Nakagiri's Monodromy Representations Associated with Appell's Hypergeometric Functions F_2 and F_3 *

By

Katsuhisa MIMACHI

(Osaka University, Japan)

Abstract. We study monodromy representations associated with Appell's hypergeometric functions F_2 and F_3 by using integrals of multivalued functions. In particular, we derive Nakagiri's matrix elements of the representations in a different manner and relax his conditions on the parameters. As an application, we give a simple and elementary derivation of sufficient conditions that the systems of differential equations satisfied by F_2 and F_3 are irreducible.

Key Words and Phrases. Appell's hypergeometric functions F_2 and F_3 , Hypergeometric integrals, Nakagiri's monodromy representations, Irreducibility.

2010 *Mathematics Subject Classification Numbers.* Primary 33C65; Secondary 32S40, 33C60.

Contents

0.	Introduction	78
1.	Auxiliary integrals	81
1.1.	A fundamental set of solutions	81
1.2.	Proof of Theorem 1.1	84
1.3.	Proof of Theorem 1.2	93
2.	Fundamental sets of solutions for E_2 and E_3	99
2.1.	E_2 case	99
2.2.	E_3 case	102
3.	A monodromy representation associated with the auxiliary integrals	105
4.	Monodromy representations associated with F_2 and F_3	118
4.1.	F_2 case	118
4.2.	F_3 case	123
5.	Irreducibility	127
6.	Appendix	130
	References	130

* Supported by JSPS Grant-in-Aid for Scientific Research (B) 23340002.

0. Introduction

We denote by E_2 and E_3 the systems of differential equations satisfied by Appell's hypergeometric functions F_2 and F_3 , respectively. The first purpose of the present paper is to derive Nakagiri's matrix elements of the five generators of the monodromy representations associated with F_2 and F_3 and to relax his conditions on the parameters [Na][ST]. Our approach is by using integrals of multivalued functions, and by deforming the chains on which the integrals are defined. Especially, when we manipulate the deformation, we depict each of the chains in the two dimensional complex space by the product of the paths in the complex plane, just like we do a multiple integral by an iterated integral, while Nakagiri depicted only real three dimensional sections of the chains, for instance, the section determined by the variables $\text{Re } t_1$, $\text{Im } t_1$ and $\text{Re } t_2$ for the real four dimensional space by t_1 , t_2 . On the other hand, a necessary and sufficient condition for each of the systems E_2 and E_3 being irreducible is obtained by Bod [Bo] and Kato [Ka]. To obtain the conditions, Bod applies the result by Beukers for the A -hypergeometric systems [Be], and Kato uses his connection relations among several kinds of solutions of E_2 . The second purpose is to give a simple and elementary derivation of sufficient conditions for the systems E_2 and E_3 being irreducible. In [MS4], a simple and elementary derivation of sufficient conditions for the systems E_2 and E_3 being reducible. Therefore, combining these results, we also arrive at the necessary and sufficient conditions obtained by Bod and Kato.

Appell's hypergeometric functions F_2 and F_3 are the analytic continuations of Appell's hypergeometric series

$$F_2(\alpha, \beta, \beta', \gamma; x, y) = \sum_{m, n \geq 0} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m! n! (\gamma)_m (\gamma')_n} x^m y^n, \quad |x| + |y| < 1,$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \sum_{m, n \geq 0} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{m! n! (\gamma)_{m+n}} x^m y^n, \quad |x| < 1, |y| < 1,$$

where $(a)_n = a(a+1) \dots (a+n-1)$. The functions F_2 and F_3 satisfy the following systems E_2 and E_3 , respectively:

$$(0.1) \quad \begin{cases} (E_2) \quad \left\{ \begin{aligned} & \left[x(1-x) \frac{\partial^2}{\partial x^2} - xy \frac{\partial^2}{\partial x \partial y} + \{\gamma - (\alpha + \beta + 1)x\} \frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} - \alpha \beta \right] F = 0, \\ & \left[y(1-y) \frac{\partial^2}{\partial y^2} - xy \frac{\partial^2}{\partial x \partial y} + \{\gamma' - (\alpha + \beta' + 1)y\} \frac{\partial}{\partial y} - \beta' x \frac{\partial}{\partial x} - \alpha \beta' \right] F = 0 \end{aligned} \right. \end{cases}$$

on the space

$$\mathcal{C}^2 \setminus \{\{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\} \cup \{x + y = 1\}\} \subset (\mathbf{P}^1)^2,$$

and

$$(0.2) \quad (E_3) \quad \begin{cases} \left[x(1-x) \frac{\partial^2}{\partial x^2} + y \frac{\partial^2}{\partial x \partial y} + \{\gamma - (\alpha + \beta + 1)x\} \frac{\partial}{\partial x} - \alpha\beta \right] F = 0, \\ \left[y(1-y) \frac{\partial^2}{\partial y^2} + x \frac{\partial^2}{\partial x \partial y} + \{\gamma - (\alpha' + \beta' + 1)y\} \frac{\partial}{\partial y} - \alpha'\beta' \right] F = 0 \end{cases}$$

on the space

$$\mathcal{C}^2 \setminus \{\{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\} \cup \{x + y = 1\}\} \subset (\mathbf{P}^1)^2.$$

Each of the systems E_2 and E_3 has rank 4. Necessary and sufficient conditions for the systems E_2 and E_3 being irreducible are the following:

$$(0.3) \quad (E_2) \quad \alpha, \beta, \beta', \gamma - \beta, \gamma' - \beta', \gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha \notin \mathbf{Z},$$

$$(0.4) \quad (E_3) \quad \alpha, \alpha', \beta, \beta', \gamma - \beta - \beta', \gamma - \alpha - \beta', \gamma - \alpha' - \beta, \gamma - \alpha - \alpha' \notin \mathbf{Z}.$$

On the other hand, in [Na], Nakagiri studies monodromy representations associated with F_2 and F_3 and gives explicit formulas about the matrix elements of the representations (See also [ST]). He imposes the condition

$$(0.5) \quad \begin{cases} \operatorname{Re}(1 - \alpha), \operatorname{Re}(\beta), \operatorname{Re}(\beta'), \operatorname{Re}(\gamma - \beta), \operatorname{Re}(\gamma' - \beta') > 0, \\ \gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha - 1 \notin \mathbf{Z}_{\leq 0}, \quad \text{and} \quad \gamma, \gamma' \notin \mathbf{Z} \end{cases}$$

to obtain the formula in E_2 case, and imposes the condition

$$(0.6) \quad \begin{cases} \operatorname{Re}(1 - \alpha), \operatorname{Re}(1 - \alpha'), \operatorname{Re}(\beta), \operatorname{Re}(\beta'), \operatorname{Re}(\gamma - \beta - \beta') > 0, \\ \gamma - \alpha - \alpha', \gamma - \alpha - \beta', \gamma - \alpha' - \beta \notin \mathbf{Z}_{\leq 0}, \quad \text{and} \quad \beta - \alpha, \beta' - \alpha' \notin \mathbf{Z} \end{cases}$$

to obtain the formula in E_3 case. We are interested in relaxing his conditions. The result is the following: In E_2 case, the condition is relaxed to

$$(0.7) \quad \begin{cases} -\alpha, \beta - 1, \beta' - 1, \gamma - \beta - 1, \gamma' - \beta' - 1 \notin \mathbf{Z}_{<0}, \\ \gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha - 1 \notin \mathbf{Z}_{\leq 0}, \end{cases}$$

and, in E_3 case, the condition is relaxed to

$$(0.8) \quad \begin{cases} -\alpha, -\alpha', \beta - 1, \beta' - 1, \gamma - \beta - \beta' - 1 \notin \mathbf{Z}_{<0}, \\ \gamma - \alpha - \beta', \gamma - \alpha' - \beta, \gamma - \alpha - \alpha' - 1 \notin \mathbf{Z}_{\leq 0}. \end{cases}$$

The contents of this paper is as follows.

In Sections 1, we introduce four integrals, which we call *the auxiliary integrals*, in the form

$$(0.9) \quad \iint_C t_1^{\lambda_1} (t_1 - z_1)^{\lambda_2} t_2^{\lambda_3} (t_2 - z_2)^{\lambda_4} (t_1 + t_2 - 1)^{\lambda_5} dt_1 dt_2,$$

where C is an appropriate chain. The properties of the auxiliary integrals give rise to the results for F_2 and F_3 . When none of λ_j is an integer, all the chains C which give the four integrals turn out to be cycles, and thus it is easily shown that each of the four integrals satisfies a system of differential equations. By contrast, when at least one of λ_j is an integer, it is subtle and tedious to show that each of the four integrals satisfies the system of differential equations. The condition for each of the four integrals satisfying the system is given in Theorem 1.1. Generally speaking, the chains which are not cycles play an essential role in the study of reducible systems (Theorem 5.1 says that when at least one of λ_j is an integer the system satisfied by the auxiliary integrals is reducible). Next, the condition for the four integrals being linearly independent is given in Theorem 1.2. The condition is derived by calculating a Wronskian formula for the four integrals. Combining Theorem 1.1 and Theorem 1.2, we obtain Corollary 1.3 to show the condition for the four integrals constituting a set of fundamental solutions. In Section 2, we give corresponding results for E_2 and E_3 . We show that four integrals in the form

$$(0.10) \quad \iint_C t_1^{\beta-1} t_2^{\beta'-1} (1-t_1)^{\gamma-\beta-1} (1-t_2)^{\gamma'-\beta'-1} (1-xt_1-yt_2)^{-\alpha} dt_1 dt_2$$

satisfy E_2 under the condition

$$(0.11) \quad -\alpha, \beta-1, \beta'-1, \gamma-\beta-1, \gamma'-\beta'-1 \notin \mathbf{Z}_{<0},$$

and that the four integrals are linearly independent and constitute a fundamental set of solutions of E_2 if (0.7). Similarly, we show that four integrals in the form

$$(0.12) \quad \iint_C t_1^{\beta-1} t_2^{\beta'-1} (1-t_1-t_2)^{\gamma-\beta-\beta'-1} (1-xt_1)^{-\alpha} (1-yt_2)^{-\alpha'} dt_1 dt_2$$

satisfy E_3 under the condition

$$(0.13) \quad -\alpha, -\alpha', \beta-1, \beta'-1, \gamma-\beta-\beta'-1 \notin \mathbf{Z}_{<0},$$

and that the four integrals are linearly independent and constitute a fundamental set of solutions of E_3 if (0.8). In Section 3, we study the monodromy representation associated with the auxiliary integrals (0.9) and calculate matrix elements of the five generators of the monodromy representation with respect

to the four integrals by deforming the chains appropriately, as in the cases of [Mi1, Mi2, MS1, MS2, MS3]. The result is given in Theorem 3.1, which gives rise to the matrix elements in the cases E_2 and E_3 . Nakagiri's matrix elements in E_2 case are given in Theorem 4.1 under the condition (0.11), and the matrix elements in E_3 case are given in Theorem 4.2 under the condition (0.13). In Section 5, by using the explicit expressions of the matrix elements, we derive sufficient conditions for the systems E_2 and E_3 being irreducible. Finally, in Section 6, we make a comment from the viewpoint of twisted homology theory: Our representations can be considered as monodromy representations on the twisted homology groups.

