{3/2} |\nabla \phi^{1/2}| dx \right)^2 + C(\varepsilon) \int_{\mathbf{R}^2} |f| \phi \, dx, \end{aligned}$$

where ε is any positive number and $C(\varepsilon) = O(e^{3\varepsilon^{-1}})$ as $\varepsilon \rightarrow 0$.

Proof. Without losing generality, we assume that f is a nonnegative function. Now since $\phi(x) \geq 0$, setting $d\mu(x) = \phi(x)dx$ and letting $\mu_f(t) = \mu(\{x; f(x) > t\})$ for $f \geq 0$, we have

$$\begin{aligned} (2.5) \quad \int_{\mathbf{R}^2} |f(x)|^2 d\mu &= 2 \int_{\alpha}^{\infty} t \mu_f(t) dt + 2 \int_0^{\alpha} t \mu_f(t) dt \\ &\leq 2 \int_0^{\infty} (t + \alpha) \mu_f(t + \alpha) dt + 2 \int_0^{\alpha} t \mu_f(t) dt \\ &\leq 2 \int_0^{\infty} t \mu_{f-\alpha}(t) dt + 2\alpha \int_0^{\infty} \mu_{f-\alpha}(t) dt + 2 \int_0^{\alpha} t \mu_f(t) dt \\ &\leq 2 \int_{f>\alpha} |f - \alpha|^2 d\mu + 2\alpha \int_{f>\alpha} |f - \alpha| d\mu + 2\alpha \int_0^{\alpha} \mu_f(t) dt. \end{aligned}$$

By applying (2.2) as $g = (f - \alpha)_+ \phi^{1/2}$, where $(f - \alpha)_+$ is the positive part of $f - \alpha$, it follows that

$$\begin{aligned} (2.6) \quad \int_{\mathbf{R}^2} |f(x)|^2 \phi \, dx &\leq 2 \int_{f>\alpha} |f - \alpha|^2 \phi \, dx + 4\alpha \int_{\mathbf{R}^2} |f| \phi \, dx \\ &\leq \left(\int_{f>\alpha} |\nabla f| \phi^{1/2} \, dx \right)^2 \\ &\quad + \left(\int_{f>\alpha} |f| |\nabla \phi^{1/2}| \, dx \right)^2 + 4\alpha \int_{\mathbf{R}^2} |f| \phi \, dx. \end{aligned}$$

To show (2.3), we take $\alpha = 1$ in (2.6) and use Hölder's inequality to get

$$\begin{aligned} \left(\int_{f>1} |\nabla f| \phi^{1/2} \, dx \right)^2 &\leq \left(\int_{\{f>1\} \cap \text{supp}[\phi]} f \, dx \right) \left(\int_{f>1} \frac{1}{f} |\nabla f|^2 \phi \, dx \right) \\ &\leq 2 \left(\int_{\{f>1\} \cap \text{supp}[\phi]} f \, dx \right) \left(\int_{f>1} \frac{|\nabla f|^2}{1+f} \phi \, dx \right). \end{aligned}$$

Hence we have (2.3).

Next, to show (2.4), by Hölder's inequality we calculate as follows.

$$\begin{aligned}
& \left(\int_{f>\alpha} |\nabla f| \phi^{1/2} dx \right)^2 \\
& \leq \left(\int_{\{f>\alpha\} \cap \text{supp}[\phi]} (1+f) dx \right) \left(\int_{f>\alpha} \frac{|\nabla f|^2}{1+f} \phi dx \right) \\
& \leq \frac{1}{\log(1+\alpha)} \left(\int_{\{f>\alpha\} \cap \text{supp}[\phi]} (1+f) \log(1+f) dx \right) \left(\int_{f>\alpha} \frac{|\nabla f|^2}{1+f} \phi dx \right) \\
& \leq \frac{1}{\log(1+\alpha)} \left(\int_{\text{supp}[\phi]} (1+f) \log(1+f) dx \right) \left(\int_{\mathbf{R}^2} \frac{|\nabla f|^2}{1+f} \phi dx \right).
\end{aligned}$$

Putting $\varepsilon^{-1} = \log(1+\alpha)$, we conclude (2.4).

Using calculations similar to (2.5), we have

$$\int_{\mathbf{R}^2} |f(x)|^3 \phi(x) dx \leq \frac{5}{3} \int_{f>\alpha} |f-\alpha|^3 \phi dx + 18\alpha^2 \int_{f>\alpha} |f| \phi dx.$$

Applying (2.2) as $g = (f-\alpha)_+^{3/2} \phi^{1/2}$ yields that

$$\begin{aligned}
& \int_{f>\alpha} |f-\alpha|^3 \phi dx \\
& \leq \frac{1}{2} \left(\int_{f>\alpha} |\nabla \{(f-\alpha)^{3/2} \phi^{1/2}\}| dx \right)^2 \\
& \leq \frac{9}{4} \left(\int_{f>\alpha} |f-\alpha|^{1/2} |\nabla f| \phi^{1/2} dx \right)^2 + \left(\int_{f>\alpha} |f-\alpha|^{3/2} |\nabla \phi^{1/2}| dx \right)^2 \\
& \leq 3 \left(\int_{\{f>\alpha\} \cap \text{supp}[\phi]} |f-\alpha| dx \right) \left(\int_{f>\alpha} |\nabla f|^2 \phi dx \right) + \left(\int_{f>\alpha} |f|^{3/2} |\nabla \phi^{1/2}| dx \right)^2 \\
& \leq \frac{3}{\log(1+\alpha)} \left(\int_{\{f>\alpha\} \cap \text{supp}[\phi]} (1+f) \log(1+f) dx \right) \left(\int_{f>\alpha} |\nabla f|^2 \phi dx \right) \\
& \quad + \left(\int_{f>\alpha} |f|^{3/2} |\nabla \phi^{1/2}| dx \right)^2.
\end{aligned}$$

This implies the desired estimate by putting $\varepsilon = 3/\log(1+\alpha)$. $\square$

Lemma 2.3. *Let f be a nonnegative measurable function on a measurable set Ω in $\mathbf{R}^n$ satisfying $f \log(2+|x|) \in L^1(\Omega)$. The following conditions (i) and (ii) are equivalent.*

- (i) $f \log f \in L^1(\Omega)$,
- (ii) $(f + 1) \log(f + 1) \in L^1(\Omega)$.

More precisely,

$$(2.7) \quad \int_{\Omega} (f + 1) \log(f + 1) dx \leq 2 \int_{\Omega} f |\log f| dx + (2 \log 2) \int_{\Omega} f dx,$$

$$(2.8) \quad \int_{\Omega} f |\log f| dx \leq \int_{\Omega} (f + 1) \log(f + 1) dx \\ + 2\alpha \int_{\Omega} f \log(2 + |x|) dx + \frac{1}{e} \int_{\Omega} \frac{1}{(2 + |x|)^{\alpha}} dx,$$

where $n < \alpha < \infty$.

Proof. By the inequality

$$(1 + a) \log(1 + a) \leq 2a |\log a| + (2 \log 2)a, \quad a \geq 0,$$

we have

$$(1 + f(x)) \log(1 + f(x)) \leq 2f(x) |\log f(x)| + (2 \log 2)f(x).$$

Integrating this on Ω implies (2.7).

We claim the following inequality: For $a \geq 0$, $b > 0$,

$$(2.9) \quad a |\log a| \leq (a + 1) \log(a + 1) + 2a |\log b| + e^{-1}b.$$

In fact, $|(a/b) \log(a/b)| \leq e^{-1}$ for $a/b \leq 1$. Hence $a |\log a| \leq e^{-1}b + a |\log b|$. Next, for $a/b > 1$, $|\log(a/b)| \leq |\log((a + 1)/b)|$, from which it follows that $|\log a| \leq \log(a + 1) + 2 |\log b|$. Hence $a |\log a| \leq (a + 1) \log(a + 1) + 2a |\log b|$. Thus we obtain the claim.

Put $a = f(x)$, $b = (2 + |x|)^{-\alpha}$ ($n < \alpha < \infty$) in (2.9). Then

$$f(x) |\log f(x)| \leq (f(x) + 1) \log(f(x) + 1) + 2\alpha f(x) \log(2 + |x|) + e^{-1}(2 + |x|)^{-\alpha}.$$

Integrating this on Ω implies (2.8). □

We next consider the Poisson equation

$$\begin{cases} -\Delta \psi = f & \text{in } \mathbf{R}^2, \\ f \in L^1(\mathbf{R}^2) \end{cases}$$

and define the solution of the Poisson equation by the Newtonian potential

$$(2.10) \quad \psi = \frac{1}{2\pi} \log|x|^{-1} * f \quad \text{in } \mathcal{S}^*,$$

where $\mathcal{S}^*$ is the space of tempered distributions on $\mathbf{R}^2$. Let $L \log L(\mathbf{R}^n)$ be the Orlicz class of functions defined by

$$\int_{\mathbf{R}^n} (1 + |f(x)|) \log(1 + |f(x)|) dx < \infty,$$

and $BMO(\mathbf{R}^n)$ the class of functions of the bounded mean oscillations defined by

$$\|f\|_{BMO} \equiv \sup_{R>0, x} \frac{1}{|B_R|} \int_{B_R(x)} |f(y) - [f]_{B_R(x)}| dy < \infty,$$

where $[f]_{B_R(x)}$ is the average of f over the ball $B_R(x)$ of radius R centered at x and $|B_R|$ is the Lebesgue measure of B_R . For simplicity we denote $L \log L(\mathbf{R}^2)$ and $BMO(\mathbf{R}^2)$ by $L \log L$ and BMO , respectively. If f satisfies $f \in L \log L$ then the solution ψ becomes a locally bounded function and if $f \log(1 + |x|) \in L^1(\mathbf{R}^2)$ then the solution is locally integrable, and in both case, ψ satisfies the Poisson equation in the sense of distribution. Note that in this case

$$\nabla \psi(x) = -\frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{x-y}{|x-y|^2} f(y) dy \quad \text{a.e. in } \mathbf{R}^2.$$

The following proposition is well known for the solution of two dimensional Poisson equation:

Proposition 2.4 (BMO estimate). *Let $f \log(2 + |x|) \in L^1(\mathbf{R}^2)$ and ψ be a corresponding solution of the Poisson equation given by (2.10). Then we have that $\psi \in BMO$ and*

$$\|\psi\|_{BMO} \leq C \|f\|_1,$$

where C is a constant independent of f .

Remark. This is well known and can be generalize such that f is in a class of the Radon measure or the higher dimension cases $n \geq 3$ with $f \in L^{n/2}(\mathbf{R}^n)$. The Brezis-Merle inequality for the elliptic case is a corollary of the above theorem and the well known John-Nirenberg inequality.

The following lemma is frequently used in the course of the proofs. For the proof of the lemma, see [63] for example.

Lemma 2.5. *Let $\psi \in BMO(\mathbf{R}^2)$ and $\psi_m = \psi - [\psi]_{B_R}$, where $[\psi]_{B_R}$ is the average of ψ over a ball B_R . Then, for any $1 \leq p < \infty$,*

$$\left(R^{-2} \int_{B_R} |\psi_m|^p dx \right)^{1/p} \leq C \|\psi\|_{BMO},$$

where C is a positive constant depending only on p .

In order to obtain a priori estimates for $\nabla\psi$, we need the Hardy-Littlewood-Sobolev inequality (see [63] for example).

Lemma 2.6 (The Hardy-Littlewood-Sobolev inequality). *Let $0 < \mu < n$ and $1 < p, q < \infty$ with $1/q = 1/p - \mu/n$. Then there exists a positive constant C depending only on p, q, n, μ such that for any $f \in L^p(\mathbf{R}^n)$,*

$$\| |x|^{-n+\mu} * f \|_{L^q(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)},$$

where the symbol $*$ stands for the convolution.

The following lemma about the Poisson equation in $\mathbf{R}^2$ is a consequence of the Brezis-Merle inequality under zero Dirichlet boundary conditions.

