Existence and Non-Existence Results for Nonlocal Elliptic Systems via Sub-Supersolution Method

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Abstract. In this paper, using a sub-supersolution argument, we prove an existence result on a positive weak solution for a class of nonlocal elliptic systems in bounded domains.

Key Words and Phrases. Nonlocal elliptic systems, Existence, Non-Existence, Positive solution, Sub-supersolution method.

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1. Introduction and Preliminaries

In this paper, we are interested in a class of nonlinear elliptic systems of the form

$$(1.1) \quad \begin{cases} -A\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = \lambda f(v) & \text{in } \Omega, \\ -B\left(\int_{\Omega} |\nabla v|^2 dx\right) \Delta v = \lambda g(u) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $A, B: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ are continuous functions, $1 < p < N$ and λ is a positive parameter.

Since the first equation in (1.1) contains an integral over Ω , it is no longer a pointwise identity; therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [3]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$(1.2) \quad \rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff in 1883, see [8]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [1, 2, 4, 5, 9, 11, 12], in which the authors have used different methods to get the existence of solutions for (1.1) in the single equation case. In the papers [10, 13], Z. Zhang et al. studied the existence of nontrivial sign-changing solutions for this type of problem. In this paper, we are interested in finding positive solutions for system (1.1) via a sub-supersolution method. Our paper is motivated by the recent results in [1, 2, 6, 7]. In the paper [2, Theorem 2], Azzouz and Bensedik studied the existence of a positive solution for the nonlocal problem of the form

$$(1.3) \quad \begin{cases} -M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = |u|^{p-1}u + \lambda f(x) & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$, $N \geq 3$ and $p > 1$, i.e. the nonlinear term is superlinear at infinity and f is a sign-changing function. Using the sub-supersolution method combining a comparison principle introduced in [1], the authors established the existence of a positive solution for (1.3) when the parameter $\lambda > 0$ is small enough. It was also proved in [2] that problem (1.3) has no positive solution when λ large enough. In the present paper, we consider system (1.1) in the case when the nonlinearities are "sublinear" at infinity, see the condition (H_3) . We are inspired by the ideas in the interesting paper [6], in which the authors considered system (1.1) in the case $A(t) = B(t) \equiv 1$. More precisely, under suitable conditions on f , g , we shall show that system (1.1) has a positive solution for $\lambda > 0$ large enough. Moreover, with particular nonlinearities f , g we show the (1.1) has no positive for $\lambda > 0$ small enough. To our best knowledge, this is a new research topic for nonlocal problems, see [7]. In the rest of this section, we recall a comparison principle involving Kirchhoff type operators. The readers may concern its proof in the paper [7].

Theorem 1.1. *Assume that $M : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is a continuous and increasing function satisfying*

$$(1.4) \quad \lim_{t \rightarrow 0^+} M(t) = m_0 > 0.$$

where m_0 is a positive constant.

Assume that u, v are two non-negative functions such that

$$(1.5) \quad \begin{cases} -M \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u \geq -M \left(\int_{\Omega} |\nabla v|^2 dx \right) \Delta v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \geq v$ a.e. in Ω .

2. Existence result

In this section, we shall state and prove the main result of this paper. Let us assume the following assumptions:

(H_1) Assume that $A, B: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ are two continuous and increasing functions and there exists $a_i, b_i > 0$, $i = 1, 2$, such that

$$a_1 \leq A(t) \leq a_2, \quad b_1 \leq B(t) \leq b_2 \quad \text{for all } t \in \mathbf{R}^+;$$

(H_2) f, g are continuous on $[0, +\infty)$, C^1 on $(0, +\infty)$, and increasing functions such that

$$\lim_{t \rightarrow +\infty} f(t) = \lim_{t \rightarrow +\infty} g(t) = +\infty;$$

(H_3) It holds that

$$\lim_{t \rightarrow +\infty} \frac{f(Kg(t))}{t} = 0$$

for every constant $K > 0$.

Theorem 2.1. Assume that the conditions (H_1)–(H_3) hold. Then there exists $\lambda^* > 0$ such that problem (1.1) has a large positive weak solution (u, v) for $\lambda > \lambda^*$.