We refer to [AK, Ba, E, HK, Kim] for related results of F_2 , F_3 or E_2 , E_3 , and to [Ao1, Ao2, C, MY1] for the twisted de Rham theory.

In this paper, we frequently use the symbols

$$e(A) = \exp(2\pi\sqrt{-1}A), \quad \lambda_{ijk\dots l} = \lambda_i + \lambda_j + \lambda_k + \dots + \lambda_l, \quad e_{ijk\dots l} = e(\lambda_{ijk\dots l}),$$

$$\partial_x = \frac{\partial}{\partial x}, \quad \theta_x = x\partial_x, \quad t = (t_1, t_2), \quad dt = dt_1 dt_2$$

for simplicity.

1. Auxiliary integrals

1.1. A fundamental set of solutions

Let $z = (z_1, z_2)$ be a point of

$$Z = \mathcal{C}^2 \setminus \{z_1 = 0\} \cup \{z_1 = 1\} \cup \{z_2 = 0\} \cup \{z_2 = 1\} \cup \{z_1 + z_2 = 1\} \subset (\mathbf{P}^1)^2,$$

and $u(t) = u(z; t)$ be a function

$$(1.1) \quad u(z; t) = \prod_i f_i(t)^{\lambda_i} = t_1^{\lambda_1} (t_1 - z_1)^{\lambda_2} t_2^{\lambda_3} (t_2 - z_2)^{\lambda_4} (t_1 + t_2 - 1)^{\lambda_5}$$

on $T_z = \mathcal{C}^2 \setminus \bigcup_{i, \lambda_i \notin \mathbf{Z}_{\geq 0}} \{f_i(t) = 0\}$, where $f_1(t) = t_1, f_2(t) = t_1 - z_1, \dots, f_5(t) = t_1 + t_2 - 1$. When a pair of complex variables $z = (z_1, z_2)$ are real and sufficiently small positive numbers, assign the name D_j for $1 \leq j \leq 4$ to each domain of the real manifold T_R (the real locus of $T = T_z$) as in Figure 1:

$$D_1 = \{0 < t_1 < z_1, 0 < t_2 < z_2\}, \quad D_2 = \{z_1 < t_1, z_2 < t_2, t_1 + t_2 < 1\},$$

$$D_3 = \{0 < t_2 < z_2, z_1 < t_1, t_1 + t_2 < 1\}, \quad D_4 = \{0 < t_1 < z_1, z_2 < t_2, t_1 + t_2 < 1\},$$

where the orientation of D_j is fixed to be natural one induced from T_R .

Fig. 1

For a domain D_j , we let $u_{D_j}(z; t)$ denote the function

$$u_{D_j}(z; t) = \prod_i (\varepsilon_i f_i(t))^{\lambda_i},$$

where $\varepsilon_i = \pm$ is so determined that each $\varepsilon_i f_i(t)$ is positive and the argument of $\varepsilon_i f_i(t)$ is assigned to be zero on D_j .

In case $\lambda_i \notin \mathbf{Z}_{<0}$, $1 \leq i \leq 5$, we consider the four integrals $I_j(z) = \int_{\text{reg } D_j} u_{D_j}(t; z) dt$, $1 \leq j \leq 4$, over the regularization of D_j defined by

$$\begin{aligned} \int_{\text{reg } D_1} u_{D_1}(t) dt &= \int_{\text{reg}\{0 < t_1 < z_1\}} dt_1 \int_{\text{reg}\{0 < t_2 < z_2\}} dt_2 u_{D_1}(t), \\ \int_{\text{reg } D_2} u_{D_2}(t) dt &= \int_{\text{reg}\{z_1 < t_1 < 1 - z_2\}} dt_1 \int_{\text{reg}\{z_2 < t_2 < 1 - t_1\}} dt_2 u_{D_2}(t) \\ &= \int_{\text{reg}\{z_2 < t_2 < 1 - z_1\}} dt_2 \int_{\text{reg}\{z_1 < t_1 < 1 - t_2\}} dt_1 u_{D_2}(t), \\ \int_{\text{reg } D_3} u_{D_3}(t) dt &= \int_{\text{reg}\{0 < t_2 < z_2\}} dt_2 \int_{\text{reg}\{z_1 < t_1 < 1 - t_1\}} dt_1 u_{D_3}(t), \\ \int_{\text{reg } D_4} u_{D_4}(t) dt &= \int_{\text{reg}\{0 < t_1 < z_1\}} dt_1 \int_{\text{reg}\{z_2 < t_2 < 1 - t_2\}} dt_2 u_{D_4}(t), \end{aligned}$$

where the regularization of the path $\{0 < t_1 < z_1\}$ associated with the function $u_{D_1}(t)$ is given by

$$\text{reg}(0, z_1) = \begin{cases} \frac{1}{d_{\lambda_1}} S(0 + \varepsilon) + \overrightarrow{[0 + \varepsilon, z_1 - \varepsilon]} - \frac{1}{d_{\lambda_2}} S(z_1 - \varepsilon), & \text{if } \lambda_1, \lambda_2 \notin \mathbf{Z}, \\ \frac{1}{d_{\lambda_1}} S(0 + \varepsilon) + \overrightarrow{[0 + \varepsilon, z_1]}, & \text{if } \lambda_1 \notin \mathbf{Z}, \lambda_2 \in \mathbf{Z}_{\geq 0}, \\ \overrightarrow{[0, z_1 - \varepsilon]} - \frac{1}{d_{\lambda_2}} S(z_1 - \varepsilon), & \text{if } \lambda_1 \in \mathbf{Z}_{\geq 0}, \lambda_2 \notin \mathbf{Z}, \\ \overrightarrow{[0, z_1]}, & \text{if } \lambda_1, \lambda_2 \in \mathbf{Z}_{\geq 0}. \end{cases}$$

and the other regularizations of the paths are similarly given. Here d_λ denotes the value $e(\lambda) - 1$, and $S(z_k + \varepsilon)$ or $S(z_k - \varepsilon)$ stands for the positively oriented circle that is centered at the point z_k and that starts and ends at the point $z_k + \varepsilon$ or $z_k - \varepsilon$, as is depicted in Figure 2.

Fig. 2. $S(z_k + \varepsilon)$ and $S(z_k - \varepsilon)$

We refer to [Kit] for a way of constructing the cycle $\text{reg } D_j$ when $\lambda_i \notin \mathbf{Z}$ for all $1 \leq i \leq 5$, and to [MS1] for the regularization of the chain when some of λ_i are integers.

Then we have the following.

Theorem 1.1. *If*

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \notin \mathbf{Z}_{<0},$$

then each integral $I_j(z) = \int_{\text{reg } D_j} u_{D_j}(x, y; t) dt$ for $1 \leq j \leq 4$ satisfies the system of differential equations

$$(1.2) \quad \begin{cases} [\theta_{z_1}(\theta_{z_1} - \lambda_{12} - 1) - z_1(\theta_{z_1} - \lambda_2)(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})]F = 0, \\ [\theta_{z_2}(\theta_{z_2} - \lambda_{34} - 1) - z_2(\theta_{z_2} - \lambda_4)(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})]F = 0. \end{cases}$$

Theorem 1.2. *If*

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \notin \mathbf{Z}_{<0} \quad \text{and} \quad \lambda_{125} + 2, \lambda_{345} + 2, \lambda_{12345} + 3 \notin \mathbf{Z}_{\leq 0},$$

then the integrals $I_j(z) = \int_{\text{reg } D_j} u_{D_j}(z; t) dt$ for $1 \leq j \leq 4$ are linearly independent.

Therefore, combining Theorems 1.1 with 1.2 implies

Corollary 1.3. *If*

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \notin \mathbf{Z}_{<0} \quad \text{and} \quad \lambda_{125} + 2, \lambda_{345} + 2, \lambda_{12345} + 3 \notin \mathbf{Z}_{\leq 0},$$

then the integrals $I_j(z) = \int_{\text{reg } D_j} u_{D_j}(z; t) dt$ for $1 \leq j \leq 4$ constitute a fundamental set of solutions of (1.2).

Theorems 1.1 and 1.2 are proved in Subsections 1.2 and 1.3.

1.2. Proof of Theorem 1.1

We show that each of the integrals $\int_{\text{reg } D_j} u_{D_j}(x, y; t) dt$, $1 \leq j \leq 4$, satisfies the system of equations (1.2). We start with the following.