Lemma 2.7. *Let Ω be a bounded domain in $\mathbf{R}^2$ with smooth boundary. For $g \in L^2(\Omega)$, let $v \in W^{2,2}(\Omega)$ be a solution of $-\Delta v = g$ in Ω . If $\|g\|_{L^1(\Omega)} < 4\pi$, then*

$$\int_{\Omega} e^{|v(x)|} dx \leq \frac{4\pi^2}{4\pi - \|g\|_{L^1(\Omega)}} d(\Omega)^2 \exp \left\{ \sup_{\partial\Omega} |v(x)| \right\},$$

where $d(\Omega)$ is the diameter of Ω .

Proof. Let $w \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ be a solution of

$$-\Delta w = g \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

We recall the Brezis-Merle inequality (Theorem 1, [16]): For every $\delta \in (0, 4\pi)$,

$$\int_{\Omega} \exp \left\{ \frac{(4\pi - \delta)|w(x)|}{\|g\|_{L^1(\Omega)}} \right\} dx \leq \frac{4\pi^2}{\delta} d(\Omega)^2.$$

Taking $\delta = 4\pi - \|g\|_{L^1(\Omega)} (> 0)$ implies

$$\int_{\Omega} e^{|w(x)|} dx \leq \frac{4\pi^2}{4\pi - \|g\|_{L^1(\Omega)}} d(\Omega)^2.$$

Since $\Delta(v - w) = 0$ in Ω , the maximum principle implies

$$|v(x) - w(x)| \leq \sup_{\partial\Omega} |v - w| = \sup_{\partial\Omega} |v|, \quad x \in \Omega.$$

Hence

$$\int_{\Omega} e^{|v(x)|} dx \leq \int_{\Omega} e^{|w(x)|} e^{|v(x)-w(x)|} dx \leq \frac{4\pi^2}{4\pi - \|g\|_{L^1(\Omega)}} d(\Omega)^2 \exp \left\{ \sup_{\partial\Omega} |v(x)| \right\}. \quad \square$$

3. A priori estimates for the Cauchy problem (1.1)

In this section, we assume that for given nontrivial initial data u_0 satisfying

$$(3.1) \quad u_0 \geq 0, \quad u_0 \in L^1(\mathbf{R}^2), \quad \int_{\mathbf{R}^2} u_0 \log(1 + |x|) dx < \infty,$$

there exists a time local nonnegative solution u for the Cauchy problem (1.1) over arbitrary time interval $[0, T)$ with ψ defined by (1.4) such that (u, ψ) is smooth and have a proper integrability. We prove the following theorem, which implies (1.12) and hence, the global existence of the nonnegative solution (u, ψ) .

Theorem 3.1. *Under the assumption*

$$\int_{\mathbf{R}^2} u_0 dx \leq 8\pi,$$

it holds that for $0 < t_0 < T$,

$$(3.2) \quad \sup_{t_0 \leq t < T} \int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) dx < \infty.$$

As we mentioned in the introduction, the proof of the a priori estimate (3.2) requires a careful treatment on the behavior of the solution ψ . In order to do this, we split the derivation of the crucial a priori estimate into two parts, namely, exterior region and interior region. We first prove (3.2) for the exterior regions (see Proposition 3.2) and then for the interior regions (see Proposition 3.16), and hence conclude Theorem 3.1.

Since the nontrivial nonnegative solution u is positive on $\mathbf{R}^2$ for $0 < t < T$ and has smooth properties mentioned in Proposition 1.1 and Remark, taking $t = t_0 \in (0, T)$ as the initial time, throughout this section we may assume, in addition to (3.1), that u_0 is positive on $\mathbf{R}^2$ and

$$(1 + u_0) \log(1 + u_0) \in L^1(\mathbf{R}^2), \quad u_0 \in W^{k,p}(\mathbf{R}^2)$$

for all $k \geq 0$ and all $1 < p \leq \infty$.

3.1. Entropy estimates for exterior regions

We begin with the estimate for large $|x|$. In this case, the a priori estimate can be obtained by the smallness of the solution u for sufficiently large $|x|$. For the solution u , let

$$H_{ext}(t; R) \equiv \int_{|x| \geq R} ((1 + u(t)) \log(1 + u(t)) - u(t)) dx.$$

Proposition 3.2. *There exists a sufficiently large number R_0 depending only on T and $\|u_0\|_1$ such that*

$$(3.3) \quad \sup_{0 \leq t < T} H_{ext}(t; R_0) + \int_0^T \int_{|x| \geq R_0} \frac{|\nabla u(t)|^2}{1+u(t)} dx dt \leq 2H_{ext}(0; R_0/2) + c(\|u_0\|_1)T,$$

$$(3.4) \quad \int_0^T \int_{|x| \geq R_0} u^2(t) dx dt \leq C\|u_0\|_1 H_{ext}(0; R_0/2) + c(\|u_0\|_1)T,$$

where $c(\|u_0\|_1)$ is a positive constant depending on $\|u_0\|_1$ and $C > 0$ a constant independent of u_0 and T .

Remark 3.3. The assumption $\int_{\mathbf{R}^2} u_0 dx \leq 8\pi$ is not required to prove Proposition 3.2.

Before proving this proposition, we introduce the Littlewood-Paley partition of the unity.

Definition 3.1. Let $\phi(r)$ be a function in $C_0^\infty(0, \infty)$ such that

$$\phi(r) = \begin{cases} 0, & 0 \leq r \leq 1/2, \\ \leq 1, \text{ smooth, positive,} & 1/2 < r < 2. \end{cases}$$

Then a family of the cut off functions $\{\Phi_n(x)\}_{n=0}^\infty$ is called as the Littlewood-Paley partition of the unity if they satisfies

$$\Phi_n(x) = \phi(2^{-n}|x|), \quad x \in \mathbf{R}^2, n = 0, 1, 2, \dots$$

and

$$\sum_{n=0}^\infty \Phi_n(x) = \begin{cases} 1, & 1 < |x|, \\ \text{smooth, positive,} & 1/2 < |x| \leq 1, \\ 0, & |x| < 1/2. \end{cases}$$

The Littlewood-Paley decomposition of the unity has the following properties: Observing that $\phi^{1/6}(r)$ has the same support as $\phi(r)$, we have

$$|\phi'(r)| = 6\phi^{5/6}(r)|(\phi^{1/6})'(r)| \leq C\phi(r)^{5/6}, \quad |\phi''(r)| \leq C\phi(r)^{2/3} \quad (r \geq 0).$$

Then it follows from the construction of $\{\Phi_n\}_n$ that

$$\Phi_n(x) = \begin{cases} 0, & |x| \leq 2^{n-1}, |x| \geq 2^{n+1}, \\ \leq 1, \text{ smooth, positive,} & 2^{n-1} < |x| < 2^{n+1}, \end{cases}$$

and

$$(3.5) \quad \begin{aligned} |\partial_x \Phi_n| &\leq C 2^{-n} \Phi_n^{5/6}, & |\partial_x \Phi_n^{1/2}| &\leq C 2^{-n} \Phi_n^{1/3}, \\ |\partial_x^2 \Phi_n| &\leq C 2^{-2n} \Phi_n^{2/3}, \end{aligned}$$

where $C > 0$ is a constant independent of n .

We also denote the annulus by

$$A_n = \{x \in \mathbf{R}^2; 2^{n-1} \leq |x| \leq 2^{n+1}\}$$

so that $\text{supp } \Phi_n \subset A_n$. If we set

$$\tilde{\Phi}_n(x) = \Phi_{n-1}(x) + \Phi_n(x) + \Phi_{n+1}(x),$$

then we have by its construction, that

$$\tilde{\Phi}_n(x) \Phi_n(x) = \Phi_n(x)$$

and

$$(3.6) \quad \tilde{\Phi}_n(x) = 1, \quad x \in A_n.$$

Proof of Proposition 3.2. For fixed n , we let $\psi_m(t) = \psi(t) - [\psi(t)]_B$, where $[\psi(t)]_B = (1/|B|) \int_B \psi(t) dx$, $B = B_{2^{n+2}}(0)$. The equation exchanging $\psi(t)$ into $\psi_m(t)$ also holds, that is, $-\Delta \psi_m = u$. We observe that by the equation of u ,

$$\partial_t \{((1+u) \log(1+u) - u) \Phi_n\} = \Delta u \log(1+u) \Phi_n - (\nabla \cdot (u \nabla \psi_m)) \log(1+u) \Phi_n,$$

and that

$$\begin{aligned} \Delta u \log(1+u) \Phi_n &= \nabla \cdot \{ \nabla u \log(1+u) \Phi_n - ((1+u) \log(1+u) - u) \nabla \Phi_n \} \\ &\quad + ((1+u) \log(1+u) - u) \Delta \Phi_n - \frac{|\nabla u|^2}{1+u} \Phi_n, \\ -(\nabla \cdot (u \nabla \psi_m)) \log(1+u) \Phi_n &= -\nabla \cdot \{ u \log(1+u) \nabla \psi_m \Phi_n \} \\ &\quad + u \log(1+u) \nabla \psi_m \cdot \nabla \Phi_n \\ &\quad + u \nabla \log(1+u) \cdot \nabla \psi_m \Phi_n. \end{aligned}$$

Hence

$$\begin{aligned} (3.7) \quad \partial_t \{((1+u) \log(1+u) - u) \Phi_n\} &+ \frac{|\nabla u|^2}{1+u} \Phi_n \\ &= \nabla \cdot \{ \nabla u \log(1+u) \Phi_n - ((1+u) \log(1+u) - u) \nabla \Phi_n \\ &\quad - u \log(1+u) \nabla \psi_m \Phi_n \} \\ &\quad + ((1+u) \log(1+u) - u) \Delta \Phi_n + u \log(1+u) \nabla \psi_m \cdot \nabla \Phi_n \\ &\quad + u \nabla \log(1+u) \cdot \nabla \psi_m \Phi_n. \end{aligned}$$

We rewrite the last two terms on the right hand side of (3.7) as follows.

$$\begin{aligned}
(3.8) \quad & u \log(1+u) \nabla \psi_m \cdot \nabla \Phi_n + u \nabla \log(1+u) \cdot \nabla \psi_m \Phi_n \\
&= u \log(1+u) \nabla \psi_m \cdot \nabla \Phi_n + \nabla(u - \log(1+u)) \cdot \nabla \psi_m \Phi_n \\
&= \nabla \cdot \{(u - \log(1+u)) \nabla \psi_m \Phi_n\} + ((1+u) \log(1+u) - u) \nabla \psi_m \cdot \nabla \Phi_n \\
&\quad - (u - \log(1+u)) \Delta \psi_m \Phi_n \\
&= \nabla \cdot \{(u - \log(1+u)) \nabla \psi_m \Phi_n + ((1+u) \log(1+u) - u) \psi_m \nabla \Phi_n\} \\
&\quad - \log(1+u) \psi_m \nabla u \cdot \nabla \Phi_n - ((1+u) \log(1+u) - u) \psi_m \Delta \Phi_n \\
&\quad - (u - \log(1+u)) \Delta \psi_m \Phi_n.
\end{aligned}$$

Put $-\Delta \psi_m = u$ to the last term on the right hand side of (3.8), substitute (3.8) into (3.7) and then integrate (3.7) on $\mathbf{R}^2$. We then have

$$\begin{aligned}
(3.9) \quad & \frac{d}{dt} \int_{\mathbf{R}^2} ((1+u(t)) \log(1+u(t)) - u(t)) \Phi_n \, dx + \int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n \, dx \\
&= \int_{\mathbf{R}^2} (u^2(t) - u(t) \log(1+u(t))) \Phi_n \, dx \\
&\quad + \int_{\mathbf{R}^2} ((1+u(t)) \log(1+u(t)) - u(t)) \Delta \Phi_n \, dx \\
&\quad - \int_{\mathbf{R}^2} \log(1+u(t)) \psi_m(t) \nabla u(t) \cdot \nabla \Phi_n \, dx \\
&\quad - \int_{\mathbf{R}^2} ((1+u(t)) \log(1+u(t)) - u(t)) \psi_m(t) \Delta \Phi_n \, dx \\
&= I + II + III + IV.
\end{aligned}$$