Define $f(t) = 0$ and $g(t) = 0$ for all $t < 0$. We shall establish Theorem 2.1 by constructing a positive subsolution $(\underline{u}, \underline{v}) \in H_0^1(\Omega, \mathbf{R}^2)$ and a positive supersolution $(\bar{u}, \bar{v}) \in H_0^1(\Omega, \mathbf{R}^2)$ of problem (1.1) such that $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$ in Ω , that is, $\underline{u}, \bar{u}, \underline{v}, \bar{v}$ satisfy

$$\begin{aligned} A \left(\int_{\Omega} |\nabla \underline{u}|^2 dx \right) \int_{\Omega} \nabla \underline{u} \cdot \nabla \varphi dx &\leq \lambda \int_{\Omega} f(\underline{v}) \varphi dx \\ B \left(\int_{\Omega} |\nabla \underline{v}|^2 dx \right) \int_{\Omega} \nabla \underline{v} \cdot \nabla \varphi dx &\leq \lambda \int_{\Omega} g(\underline{u}) \varphi dx \end{aligned}$$

and

$$A \left(\int_{\Omega} |\nabla \bar{u}|^2 dx \right) \int_{\Omega} \nabla \bar{u} \cdot \nabla \varphi dx \geq \lambda \int_{\Omega} f(\bar{v}) \varphi dx$$

$$B \left(\int_{\Omega} |\nabla \bar{v}|^2 dx \right) \int_{\Omega} \nabla \bar{v} \cdot \nabla \varphi dx \geq \lambda \int_{\Omega} g(\bar{u}) \varphi dx$$

for all $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$.

Proof of Theorem 2.1. Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction with $\|\phi_1\|_{\infty} = 1$. Let $k_0, m_0, \delta > 0$ such that $f(t), g(t) \geq -k_0$ for all $t \in \mathbf{R}^+$ and $|\nabla \phi_1|^2 - \lambda_1 \phi_1^2 \geq m_0$ on $\bar{\Omega}_{\delta} := \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$.

For each $\lambda > 0$, let us define $\underline{u} = (\lambda k_0 / (2m_0 a_1)) \phi_1^2$ and $\underline{v} = (\lambda k_0 / (2m_0 b_1)) \phi_1^2$, where a_1, b_1 are given by the condition (H_1) . We shall verify that $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1) for λ large enough. Indeed, let $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$ in Ω . By (H_1) , a simple calculation shows that

$$\begin{aligned} & A \left(\int_{\bar{\Omega}_{\delta}} |\nabla \underline{u}|^2 dx \right) \int_{\bar{\Omega}_{\delta}} \nabla \underline{u} \cdot \nabla \varphi dx \\ &= A \left(\int_{\bar{\Omega}_{\delta}} |\nabla \underline{u}|^2 dx \right) \frac{\lambda k_0}{m_0 a_1} \int_{\bar{\Omega}_{\delta}} \phi \nabla \phi \cdot \nabla \varphi dx \\ &= \frac{\lambda k_0}{m_0 a_1} A \left(\int_{\bar{\Omega}_{\delta}} |\nabla \underline{u}|^2 dx \right) \left\{ \int_{\bar{\Omega}_{\delta}} \nabla \phi \cdot \nabla (\phi \varphi) dx - \int_{\bar{\Omega}_{\delta}} |\nabla \phi|^2 \varphi dx \right\} \\ &= \frac{\lambda k_0}{m_0 a_1} A \left(\int_{\bar{\Omega}_{\delta}} |\nabla \underline{u}|^2 dx \right) \int_{\bar{\Omega}_{\delta}} (\lambda_1 \phi^2 - |\nabla \phi|^2) \varphi dx. \end{aligned}$$

On $\bar{\Omega}_{\delta}$ we have $|\nabla \phi|^2 - \lambda_1 \phi^2 \geq m_0$, which implies by using (H_1) that

$$\begin{aligned} (2.1) \quad & A \left(\int_{\bar{\Omega}_{\delta}} |\nabla \underline{u}|^2 dx \right) \int_{\bar{\Omega}_{\delta}} \nabla \underline{u} \cdot \nabla \varphi dx \leq \frac{\lambda k_0}{m_0} \int_{\bar{\Omega}_{\delta}} (\lambda_1 \phi^2 - |\nabla \phi|^2) \varphi dx \\ & \leq \lambda \int_{\bar{\Omega}_{\delta}} f(\underline{v}) \varphi dx. \end{aligned}$$