Lemma 1.4. (1)

$$(1.3) \quad [\theta_{z_1}(\theta_{z_1} - \lambda_{12} - 1) - z_1(\theta_{z_1} - \lambda_2)(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})]u(t) \\ = -\partial_{t_1} \left\{ \frac{\lambda_2 z_1 t_1 (t_1 - 1)}{t_1 - z_1} u(t) \right\} - \partial_{t_2} \left\{ \frac{\lambda_2 z_1 t_1 t_2}{t_1 - z_1} u(t) \right\}.$$

(2)

$$(1.4) \quad [\theta_{z_2}(\theta_{z_2} - \lambda_{34} - 1) - z_2(\theta_{z_2} - \lambda_4)(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})]u(t) \\ = -\partial_{t_2} \left\{ \frac{\lambda_4 z_2 t_2 (t_2 - 1)}{t_2 - z_2} u(t) \right\} - \partial_{t_1} \left\{ \frac{\lambda_4 z_2 t_1 t_2}{t_2 - z_2} u(t) \right\}.$$

Proof. (1) We have

$$(1.5) \quad \theta_{z_1} u(t) = \frac{-\lambda_2 z_1}{t_1 - z_1} u(t), \quad \theta_{z_2} u(t) = \frac{-\lambda_4 z_2}{t_2 - z_2} u(t), \\ \theta_{t_1} u(t) = \left[\lambda_1 + \frac{\lambda_2 t_1}{t_1 - z_1} + \frac{\lambda_5 t_1}{t_1 + t_2 - 1} \right] u(t), \\ \theta_{t_2} u(t) = \left[\lambda_3 + \frac{\lambda_4 t_2}{t_2 - z_2} + \frac{\lambda_5 t_2}{t_1 + t_2 - 1} \right] u(t),$$

and

$$(1.6) \quad (\theta_{z_1} - \lambda_2)u(t) = \frac{-\lambda_2 t_1}{t_1 - z_1} u(t),$$

which imply

$$(\theta_{z_1} + \theta_{t_1})u(t) = \left\{ \lambda_{12} + \frac{\lambda_5 t_1}{t_1 + t_2 - 1} \right\} u(t), \\ (\theta_{z_1} + \theta_{z_2} + \theta_{t_1} + \theta_{t_2})u(t) = \left\{ \lambda_{12345} + \frac{\lambda_5}{t_1 + t_2 - 1} \right\} u(t),$$

and hence

$$(1.7) \quad (\theta_{z_1} - \lambda_{12} - 1)u(t) = -\partial_{t_1} \{ t_1 u(t) \} + \frac{\lambda_5 t_1}{t_1 + t_2 - 1} u(t),$$

$$(1.8) \quad (\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})u(t) = -\partial_{t_1}\{t_1 u(t)\} - \partial_{t_2}\{t_2 u(t)\} + \frac{\lambda_5}{t_1 + t_2 - 1}u(t),$$

because of $\theta_{t_i}u(t) = -u(t) + \partial_{t_i}\{t_i u(t)\}$.

By (1.5), the action of θ_{z_1} on the equality (1.7) implies

$$(1.9) \quad \begin{aligned} \theta_{z_1}(\theta_{z_1} - \lambda_{12} - 1)u(t) &= -\partial_{t_1}\{t_1 \theta_{z_1} u(t)\} + \frac{\lambda_5 t_1}{t_1 + t_2 - 1} \theta_{z_1} u(t) \\ &= -\partial_{t_1} \left\{ t_1 \frac{-\lambda_2 z_1}{t_1 - z_1} u(t) \right\} + \frac{\lambda_5 t_1}{t_1 + t_2 - 1} \frac{-\lambda_2 z_1}{t_1 - z_1} u(t). \end{aligned}$$

By (1.6), the action of $(\theta_{z_1} - \lambda_2)$ on the equality (1.8) implies

$$(1.10) \quad \begin{aligned} (\theta_{z_1} - \lambda_2)(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})u(t) \\ &= -\partial_{t_1}\{t_1(\theta_{z_1} - \lambda_2)u(t)\} - \partial_{t_2}\{t_2(\theta_{z_1} - \lambda_2)u(t)\} + \frac{\lambda_5}{t_1 + t_2 - 1}(\theta_{z_1} - \lambda_2)u(t) \\ &= -\partial_{t_1} \left\{ t_1 \frac{-\lambda_2 t_1}{t_1 - z_1} u(t) \right\} - \partial_{t_2} \left\{ t_2 \frac{-\lambda_2 t_1}{t_1 - z_1} u(t) \right\} + \frac{\lambda_5}{t_1 + t_2 - 1} \frac{-\lambda_2 t_1}{t_1 - z_1} u(t). \end{aligned}$$

Combination of (1.9) and (1.10) leads to (1.3).

(2) Equality (1.4) follows from (1.3) by changing the role of z_1 with that of z_2 , and changing the role of (λ_1, λ_2) with that of (λ_3, λ_4) . $\square$

If $\lambda_i \notin \mathbf{Z}$ for all i ($1 \leq i \leq 5$), all of the integrals

$$\int_{\text{reg } D_j} \partial_{t_1} \left\{ \frac{\lambda_2 z_1 t_1 (t_1 - 1)}{t_1 - z_1} u(z; t) \right\} dt \quad \text{and} \quad \int_{\text{reg } D_j} \partial_{t_2} \left\{ \frac{\lambda_2 z_1 t_1 t_2}{t_1 - z_1} u(z; t) \right\} dt$$

for $j = 1, 2, 3, 4$ are equal to zero, since each $\text{reg } D_j$ is a cycle. Therefore Lemma 1.4 implies that each of the integrals $\int_{\text{reg } D_j} U(t) dt$ for $1 \leq j \leq 4$ satisfies the system of differential equations (1.2). On the other hand, if some of λ_i are integers,

$$\int_{\text{reg } D_j} \partial_{t_1} \left\{ \frac{\lambda_2 z_1 t_1 (t_1 - 1)}{t_1 - z_1} u(z; t) \right\} dt \quad \text{or} \quad \int_{\text{reg } D_j} \partial_{t_2} \left\{ \frac{\lambda_2 z_1 t_1 t_2}{t_1 - z_1} u(z; t) \right\} dt$$

might not be zero, and thus we need more argument to obtain the assertion.

The assertion for the second equation of (1.2) follows from the first equation of (1.2) by changing the role of $(z_1, \lambda_1, \lambda_2)$ with that of $(z_2, \lambda_3, \lambda_4)$. Hence in what follows we only prove that each of the integrals $\int_{\text{reg } D_j} u_{D_j}(t) dt$ satisfies the first equation of (1.2).

In what follows, we assume that

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \notin \mathbf{Z}_{<0},$$

and divide the case $\lambda_1, \dots, \lambda_5 \notin \mathbf{Z}_{<0}$ into three subcases (a) $\lambda_2 = 0$, (b) $\lambda_2 = 1$, (c) $\lambda_2 \neq 0, 1$ ($\lambda_2 \notin \mathbf{Z}_{\leq 1}$).

(a) Case $\lambda_2 = 0$: The first equation of (1.2) reduces to

$$[(\theta_{z_1} - \lambda_1 - 1) - z_1(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})]\theta_{z_1}F = 0.$$

Moreover, the equalities

$$\begin{aligned} \theta_{z_1} \int_{\text{reg } D_1} u_{D_1}(z; t) dt &= z_1^{\lambda_1+1} \int_{\text{reg}(0, z_2)} t_2^{\lambda_3} (z_2 - t_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt_2 \\ &= z_1^{\lambda_1+1} z_2^{\lambda_{34}+1} \int_{\text{reg}(0, 1)} t_2^{\lambda_3} (1 - t_2)^{\lambda_4} (1 - z_1 - z_2 t_2)^{\lambda_5} dt_2, \\ \theta_{z_1} \int_{\text{reg } D_2} u_{D_2}(z; t) dt &= -z_1^{\lambda_1+1} \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} (t_2 - z_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt_2 \\ &= -z_1^{\lambda_1+1} z_2^{\lambda_{34}+1} \int_{\text{reg}(1, (1-z_1)/z_2)} t_2^{\lambda_3} (t_2 - 1)^{\lambda_4} (1 - z_1 - z_2 t_2)^{\lambda_5} dt_2, \\ \theta_{z_1} \int_{\text{reg } D_3} u_{D_3}(z; t) dt &= -z_1^{\lambda_1+1} \int_{\text{reg}(0, z_2)} t_2^{\lambda_3} (z_2 - t_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt_2 \\ &= -z_1^{\lambda_1+1} z_2^{\lambda_{34}+1} \int_{\text{reg}(0, 1)} t_2^{\lambda_3} (1 - t_2)^{\lambda_4} (1 - z_1 - z_2 t_2)^{\lambda_5} dt_2, \\ \theta_{z_1} \int_{\text{reg } D_4} u_{D_4}(z; t) dt &= z_1^{\lambda_1+1} \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} (z_2 - t_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt_2 \\ &= z_1^{\lambda_1+1} z_2^{\lambda_{34}+1} \int_{\text{reg}(1, (1-z_1)/z_2)} t_2^{\lambda_3} (t_2 - 1)^{\lambda_4} (1 - z_1 - z_2 t_2)^{\lambda_5} dt_2 \end{aligned}$$

and

$$\begin{aligned} &[(\theta_{z_1} - \lambda_1 - 1) - z_1(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})]z_1^{\lambda_1+1}z_2^{\lambda_{34}+1} \\ &= z_1^{\lambda_1+1}z_2^{\lambda_{34}+1}[(1 - z_1)\theta_{z_1} - z_1(\theta_{z_2} - \lambda_5)] \end{aligned}$$

make our work reduce to check the equalities

$$(1.11) \quad [(1 - z_1)\theta_{z_1} - z_1(\theta_{z_2} - \lambda_5)] \int_{\text{reg}(0, 1)} t_2^{\lambda_3} (1 - t_2)^{\lambda_4} (1 - z_1 - z_2 t_2)^{\lambda_5} dt_2 = 0$$

and

$$\begin{aligned} (1.12) \quad &[(1 - z_1)\theta_{z_1} - z_1(\theta_{z_2} - \lambda_5)] \\ &\times \int_{\text{reg}(1, (1-z_1)/z_2)} t_2^{\lambda_3} (t_2 - 1)^{\lambda_4} (1 - z_1 - z_2 t_2)^{\lambda_5} dt_2 = 0. \end{aligned}$$

In case $\lambda_5 \neq 0$, the equality

$$[(1 - z_1)\theta_{z_1} - z_1(\theta_{z_2} - \lambda_5)](1 - z_1 - z_2 t_2)^{\lambda_5} = 0$$

implies (1.11) and (1.12).