For the first term on the right hand side of (3.9), noting $0 \leq u^2 - u \log(1+u)$, we apply (2.3) for the solution u to have

$$\begin{aligned}
(3.10) \quad & |I| \leq \int_{\mathbf{R}^2} u^2(t) \Phi_n \, dx \\
&\leq 2 \left(\int_{A_n} u(t) \, dx \right) \left(\int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n \, dx \right) + \left(\int_{\mathbf{R}^2} u(t) |\nabla \Phi_n^{1/2}| \, dx \right)^2 \\
&\quad + 4 \int_{\mathbf{R}^2} u(t) \Phi_n \, dx,
\end{aligned}$$

where

$$A_n = \{x \in \mathbf{R}^2; 2^{n-1} \leq |x| \leq 2^{n+1}\}.$$

Using (3.5) and the non-negativeness of the solution, we have

$$(3.11) \quad \left(\int_{\mathbf{R}^2} u(t) |\nabla \Phi_n^{1/2}| dx \right)^2 \leq C 2^{-2n} \left(\int_{\mathbf{R}^2} u(t) dx \right)^2 = C 2^{-2n} \|u_0\|_1^2.$$

Hence by (3.10) and (3.11),

$$(3.12) \quad |I| \leq 2 \left(\int_{A_n} u(t) dx \right) \left(\int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n dx \right) + C 2^{-2n} \|u_0\|_1^2 + 4 \int_{\mathbf{R}^2} u(t) \Phi_n dx.$$

For the second term on the right hand side of (3.9), since

$$0 \leq (1+u) \log(1+u) - u \leq C \begin{cases} u & (0 \leq u \leq 1), \\ u^{4/3} & (u \geq 1), \end{cases}$$

by (3.5) and Hölder's inequality we have

$$(3.13) \quad \begin{aligned} |II| &\leq C 2^{-2n} \int_{0 \leq u \leq 1} u(t) \Phi_n^{2/3} dx + C 2^{-2n} \int_{u \geq 1} u^{4/3}(t) \Phi_n^{2/3} dx \\ &\leq C 2^{-2n} \|u_0\|_1 + C 2^{-2n} |A_n|^{1/3} \left(\int_{\mathbf{R}^2} u^2(t) \Phi_n dx \right)^{2/3} \\ &\leq \int_{\mathbf{R}^2} u^2(t) \Phi_n dx + C 2^{-2n} \|u_0\|_1 + C 2^{-4n}. \end{aligned}$$

By the same estimate of $\int_{\mathbf{R}^2} u^2(t) \Phi_n dx$ as in the estimate of $|I|$, it follows from (3.13) that

$$(3.14) \quad \begin{aligned} |II| &\leq 2 \left(\int_{A_n} u(t) dx \right) \left(\int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n dx \right) \\ &\quad + C 2^{-2n} (\|u_0\|_1 + \|u_0\|_1^2) + 4 \int_{\mathbf{R}^2} u(t) \Phi_n dx + C 2^{-4n}. \end{aligned}$$

The third term on the right hand side of (3.9) is estimated as follows.

$$(3.15) \quad \begin{aligned} |III| &\leq C 2^{-n} \int_{A_n} \frac{|\nabla u(t)|}{\sqrt{1+u(t)}} (\sqrt{1+u(t)} \log(1+u(t))) |\psi_m(t)| \Phi_n^{5/6} dx \\ &\leq C 2^{-n} \left(\int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n dx \right)^{1/2} \\ &\quad \times \left(\int_{\mathbf{R}^2} (1+u(t))^{3/2} (\log(1+u(t)))^3 \Phi_n dx \right)^{1/3} \left(\int_{A_n} |\psi_m(t)|^6 dx \right)^{1/6} \\ &\leq \frac{\varepsilon}{2} \int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n dx + \frac{\varepsilon}{3} \int_{\mathbf{R}^2} (1+u(t))^{3/2} (\log(1+u(t)))^3 \Phi_n dx \\ &\quad + C \varepsilon^{-5} 2^{-6n} \int_{A_n} |\psi_m(t)|^6 dx, \end{aligned}$$

where ε is an arbitrary number with $0 < \varepsilon < 1$. Noting that

$$(1+u)^{3/2}(\log(1+u))^3 \leq C \begin{cases} u & (0 \leq u \leq 1), \\ u^2 & (u \geq 1), \end{cases}$$

we have

$$\begin{aligned} (3.16) \quad & \int_{\mathbf{R}^2} (1+u(t))^{3/2}(\log(1+u(t)))^3 \Phi_n \, dx \\ & \leq C \int_{\mathbf{R}^2} u^2(t) \Phi_n \, dx + C \int_{\mathbf{R}^2} u(t) \Phi_n \, dx \\ & \leq C \left(\int_{A_n} u(t) \, dx \right) \left(\int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n \, dx \right) + C 2^{-2n} \left(\int_{\mathbf{R}^2} u_0 \, dx \right)^2 \\ & \quad + C(1+\varepsilon) \int_{\mathbf{R}^2} u(t) \Phi_n \, dx \end{aligned}$$

and by Lemma 2.5,

$$(3.17) \quad 2^{-6n} \int_{A_n} |\psi_m(t)|^6 \, dx \leq C 2^{-4n} \|\psi(t)\|_{BMO}^6.$$

Hence, by (3.15)–(3.17),

$$\begin{aligned} (3.18) \quad |III| & \leq \left(\frac{\varepsilon}{2} + C \int_{A_n} u(t) \, dx \right) \int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n \, dx \\ & \quad + C 2^{-2n} \left(\int_{\mathbf{R}^2} u_0 \, dx \right)^2 + C(1+\varepsilon) \int_{\mathbf{R}^2} u(t) \Phi_n \, dx \\ & \quad + C \varepsilon^{-5} 2^{-4n} \|\psi(t)\|_{BMO}^6. \end{aligned}$$

We proceed to estimate IV as follows. By Lemma 2.5 and the analogous estimate in (3.16),

$$\begin{aligned} (3.19) \quad |IV| & \leq C 2^{-2n} \int_{A_n} ((1+u(t)) \log(1+u(t)) - u(t)) |\psi_m(t)| \Phi_n^{2/3} \, dx \\ & \leq C 2^{-2n} \left(\int_{A_n} ((1+u(t)) \log(1+u(t)) - u(t))^{3/2} \Phi_n \, dx \right)^{2/3} \\ & \quad \times \left(\int_{A_n} |\psi_m(t)|^3 \, dx \right)^{1/3} \end{aligned}$$

$$\begin{aligned}
&\leq C2^{-2n+2n/3}\|\psi(t)\|_{BMO} \\
&\quad \times \left(\int_{\mathbf{R}^2} ((1+u(t)) \log(1+u(t)) - u(t))^{3/2} \Phi_n dx \right)^{2/3} \\
&\leq \int_{\mathbf{R}^2} ((1+u(t)) \log(1+u(t)) - u(t))^{3/2} \Phi_n dx + C2^{-4n}\|\psi(t)\|_{BMO}^3 \\
&\leq \left(\int_{A_n} u(t) dx \right) \left(\int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n dx \right) \\
&\quad + C\varepsilon 2^{-2n} \left(\int_{\mathbf{R}^2} u_0 dx \right)^2 + C\varepsilon(1+\varepsilon) \int_{\mathbf{R}^2} u(t) \Phi_n dx \\
&\quad + C2^{-4n}\|\psi(t)\|_{BMO}^3.
\end{aligned}$$

Combining (3.12), (3.14), (3.18) and (3.19) yields that

$$\begin{aligned}
(3.20) \quad &|I| + |II| + |III| + |IV| \\
&\leq \left\{ 9\varepsilon + C \int_{A_n} u(t) dx \right\} \int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n dx \\
&\quad + C2^{-2n}(\|u_0\|_1 + \|u_0\|_1^2) + C(1+\varepsilon)^2 \int_{\mathbf{R}^2} u(t) \Phi_n dx \\
&\quad + C\varepsilon^{-5} 2^{-4n} \|\psi(t)\|_{BMO}^6 + C2^{-4n} \|\psi(t)\|_{BMO}^3 + C2^{-4n},
\end{aligned}$$

where ε is an arbitrary number with $0 < \varepsilon < 1$.

Now we estimate $\int_{A_n} u(t) dx$ by using $\tilde{\Phi}_n(x) = \Phi_{n-1}(x) + \Phi_n(x) + \Phi_{n+1}(x)$. Observing that

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbf{R}^2} u(t) \tilde{\Phi}_n dx &= \int_{\mathbf{R}^2} u(t) \Delta \tilde{\Phi}_n dx + \int_{\mathbf{R}^2} u(t) \nabla \psi(t) \cdot \nabla \tilde{\Phi}_n dx \\
&= \int_{\mathbf{R}^2} u(t) \Delta \tilde{\Phi}_n dx - \frac{1}{4\pi} \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} u(t, x) u(t, y) \frac{x-y}{|x-y|^2} \\
&\quad \cdot \{ \nabla \tilde{\Phi}_n(x) - \nabla \tilde{\Phi}_n(y) \} dy dx \\
&\leq \|\Delta \tilde{\Phi}_n\|_\infty \|u_0\|_1 + \frac{1}{4\pi} \|u_0\|_1^2 \|\partial_x^2 \tilde{\Phi}_n\|_\infty,
\end{aligned}$$

we have

$$\int_{\mathbf{R}^2} u(t) \tilde{\Phi}_n dx \leq \int_{\mathbf{R}^2} u_0 \tilde{\Phi}_n dx + C2^{-2n} t \{ \|u_0\|_1 + \|u_0\|_1^2 \}.$$

By this inequality and (3.6),

$$(3.21) \quad \int_{A_n} u(t) dx \leq \int_{\mathbf{R}^2} u_0 \tilde{\Phi}_n dx + C 2^{-2n} t \{ \|u_0\|_1 + \|u_0\|_1^2 \}.$$

Take $\varepsilon = 1/36$ in (3.20). Given T , by (3.21) we can choose N sufficiently large such that for all $n \geq N$,

$$9\varepsilon + C \sup_{0 \leq t < T} \int_{A_n} u(t) dx \leq \frac{1}{2}.$$

Then, by (3.9) and (3.20),

$$(3.22) \quad \begin{aligned} & \frac{d}{dt} \left(\int_{\mathbf{R}^2} ((1 + u(t)) \log(1 + u(t)) - u(t)) \Phi_n dx \right) + \frac{1}{2} \int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1 + u(t)} \Phi_n dx \\ & \leq C(1 + \varepsilon)^2 \int_{\mathbf{R}^2} u(t) \Phi_n dx \\ & \quad + C 2^{-2n} (\|\psi(t)\|_{BMO}^6 + \|\psi(t)\|_{BMO}^3 + \|u_0\|_1^2 + \|u_0\|_1 + 1). \end{aligned}$$

By using $\|\psi(t)\|_{BMO} \leq C\|u(t)\|_1 = C\|u_0\|_1$ by virtue of Proposition 2.4, from (3.22) it follows that for every $n \geq N$ and for any $0 < t < T$,

$$(3.23) \quad \begin{aligned} & \int_{\mathbf{R}^2} ((1 + u(t)) \log(1 + u(t)) - u(t)) \Phi_n dx + \frac{1}{2} \int_0^t \int_{\mathbf{R}^2} \frac{|\nabla u|^2}{1 + u} \Phi_n dx ds \\ & \leq \int_{\mathbf{R}^2} ((1 + u_0) \log(1 + u_0) - u_0) \Phi_n dx \\ & \quad + C \int_0^T \int_{\mathbf{R}^2} u(t) \Phi_n dx dt \\ & \quad + C 2^{-4n} (\|u_0\|_1^6 + \|u_0\|_1^3 + \|u_0\|_1^2 + \|u_0\|_1 + 1) T. \end{aligned}$$

Adding these for $n \geq N$, by $\sum_{n=N}^{\infty} \Phi_n(x) = 1$ for $|x| \geq 2^N$, we have

$$\begin{aligned} & \int_{|x| \geq 2^N} ((1 + u(t)) \log(1 + u(t)) - u(t)) dx + \frac{1}{2} \int_0^t \int_{|x| \geq 2^N} \frac{|\nabla u|^2}{1 + u} dx dt \\ & \leq \int_{|x| \geq 2^{N-1}} ((1 + u_0) \log(1 + u_0) - u_0) dx \\ & \quad + C(\|u_0\|_1 + \|u_0\|_1^2 + \|u_0\|_1^3 + \|u_0\|_1^6 + 1) T. \end{aligned}$$

Putting $R_0 = 2^N$, (3.3) is deduced. Next, by calculations similar to those in (3.12),

$$(3.24) \quad \int_{\mathbf{R}^2} u^2(t) \Phi_n dx \leq 2 \|u_0\|_1 \left(\int_{\mathbf{R}^2} \frac{|\nabla u(t)|^2}{1+u(t)} \Phi_n dx \right) \\ + C 2^{-2n} \|u_0\|_1^2 + 4 \int_{\mathbf{R}^2} u(t) \Phi_n dx.$$

Using (3.23) to estimate the first term on the right hand side of (3.24) and adding these for $n \geq N$, we conclude (3.4). $\square$

3.2. A priori estimates for exterior regions

Let R_0 be the positive number determined in Proposition 3.2. Then

$$(3.25) \quad \sup_{0 < t < T} \int_{|x| \geq R_0} (1+u(t)) \log(1+u(t)) dx \leq c(u_0, T),$$

$$(3.26) \quad \int_0^T \int_{|x| \geq R_0} \frac{|\nabla u|^2}{1+u} dx dt \leq c(u_0, T),$$

$$(3.27) \quad \int_0^T \int_{|x| \geq R_0} u^2 dx dt \leq c(u_0, T),$$

where $c(u_0, T)$ is a positive constant depending on $\|(2+u_0) \log(2+u_0)\|_1$, T as mentioned in the introduction.