Next, on $\Omega \setminus \bar{\Omega}_{\delta}$ we have $\phi \geq r$ for some $r > 0$, and therefore by the conditions (H_1) , (H_2) and the definition of $\underline{v}$, it follows that

$$\begin{aligned} (2.2) \quad & \lambda \int_{\Omega \setminus \bar{\Omega}_{\delta}} f(\underline{v}) \varphi dx \geq \lambda \frac{k_0 a_2}{m_0 a_1} \lambda_1 \int_{\Omega \setminus \bar{\Omega}_{\delta}} \varphi dx \\ & \geq \lambda \frac{k_0}{m_0 a_1} A \left(\int_{\Omega \setminus \bar{\Omega}_{\delta}} |\nabla \underline{u}|^2 dx \right) \lambda_1 \int_{\Omega \setminus \bar{\Omega}_{\delta}} \varphi dx \end{aligned}$$

$$\begin{aligned}
&\geq \lambda \frac{k_0}{m_0 a_1} A \left(\int_{\Omega \setminus \bar{\Omega}_\delta} |\nabla \underline{u}|^2 dx \right) \int_{\Omega \setminus \bar{\Omega}_\delta} (\lambda_1 \phi^2 - |\nabla \phi|^2) \phi \, dx \\
&= A \left(\int_{\Omega \setminus \bar{\Omega}_\delta} |\nabla \underline{u}|^2 dx \right) \int_{\Omega \setminus \bar{\Omega}_\delta} \nabla \underline{u} \cdot \nabla \phi \, dx
\end{aligned}$$

for $\lambda > 0$ large enough.

Relations (2.1) and (2.2) imply that

$$(2.3) \quad A \left(\int_{\Omega} |\nabla \underline{u}|^2 dx \right) \int_{\Omega} \nabla \underline{u} \cdot \nabla \phi \, dx \leq \lambda \int_{\Omega} f(\underline{v}) \phi \, dx$$

for $\lambda > 0$ large enough and any $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω . Similarly,

$$(2.4) \quad B \left(\int_{\Omega} |\nabla \underline{v}|^2 dx \right) \int_{\Omega} \nabla \underline{v} \cdot \nabla \phi \, dx \leq \lambda \int_{\Omega} g(\underline{u}) \phi \, dx$$

for $\lambda > 0$ large enough and any $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω . From (2.3) and (2.4), $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1). Moreover, we have $\underline{u} > 0$ and $\underline{v} > 0$ in Ω , $\underline{u} \rightarrow +\infty$ and $\underline{v} \rightarrow +\infty$ as $\lambda \rightarrow +\infty$.

Next, we shall construct a supersolution of problem (1.1). For this purpose, we fix $\lambda > 0$ for which the subsolution is ensured. Let ϕ_0 be the solution of the following problem

$$(2.5) \quad \begin{cases} -\Delta \phi = 1 & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases}$$

Let

$$\bar{u} := \frac{C\lambda}{R} \phi_0, \quad \bar{v} := \frac{\lambda}{b_1} g(C\lambda) \phi_0,$$

where $R = \|\phi_0\|_\infty > 0$ and $C > 0$ is a large positive real number to be chosen later. We shall verify that $(\bar{u}, \bar{v})$ is a supersolution of problem (1.1) for $\lambda > 0$ large. To this end, let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω , from (H_1) , we have

$$\begin{aligned}
(2.6) \quad A \left(\int_{\Omega} |\nabla \bar{u}|^2 dx \right) \int_{\Omega} \nabla \bar{u} \cdot \nabla \phi \, dx &= A \left(\int_{\Omega} |\nabla \bar{u}|^2 dx \right) \cdot \lambda \frac{C}{R} \int_{\Omega} \nabla \phi_0 \cdot \nabla \phi \, dx \\
&= \lambda \frac{C}{R} A \left(\int_{\Omega} |\nabla \bar{u}|^2 dx \right) \int_{\Omega} \phi \, dx \\
&\geq \lambda a_1 \frac{C}{R} \int_{\Omega} \phi \, dx.
\end{aligned}$$

Using the condition (H_3) , we can choose a number $C > 0$ large enough so that

$$C\lambda \geq \frac{\lambda R}{a_1} f\left(\frac{R\lambda}{b_1} g(C\lambda)\right)$$

and therefore

$$\begin{aligned} (2.7) \quad A\left(\int_{\Omega} |\nabla \bar{u}|^2 dx\right) \int_{\Omega} \nabla \bar{u} \cdot \nabla \varphi \, dx &\geq \lambda a_1 \frac{C}{R} \int_{\Omega} \varphi \, dx \\ &\geq \lambda \int_{\Omega} f\left(\frac{R\lambda}{b_1} g(C\lambda)\right) \varphi \, dx \\ &\geq \lambda \int_{\Omega} f(\bar{v}) \varphi \, dx. \end{aligned}$$