In case $\lambda_5 = 0$, (1.11) trivially holds, and (1.12) follows from

$$\theta_{z_1} \int_{\text{reg}(1, (1-z_1)/z_2)} t_2^{\lambda_3} (t_2 - 1)^{\lambda_4} dt_2 = -z_1 z_2^{-1-\lambda_{34}} (1 - z_1)^{\lambda_3} (1 - z_1 - z_2)^{\lambda_4},$$

and

$$\theta_{z_2} \int_{\text{reg}(1, (1-z_1)/z_2)} t_2^{\lambda_3} (t_2 - 1)^{\lambda_4} dt_2 = -z_2^{-1-\lambda_{34}} (1 - z_1)^{\lambda_3+1} (1 - z_1 - z_2)^{\lambda_4}.$$

We arrive at the required result.

(b) Case $\lambda_2 = 1$: The left side of equality (1.3) reduces to

$$\begin{cases} [-\lambda_{12}\theta_{z_1} - z_1(\theta_{z_1} - 1)(\theta_{z_2} - 2 - \lambda_{12345})]u(z; t) & \text{if } \lambda_4 \neq 0, \\ [-\lambda_{12}\theta_{z_1} + z_1(\theta_{z_1} - 1)(2 + \lambda_{12345})]u(z; t) & \text{if } \lambda_4 = 0, \end{cases}$$

since $\theta_{z_1}^2 u(z; t) = \theta_{z_1} u(z; t)$ for $\lambda_2 = 1$ and $\theta_{z_2} u(z; t) = 0$ for $\lambda_4 = 0$. We prove the equality

$$(1.13) \quad \int_{\text{reg } D_j} \partial_{t_1} \left\{ \frac{z_1 t_1 (t_1 - 1)}{t_1 - z_1} u_{D_j}(z; t) \right\} dt + \int_{\text{reg } D_j} \partial_{t_2} \left\{ \frac{z_1 t_1 t_2}{t_1 - z_1} u_{D_j}(z; t) \right\} dt$$

$$= \begin{cases} (1 - z_1)(\theta_{z_1}^2 - \theta_{z_1}) \int_{\text{reg } D_j} u_{D_j}(z; t) dt & \text{if } \lambda_4 \neq 0, \\ (1 - z_1)(\theta_{z_1}^2 - \theta_{z_1}) \int_{\text{reg } D_j} u_{D_j}(z; t) dt \\ \quad - z_1(\theta_{z_1} - 1)\theta_{z_2} \int_{\text{reg } D_j} u_{D_j}(z; t) dt & \text{if } \lambda_4 = 0. \end{cases}$$

(b-1) In case $j = 1$: We have

$$\begin{aligned} & \int_{\text{reg } D_1} \partial_{t_1} \left\{ \frac{z_1 t_1 (t_1 - 1)}{t_1 - z_1} u_{D_1}(z; t) \right\} dt \\ &= \int_{\text{reg}(0, z_2)} dt_2 \int_{\text{reg}(0, z_1)} dt_1 \partial_{t_1} \{ z_1 t_1^{\lambda_1+1} (1 - t_1) t_2^{\lambda_3} (z_2 - t_2)^{\lambda_4} (1 - t_1 - t_2)^{\lambda_5} \} \\ &= (1 - z_1) z_1^{\lambda_1+2} \int_{\text{reg}(0, z_2)} t_2^{\lambda_3} (z_2 - t_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt_2 \end{aligned}$$

and

$$\begin{aligned}
& \int_{\text{reg } D_1} \partial_{t_2} \left\{ \frac{z_1 t_1 t_2}{t_1 - z_1} u_{D_1}(z; t) \right\} dt_1 dt_2 \\
&= \int_{\text{reg}(0, z_1)} dt_1 \int_{\text{reg}(0, z_2)} dt_2 \partial_{t_2} \{ -z_1 t_1^{\lambda_1+1} t_2^{\lambda_3+1} (z_2 - t_2)^{\lambda_4} (1 - t_1 - t_2)^{\lambda_5} \} \\
&= \begin{cases} 0 & \text{if } \lambda_4 \neq 0, \\ -z_1 z_2^{\lambda_3+1} \int_{\text{reg}(0, z_1)} t_1^{\lambda_1+1} (1 - t_1 - z_2)^{\lambda_5} dt_1 & \text{if } \lambda_4 = 0. \end{cases}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& (\theta_{z_1}^2 - \theta_{z_1}) \int_{\text{reg } D_1} u_{D_1}(z; t) dt \\
&= (\theta_{z_1} - 1) \int_{\text{reg } D_1} \{ z_1 t_1^{\lambda_1} t_2^{\lambda_3} (z_2 - t_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} \} dt \\
&= z_1 \theta_{z_1} \int_{\text{reg } D_1} t_1^{\lambda_1} t_2^{\lambda_3} (z_2 - t_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt \\
&= z_1^{\lambda_1+2} \int_{\text{reg}(0, z_2)} t_2^{\lambda_3} (z_2 - t_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt_2
\end{aligned}$$

and, if $\lambda_4 = 0$,

$$z_1 (\theta_{z_1} - 1) \theta_{z_2} \int_{\text{reg } D_1} u_{D_1}(z; t) dt = z_1 z_2^{\lambda_3+1} \int_{\text{reg}(0, z_1)} t_1^{\lambda_1+1} (1 - z_2 - t_1)^{\lambda_5} dt_1.$$

Combination of these equalities implies (1.13).

(b-2) In case $j = 2$: We have

$$\begin{aligned}
& \int_{\text{reg } D_2} \partial_{t_1} \left\{ \frac{z_1 t_1 (t_1 - 1)}{t_1 - z_1} u_{D_2}(z; t) \right\} dt \\
&= \begin{cases} -z_1^{\lambda_1+2} (z_1 - 1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} (t_2 - z_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt_2 & \text{if } \lambda_5 \neq 0, \\ -z_1^{\lambda_1+2} (z_1 - 1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} (t_2 - z_2)^{\lambda_4} dt_2 \\ \quad - z_1 \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3+1} (1 - t_2)^{\lambda_1+1} (t_2 - z_2)^{\lambda_4} dt_2 & \text{if } \lambda_5 = 0 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\text{reg } D_2} \partial_{t_2} \left\{ \frac{z_1 t_1 t_2}{t_1 - z_1} u_{D_2}(z; t) \right\} dt \\
&= \begin{cases} 0 & \text{if } \lambda_4, \lambda_5 \neq 0, \\
z_1 \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (1-t_1)^{\lambda_3+1} (1-t_1-z_2)^{\lambda_4} dt_1 & \text{if } \lambda_4 \neq 0, \lambda_5 = 0, \\
-z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (1-t_1-z_2)^{\lambda_5} dt_1 & \text{if } \lambda_4 = 0, \lambda_5 \neq 0, \\
z_1 \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (1-t_1)^{\lambda_3+1} dt_1 \\
\quad - z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} dt_1 & \text{if } \lambda_4 = \lambda_5 = 0, \end{cases}
\end{aligned}$$

and hence

$$\begin{aligned}
& \int_{\text{reg } D_2} \partial_{t_1} \left\{ \frac{z_1 t_1 (t_1 - 1)}{t_1 - z_1} u_{D_2}(z; t) \right\} dt + \int_{\text{reg } D_2} \partial_{t_2} \left\{ \frac{z_1 t_1 t_2}{t_1 - z_1} u_{D_2}(z; t) \right\} dt \\
&= \begin{cases} -z_1^{\lambda_1+2} (z_1 - 1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} (t_2 - z_2)^{\lambda_4} (1 - z_1 - t_2)^{\lambda_5} dt_2 & \text{if } \lambda_4, \lambda_5 \neq 0, \\
\left\{ -z_1^{\lambda_1+2} (z_1 - 1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} (t_2 - z_2)^{\lambda_4} dt_2 \right. \\
\quad \left. - z_1 \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3+1} (1-t_2)^{\lambda_1+1} (t_2 - z_2)^{\lambda_4} dt_2 \right\} \\
\quad + z_1 \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (1-t_1)^{\lambda_3+1} (1-t_1-z_2)^{\lambda_4} dt_1 & \text{if } \lambda_4 \neq 0, \lambda_5 = 0, \\
-z_1^{\lambda_1+2} (z_1 - 1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} (1 - z_1 - t_2)^{\lambda_5} dt_2 \\
\quad - z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (1-t_1-z_2)^{\lambda_5} dt_1 & \text{if } \lambda_4 = 0, \lambda_5 \neq 0, \\
\left\{ -z_1^{\lambda_1+2} (z_1 - 1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} dt_2 - z_1 \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3+1} (1-t_2)^{\lambda_1+1} dt_2 \right\} \\
\quad + \left\{ z_1 \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (1-t_1)^{\lambda_3+1} dt_1 \right. \\
\quad \left. - z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} dt_1 \right\} & \text{if } \lambda_4 = \lambda_5 = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
&= \begin{cases} -z_1^{\lambda_1+2}(z_1-1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3}(t_2-z_2)^{\lambda_4}(1-z_1-t_2)^{\lambda_5} dt_2 & \text{if } \lambda_4, \lambda_5 \neq 0, \\ -z_1^{\lambda_1+2}(z_1-1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3}(t_2-z_2)^{\lambda_4} dt_2 & \text{if } \lambda_4 \neq 0, \lambda_5 = 0, \\ -z_1^{\lambda_1+2}(z_1-1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3}(1-z_1-t_2)^{\lambda_5} dt_2 \\ \quad - z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1}(1-t_1-z_2)^{\lambda_5} dt_1 & \text{if } \lambda_4 = 0, \lambda_5 \neq 0, \\ -z_1^{\lambda_1+2}(z_1-1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3} dt_2 - z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} dt_1 & \text{if } \lambda_4 = \lambda_5 = 0 \end{cases} \\
&= \begin{cases} -z_1^{\lambda_1+2}(z_1-1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3}(t_2-z_2)^{\lambda_4}(1-z_1-t_2)^{\lambda_5} dt_2 & \text{if } \lambda_4 \neq 0, \\ -z_1^{\lambda_1+2}(z_1-1) \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3}(1-z_1-t_2)^{\lambda_5} dt_2 \\ \quad - z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1}(1-t_1-z_2)^{\lambda_5} dt_1 & \text{if } \lambda_4 = 0. \end{cases}
\end{aligned}$$

The second equality above follows from

$$\begin{aligned}
&\int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3+1}(1-t_2)^{\lambda_1+1}(t_2-z_2)^{\lambda_4} dt_2 \\
&= \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1}(1-t_1)^{\lambda_3+1}(1-t_1-z_2)^{\lambda_4} dt_1,
\end{aligned}$$

which is derived by the change of integration variable $1-t_2 = t_1$ ($-dt_2 = dt_1$).