For $R > 1$, let $\Phi_R \in C_0^\infty(\mathbf{R}^2)$ be such that

$$\Phi_R(x) = \begin{cases} 0, & |x| \leq R/2, \\ \text{less than 1, positive,} & R/2 < |x| < R, \\ 1, & |x| \geq R, \end{cases}$$

and

$$|\partial_x \Phi_R| \leq CR^{-1} \Phi_R^{5/6}, \quad |\partial_x \Phi_R^{1/2}| \leq CR^{-1} \Phi_R^{1/3}, \quad |\partial_x^2 \Phi_R| \leq CR^{-2} \Phi_R^{2/3},$$

where C is a positive constant independent of R . We put

$$\tilde{A}_R = \{x \in \mathbf{R}^2; R/2 \leq |x| \leq R\}.$$

Lemma 3.4. *For any $R \geq 2R_0$,*

$$(3.28) \quad \sup_{0 < t < T} \int_{|x| \geq R} u^2(t) dx + \frac{1}{2} \int_0^T \int_{|x| \geq R} |\nabla u|^2 dx dt \leq \int_{\mathbf{R}^2} u_0 dx + c(u_0, T, R).$$

Proof. Let $\psi_m(t) = \psi(t) - [\psi(t)]_B$, where $[\psi(t)]_B = (1/|B|) \int_B \psi(t) dx$, $B = B_{2R}(0)$. Multiply the equation of u by $u\Phi_R$ and integrate on $\mathbf{R}^2$. Integrating by parts, we have

$$\begin{aligned}
 (3.29) \quad & \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} u^2(t) \Phi_R dx + \int_{\mathbf{R}^2} |\nabla u(t)|^2 \Phi_R dx \\
 &= - \int_{\mathbf{R}^2} u \nabla u \cdot \nabla \Phi_R dx - \frac{1}{2} \int_{\mathbf{R}^2} \Delta \psi_m u^2 \Phi_R dx + \frac{1}{2} \int_{\mathbf{R}^2} u^2 \nabla \psi_m \cdot \nabla \Phi_R dx \\
 &= \frac{1}{2} \int_{\mathbf{R}^2} u^2 \Delta \Phi_R dx + \frac{1}{2} \int_{\mathbf{R}^2} u^3 \Phi_R dx - \frac{1}{2} \int_{\mathbf{R}^2} u^2 \psi_m \Delta \Phi_R dx \\
 &\quad - \frac{1}{2} \int_{\mathbf{R}^2} \nabla u^2 \cdot \nabla \Phi_R \psi_m dx \\
 &= \frac{1}{2} \int_{\mathbf{R}^2} u^3 \Phi_R dx + \frac{1}{2} \int_{\mathbf{R}^2} u^2 (1 - \psi_m) \Delta \Phi_R dx - \int_{\mathbf{R}^2} u \nabla u \cdot \nabla \Phi_R \psi_m dx.
 \end{aligned}$$

Here we used $-\Delta \psi_m = u$. Now, by $|\Delta \Phi_R| \leq CR^{-2} \Phi_R^{2/3}$ and Lemma 2.5, we have

$$\begin{aligned}
 \frac{1}{2} \int_{\mathbf{R}^2} u^2 (1 - \psi_m) \Delta \Phi_R dx &\leq CR^{-2} \left(\int_{\tilde{A}_R} (1 + |\psi_m|)^3 dx \right)^{1/3} \left(\int_{\mathbf{R}^2} u^3 \Phi_R dx \right)^{2/3} \\
 &\leq \frac{1}{4} \int_{\mathbf{R}^2} u^3 \Phi_R dx + CR^{-6} \int_{\tilde{A}_R} (1 + |\psi_m|)^3 dx \\
 &\leq \frac{1}{4} \int_{\mathbf{R}^2} u^3 \Phi_R dx + CR^{-4} (\|\psi(t)\|_{BMO}^3 + 1),
 \end{aligned}$$

and by $|\nabla \Phi_R| \leq CR^{-1} \Phi_R^{5/6}$,

$$\begin{aligned}
 - \int_{\mathbf{R}^2} u \nabla u \cdot \nabla \Phi_R \psi_m dx &\leq CR^{-1} \|\psi_m\|_{L^6(\tilde{A}_R)} \left(\int_{\mathbf{R}^2} u^3 \Phi_R dx \right)^{1/3} \left(\int_{\mathbf{R}^2} |\nabla u|^2 \Phi_R dx \right)^{1/2} \\
 &\leq \frac{1}{3} \int_{\mathbf{R}^2} u^3 \Phi_R dx + \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u(t)|^2 \Phi_R dx + CR^{-4} \|\psi(t)\|_{BMO}^6.
 \end{aligned}$$

Hence

$$\begin{aligned}
 (3.30) \quad & \frac{d}{dt} \int_{\mathbf{R}^2} u^2 \Phi_R dx + \int_{\mathbf{R}^2} |\nabla u|^2 \Phi_R dx \\
 &\leq 2 \int_{\mathbf{R}^2} u^3 \Phi_R dx + CR^{-4} (\|\psi(t)\|_{BMO}^6 + \|\psi(t)\|_{BMO}^3 + 1).
 \end{aligned}$$

We apply (3) of Lemma 2.2 to the first term on the right hand side of (3.30) and get

$$\begin{aligned}
(3.31) \quad & \frac{d}{dt} \int_{\mathbf{R}^2} u^2 \Phi_R dx + \left(1 - \varepsilon \int_{\tilde{A}_R} (1+u) \log(1+u) dx \right) \int_{\mathbf{R}^2} |\nabla u|^2 \Phi_R dx \\
& \leq C \left(\int_{\mathbf{R}^2} u^{3/2} |\nabla \Phi_R^{1/2}| dx \right)^2 + C(\varepsilon) \int_{\mathbf{R}^2} u \Phi_R dx \\
& \quad + CR^{-4} (\|\psi(t)\|_{BMO}^6 + \|\psi(t)\|_{BMO}^3 + 1).
\end{aligned}$$

Let $R \geq 2R_0$. We observe that

$$\begin{aligned}
\left(\int_{\mathbf{R}^2} u^{3/2} |\nabla \Phi_R^{1/2}| dx \right)^2 & \leq CR^{-2} \left(\int_{|x| \geq R/2} u^{3/2} dx \right)^2 \\
& \leq CR^{-2} \left(\int_{\mathbf{R}^2} u dx \right) \left(\int_{|x| \geq R/2} u^2 dx \right) \\
& \leq CR^{-2} \|u_0\|_1 \int_{|x| \geq R/2} u^2 dx,
\end{aligned}$$

and that by (3.27),

$$\begin{aligned}
(3.32) \quad & \int_0^T \left(\int_{\mathbf{R}^2} u^{3/2} |\nabla \Phi_R^{1/2}| dx \right)^2 dt \leq CR^{-2} \|u_0\|_1 \int_0^T \int_{|x| \geq R/2} u^2 dx dt \\
& \leq c(u_0, T) R^{-2}.
\end{aligned}$$

Choosing $\varepsilon > 0$ in (3.31) such that

$$1 - \varepsilon \sup_{0 \leq t < T} \int_{A_R} (1+u(t)) \log(1+u(t)) dx = \frac{1}{2}$$

and integrating the resulting inequality from 0 to t with respect to the time variable, by (3.32) we have

$$\int_{\mathbf{R}^2} u^2(t) \Phi_R dx + \frac{1}{2} \int_0^t \int_{\mathbf{R}^2} |\nabla u|^2 \Phi_R dx d\sigma \leq \int_{\mathbf{R}^2} u_0^2 \Phi_R dx + c(u_0, T, R).$$

This implies (3.28). □

The following lemma is a consequence of (1) of Lemma 2.2 and Lemma 3.4.

Lemma 3.5. *For any $R \geq 2^2 R_0$,*

$$(3.33) \quad \int_0^T \int_{|x| \geq R} u^4 dx dt \leq c(u_0, \|u_0\|_2, T, R).$$

Proof. Apply (1) of Lemma 2.2 as $f = u^2$. Then

$$(3.34) \quad \int_{\mathbf{R}^2} u^4 \Phi_R dx \leq 2 \left(\int_{|x| \geq R/2} u^2 dx \right) \left(\int_{u>1} \frac{|\nabla u^2|^2}{1+u^2} \Phi_R dx \right) \\ + \left(\int_{\mathbf{R}^2} u^2 |\nabla \Phi_R^{1/2}| dx \right)^2 + 4 \int_{\mathbf{R}^2} u^2 \Phi_R dx.$$

We estimate the first two terms on the right hand side of (3.34) as follows.

$$\left(\int_{|x| \geq R/2} u^2 dx \right) \left(\int_{u>1} \frac{|\nabla u^2|^2}{1+u^2} \Phi_R dx \right) \\ \leq 4 \left(\int_{|x| \geq R/2} u^2(t) dx \right) \left(\int_{|x| \geq R/2} |\nabla u|^2 \Phi_R dx \right), \\ \left(\int_{\mathbf{R}^2} u^2 |\nabla \Phi_R^{1/2}| dx \right)^2 \leq CR^{-2} \left(\int_{|x| \geq R/2} u^2 dx \right)^2.$$

Hence, by Lemma 3.4, we deduce (3.33). $\square$

Lemma 3.6. For any $R \geq 2^3 R_0$,

$$(3.35) \quad \sup_{0 < t < T} \int_{|x| \geq R} u^3(t) dx + \int_0^T \int_{|x| \geq R} |\nabla u^{3/2}|^2 \Phi_R dx dt \\ \leq \int_{\mathbf{R}^2} u_0^3 dx + c(u_0, \|u_0\|_2, T, R).$$

Proof. Multiply the equation of u by $u^2 \Phi_R$ and integrate on $\mathbf{R}^2$. Then, by integration by parts, we have

$$\frac{1}{3} \frac{d}{dt} \int_{\mathbf{R}^2} u^3 \Phi_R dx + \frac{8}{9} \int_{\mathbf{R}^2} |\nabla u^{3/2}|^2 \Phi_R dx \\ = \frac{1}{3} \int_{\mathbf{R}^2} u^3 \Delta \Phi_R dx - \int_{\mathbf{R}^2} u^2 \nabla \cdot (u \nabla \psi_m) \Phi_R dx.$$

We rewrite the second term on the right hand side of the equality above as follows.