Next, from the definition of $\bar{v}$, the condition (H_1) and the fact that g is increasing, we also deduce that

$$\begin{aligned} (2.8) \quad B\left(\int_{\Omega} |\nabla \bar{v}|^2 dx\right) \int_{\Omega} \nabla \bar{v} \cdot \nabla \varphi \, dx &= B\left(\int_{\Omega} |\nabla \bar{v}|^2 dx\right) \frac{\lambda}{b_1} g(C\lambda) \int_{\Omega} \nabla \phi_0 \cdot \nabla \varphi \, dx \\ &= B\left(\int_{\Omega} |\nabla \bar{v}|^2 dx\right) \frac{\lambda}{b_1} g(C\lambda) \int_{\Omega} \varphi \, dx \\ &\geq \lambda g(C\lambda) \int_{\Omega} \varphi \, dx \\ &\geq \lambda \int_{\Omega} g\left(\frac{C\lambda}{R} \phi_0\right) \varphi \, dx \\ &= \lambda \int_{\Omega} g(\bar{u}) \varphi \, dx. \end{aligned}$$

From (2.7) and (2.8), $(\bar{u}, \bar{v})$ is a supersolution of problem (1.1) with $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$ for $C > 0$ large.

In order to obtain a weak solution of problem (1.1) we shall use the arguments by Azzouz and Bensedik [2] (observe that f, g does not depend on x). For this purpose, we define a sequence $\{(u_n, v_n)\} \subset H_0^1(\Omega, \mathbf{R}^2)$ as follows: $u_0 := \bar{u}$, $v_0 := \bar{v}$ and (u_n, v_n) is the unique solution of the system

$$(2.9) \quad \begin{cases} -A\left(\int_{\Omega} |\nabla u_n|^2 dx\right) \Delta u_n = \lambda f(v_{n-1}) & \text{in } \Omega, \\ -B\left(\int_{\Omega} |\nabla v_n|^2 dx\right) \Delta v_n = \lambda g(u_{n-1}) & \text{in } \Omega, \\ u_n = v_n = 0, & \text{on } \partial\Omega. \end{cases}$$

System (2.9) is (A, B) -linear in the sense that, if $(u_{n-1}, v_{n-1}) \in H_0^1(\Omega, \mathbf{R}^2)$ is given, the right hand sides of (2.9) is independent of u_n, v_n . Set $\mathcal{A}(t) = tA(t^2)$, $\mathcal{B}(t) = tB(t^2)$. Then since $\mathcal{A}(\mathbf{R}) = \mathbf{R}$, $\mathcal{B}(\mathbf{R}) = \mathbf{R}$, and $f(v_{n-1}) \in L^2(\Omega)$, $g(u_{n-1}) \in L^2(\Omega)$ (in x), we deduce from a result in [1] that system (2.9) has a unique solution $(u_n, v_n) \in H_0^1(\Omega, \mathbf{R}^2)$.

Using (2.9) and the fact that (u_0, v_0) is a supersolution of (1.1) we have

$$\begin{aligned} -A\left(\int_{\Omega} |\nabla u_0|^2 dx\right) \Delta u_0 &\geq \lambda f(v_0) = -A\left(\int_{\Omega} |\nabla u_1|^2 dx\right) \Delta u_1, \\ -B\left(\int_{\Omega} |\nabla v_0|^2 dx\right) \Delta v_0 &\geq \lambda g(u_0) = -B\left(\int_{\Omega} |\nabla v_1|^2 dx\right) \Delta v_1 \end{aligned}$$

and by Theorem 1.1, $u_0 \geq u_1$ and $v_0 \geq v_1$. Also, since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$ and the monotonicity of f, g one has

$$\begin{aligned} -A\left(\int_{\Omega} |\nabla u_1|^2 dx\right) \Delta u_1 &= \lambda f(v_0) \geq \lambda f(\underline{v}) \geq -A\left(\int_{\Omega} |\nabla \underline{u}|^2 dx\right) \Delta \underline{u}, \\ -B\left(\int_{\Omega} |\nabla v_1|^2 dx\right) \Delta v_1 &= \lambda g(u_0) \geq \lambda g(\underline{u}) \geq -B\left(\int_{\Omega} |\nabla \underline{v}|^2 dx\right) \Delta \underline{v} \end{aligned}$$