On the other hand, we have

$$\begin{aligned}
&(1-z_1)(\theta_{z_1}^2 - \theta_{z_1}) \int_{\text{reg } D_2} u_{D_2}(z; t) dt \\
&= (1-z_1) z_1^{\lambda_1+2} \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3}(t_2-z_2)^{\lambda_4}(1-z_1-t_2)^{\lambda_5} dt_2
\end{aligned}$$

and

$$\begin{aligned}
&z_1(\theta_{z_1} - \lambda_2)\theta_{z_2} \int_{\text{reg } D_2} u_{D_2}(z; t) dt \\
&= z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1}(1-z_2-t_1)^{\lambda_5} dt_1 \quad \text{for } \lambda_4 = 0.
\end{aligned}$$

Thus we obtain (1.13).

A similar argument leads to (1.13) for $j = 3$ and $j = 4$.

(c) Case $\lambda_2 \neq 0, 1$: If $\lambda_4 = 0$, then $\theta_{z_2}u(z; t) = 0$, and the left side of equality (1.3) reduces to

$$[-\lambda_{12}\theta_{z_1} + z_1(\theta_{z_1} - 1)(2 + \lambda_{12345})]u(z; t).$$

Hence we prove

$$(1.14) \quad \int_{\text{reg } D_j} \partial_{t_1} \left\{ \frac{\lambda_2 z_1 t_1 (t_1 - 1)}{t_1 - z_1} u_{D_j}(z; t) \right\} dt + \int_{\text{reg } D_j} \partial_{t_2} \left\{ \frac{\lambda_2 z_1 t_1 t_2}{t_1 - z_1} u_{D_j}(z; t) \right\} dt$$

$$= \begin{cases} 0 & \text{if } \lambda_4 \neq 0, \\ -z_1(\theta_{z_1} - \lambda_2)\theta_{z_2} \int_{\text{reg } D_j} u_{D_j}(z; t) dt & \text{if } \lambda_4 = 0. \end{cases}$$

(c-1) In case $j = 1$: We have

$$\int_{\text{reg } D_1} \partial_{t_1} \left\{ \frac{\lambda_2 z_1 t_1 (t_1 - 1)}{t_1 - z_1} u_{D_1}(z; t) \right\} dt = 0$$

and

$$\int_{\text{reg } D_1} \partial_{t_2} \left\{ \frac{\lambda_2 z_1 t_1 t_2}{t_1 - z_1} u_{D_1}(z; t) \right\} dt$$

$$= \begin{cases} 0 & \text{if } \lambda_4 \neq 0, \\ -\lambda_2 z_1 z_2^{\lambda_3+1} \int_{\text{reg}(0, z_1)} t_1^{\lambda_1+1} (1 - t_1 - z_2)^{\lambda_5} dt_1 & \text{if } \lambda_4 = 0. \end{cases}$$

On the other hand, we have

$$z_1(\theta_{z_1} - \lambda_2)\theta_{z_2} \int_{\text{reg } D_1} u_{D_1}(z; t) dt$$

$$= \lambda_2 z_1 z_2^{\lambda_3+1} \int_{\text{reg}(0, z_1)} t_1^{\lambda_1+1} (z_1 - t_1)^{\lambda_2-1} (1 - z_2 - t_1)^{\lambda_5} dt_1 \quad \text{for } \lambda_4 = 0.$$

Combination of these equalities implies (1.14).

(c-2) In case $j = 2$: We have

$$\int_{\text{reg } D_2} \partial_{t_1} \left\{ \frac{\lambda_2 z_1 t_1 (t_1 - 1)}{t_1 - z_1} u_{D_2}(z; t) \right\} dt$$

$$= \begin{cases} 0 & \text{if } \lambda_5 \neq 0, \\ -\lambda_2 z_1 \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3+1} (1 - t_2)^{\lambda_1+1} (t_2 - z_2)^{\lambda_4} \\ \quad \times (1 - t_2 - z_1)^{\lambda_2-1} dt_2 & \text{if } \lambda_5 = 0 \end{cases}$$

and

$$\begin{aligned} & \int_{\text{reg } D_2} \partial_{t_2} \left\{ \frac{\lambda_2 z_1 t_1 t_2}{t_1 - z_1} u_{D_2}(z; t) \right\} dt \\ &= \begin{cases} 0 & \text{if } \lambda_4, \lambda_5 \neq 0, \\ \lambda_2 z_1 \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (t_1 - z_1)^{\lambda_2-1} (1 - t_1)^{\lambda_3+1} \\ \quad \times (1 - t_1 - z_2)^{\lambda_4} dt_1 & \text{if } \lambda_4 \neq 0, \lambda_5 = 0, \\ -\lambda_2 z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (t_1 - z_1)^{\lambda_2-1} \\ \quad \times (1 - t_1 - z_2)^{\lambda_5} dt_1 & \text{if } \lambda_4 = 0, \lambda_5 \neq 0, \\ \lambda_2 z_1 \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (t_1 - z_1)^{\lambda_2-1} (1 - t_1)^{\lambda_3+1} dt_1 \\ \quad - \lambda_2 z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (t_1 - z_1)^{\lambda_2-1} dt_1 & \text{if } \lambda_4 = \lambda_5 = 0. \end{cases} \end{aligned}$$

Using the equality

$$\begin{aligned} & \int_{\text{reg}(z_2, 1-z_1)} t_2^{\lambda_3+1} (1 - t_2)^{\lambda_1+1} (t_2 - z_2)^{\lambda_4} (1 - t_2 - z_1)^{\lambda_2-1} dt_2 \\ &= \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (t_1 - z_1)^{\lambda_2-1} (1 - t_1)^{\lambda_3+1} (1 - t_1 - z_2)^{\lambda_4} dt_1, \end{aligned}$$

which follows from the change of integration variable $1 - t_2 = t_1$ ($-dt_2 = dt_1$), we find that

$$\begin{aligned} & \int_{\text{reg } D_2} \partial_{t_1} \left\{ \frac{\lambda_2 z_1 t_1 (t_1 - 1)}{t_1 - z_1} u_{D_2}(z; t) \right\} dt + \int_{\text{reg } D_2} \partial_{t_2} \left\{ \frac{\lambda_2 z_1 t_1 t_2}{t_1 - z_1} u_{D_2}(z; t) \right\} dt \\ &= \begin{cases} 0 & \text{if } \lambda_4 \neq 0, \\ -\lambda_2 z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (t_1 - z_1)^{\lambda_2-1} (1 - t_1 - z_2)^{\lambda_5} dt_1 & \text{if } \lambda_4 = 0. \end{cases} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & z_1(\theta_{z_1} - \lambda_2)\theta_{z_2} \int_{\text{reg } D_2} u_{D_2}(z; t) dt \\ &= \lambda_2 z_1 z_2^{\lambda_3+1} \int_{\text{reg}(z_1, 1-z_2)} t_1^{\lambda_1+1} (t_1 - z_1)^{\lambda_2-1} (1 - z_2 - t_1)^{\lambda_5} dt_1 \quad \text{for } \lambda_4 = 0. \end{aligned}$$

Combination of these equalities implies (1.14).

A similar argument leads to (1.14) for $j = 3$ and $j = 4$. This completes the proof of Theorem 1.1.

1.3. Proof of Theorem 1.2

In this subsection we derive a Wronskian formula for $\int_{\text{reg } D_j} u_{D_j}(t)dt$, $1 \leq j \leq 4$, to obtain the condition for a set $\{\int_{\text{reg } D_j} u_{D_j}(t)dt \mid 1 \leq j \leq 4\}$ being linearly independent.

Lemma 1.5. *Suppose that each of f_j for $1 \leq j \leq 4$ is a solution of the system of differential equations (1.2). Let $W(z)$ denote the determinant*

$$W(z) = \det \begin{pmatrix} f_1 & (f_1)_{z_1} & (f_1)_{z_2} & (f_1)_{z_1 z_2} \\ f_2 & (f_2)_{z_1} & (f_2)_{z_2} & (f_2)_{z_1 z_2} \\ f_3 & (f_3)_{z_1} & (f_3)_{z_2} & (f_3)_{z_1 z_2} \\ f_4 & (f_4)_{z_1} & (f_4)_{z_2} & (f_4)_{z_1 z_2} \end{pmatrix},$$

where f_{z_1} denotes $\partial_{z_1} f$ etc. Then we have the following.