$$- \int_{\mathbf{R}^2} u^2 \nabla \cdot (u \nabla \psi_m) \Phi_R dx = -\frac{2}{3} \int_{\mathbf{R}^2} u^3 \Delta \psi_m \Phi_R dx + \frac{1}{3} \int_{\mathbf{R}^2} u^3 \nabla \psi_m \cdot \nabla \Phi_R dx \\ = \frac{2}{3} \int_{\mathbf{R}^2} u^4 \Phi_R dx + \frac{1}{3} \int_{\mathbf{R}^2} u^3 \nabla \psi_m \cdot \nabla \Phi_R dx.$$

Hence

$$\begin{aligned}
 (3.36) \quad & \frac{d}{dt} \int_{\mathbf{R}^2} u^3 \Phi_R dx + \frac{8}{3} \int_{\mathbf{R}^2} |\nabla u^{3/2}|^2 \Phi_R dx \\
 &= \int_{\mathbf{R}^2} u^3 \Delta \Phi_R dx + \int_{\mathbf{R}^2} u^3 \nabla \psi_m \cdot \nabla \Phi_R dx + 2 \int_{\mathbf{R}^2} u^4 \Phi_R dx \\
 &\leq CR^{-2} \int_{|x| \geq R/2} u^3 dx + \int_{\mathbf{R}^2} u^3 \nabla \psi_m \cdot \nabla \Phi_R dx + 2 \int_{|x| \geq R/2} u^4 dx.
 \end{aligned}$$

The second term on the right hand side of (3.36) is written as

$$\begin{aligned}
 \int_{\mathbf{R}^2} u^3 \nabla \psi_m \cdot \nabla \Phi_R dx &= -2 \int_{\mathbf{R}^2} u^{3/2} (\nabla u^{3/2} \cdot \nabla \Phi_R) \psi_m dx - \int_{\mathbf{R}^2} u^3 \psi_m \Delta \Phi_R dx \\
 &= I + II.
 \end{aligned}$$

By Hölder's inequality and Lemma 2.5, I is estimated as follows.

$$\begin{aligned}
 I &\leq CR^{-1} \|\psi_m(t)\|_{L^8(\tilde{A}_R)} \left(\int_{|x| \geq R/2} u^4 dx \right)^{3/8} \left(\int_{\mathbf{R}^2} |\nabla u^{3/2}|^2 \Phi_R dx \right)^{1/2} \\
 &\leq \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u^{3/2}|^2 \Phi_R dx + \frac{3}{8} \int_{|x| \geq R/2} u^4 dx + CR^{-8} \|\psi_m(t)\|_{L^8(\tilde{A}_R)}^8 \\
 &\leq \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u^{3/2}|^2 \Phi_R dx + \frac{3}{8} \int_{|x| \geq R/2} u^4 dx + CR^{-6} \|\psi(t)\|_{BMO}^8.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 II &\leq CR^{-2} \|\psi_m(t)\|_{L^4(\tilde{A}_R)} \left(\int_{|x| \geq R/2} u^4 dx \right)^{3/4} \\
 &\leq \frac{1}{4} \int_{|x| \geq R/2} u^4 dx + CR^{-6} \|\psi(t)\|_{BMO}^4.
 \end{aligned}$$

From (3.36) and the estimates on I and II , we have

$$\begin{aligned}
 (3.37) \quad & \frac{d}{dt} \int_{\mathbf{R}^2} u^3 \Phi_R dx + \int_{\mathbf{R}^2} |\nabla u^{3/2}|^2 \Phi_R dx \\
 &\leq CR^{-2} \int_{|x| \geq R/2} u^3 dx + C \int_{|x| \geq R/2} u^4 dx \\
 &\quad + CR^{-6} (\|\psi(t)\|_{BMO}^8 + \|\psi(t)\|_{BMO}^4) \\
 &= F(t).
 \end{aligned}$$

Since $\|\psi(t)\|_{BMO} \leq C\|u(t)\|_1 = C\|u_0\|_1$ and

$$\int_{|x| \geq R/2} u^3 dx \leq \int_{\mathbf{R}^2} u dx + \int_{|x| \geq R/2} u^4 dx,$$

by Lemmas 3.4 and 3.5 we have

$$\int_0^T F(t) dt \leq c(u_0, \|u_0\|_2, T, R).$$

Therefore it follows from (3.37) that for $0 < t < T$,

$$\int_{\mathbf{R}^2} u^3(t) \Phi_R dx + \int_0^t \int_{\mathbf{R}^2} |\nabla u^{3/2}|^2 \Phi_R dx ds \leq \int_{\mathbf{R}^2} u_0^3 \Phi_R dx + c(u_0, \|u_0\|_2, T, R),$$

which implies (3.35). $\square$

3.3. A priori estimates for annuli

Let R_0 be the positive number determined in Proposition 3.2 and put

$$A_R = \{x \in \mathbf{R}^2; R/2 \leq |x| \leq 2R\}.$$

Lemma 3.7. *There exists $R_1 \geq 2^5 R_0$ such that for any $R \geq R_1$, $\partial_x \psi$ and ψ_m are bounded on $(0, T) \times A_R$, where $\psi_m(t) = \psi(t) - [\psi(t)]_B$, $[\psi(t)]_B = (1/|B|) \int_B \psi(t) dx$, $B = B_{2R}(0)$.*

Proof. By the expression of ψ , we have

$$2\pi |\partial_x \psi(t, x)| \leq \left(\int_{|y| < R/3} + \int_{R/3 < |y| < 3R} + \int_{|y| > 3R} \right) \frac{u(t, y)}{|x - y|} dy.$$

Let $x \in A_R$, that is, $R/2 \leq |x| \leq 2R$. Then, $|x - y| \geq R/6$ for $|y| < R/3$ and $|x - y| \geq R$ for $|y| > 3R$, and hence

$$\begin{aligned} \int_{|y| < R/3} \frac{u(t, y)}{|x - y|} dy + \int_{|y| > 3R} \frac{u(t, y)}{|x - y|} dy &\leq \frac{6}{R} \int_{|y| < R/3} u(t, y) dy + \frac{1}{R} \int_{|y| > 3R} u(t, y) dy \\ &\leq \frac{7}{R} \int_{\mathbf{R}^2} u(t, y) dy = \frac{7}{R} \|u_0\|_1. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} \int_{R/3 < |y| < 3R} \frac{u(t, y)}{|x - y|} dy &\leq \left(\int_{R/3 < |y| < 3R} |x - y|^{-3/2} dy \right)^{2/3} \left(\int_{R/3 < |y| < 3R} u^3(t, y) dy \right)^{1/3} \\ &\leq CR^{1/3} \left(\int_{R/3 < |y| < 3R} u^3(t, y) dy \right)^{1/3}. \end{aligned}$$

Hence

$$|\partial_x \psi(t, x)| \leq \frac{C}{R} \|u_0\|_1 + CR^{1/3} \left(\int_{R/3 < |y| < 3R} u^3(t, y) dy \right)^{1/3},$$

from which with Lemma 3.6 it follows that $\partial_x \psi$ is bounded on $(0, T) \times A_R$ for any $R \geq 2^5 R_0$.

By Lemma 2.5 and Proposition 2.4, for $1 < p < \infty$,

$$\|\psi_m(t)\|_{L^p(A_R)} \leq C(2R)^{-2p} \|\psi(t)\|_{BMO} \leq C(2R)^{-2p} \|u(t)\|_1 = C(2R)^{-2p} \|u_0\|_1,$$

and hence, by the boundedness of $\partial_x \psi$ on $(0, T) \times A_R$ for any $R \geq 2^5 R_0$, $\psi_m(t)$ is in $W^{1,p}(A_R)$ for $2 < p < \infty$ and for each $0 < t < T$. Then the boundedness of ψ_m on $(0, T) \times A_R$ follows from the Sobolev inequalities. $\square$

For $R \geq 1$, let $\Phi_R \in C_0^\infty(\mathbf{R}^2)$ be such that

$$\Phi_R(x) = \begin{cases} 0, & |x| \leq R/3, |x| \geq 3R, \\ 1, & R/2 \leq |x| \leq 2R, \\ \text{less than 1, positive,} & R/3 < |x| < R/2, 2R < |x| < 3R, \end{cases}$$

and

$$|\partial_x \Phi_R| \leq CR^{-1} \Phi_R^{5/6}, \quad |\partial_x \Phi_R^{1/2}| \leq CR^{-1} \Phi_R^{1/3}, \quad |\partial_x^2 \Phi_R| \leq CR^{-2} \Phi_R^{2/3},$$

where C is a positive constant independent of R . We put $w(t, x) = u(t, x) \Phi_R(x)$, which satisfies $w \in C([0, T]; L^2(\mathbf{R}^2)) \cap L^\infty(0, T; L^2(\mathbf{R}^2))$ and

$$(3.38) \quad \partial_t w - \Delta w = f, \quad 0 < t < T, x \in \mathbf{R}^2,$$

where

$$(3.39) \quad f = -(2\nabla \Phi_R + \Phi_R \nabla \psi) \cdot \nabla u - (\Delta \Phi_R) u + \Phi_R u^2.$$

Lemma 3.8. *There exists $R_2 \geq 3R_1$ such that for any $R \geq R_2$, u is bounded on $(0, T) \times A_R$.*

Proof. By Lemmas in Subsection 3.2 and Lemma 3.7, for $R \geq 3R_1$, f defined by (3.39) is in $L^2(0, T; L^2(\mathbf{R}^2))$, and then by regularity theorems for parabolic equations,

$$w \in L^2(0, T; H^2(\mathbf{R}^2)), \quad \partial_t w \in L^2(0, T; L^2(\mathbf{R}^2)).$$

By the Sobolev inequalities,

$$w \in L^\infty(0, T; L^2(\mathbf{R}^2)) \cap L^2(0, T; W^{1,p}(\mathbf{R}^2)) \quad \text{for all } 1 < p < \infty.$$

Hence for $R \geq 3R_1$, we have

$$(3.40) \quad f \in L^2(0, T; L^p(\mathbf{R}^2)) \quad \text{for all } 1 < p < \infty.$$

Since w satisfies (3.38), w is represented as

$$(3.41) \quad w(t) = e^{tA}w_0 + \int_0^t e^{(t-s)A}f(s)ds, \quad 0 < t < T,$$

where $w_0 = u_0\Phi_R$. By $L^\infty - L^q$ estimates for e^{tA} , for $2 < q < \infty$ we have

$$(3.42) \quad \begin{aligned} \|w(t)\|_\infty &\leq \|w_0\|_\infty + C \int_0^t (t-s)^{-1/q} \|f(s)\|_q ds \\ &\leq \|w_0\|_\infty + C \left(\int_0^t (t-s)^{-2/q} ds \right)^{1/2} \left(\int_0^t \|f(s)\|_q^2 ds \right)^{1/2} \\ &\leq \|w_0\|_\infty + C \frac{q}{q-2} t^{1/2-1/q} \left(\int_0^t \|f(s)\|_q^2 ds \right)^{1/2}. \end{aligned}$$

This implies the boundedness of u on A_R from the definition of w . $\square$

Lemma 3.9. *There exists $R_3 \geq 3^3 R_2$ such that for any $R \geq R_3$, $\partial_x u$ is bounded on $(0, T) \times A_R$.*

Proof. We begin with proving the following: For any $R \geq 3R_1$,

$$(3.43) \quad \sup_{0 < t < T} \|\partial_x u(t)\|_{L^2(A_R)} < \infty.$$

To prove this, by integration by parts we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} |\nabla u|^2 \Phi_R dx = - \int_{\mathbf{R}^2} \partial_t u \Delta u \Phi_R dx - \int_{\mathbf{R}^2} \partial_t u \nabla u \cdot \nabla \Phi_R dx.$$