from which, according to Theorem 1.1, $u_1 \geq \underline{u}$, $v_1 \geq \underline{v}$. For u_2, v_2 we write

$$\begin{aligned} -A\left(\int_{\Omega} |\nabla u_1|^2 dx\right) \Delta u_1 &= \lambda f(v_0) \geq \lambda f(v_1) = -A\left(\int_{\Omega} |\nabla u_2|^2 dx\right) \Delta u_2, \\ -B\left(\int_{\Omega} |\nabla v_1|^2 dx\right) \Delta v_1 &= \lambda g(u_0) \geq \lambda g(u_1) = -B\left(\int_{\Omega} |\nabla v_2|^2 dx\right) \Delta v_2, \end{aligned}$$

and then $u_1 \geq u_2$, $v_1 \geq v_2$. Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$\begin{aligned} -A\left(\int_{\Omega} |\nabla u_2|^2 dx\right) \Delta u_2 &= \lambda f(v_1) \geq \lambda f(\underline{v}) \geq -A\left(\int_{\Omega} |\nabla \underline{u}|^2 dx\right) \Delta \underline{u}, \\ -B\left(\int_{\Omega} |\nabla v_2|^2 dx\right) \Delta v_2 &= \lambda g(u_1) \geq \lambda g(\underline{u}) \geq -B\left(\int_{\Omega} |\nabla \underline{v}|^2 dx\right) \Delta \underline{v}. \end{aligned}$$

Repeating this argument we get a bounded monotone sequence $\{(u_n, v_n)\} \subset H_0^1(\Omega, \mathbf{R}^2)$ satisfying

$$(2.10) \quad \bar{u} = u_0 \geq u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots \geq \underline{u} > 0,$$

$$(2.11) \quad \bar{v} = v_0 \geq v_1 \geq v_2 \geq \cdots \geq v_n \geq \cdots \geq \underline{v} > 0.$$

Using the continuity of the functions f, g and the definition of the sequences $\{u_n\}$, $\{v_n\}$, there exist two constant $C_1, C_2 > 0$ independent of n

such that

$$(2.12) \quad |f(v_{n-1})| \leq C_1, \quad |g(u_{n-1})| \leq C_2 \quad \text{for all } n.$$

From (2.12), multiplying the first equation of (2.9) by u_n , integrating, using the Hölder inequality and the Sobolev embedding we can show that

$$\begin{aligned} a_1 \int_{\Omega} |\nabla u_n|^2 dx &\leq A \left(\int_{\Omega} |\nabla u_n|^2 dx \right) \int_{\Omega} |\nabla u_n|^2 dx \\ &= \lambda \int_{\Omega} f(v_{n-1}) u_n dx \\ &\leq \lambda \int_{\Omega} |f(v_{n-1})| |u_n| dx \\ &\leq \lambda C_1 \int_{\Omega} |u_n| dx \\ &\leq C_3 \|u_n\|, \end{aligned}$$

or

$$(2.13) \quad \|u_n\| \leq C_3, \quad \forall n$$

where $C_3 > 0$ is a constant independent of n . Similarly, there exists $C_4 > 0$ independent of n such that

$$(2.14) \quad \|v_n\| \leq C_4, \quad \forall n.$$

From (2.13), (2.14) we infer that $\{(u_n, v_n)\}$ has a subsequence which weakly converges in $H_0^1(\Omega, \mathbf{R}^2)$ to a limit (u, v) with the properties $u \geq \underline{u} > 0$ and $v \geq \underline{v} > 0$. Being monotone and also using a standard regularity argument, $\{(u_n, v_n)\}$ converges itself to (u, v) . Now, letting $n \rightarrow \infty$ in (2.9), we deduce that (u, v) is a positive solution of system (1.1). The proof of Theorem 2.1 is now completed. $\square$

3. Non-existence result

In this section, we are interested in the non-existence of a positive solution for system (1.1). We consider system (1.1) under the particular nonlinearities:

$$(3.1) \quad f(t) := c_1 t^q - d_1, \quad g(t) := c_2 t^r - d_2, \quad t \in \mathbf{R}^+,$$

where $c_i, d_i > 0$, $i = 1, 2$ and $q, r \in (0, 1)$. It should be noticed that the above nonlinearities satisfy the conditions (H_2) and (H_3) . Moreover, the functions f and g defined as in (3.1) can change sign and they are sublinear at infinity in the sense that $\lim_{t \rightarrow +\infty} f(t)/t = \lim_{t \rightarrow +\infty} g(t)/t = 0$. For this reason, our situation

is different from [2] even in the single equation case. Now, system (1.1) can be read as follows

$$(3.2) \quad \begin{cases} -A \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u = \lambda(c_1 v^q - d_1) & \text{in } \Omega, \\ -B \left(\int_{\Omega} |\nabla v|^2 dx \right) \Delta v = \lambda(c_2 u^r - d_2) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases}$$