(1) *The determinant $W(z)$ satisfies the system of differential equations*

$$(1.15) \quad \begin{cases} \partial_{z_1} W(z) = \left\{ \frac{2\lambda_{12}}{z_1} + \frac{\lambda_{235}}{z_1 - 1} + \frac{\lambda_{245}}{z_1 + z_2 - 1} \right\} W(z), \\ \partial_{z_2} W(z) = \left\{ \frac{2\lambda_{34}}{z_2} + \frac{\lambda_{145}}{z_2 - 1} + \frac{\lambda_{245}}{z_1 + z_2 - 1} \right\} W(z). \end{cases}$$

(2) *The determinant $W(z)$ is a constant multiple of*

$$z_1^{2\lambda_{12}} z_2^{2\lambda_{34}} (1 - z_1)^{\lambda_{235}} (1 - z_2)^{\lambda_{145}} (1 - z_1 - z_2)^{\lambda_{245}}.$$

Proof. (1) We write (1.2) as

$$(1.16) \quad \begin{cases} p_1 F_{z_1 z_1} + p_2 F_{z_1 z_2} + p_3 F_{z_1} + p_4 F_{z_2} + p_5 F = 0, \\ q_1 F_{z_2 z_2} + q_2 F_{z_1 z_2} + q_3 F_{z_2} + q_4 F_{z_1} + q_5 F = 0, \end{cases}$$

where

$$\begin{aligned} p_1 &= z_1(1 - z_1), & p_2 &= -z_1 z_2, & p_3 &= -\lambda_{12} + (1 + \lambda_2 + \lambda_{12345})z_1, \\ p_4 &= \lambda_2 z_2, & p_5 &= -\lambda_2(2 + \lambda_{12345}), \\ q_1 &= z_2(1 - z_2), & q_2 &= -z_1 z_2, & q_3 &= -\lambda_{34} + (1 + \lambda_4 + \lambda_{12345})z_2, \\ q_4 &= \lambda_4 z_1, & q_5 &= -\lambda_4(2 + \lambda_{12345}). \end{aligned}$$

Substituting

$$F_{z_2 z_2} = -\frac{1}{q_1} \{q_2 F_{z_1 z_2} + q_3 F_{z_2} + q_4 F_{z_1} + q_5 F\}$$

into the z_2 -derivative of the first equation of (1.16)

$$p_1 F_{z_1 z_1 z_2} + (p_2)_{z_2} F_{z_1 z_2} + p_2 F_{z_1 z_2 z_2} + p_3 F_{z_1 z_2} + (p_4)_{z_2} F_{z_2} + p_4 F_{z_2 z_2} + p_5 F_{z_2} = 0,$$

we find that

$$(1.17) \quad p_1 F_{z_1 z_1 z_2} + p_2 F_{z_1 z_2 z_2} + \left\{ (p_2)_{z_2} + p_3 - \frac{q_2}{q_1} p_4 \right\} F_{z_1 z_2} + \cdots = 0.$$

Similarly, substituting

$$(1.18) \quad F_{z_1 z_1} = -\frac{1}{p_1} \{p_2 F_{z_1 z_2} + p_3 F_{z_1} + p_4 F_{z_2} + p_5 F\}$$

into the z_1 -derivative of the second equation of (1.16)

$$q_1 F_{z_2 z_2 z_1} + (q_2)_{z_1} F_{z_1 z_2} + q_2 F_{z_1 z_2 z_1} + q_3 F_{z_2 z_1} + (q_4)_{z_1} F_{z_1} + q_4 F_{z_1 z_1} + q_5 F_{z_1} = 0,$$

we find that

$$(1.19) \quad q_1 F_{z_2 z_2 z_1} + q_2 F_{z_1 z_2 z_1} + \left\{ (q_2)_{z_1} + q_3 - \frac{p_2}{p_1} q_4 \right\} F_{z_1 z_2} + \cdots = 0.$$

Canceling out the terms of $F_{z_1 z_2 z_2}$ from (1.17) and (1.19), we have

$$(p_1 q_1 - p_2 q_2) F_{z_1 z_1 z_2} + \left\{ q_1 \left((p_2)_{z_2} + p_3 - \frac{q_2}{q_1} p_4 \right) - p_2 \left((q_2)_{z_1} + q_3 - \frac{p_2}{p_1} q_4 \right) \right\} F_{z_1 z_2} + \cdots = 0,$$

so that

$$(1.20) \quad F_{z_1 z_1 z_2} = -\frac{1}{p_1 q_1 - p_2 q_2} \left\{ q_1 \left((p_2)_{z_2} + p_3 - \frac{q_2}{q_1} p_4 \right) - p_2 \left((q_2)_{z_1} + q_3 - \frac{p_2}{p_1} q_4 \right) \right\} F_{z_1 z_2} + \cdots.$$

Therefore, using (1.18) and (1.20), we have

$$\begin{aligned}
& \partial_{z_1} W(z) \\
&= \det \begin{pmatrix} f_1 & (f_1)_{z_1 z_1} & (f_1)_{z_2} & (f_1)_{z_1 z_2} \\ f_2 & (f_2)_{z_1 z_1} & (f_2)_{z_2} & (f_2)_{z_1 z_2} \\ f_3 & (f_3)_{z_1 z_1} & (f_3)_{z_2} & (f_3)_{z_1 z_2} \\ f_4 & (f_4)_{z_1 z_1} & (f_4)_{z_2} & (f_4)_{z_1 z_2} \end{pmatrix} + \det \begin{pmatrix} f_1 & (f_1)_{z_1} & (f_1)_{z_2} & (f_1)_{z_1 z_2 z_1} \\ f_2 & (f_2)_{z_1} & (f_2)_{z_2} & (f_2)_{z_1 z_2 z_1} \\ f_3 & (f_3)_{z_1} & (f_3)_{z_2} & (f_3)_{z_1 z_2 z_1} \\ f_4 & (f_4)_{z_1} & (f_4)_{z_2} & (f_4)_{z_1 z_2 z_1} \end{pmatrix} \\
&= \left[-\frac{p_3}{p_1} - \frac{1}{p_1 q_1 - p_2 q_2} \left\{ q_1 \left((p_2)_{z_2} + p_3 - \frac{q_2}{q_1} p_4 \right) \right. \right. \\
&\quad \left. \left. - p_2 \left((q_2)_{z_1} + q_3 - \frac{p_2}{p_1} q_4 \right) \right\} \right] W(z) \\
&= \left[\frac{2\lambda_{12}}{z_1} + \frac{\lambda_{235}}{z_1 - 1} + \frac{\lambda_{245}}{z_1 + z_2 - 1} \right] W(z).
\end{aligned}$$

This is the first equation of (1.15). The second equation of (1.15) follows from the change of variables and parameters such that $(z_1, z_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (z_2, z_1, \lambda_3, \lambda_4, \lambda_1, \lambda_2)$.

(2) Solving differential equations (1.15), we reach the second assertion.

This completes the proof of Lemma 1.5. $\square$

Next we determine the multiplicative constant stated in (2) of Lemma 1.5. For this purpose, we prepare some formulas concerning the asymptotics of $I_j(z)$ around $(0, 0)$.

Lemma 1.6. *Let $I_j(z) = \int_{\text{reg } D_j} u_{D_j}(z; t) dt$ for $1 \leq j \leq 4$. Then we have*

(1) *If $\text{Re}(\lambda_i + 1) > 0$ ($i = 1, \dots, 4$),*

$$I_1(z) \sim B(\lambda_1 + 1, \lambda_2 + 1) B(\lambda_3 + 1, \lambda_4 + 1) z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1}$$

as $(z_1, z_2) \rightarrow (0, 0)$.

(2) *If $\text{Re}(\lambda_{12} + 1) > 0, \text{Re}(\lambda_{34} + 1) > 0, \text{Re}(\lambda_i + 1) > 0$ ($i = 2, 4, 5$),*

$$I_2(z) \sim \frac{\Gamma(\lambda_{12} + 1) \Gamma(\lambda_{34} + 1) \Gamma(\lambda_5 + 1)}{\Gamma(\lambda_{12345} + 3)}$$

as $(z_1, z_2) \rightarrow (0, 0)$.

(3) *If $\text{Re}(\lambda_{12} + 1) > 0, \text{Re}(\lambda_i + 1) > 0$ ($i = 2, 3, 4, 5$),*

$$I_3(z) \sim B(\lambda_{12} + 1, \lambda_5 + 1) B(\lambda_3 + 1, \lambda_4 + 1) z_2^{\lambda_{34}+1}$$

as $(z_1, z_2) \rightarrow (0, 0)$.

$$(4) \quad \text{If } \operatorname{Re}(\lambda_{34} + 1) > 0, \operatorname{Re}(\lambda_i + 1) > 0 \quad (i = 1, 2, 4, 5),$$

$$I_4(z) \sim B(\lambda_1 + 1, \lambda_2 + 1)B(\lambda_{34} + 1, \lambda_5 + 1)z_1^{\lambda_{12}+1}$$

as $(z_1, z_2) \rightarrow (0, 0)$.

Here $f(z_1, z_2) \sim g(z_1, z_2)$ means $f(z_1, z_2)/g(z_1, z_2) \rightarrow 1$ as $(z_1, z_2) \rightarrow (0, 0)$.

Proof. (1) The change of integration variables $t_1 \rightarrow z_1 t_1$ and $t_2 \rightarrow z_2 t_2$ implies

$$\begin{aligned} I_1(z) &= \int_{\operatorname{reg}(0, z_2)} \int_{\operatorname{reg}(0, z_1)} t_1^{\lambda_1} t_2^{\lambda_3} (z_1 - t_1)^{\lambda_2} (z_2 - t_2)^{\lambda_4} (1 - t_1 - t_2)^{\lambda_5} dt_1 dt_2 \\ &= z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1} \int_{\operatorname{reg}(0, 1)} \int_{\operatorname{reg}(0, 1)} t_1^{\lambda_1} t_2^{\lambda_3} (1 - t_1)^{\lambda_2} (1 - t_2)^{\lambda_4} (1 - z_1 t_1 - z_2 t_2)^{\lambda_5} dt_1 dt_2 \\ &\sim z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1} \int_{\operatorname{reg}(0, 1)} \int_{\operatorname{reg}(0, 1)} t_1^{\lambda_1} t_2^{\lambda_3} (1 - t_1)^{\lambda_2} (1 - t_2)^{\lambda_4} dt_1 dt_2 \\ &\quad \text{as } (z_1, z_2) \rightarrow (0, 0). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_{\operatorname{reg}(0, 1)} \int_{\operatorname{reg}(0, 1)} t_1^{\lambda_1} t_2^{\lambda_3} (1 - t_1)^{\lambda_2} (1 - t_2)^{\lambda_4} dt_1 dt_2 \\ &= \int_{\operatorname{reg}(0, 1)} t_1^{\lambda_1} (1 - t_1)^{\lambda_2} dt_1 \int_{\operatorname{reg}(0, 1)} t_2^{\lambda_3} (1 - t_2)^{\lambda_4} dt_2 \\ &= B(\lambda_1 + 1, \lambda_2 + 1)B(\lambda_3 + 1, \lambda_4 + 1). \end{aligned}$$

These lead to the result.