We put $\psi_m(t) = \psi(t) - [\psi(t)]_B$, $B = B_{2R}(0)$ and

$$\tilde{A}_R = \{x \in \mathbf{R}^2; R/3 \leq |x| \leq 3R\}.$$

By $\partial_t u = \Delta u - \nabla u \cdot \nabla \psi_m + u^2$ and Hölder's inequality, we have

$$\begin{aligned} - \int_{\mathbf{R}^2} \partial_t u \Delta u \Phi_R dx &= - \int_{\mathbf{R}^2} |\Delta u|^2 \Phi_R dx + \int_{\mathbf{R}^2} \nabla u \cdot \nabla \psi_m \Delta u \Phi_R dx - \int_{\mathbf{R}^2} u^2 \Delta u \Phi_R dx \\ &\leq - \frac{3}{4} \int_{\mathbf{R}^2} |\Delta u|^2 \Phi_R dx + C \|\nabla \psi_m(t)\|_{L^\infty(\tilde{A}_R)}^2 \int_{\tilde{A}_R} |\nabla u|^2 dx \\ &\quad + C \int_{\tilde{A}_R} u^4 dx. \end{aligned}$$

Similarly,

$$\begin{aligned}
-\int_{\mathbf{R}^2} \partial_t u \nabla u \cdot \nabla \Phi_R \, dx &= -\int_{\mathbf{R}^2} \Delta u \nabla u \cdot \nabla \Phi_R \, dx + \int_{\mathbf{R}^2} \nabla u \cdot \nabla \psi_m \nabla u \cdot \nabla \Phi_R \, dx \\
&\quad - \int_{\mathbf{R}^2} u^2 \nabla u \cdot \nabla \Phi_R \, dx \\
&\leq CR^{-1} \int_{\mathbf{R}^2} |\Delta u| |\nabla u| \Phi_R^{5/6} \, dx \\
&\quad + CR^{-1} \|\nabla \psi_m(t)\|_{L^\infty(\tilde{A}_R)} \int_{\tilde{A}_R} |\nabla u|^2 \, dx + CR^{-1} \int_{\tilde{A}_R} u^2 |\nabla u| \, dx \\
&\leq \frac{1}{4} \int_{\mathbf{R}^2} |\Delta u|^2 \Phi_R \, dx + C(\|\nabla \psi_m(t)\|_{L^\infty(\tilde{A}_R)} + 1) \int_{\tilde{A}_R} |\nabla u|^2 \, dx \\
&\quad + C \int_{\tilde{A}_R} u^4 \, dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbf{R}^2} |\nabla u|^2 \Phi_R \, dx + \int_{\mathbf{R}^2} |\Delta u|^2 \Phi_R \, dx \\
&\leq C(\|\nabla \psi_m(t)\|_{L^\infty(\tilde{A}_R)}^2 + \|\nabla \psi_m(t)\|_{L^\infty(\tilde{A}_R)} + 1) \int_{\tilde{A}_R} |\nabla u|^2 \, dx + C \int_{\tilde{A}_R} u^4 \, dx \\
&= F(t).
\end{aligned}$$

Then (3.43) follows from $\int_0^T F(t) \, dt < \infty$ for any $R \geq 3R_1$ by virtue of Lemmas 3.4, 3.7 and 3.8.

Next, by (3.43) we see that for any $R \geq 3^2 R_2$,

$$\sup_{0 < t < T} \|f(t)\|_2 < \infty.$$

We observe the following expression

$$(3.44) \quad \partial_x w(t) = e^{tA} \partial_x w_0 + \int_0^t \partial_x e^{(t-s)A} f(s) \, ds, \quad 0 < t < T.$$

Let $2 \leq p < \infty$. Then

$$\begin{aligned}
\|\partial_x w(t)\|_p &\leq \|\partial_x w_0\|_p + C \int_0^t (t-s)^{-1+1/p} \|f(s)\|_2 \, ds \\
&\leq \|\partial_x w_0\|_p + C p t^{1/p} \sup_{0 < s < T} \|f(s)\|_2.
\end{aligned}$$

Hence we obtain $\sup_{0 < t < T} \|\partial_x w(t)\|_p < \infty$, and hence

$$\sup_{0 < t < T} \|\partial_x u(t)\|_{L^p(A_R)} < \infty.$$

By this estimate, for any $R \geq 3^3 R_2$, we also have

$$\sup_{0 < t < T} \|f(t)\|_p < \infty \quad \text{for } 2 \leq p < \infty.$$

Similarly, the L^∞ -norm of the integral term on the right hand side of (3.44) is estimated by

$$\int_0^t (t-s)^{-1/q-1/2} \|f(s)\|_q ds \leq \frac{1}{1/2-1/q} t^{1/2-1/q} \sup_{0 < t < T} \|f(t)\|_q,$$

where $2 < q < \infty$. Hence we get $\sup_{0 < t < T} \|\partial_x w(t)\|_\infty < \infty$, which implies the boundedness of $\partial_x u$ on $(0, T) \times A_R$ for any $R \geq 3^3 R_2$. $\square$

Lemma 3.10. *There exists $R_4 \geq 3R_3$ such that for any $R \geq R_4$, $\partial_x^2 \psi$ is bounded on $(0, T) \times A_R$.*

Proof. Let $R \geq 3R_3$. With Φ_R we write $\partial_x \psi$ as follows.

$$\begin{aligned} 2\pi \partial_{x_j} \psi(t, x) &= - \int_{\mathbf{R}^2} \frac{x_j - y_j}{|x - y|^2} u(t, y) dy \\ &= - \int_{\mathbf{R}^2} \frac{x_j - y_j}{|x - y|^2} u(t, y) (1 - \Phi_R(y)) dy - \int_{\mathbf{R}^2} \frac{x_j - y_j}{|x - y|^2} u(t, y) \Phi_R(y) dy \\ &= I(t, x) + II(t, x). \end{aligned}$$

Let $x \in A_R$, that is, $R/2 \leq |x| \leq 2R$. Observing

$$\partial_{x_k} I(t, x) = - \int_{\mathbf{R}^2} \left(\frac{\delta_{jk}}{|x - y|^2} - 2 \frac{(x_j - y_j)(x_k - y_k)}{|x - y|^4} \right) u(t, y) (1 - \Phi_R(y)) dy,$$

we have

$$\begin{aligned} |\partial_{x_k} I(t, x)| &\leq \left(\int_{|y| < R/3} + \int_{|y| > 3R} \right) \frac{3}{|x - y|^2} u(t, y) dy \\ &\leq CR^{-2} \int_{\mathbf{R}^2} u(t, y) dy = CR^{-2} \|u_0\|_1. \end{aligned}$$

By the expression

$$\partial_{x_k} II(t, x) = - \int_{\mathbf{R}^2} \frac{x_j - y_j}{|x - y|^2} \partial_{y_k} (u(t, y) \Phi_R(y)) dy,$$

we estimate as follows.

$$\begin{aligned} |\partial_{x_k} H(t, x)| &\leq \int_{R/3 < |y| < 3R} \frac{1}{|x - y|} |\partial_{y_k}(u(t, y) \Phi_R(y))| dy \\ &\leq \left(\int_{|x| < 3R} \frac{1}{|x - y|} dy \right) \sup_{0 < t < T} \|u(t), \partial_x u(t)\|_{L^\infty(R/3 < |x| < 3R)}. \end{aligned}$$

Hence we obtain the boundedness of $\partial_x^2 \psi$ on $(0, T) \times A_R$ by Lemmas 3.8 and 3.9. $\square$

Lemma 3.11. *There exists $R_5 \geq 3^2 R_4$ such that for any $R \geq R_5$, $\partial_x^2 u$ is bounded on $(0, T) \times A_R$.*

Proof. By the preceding lemmas, for any $R \geq R_4$ the function f defined by (3.39) is estimated as

$$\sup_{0 < t < T} \|f(t)\|_\infty < \infty.$$

Applying regularity theorems for parabolic equations to w satisfying (3.38), we have

$$w \in L^p(0, T; W^{2,p}(\mathbf{R}^2)) \cap W^{1,p}(0, T; L^p(\mathbf{R}^2)), \quad 1 < p < \infty,$$

and hence, for any $R \geq 3R_4$,

$$\int_0^T \|\partial_x f(t)\|_p^p dt < \infty, \quad 1 < p < \infty.$$

We observe

$$\partial_x^2 w(t) = e^{tA} \partial_x^2 w_0 - \int_0^t \partial_x e^{(t-s)A} \partial_x f(s) ds.$$

Let $2 < p < \infty$. We estimate L^p -norms for the integral term as follows.

$$\begin{aligned} \int_0^t \|\partial_x e^{(t-s)A} \partial_x f(s)\|_p ds &\leq C \int_0^t (t-s)^{-1/2} \|\partial_x f(s)\|_p ds \\ &\leq C \left(\int_0^t (t-s)^{-p'/2} ds \right)^{1/p'} \left(\int_0^t \|\partial_x f(s)\|_p^p ds \right)^{1/p}, \end{aligned}$$

where $1 < p' < 2$, $1/p + 1/p' = 1$. Hence

$$\sup_{0 < t < T} \|\partial_x^2 u(t)\|_{L^p(A_R)} < \infty \quad \text{for } 2 < p < \infty.$$

By this estimates, for any $R \geq 3^2 R_4$

$$\sup_{0 < t < T} \|\partial_x f(t)\|_p < \infty, \quad 1 < p < \infty.$$

Similarly, by this estimate,

$$\int_0^t \|\partial_x e^{(t-s)A} \partial_x f(s)\|_\infty ds \leq C \int_0^t (t-s)^{-1/p-1/2} ds \sup_{0 < t < T} \|\partial_x f(t)\|_p,$$

where $2 < p < \infty$. Hence we obtain $\sup_{0 < t < T} \|\partial_x^2 w(t)\|_\infty < \infty$, and hence the boundedness of $\partial_x^2 u$ on $(0, T) \times A_R$. $\square$

Proposition 3.12. *There exists $\tilde{R}_0 > R_0$ such that for any $R \geq \tilde{R}_0$, the following hold:*

- (1) $\partial_t^k \partial_x^\ell u$ ($0 \leq 2k + \ell \leq 2$) are bounded on $(0, T) \times A_R$,
- (2) ψ_m and $\partial_x^\ell \psi$ ($1 \leq \ell \leq 2$) are bounded on $(0, T) \times A_R$, where $\psi_m(t) = \psi(t) - [\psi(t)]_B$, $B = B_{2R}(0)$,
- (3) there exist x_0 satisfying $R < |x_0| < 2R$, and $\varepsilon_0 > 0$, $\delta > 0$ such that

$$(3.45) \quad u(t, x) \geq \delta \quad \text{for } |x - x_0| \leq \varepsilon_0, 0 \leq t < T.$$

Proof. The assertions (1) and (2) follow from the preceding lemmas.

To prove (3), we first claim that $\partial_x^\ell u$ ($0 \leq \ell \leq 2$) are uniformly Hölder continuous on $[T/2, T) \times \tilde{A}_R$, where $\tilde{A}_R := \{x \in \mathbf{R}^2; 2R/3 \leq |x| \leq 8R/3\}$. In fact, since u satisfies $\partial_t u - \Delta u = -\nabla \cdot (u \nabla \psi)$ and $u \nabla \psi$ is bounded on $(0, T) \times \{x \in \mathbf{R}^2; R/3 < |x| < 3R\}$, applying the Hölder estimates for parabolic equations (see Theorem 10.1 in [38] for example), we see that u is uniformly Hölder continuous on $[T/2, T) \times \tilde{A}_R$. Similarly, by $\partial_t(\partial_x u) - \Delta(\partial_x u) = -\nabla \cdot (\partial_x(u \nabla \psi))$ and the boundedness of $\partial_x(u \nabla \psi)$ in $(0, T) \times \{x \in \mathbf{R}^2; R/3 < |x| < 3R\}$, the uniformly Hölder continuity of $\partial_x u$ in $[T/2, T) \times \tilde{A}_R$ is deduced. As for $\partial_{x_k} \partial_{x_j} u$, putting $v = \partial_{x_k} \partial_{x_j} u$, we observe that

$$\partial_t v - \Delta v + \nabla \psi \cdot \nabla v - 2uv = \partial_{x_k} g_1 + g_2,$$

where $g_1 = -\nabla u \cdot \nabla \partial_{x_j} \psi$, $g_2 = -\nabla \partial_{x_j} u \cdot \nabla \partial_{x_k} \psi + 2\partial_{x_k} u \partial_{x_j} u$. Hence, by the boundedness of g_1 and g_2 on $(0, T) \times \{x \in \mathbf{R}^2; R/3 < |x| < 3R\}$, $\partial_{x_k} \partial_{x_j} u$ is uniformly Hölder continuous on $[T/2, T) \times \tilde{A}_R$.