Using Theorem 2.1, system (3.2) has at least one positive solution for λ large enough. We shall prove that system (3.2) has no positive solution for λ small enough. More precisely, we have the following result.

Theorem 3.1. *Assume that the condition (H_1) holds. Then there exists a positive constant λ_* such that for any $0 < \lambda < \lambda_*$ system (3.2) has no positive weak solution.*

Proof. Let λ_1 be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and ϕ_1 the corresponding positive eigenfunction. If $(u_\lambda, v_\lambda) \in H_0^1(\Omega, \mathbf{R}^2)$ is a positive solution of system (1.1) then

$$(3.3) \quad - \int_{\Omega} \phi_1 \Delta u_\lambda dx = \lambda_1 \int_{\Omega} u_\lambda \phi_1 dx$$

and

$$(3.4) \quad - \int_{\Omega} \phi_1 \Delta v_\lambda dx = \lambda_1 \int_{\Omega} v_\lambda \phi_1 dx.$$

Hence,

$$\begin{aligned} a_1 \lambda_1 \int_{\Omega} \phi_1 u_\lambda dx &\leq \lambda_1 A \left(\int_{\Omega} |\nabla u_\lambda|^2 dx \right) \int_{\Omega} \phi_1 u_\lambda dx \\ &= -A \left(\int_{\Omega} |\nabla u_\lambda|^2 dx \right) \int_{\Omega} \phi_1 \Delta u_\lambda dx \\ &= \lambda \int_{\Omega} f(v_\lambda) \phi_1 dx \\ &= \lambda \int_{\Omega} (c_1 v_\lambda^q - d_1) \phi_1 dx \end{aligned}$$

or

$$(3.5) \quad \int_{\Omega} \phi_1 (\lambda c_1 v_\lambda^q - a_1 \lambda_1 u_\lambda) dx \geq \lambda d_1 \int_{\Omega} \phi_1 dx.$$

Similarly, from (3.4), we deduce that

$$(3.6) \quad \int_{\Omega} \phi_1(\lambda c_2 u_{\lambda}^r - b_1 \lambda_1 v_{\lambda}) dx \geq \lambda d_2 \int_{\Omega} \phi_1 dx.$$

From (3.5) and (3.6), it follows that

$$(3.7) \quad \int_{\Omega} \phi_1[(\lambda c_2 u_{\lambda}^r - a_1 \lambda_1 u_{\lambda}) + (\lambda c_1 v_{\lambda}^q - b_1 \lambda_1 v_{\lambda})] dx \geq \lambda(d_1 + d_2) \int_{\Omega} \phi_1 dx.$$

Since $q, r \in (0, 1)$ the maximums of the functions $u_{\lambda} \mapsto \lambda c_2 u_{\lambda}^r - a_1 \lambda_1 u_{\lambda}$ and $v_{\lambda} \mapsto \lambda c_1 v_{\lambda}^q - b_1 \lambda_1 v_{\lambda}$ are achieved at $u_{\lambda} = (a_1 \lambda_1 / (\lambda r c_2))^{1/(r-1)}$ and at $v_{\lambda} = (b_1 \lambda_1 / (\lambda q c_1))^{1/(q-1)}$, respectively. Hence, by some simple calculations, it follows from (3.7) that

$$\int_{\Omega} \phi_1[L_1 \lambda^{1/(1-r)} + L_2 \lambda^{1/(1-q)}] dx \geq \lambda(d_1 + d_2) \int_{\Omega} \phi_1 dx$$

or

$$(3.8) \quad L_1 \lambda^{r/(1-r)} + L_2 \lambda^{q/(1-q)} \geq d_1 + d_2 > 0,$$

where $L_i, i = 1, 2$, are positive constants independent of λ . By (3.8), there exists a positive constant λ_* such that for any $\lambda \in (0, \lambda_*)$, system (3.2) has no positive solution. The proof of Theorem 3.1 is now completed. $\square$

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