(2) We have

$$\begin{aligned} I_2(z) &= \int_{\operatorname{reg}(z_1, 1-z_2)} \left(\int_{\operatorname{reg}(z_2, 1-t_1)} t_1^{\lambda_1} t_2^{\lambda_3} (t_1 - z_1)^{\lambda_2} (t_2 - z_2)^{\lambda_4} (1 - t_1 - t_2)^{\lambda_5} dt_2 \right) dt_1 \\ &\sim \int_{\operatorname{reg}(0, 1)} \left(\int_{\operatorname{reg}(0, 1-t_1)} t_1^{\lambda_{12}} t_2^{\lambda_{34}} (1 - t_1 - t_2)^{\lambda_5} dt_2 \right) dt_1 \quad \text{as } (z_1, z_2) \rightarrow (0, 0). \end{aligned}$$

On the other hand, Dirichlet's formula implies

$$\int_{\operatorname{reg}(0, 1)} \left(\int_{\operatorname{reg}(0, 1-t_1)} t_1^{\lambda_{12}} t_2^{\lambda_{34}} (1 - t_1 - t_2)^{\lambda_5} dt_2 \right) dt_1 = \frac{\Gamma(\lambda_{12} + 1)\Gamma(\lambda_{34} + 1)\Gamma(\lambda_5 + 1)}{\Gamma(\lambda_{12345} + 3)}.$$

These lead to the result.

(3) The change of integration variable $t_2 \rightarrow z_2 t_2$ implies

$$\begin{aligned}
 I_3(z) &= \int_{\text{reg}(0, z_2)} \int_{\text{reg}(z_1, 1-t_2)} t_1^{\lambda_1} t_2^{\lambda_3} (t_1 - z_1)^{\lambda_2} (z_2 - t_2)^{\lambda_4} (1 - t_1 - t_2)^{\lambda_5} dt_1 dt_2 \\
 &= z_2^{\lambda_{34}+1} \int_{\text{reg}(0, 1)} \int_{\text{reg}(z_1, 1-z_2 t_2)} t_1^{\lambda_1} t_2^{\lambda_3} (t_1 - z_1)^{\lambda_2} (1 - t_2)^{\lambda_4} (1 - t_1 - z_2 t_2)^{\lambda_5} dt_1 dt_2 \\
 &\sim z_2^{\lambda_{34}+1} \int_{\text{reg}(0, 1)} \int_{\text{reg}(0, 1)} t_1^{\lambda_{12}} t_2^{\lambda_3} (1 - t_1)^{\lambda_5} (1 - t_2)^{\lambda_4} dt_1 dt_2 \\
 &= z_2^{\lambda_{34}+1} B(\lambda_{12} + 1, \lambda_5 + 1) B(\lambda_3 + 1, \lambda_4 + 1) \quad \text{as } (z_1, z_2) \rightarrow (0, 0).
 \end{aligned}$$

(4) The change of variables and exponents in (3) such as

$$(z_1, z_2, t_1, t_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \rightarrow (z_2, z_1, t_2, t_1, \lambda_3, \lambda_4, \lambda_1, \lambda_2, \lambda_5)$$

implies the result. $\square$

At this stage, we reach the following.

Lemma 1.7. *Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \notin \mathbf{Z}_{<0}$ and $\lambda_{125} + 2, \lambda_{345} + 2, \lambda_{12345} + 3 \notin \mathbf{Z}_{\leq 0}$. Then we have*

$$\begin{aligned}
 &\det \begin{pmatrix} I_1(z) & (I_1(z))_{z_1} & (I_1(z))_{z_2} & (I_1(z))_{z_1 z_2} \\ I_2(z) & (I_2(z))_{z_1} & (I_2(z))_{z_2} & (I_2(z))_{z_1 z_2} \\ I_3(z) & (I_3(z))_{z_1} & (I_3(z))_{z_2} & (I_3(z))_{z_1 z_2} \\ I_4(z) & (I_4(z))_{z_1} & (I_4(z))_{z_2} & (I_4(z))_{z_1 z_2} \end{pmatrix} \\
 &= z_1^{2\lambda_{12}} z_2^{2\lambda_{34}} (1 - z_1)^{\lambda_{235}+1} (1 - z_2)^{\lambda_{145}+1} (1 - z_1 - z_2)^{\lambda_{245}} \\
 &\quad \times \frac{\Gamma(\lambda_1 + 1)^2 \Gamma(\lambda_2 + 1)^2 \Gamma(\lambda_3 + 1)^2 \Gamma(\lambda_4 + 1)^2 \Gamma(\lambda_5 + 1)^3}{\Gamma(\lambda_{125} + 2) \Gamma(\lambda_{345} + 2) \Gamma(\lambda_{12345} + 3)},
 \end{aligned}$$

where each argument of $z_1, z_2, 1 - z_1, 1 - z_2$ and $1 - z_1 - z_2$ of the right side is assigned to be zero when z_1 and z_2 are small positive numbers.

Proof. We may consider only the case when $\text{Re}(\lambda_i) > 0$ ($i = 1, \dots, 5$), $\text{Re}(\lambda_{12}) > 0$, $\text{Re}(\lambda_{34}) > 0$. Then (1) of Lemma 1.6 shows

$$\begin{aligned}
 I_1(z) &\sim B(\lambda_1 + 1, \lambda_2 + 1) B(\lambda_3 + 1, \lambda_4 + 1) z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1}, \\
 (I_1(z))_{z_1} &\sim B(\lambda_1 + 1, \lambda_2 + 1) B(\lambda_3 + 1, \lambda_4 + 1) (\lambda_{12} + 1) z_1^{\lambda_{12}} z_2^{\lambda_{34}+1}, \\
 (I_1(z))_{z_2} &\sim B(\lambda_1 + 1, \lambda_2 + 1) B(\lambda_3 + 1, \lambda_4 + 1) (\lambda_{34} + 1) z_1^{\lambda_{12}+1} z_2^{\lambda_{34}}, \\
 (I_1(z))_{z_1 z_2} &\sim B(\lambda_1 + 1, \lambda_2 + 1) B(\lambda_3 + 1, \lambda_4 + 1) (\lambda_{12} + 1) (\lambda_{34} + 1) z_1^{\lambda_{12}} z_2^{\lambda_{34}}
 \end{aligned}$$

as $(z_1, z_2) \rightarrow (0, 0)$, and hence

$$\begin{aligned} & \lim_{(z_1, z_2) \rightarrow (0, 0)} z_1^{-\lambda_{12}} z_2^{-\lambda_{34}} (I_1(z), (I_1(z))_{z_1}, (I_1(z))_{z_2}, (I_1(z))_{z_1 z_2}) \\ &= (0, 0, 0, B(\lambda_1 + 1, \lambda_2 + 1) B(\lambda_3 + 1, \lambda_4 + 1) (\lambda_{12} + 1) (\lambda_{34} + 1)) \\ &= \frac{\Gamma(\lambda_1 + 1) \Gamma(\lambda_2 + 1) \Gamma(\lambda_3 + 1) \Gamma(\lambda_4 + 1)}{\Gamma(\lambda_{12} + 1) \Gamma(\lambda_{34} + 1)} (0, 0, 0, 1). \end{aligned}$$

Similarly, (2), (3) and (4) of Lemma 1.6 imply

$$\begin{aligned} & \lim_{(z_1, z_2) \rightarrow (0, 0)} (I_2(z), (I_2(z))_{z_1}, (I_2(z))_{z_2}, (I_2(z))_{z_1 z_2}) \\ &= \frac{\Gamma(\lambda_{12} + 1) \Gamma(\lambda_{34} + 1) \Gamma(\lambda_5 + 1)}{\Gamma(\lambda_{12345} + 3)} (1, *, *, *), \\ & \lim_{(z_1, z_2) \rightarrow (0, 0)} z_2^{-\lambda_{34}} (I_3(z), (I_3(z))_{z_1}, (I_3(z))_{z_2}, (I_3(z))_{z_1 z_2}) \\ &= \frac{\Gamma(\lambda_{12} + 1) \Gamma(\lambda_5 + 1) \Gamma(\lambda_3 + 1) \Gamma(\lambda_4 + 1)}{\Gamma(\lambda_{125} + 2) \Gamma(\lambda_{34} + 1)} (0, 0, 1, *), \\ & \lim_{(z_1, z_2) \rightarrow (0, 0)} z_1^{-\lambda_{12}} (I_4(z), (I_4(z))_{z_1}, (I_4(z))_{z_2}, (I_4(z))_{z_1 z_2}) \\ &= \frac{\Gamma(\lambda_1 + 1) \Gamma(\lambda_2 + 1) \Gamma(\lambda_{34} + 1) \Gamma(\lambda_5 + 1)}{\Gamma(\lambda_{12} + 1) \Gamma(\lambda_{345} + 2)} (0, 1, 0, *), \end{aligned}$$

where each $*$ stands for a constant. As a result, we have

$$\begin{aligned} & \lim_{(z_1, z_2) \rightarrow (0, 0)} z_1^{-2\lambda_{12}} z_2^{-2\lambda_{34}} \det \begin{pmatrix} I_1(z) & (I_1(z))_{z_1} & (I_1(z))_{z_2} & (I_1(z))_{z_1 z_2} \\ I_2(z) & (I_2(z))_{z_1} & (I_2(z))_{z_2} & (I_2(z))_{z_1 z_2} \\ I_3(z) & (I_3(z))_{z_1} & (I_3(z))_{z_2} & (I_3(z))_{z_1 z_2} \\ I_4(z) & (I_4(z))_{z_1} & (I_4(z))_{z_2} & (I_4(z))_{z_1 z_2} \end{pmatrix} \\ &= \frac{\Gamma(\lambda_1 + 1)^2 \Gamma(\lambda_2 + 1)^2 \Gamma(\lambda_3 + 1)^2 \Gamma(\lambda_4 + 1)^2 \Gamma(\lambda_5 + 1)^3}{\Gamma(\lambda_{125} + 2) \Gamma(\lambda_{345} + 2) \Gamma(\lambda_{12345} + 3)}. \end{aligned}$$

This completes the proof. $\square$

Lemma 1.7 shows the linear independence of $\{\int_{\text{reg } D_j} u_{D_j}(t) dt \mid 1 \leq j \leq 4\}$. This completes the proof of Theorem 1.2.