By the uniformly Hölder continuity of $\partial_x^\ell u$ ($0 \leq \ell \leq 2$) on $[T/2, T) \times \tilde{A}_R$, the limits of $\partial_x^\ell u$ ($0 \leq \ell \leq 2$) exist on $\tilde{A}_R$ as $t \rightarrow T - 0$. Hence we may assume that $\partial_x^\ell u$ ($0 \leq \ell \leq 2$) are continuous on $(0, T] \times \tilde{A}_R$. By the positivity of u on $(t, x) \in [T/2, T) \times \{x \in \mathbf{R}^2; R \leq |x| \leq 2R\}$, $u(T, x) \geq 0$ for $R \leq |x| \leq 2R$.

We next claim that there exist x_0 satisfying $R < |x_0| < 2R$, and $\varepsilon_0 > 0$, $\delta > 0$ such that

$$(3.46) \quad u(T, x) \geq \delta \quad \text{for } |x - x_0| \leq \varepsilon_0.$$

Assume that $u(T, x) = 0$ for all $R < |x| < 2R$. Then $\partial_x^\ell u(T, x) = 0$ for all $R < |x| < 2R$ and $\ell = 1, 2$. By $\partial_t u = \Delta u - \nabla \psi \cdot \nabla u + u^2$ and the boundedness of $\nabla \psi$ on $[T/2, T) \times \{x \in \mathbf{R}^2; R < |x| < 2R\}$, we obtain that for every $R < |x| < 2R$,

$$\partial_t u(t, \tilde{x}) \rightarrow 0 \quad \text{as } (t, \tilde{x}) \rightarrow (T, x).$$

Hence we may assume that $\partial_t u$ is continuous on $(0, T] \times \tilde{A}_R$. Since $u(T, x) = 0$ for all $R < |x| < 2R$, the maximum principle implies $u(t, x) = 0$ for all $T/2 \leq t \leq T$, $R < |x| < 2R$. This contradicts the positivity of u on $(t, x) \in [0, T) \times \{x \in \mathbf{R}^2; R < |x| < 2R\}$. Therefore the claim is concluded.

By the positivity of u in $(t, x) \in [0, T) \times \{x \in \mathbf{R}^2; R < |x| < 2R\}$ and (3.46), the assertion (3) is concluded. $\square$

3.4. Entropy estimates for interior regions

We finally give the interior a priori estimate. This is a crucial estimate for proving the global existence of the solution. Let us define $\Psi(x)$ by

$$\Psi(x) = \begin{cases} 1, & |x| \leq 1, \\ \text{less than 1, positive,} & 1 < |x| < 2, \\ 0, & 2 \leq |x|, \end{cases}$$

and $\Psi_R(x)$ by

$$\Psi_R(x) = \Psi(R^{-1}x) \quad (x \in \mathbf{R}^2, R > 0).$$

Put

$$A_R \equiv \{x \in \mathbf{R}^2; R \leq |x| \leq 2R\}.$$

It follows from the definition of Ψ_R that

$$\text{supp}[\nabla \Psi_R] \subset A_R, \quad \text{supp}[\Delta \Psi_R] \subset A_R.$$

Throughout this subsection we put

$$H_{int}(t; R) = \int_{\mathbf{R}^2} ((1 + u(t)) \log(1 + u(t)) - u(t)) \Psi_R dx - \frac{1}{2} \int_{\mathbf{R}^2} u(t) \psi_m(t) \Psi_R dx,$$

where $\psi_m(t) = \psi(t) - [\psi(t)]_B$, $[\psi(t)]_B = (1/|B|) \int_B \psi(t) dx$, $B = B_{2R}(0)$.

Lemma 3.13. *It holds that*

$$\begin{aligned} & \frac{d}{dt} H_{int}(t; R) + \int_{\mathbf{R}^2} u(t) |\nabla \log(1 + u(t)) - \nabla \psi_m(t)|^2 \Psi_R \, dx \\ & \quad + \int_{\mathbf{R}^2} |\nabla \log(1 + u(t))|^2 \Psi_R \, dx \\ & = \int_{\mathbf{R}^2} u(t) \log(1 + u(t)) \Psi_R \, dx + F_1(t) + F_2(t), \end{aligned}$$

where

$$\begin{aligned} F_1(t) &= \int_{\mathbf{R}^2} ((1 + u(t)) \log(1 + u(t)) - u(t)) \Delta \Psi_R \, dx \\ & \quad - \int_{\mathbf{R}^2} u(t) \psi_m(t) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx + \int_{\mathbf{R}^2} \psi_m(t) \nabla u(t) \cdot \nabla \Psi_R \, dx \\ & \quad + \int_{\mathbf{R}^2} (u(t) \log(1 + u(t)) - \log(1 + u(t))) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx, \\ F_2(t) &= - \int_{\mathbf{R}^2} \partial_t \psi_m(t) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx - \frac{1}{4} \frac{d}{dt} \int_{\mathbf{R}^2} |\psi_m(t)|^2 \Delta \Psi_R \, dx. \end{aligned}$$

Proof. We replace Φ_n and $|\nabla u|$ in (3.7) by Ψ_R and $|\nabla(1 + u)|$, respectively, and integrate on $\mathbf{R}^2$. Then, by the relation

$$\int_{\mathbf{R}^2} \frac{|\nabla(1 + u(t))|^2}{1 + u(t)} \Psi_R \, dx = \int_{\mathbf{R}^2} (1 + u(t)) |\nabla \log(1 + u(t))|^2 \Psi_R \, dx,$$

we have

$$\begin{aligned} (3.47) \quad & \frac{d}{dt} \int_{\mathbf{R}^2} ((1 + u(t)) \log(1 + u(t)) - u(t)) \Psi_R \, dx \\ & \quad + \int_{\mathbf{R}^2} (1 + u(t)) |\nabla \log(1 + u(t))|^2 \Psi_R \, dx \\ & \quad - \int_{\mathbf{R}^2} u(t) \nabla \log(1 + u(t)) \cdot \nabla \psi_m(t) \Psi_R \, dx \\ & = \int_{\mathbf{R}^2} ((1 + u(t)) \log(1 + u(t)) - u(t)) \Delta \Psi_R \, dx \\ & \quad + \int_{\mathbf{R}^2} u(t) \log(1 + u(t)) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx. \end{aligned}$$

Multiplying the equation of u by $\psi_m \Psi_R$ and integrating on $\mathbf{R}^2$, by integration by parts we have

$$\begin{aligned}
(3.48) \quad \int_{\mathbf{R}^2} \partial_t u(t) \psi_m(t) \Psi_R \, dx &= - \int_{\mathbf{R}^2} \nabla u(t) \cdot \nabla \psi_m(t) \Psi_R \, dx \\
&\quad - \int_{\mathbf{R}^2} \psi_m(t) \nabla u(t) \cdot \nabla \Psi_R \, dx \\
&\quad + \int_{\mathbf{R}^2} u(t) |\nabla \psi_m(t)|^2 \Psi_R \, dx \\
&\quad + \int_{\mathbf{R}^2} \psi_m(t) u(t) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx.
\end{aligned}$$

Next, by $-\Delta \psi_m = u$,

$$\begin{aligned}
(3.49) \quad \int_{\mathbf{R}^2} u \partial_t \psi_m \Psi_R \, dx &= - \int_{\mathbf{R}^2} \Delta \psi_m \partial_t \psi_m \Psi_R \, dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} |\nabla \psi_m|^2 \Psi_R \, dx + \int_{\mathbf{R}^2} \partial_t \psi_m \nabla \psi_m \cdot \nabla \Psi_R \, dx.
\end{aligned}$$

Observing

$$\begin{aligned}
\int_{\mathbf{R}^2} |\nabla \psi_m|^2 \Psi_R \, dx &= - \int_{\mathbf{R}^2} \Delta \psi_m \psi_m \Psi_R \, dx - \int_{\mathbf{R}^2} \psi_m \nabla \psi_m \cdot \nabla \Psi_R \, dx \\
&= \int_{\mathbf{R}^2} u \psi_m \Psi_R \, dx + \frac{1}{2} \int_{\mathbf{R}^2} \psi_m^2 \Delta \Psi_R \, dx
\end{aligned}$$

and substituting this relation into (3.49), we obtain

$$\begin{aligned}
(3.50) \quad \int_{\mathbf{R}^2} u \partial_t \psi_m \Psi_R \, dx &= \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} u \psi_m \Psi_R \, dx + \frac{1}{4} \frac{d}{dt} \int_{\mathbf{R}^2} \psi_m^2 \Delta \Psi_R \, dx \\
&\quad + \int_{\mathbf{R}^2} \partial_t \psi_m \nabla \psi_m \cdot \nabla \Psi_R \, dx.
\end{aligned}$$

Adding (3.48) and (3.50) yields that

$$\begin{aligned}
(3.51) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} u \psi_m \Psi_R \, dx &= - \int_{\mathbf{R}^2} \nabla u(t) \cdot \nabla \psi_m(t) \Psi_R \, dx + \int_{\mathbf{R}^2} u(t) |\nabla \psi_m(t)|^2 \Psi_R \, dx \\
&\quad - \int_{\mathbf{R}^2} \psi_m(t) \nabla u(t) \cdot \nabla \Psi_R \, dx \\
&\quad + \int_{\mathbf{R}^2} u(t) \psi_m(t) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx \\
&\quad + \int_{\mathbf{R}^2} \partial_t \psi_m \nabla \psi_m \cdot \nabla \Psi_R \, dx + \frac{1}{4} \frac{d}{dt} \int_{\mathbf{R}^2} \psi_m^2 \Delta \Psi_R \, dx.
\end{aligned}$$

It follows from (3.47) and (3.51) that

$$\begin{aligned}
(3.52) \quad & \frac{d}{dt} H_{int}(t; R) + \int_{\mathbf{R}^2} |\nabla \log(1 + u(t))|^2 \Psi_R \, dx + I(t) \\
&= \int_{\mathbf{R}^2} ((1 + u(t)) \log(1 + u(t)) - u(t)) \Delta \Psi_R \, dx \\
&\quad + \int_{\mathbf{R}^2} u(t) \log(1 + u(t)) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx \\
&\quad + \int_{\mathbf{R}^2} \psi_m(t) \nabla u(t) \cdot \nabla \Psi_R \, dx - \int_{\mathbf{R}^2} u(t) \psi_m(t) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx \\
&\quad - \int_{\mathbf{R}^2} \partial_t \psi_m(t) \nabla \psi_m(t) \cdot \nabla \Psi_R \, dx - \frac{1}{4} \frac{d}{dt} \int_{\mathbf{R}^2} \psi_m^2 \Delta \Psi_R \, dx,
\end{aligned}$$

where

$$\begin{aligned}
I(t) &= \int_{\mathbf{R}^2} u(t) |\nabla \log(1 + u(t))|^2 \Psi_R \, dx \\
&\quad - \int_{\mathbf{R}^2} u(t) \nabla \psi_m(t) \cdot \nabla \log(1 + u(t)) \Psi_R \, dx \\
&\quad - \int_{\mathbf{R}^2} \nabla u(t) \cdot \nabla \psi_m(t) \Psi_R \, dx + \int_{\mathbf{R}^2} u(t) |\nabla \psi_m(t)|^2 \Psi_R \, dx.
\end{aligned}$$

We observe that

$$\begin{aligned}
I(t) &= \int_{\mathbf{R}^2} u(t) |\nabla \log(1 + u(t)) - \nabla \psi_m(t)|^2 \Psi_R \, dx \\
&\quad - \int_{\mathbf{R}^2} \nabla \psi_m(t) \cdot \nabla \log(1 + u(t)) \Psi_R \, dx,
\end{aligned}$$

and that by integration by parts and $-\Delta \psi_m = u$,

$$\begin{aligned}
- \int_{\mathbf{R}^2} \nabla \psi_m \cdot \nabla \log(1 + u) \Psi_R \, dx &= - \int_{\mathbf{R}^2} u \log(1 + u) \Psi_R \, dx \\
&\quad + \int_{\mathbf{R}^2} \log(1 + u) \nabla \psi_m \cdot \nabla \Psi_R \, dx.
\end{aligned}$$

Hence

$$\begin{aligned}
I(t) &= \int_{\mathbf{R}^2} u(t) |\nabla \log(1 + u(t)) - \nabla \psi_m(t)|^2 \Psi_R \, dx \\
&\quad - \int_{\mathbf{R}^2} u \log(1 + u) \Psi_R \, dx + \int_{\mathbf{R}^2} \log(1 + u) \nabla \psi_m \cdot \nabla \Psi_R \, dx.
\end{aligned}$$

Substituting this into (3.52) completes the proof of Lemma 3.13. $\square$

In the rest of this subsection, we take $R = 4\tilde{R}_0$, where $\tilde{R}_0$ is the one determined in Proposition 3.12.

Lemma 3.14. *For $F_1(t)$ in Lemma 3.13, it holds that*

$$\sup_{0 < t < T} |F_1(t)| < \infty.$$

Proof. Since the supports of $\nabla \Psi_R$ and $\Delta \Psi_R$ are in $A_R = \{x \in \mathbf{R}^2; R \leq |x| \leq 2R\}$, the integral regions appearing in $F_1(t)$ can be taken as A_R . The integrands in $F_1(t)$ are bounded on $[0, T) \times A_R$ by (1) and (2) of Proposition 3.12, and hence we conclude the boundedness of $F_1(t)$ on $[0, T)$. $\square$

Lemma 3.15. *For $F_2(t)$ in Lemma 3.13, it holds that*

$$\sup_{0 < t < T} \left| \int_0^t F_2(s) ds \right| < \infty.$$

Proof. Let J_R be a function in $C_0^\infty(\mathbf{R}^2)$ such that

$$J_R(x) = \begin{cases} 1, & R \leq |x| \leq 2R, \\ \text{less than 1, positive,} & R/2 < |x| < R, \ 2R < |x| \leq 3R, \\ 0, & \text{otherwise,} \end{cases}$$

$$|\nabla J_R(x)| \leq CR^{-1}, \quad |\Delta J_R(x)| \leq CR^{-2},$$

and put $\tilde{\Psi}_R(x) = \Psi_R(x)J_R(x)$. Since $\text{supp}[\nabla \Psi_R] \subset A_R$, with $\tilde{\Psi}_R$ we can rewrite $F_2(t)$ as

$$F_2(t) = - \int_{\mathbf{R}^2} \partial_t \psi_m \nabla \psi_m \cdot \nabla \tilde{\Psi}_R \, dx - \frac{1}{4} \frac{d}{dt} \int_{\mathbf{R}^2} \psi_m^2 \Delta \tilde{\Psi}_R \, dx.$$

Noting

$$\begin{aligned} -\partial_t \psi_m \nabla \psi_m \cdot \nabla \tilde{\Psi}_R &= -\nabla \cdot (\partial_t \psi_m \psi_m \nabla \tilde{\Psi}_R - (\partial_t \nabla \psi_m) \psi_m \tilde{\Psi}_R) - (\partial_t \Delta \psi_m) \psi_m \tilde{\Psi}_R \\ &\quad - \frac{1}{2} \partial_t |\nabla \psi_m|^2 \tilde{\Psi}_R + \frac{1}{2} \partial_t \psi_m^2 \Delta \tilde{\Psi}_R \end{aligned}$$

and using $-\partial_t \Delta \psi_m = \partial_t u = \Delta u - \nabla u \cdot \nabla \psi_m + u^2$, we can rewrite $F_2(t)$ as

$$\begin{aligned} F_2(t) &= \frac{1}{4} \frac{d}{dt} \left\{ \int_{\mathbf{R}^2} \psi_m^2 \Delta \tilde{\Psi}_R \, dx - 2 \int_{\mathbf{R}^2} |\nabla \psi_m|^2 \tilde{\Psi}_R \, dx \right\} \\ &\quad + \int_{\mathbf{R}^2} (\Delta u - \nabla u \cdot \nabla \psi_m + u^2) \psi_m \tilde{\Psi}_R \, dx \\ &= F_{21}(t) + F_{22}(t). \end{aligned}$$

Put $\tilde{A}_R = \{x \in \mathbf{R}^2; R/2 \leq |x| \leq 3R\}$. Since the support of $\tilde{\Psi}_R$ is included in $\tilde{A}_R$, by (2) of Proposition 3.12 we have

$$\begin{aligned} \sup_{0 < t < T} \int_{\mathbf{R}^2} \psi_m^2(t) |\Delta \tilde{\Psi}| dx &< \infty, \\ \sup_{0 < t < T} \int_{\mathbf{R}^2} |\nabla \psi_m(t)|^2 \tilde{\Psi}_R dx &< \infty, \end{aligned}$$

and hence

$$\sup_{0 < t < T} \left| \int_0^t F_{21}(s) ds \right| < \infty.$$

For $F_{22}(t)$, we rewrite $F_{22}(t)$ as

$$F_{22}(t) = -2 \int_{\mathbf{R}^2} \nabla u \cdot \nabla \psi_m \tilde{\Psi}_R dx - \int_{\mathbf{R}^2} \nabla u \cdot \nabla \tilde{\Psi}_R \psi_m dx + \int_{\mathbf{R}^2} u^2 \psi_m \tilde{\Psi}_R dx.$$

Since the integrands in $F_{22}(t)$ are bounded on $(0, T) \times \tilde{A}_R$ by (1) and (2) of Proposition 3.12, we also have

$$\sup_{0 < t < T} |F_{22}(t)| < \infty.$$

Therefore Lemma 3.15 is established. □

Proposition 3.16. *Assume $\int_{\mathbf{R}^2} u_0 dx \leq 8\pi$. Then*

$$\sup_{0 < t < T} \int_{|x| \leq 4\tilde{R}_0} (1 + u(t)) \log(1 + u(t)) dx < \infty,$$

where $\tilde{R}_0$ is the one determined in Proposition 3.12.

Proof. Put $R = 4\tilde{R}_0$. It follows from Lemma 3.13 that

$$(3.53) \quad \frac{d}{dt} H_{in}(t; R) \leq \int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) \Psi_R dx + F_1(t) + F_2(t).$$

For $0 < a < 1$,

$$\frac{1}{2} u \psi_m \leq (1 - a) u \frac{|\psi_m|}{2(1 - a)} \leq (1 - a)(1 + u) \log(1 + u) + (1 - a) \exp\left\{ \frac{|\psi_m|}{2(1 - a)} \right\},$$

from which it follows that

$$(1 + u) \log(1 + u) \leq \frac{1}{a} \left((1 + u) \log(1 + u) - \frac{1}{2} u \psi_m \right) + \frac{1 - a}{a} \exp\left\{ \frac{|\psi_m|}{2(1 - a)} \right\}.$$

Then

$$\begin{aligned}
 (3.54) \quad & \int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) \Psi_R \, dx \\
 & \leq \frac{1}{a} H_{int}(t; R) + \frac{1}{a} \int_{\mathbf{R}^2} u(t) \Psi_R \, dx + \frac{1-a}{a} \int_{\mathbf{R}^2} \exp \left\{ \frac{|\psi_m(t)|}{2(1-a)} \right\} \Psi_R \, dx \\
 & \leq \frac{1}{a} H_{int}(t; R) + \frac{1}{a} \|u_0\|_1 + F_3(t),
 \end{aligned}$$

where

$$F_3(t) = \frac{1-a}{a} \int_{\mathbf{R}^2} \exp \left\{ \frac{|\psi_m(t)|}{2(1-a)} \right\} \Psi_R \, dx.$$

By (3.53) and (3.54) we have

$$(3.55) \quad \frac{d}{dt} H_{int}(t; R) \leq \frac{1}{a} H_{int}(t; R) + F(t),$$

where

$$F(t) = F_1(t) + F_2(t) + F_3(t) + \frac{1}{a} \|u_0\|_1.$$

We claim

$$(3.56) \quad \sup_{0 < t < T} F_3(t) < \infty.$$

To prove this claim, by (3) of Proposition 3.12 we firstly note that there exist $x_0 \in \mathbf{R}^2$, $\varepsilon_0 > 0$, $\delta > 0$, satisfying $R + \varepsilon_0 < |x_0| < 2R$, such that

$$(3.57) \quad u(t, x) \geq \delta \quad \text{for } 0 \leq t < T, |x - x_0| \leq \varepsilon_0.$$

By (2) of Proposition 3.12,

$$M := \sup_{0 < t < T} \|\psi_m(t)\|_{L^\infty(R \leq |x| \leq 2R)} < \infty.$$

The function $\psi_m(t)$ satisfies

$$-\Delta \left(\frac{\psi_m(t)}{2(1-a)} \right) = \frac{u(t)}{2(1-a)}, \quad |x| < R$$

and $\sup_{|x|=R} |\psi_m(t)| \leq M$ ($0 < t < T$). By (3.57), we have

$$\begin{aligned}
 \int_{|x| \leq R} u(t) \, dx & \leq \int_{\mathbf{R}^2} u(t) \, dx - \int_{|x-x_0| \leq \varepsilon_0} u(t) \, dx \\
 & \leq 8\pi - \delta\pi\varepsilon_0^2 = \pi(8 - \delta\varepsilon_0^2)
 \end{aligned}$$

and hence

$$\int_{|x| \leq R} \frac{u(t)}{2(1-a)} dx \leq \frac{\pi}{2(1-a)} (8 - \delta \varepsilon_0^2).$$

We choose $0 < a < 1$ such that

$$\frac{\pi}{2(1-a)} (8 - \delta \varepsilon_0^2) < 4\pi$$

and apply Lemma 2.7 to get that for $0 < t < T$,

$$\begin{aligned} \int_{|x| < R} \exp \left\{ \frac{|\psi_m(t)|}{2(1-a)} \right\} dx &\leq \frac{32\pi(1-a)R^2}{\delta \varepsilon_0^2 - 8a} \exp \left\{ \sup_{|x|=R} \frac{|\psi_m(t)|}{2(1-a)} \right\} \\ &\leq \frac{32\pi(1-a)R^2}{\delta \varepsilon_0^2 - 8a} \exp \left\{ \frac{M}{2(1-a)} \right\}. \end{aligned}$$

Hence

$$\int_{\mathbf{R}^2} \exp \left\{ \frac{|\psi_m(t)|}{2(1-a)} \right\} \Psi_R dx \leq \left(\frac{32\pi(1-a)R^2}{\delta \varepsilon_0^2 - 8a} + 3\pi R^2 \right) \exp \left\{ \frac{M}{2(1-a)} \right\},$$

which implies (3.56).

By (3.55), we have

$$H_{int}(t; R) \leq H_{int}(0; R) + \frac{1}{a} \int_0^t H_{int}(s; R) ds + \sup_{0 < t < T} \left| \int_0^t F(s) ds \right|, \quad 0 < t < T$$

and by Lemmas 3.14 and 3.15 and (3.56),

$$\sup_{0 < t < T} \left| \int_0^t F(s) ds \right| < \infty.$$

Applying the Gronwall inequality, we deduce that

$$\sup_{0 < t < T} H_{int}(t; R) < \infty.$$

Therefore, by this estimate, (3.54) and (3.56), we obtain

$$\sup_{0 < t < T} \int_{\mathbf{R}^2} (1 + u(t)) \log(1 + u(t)) \Psi_R dx < \infty,$$

which completes the proof of Proposition 3.16. $\square$

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nuna adreso:

Toshitaka Nagai
Department of Mathematics
Graduate School of Science
Hiroshima University
1-3-1 Kagamiyama, Higashi-Hiroshima
Hiroshima 739-8526
Japan

Takayoshi Ogawa
Mathematical Institute
Tohoku University
6-3, Aramaki Aza-Aoba, Aoba-ku, Sendai
Miyagi 980-8578
Japan

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