2. Fundamental sets of solutions for E_2 and E_3

2.1. E_2 case

Let (x, y) be a point of

$$X^{(2)} = \mathbf{C}^2 \setminus \{x=0\} \cup \{x=1\} \cup \{y=0\} \cup \{y=1\} \cup \{x+y=1\} \subset (\mathbf{P}^1)^2,$$

and $u^{(2)}(x, y; t)$ a function

$$u^{(2)}(x, y; t) = t_1^{\beta-1} (t_1 - 1)^{\gamma-\beta-1} t_2^{\beta'-1} (t_2 - 1)^{\gamma'-\beta'-1} (1 - xt_1 - yt_2)^{-\alpha}$$

on $T_{(x,y)}^{(2)} = \mathbf{C}^2 \setminus \bigcup_{i, \lambda_i \notin \mathbf{Z}_{\geq 0}} \{f_i^{(2)}(t) = 0\}$, where $f_1^{(2)}(t) = t_1, f_2^{(2)}(t) = t_1 - 1, \dots, f_5^{(2)}(t) = 1 - xt_1 - yt_2$. When a pair of complex variables (x, y) are real and sufficiently small positive numbers, assign the name $D_j^{(2)}$ for $1 \leq j \leq 4$ to each domain of the real manifold $T_R^{(2)}$ (the real locus of $T^{(2)} = T_{(x,y)}^{(2)}$) as in Figure 3:

$$D_1^{(2)} = \{0 < t_1 < 1, 0 < t_2 < 1\},$$

$$D_2^{(2)} = \{1 < t_1, 1 < t_2, xt_1 + yt_2 < 1\},$$

$$D_3^{(2)} = \{0 < t_2 < 1, 1 < t_1, xt_1 + yt_2 < 1\},$$

$$D_4^{(2)} = \{0 < t_1 < 1, 1 < t_2, xt_1 + yt_2 < 1\},$$

where the orientation of $D_j^{(2)}$ is fixed to be natural one induced from $T_R^{(2)}$.

Fig. 3

For a domain $D_j^{(2)}$, we let $u_{D_j^{(2)}}^{(2)}(x, y; t)$ denote the function

$$u_{D_j^{(2)}}^{(2)}(x, y; t) = \prod_i (\varepsilon_i f_i^{(2)}(t))^{\lambda_i},$$

where

$$\lambda_1 = \beta - 1, \quad \lambda_2 = \gamma - \beta - 1, \quad \lambda_3 = \beta' - 1, \quad \lambda_4 = \gamma' - \beta' - 1, \quad \lambda_5 = -\alpha,$$

and $\varepsilon_i = \pm$ is so determined that each $\varepsilon_i f_i^{(2)}(t)$ is positive and the argument of $\varepsilon_i f_i^{(2)}(t)$ is assigned to be zero on $D_j^{(2)}$, as is the same way to define $u_{D_j}(z; t)$ in the previous section.

Here and in what follows, the superscript (2) is used to indicate the case of Appell's F_2 (similarly, the superscript (3) is used to indicate the case of Appell's F_3).

We have the following.

Theorem 2.1. *If*

$$\beta - 1, \gamma - \beta - 1, \beta' - 1, \gamma' - \beta' - 1, -\alpha \notin \mathbf{Z}_{<0},$$

then each of the integrals $I_j^{(2)}(x, y) = \int_{\text{reg } D_j^{(2)}} u_{D_j^{(2)}}^{(2)}(x, y; t) dt$ for $1 \leq j \leq 4$ satisfies the system of differential equations

$$(E_2) \quad \begin{cases} \left[x(1-x) \frac{\partial^2}{\partial x^2} - xy \frac{\partial^2}{\partial x \partial y} + \{\gamma - (\alpha + \beta + 1)x\} \frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} - \alpha \beta \right] F = 0, \\ \left[y(1-y) \frac{\partial^2}{\partial y^2} - xy \frac{\partial^2}{\partial x \partial y} + \{\gamma' - (\alpha + \beta' + 1)y\} \frac{\partial}{\partial y} - \beta' x \frac{\partial}{\partial x} - \alpha \beta' \right] F = 0. \end{cases}$$

Proof. Theorem 1.1 implies

$$[\theta_{z_1}(\theta_{z_1} - \lambda_{12} - 1) - z_1(\theta_{z_1} - \lambda_2)(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})] \int_{\text{reg } D_j} u_{D_j}(z; t) dt = 0,$$

$$[\theta_{z_2}(\theta_{z_2} - \lambda_{34} - 1) - z_2(\theta_{z_2} - \lambda_4)(\theta_{z_2} + \theta_{z_4} - 2 - \lambda_{12345})] \int_{\text{reg } D_j} u_{D_j}(z; t) dt = 0,$$

for $\lambda_1, \dots, \lambda_5 \notin \mathbf{Z}_{<0}$. The change of integration variables $t_i \rightarrow z_i t_i$ implies

$$\begin{aligned} & \int_{\text{reg } D_j} u_{D_j}(z; t) dt \\ &= \int_{\text{reg } D_j} (\varepsilon_1 t_1)^{\lambda_1} (\varepsilon_2(t_1 - z_1))^{\lambda_2} (\varepsilon_3 t_2)^{\lambda_3} (\varepsilon_4(t_2 - z_2))^{\lambda_4} (\varepsilon_5(t_1 + t_2 - 1))^{\lambda_5} dt \\ &= z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1} \int_{\text{reg } D_j^{(2)'}} (\varepsilon_1 t_1)^{\lambda_1} (\varepsilon_2(t_1 - 1))^{\lambda_2} (\varepsilon_3 t_2)^{\lambda_3} (\varepsilon_4(t_2 - 1))^{\lambda_4} \\ & \quad \times (\varepsilon_5(z_1 t_1 + z_2 t_2 - 1))^{\lambda_5} dt, \end{aligned}$$

where $D_j^{(2)'}$ denotes the domain $D_j^{(2)}$ after the change of variables (x, y) to (z_1, z_2) .

On the other hand, as operators, the relations

$$\theta_{z_1} z_1^{\lambda_{12}+1} = z_1^{\lambda_{12}+1} (\theta_{z_1} + \lambda_{12} + 1), \quad \text{and} \quad \theta_{z_2} z_2^{\lambda_{34}+1} = z_2^{\lambda_{34}+1} (\theta_{z_2} + \lambda_{34} + 1)$$

imply

$$\begin{aligned} & [\theta_{z_1} (\theta_{z_1} - \lambda_{12} - 1) - z_1 (\theta_{z_1} - \lambda_2) (\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})] z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1} \\ &= z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1} [(\theta_{z_1} + \lambda_{12} + 1) \theta_{z_1} - z_1 (\theta_{z_1} + \lambda_1 + 1) (\theta_{z_1} + \theta_{z_2} - \lambda_5)] \end{aligned}$$

and

$$\begin{aligned} & [\theta_{z_2} (\theta_{z_2} - \lambda_{34} - 1) - z_1 (\theta_{z_2} - \lambda_4) (\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})] z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1} \\ &= z_1^{\lambda_{12}+1} z_2^{\lambda_{34}+1} [(\theta_{z_1} + \lambda_{12} + 1) \theta_{z_2} - z_2 (\theta_{z_2} + \lambda_3 + 1) (\theta_{z_1} + \theta_{z_2} - \lambda_5)]. \end{aligned}$$

Therefore, the integral

$$\int_{\text{reg } D_j^{(2)'}} (\varepsilon_1 t_1)^{\lambda_1} (\varepsilon_2 (t_1 - 1))^{\lambda_2} (\varepsilon_3 t_2)^{\lambda_3} (\varepsilon_4 (t_2 - 1))^{\lambda_4} (\varepsilon_5 (z_1 t_1 + z_2 t_2 - 1))^{\lambda_5} dt$$

satisfies the system of differential equations

$$\begin{cases} [(\theta_{z_1} + \lambda_{12} + 1) \theta_{z_1} - z_1 (\theta_{z_1} + \lambda_1 + 1) (\theta_{z_1} + \theta_{z_2} - \lambda_5)] F = 0, \\ [(\theta_{z_2} + \lambda_{34} + 1) \theta_{z_2} - z_2 (\theta_{z_2} + \lambda_3 + 1) (\theta_{z_1} + \theta_{z_2} - \lambda_5)] F = 0. \end{cases}$$

Under the condition

$$\lambda_1 = \beta - 1, \quad \lambda_2 = \gamma - \beta - 1, \quad \lambda_3 = \beta' - 1, \quad \lambda_4 = \gamma' - \beta' - 1, \quad \lambda_5 = -\alpha,$$

with $(z_1, z_2) = (x, y)$, it follows that $\int_{\text{reg } D_j^{(2)}} u_{D_j^{(2)}}^{(2)}(x, y; t) dt$ satisfies the system of differential equations

$$\begin{cases} [(\theta_x + \gamma - 1) \theta_x - x (\theta_x + \beta) (\theta_x + \theta_y + \alpha)] F = 0, \\ [(\theta_y + \gamma' - 1) \theta_y - y (\theta_y + \beta') (\theta_x + \theta_y + \alpha)] F = 0, \end{cases}$$

which is equivalent to E_2 . This completes the proof. $\square$

Theorem 2.2. *If*

$$\beta - 1, \gamma - \beta - 1, \beta' - 1, \gamma' - \beta' - 1, -\alpha \notin \mathbf{Z}_{<0}$$

and

$$\gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha - 1 \notin \mathbf{Z}_{\leq 0},$$

then the integrals $I_j^{(2)}(x, y) = \int_{\text{reg } D_j^{(2)}} u_{D_j^{(2)}}^{(2)}(x, y; t) dt$ for $1 \leq j \leq 4$ are linearly independent.

Proof. Theorem 1.2 with

$$\lambda_1 = \beta - 1, \quad \lambda_2 = \gamma - \beta - 1, \quad \lambda_3 = \beta' - 1, \quad \lambda_4 = \gamma' - \beta' - 1, \quad \lambda_5 = -\alpha,$$

and

$$\lambda_{125} = \gamma - \alpha - 2, \quad \lambda_{345} = \gamma' - \alpha - 2, \quad \lambda_{12345} = \gamma + \gamma' - \alpha - 4$$

implies the result. $\square$

Therefore, combination of Theorems 2.1 and 2.2 implies the following.

Corollary 2.3. *If*

$$\beta - 1, \gamma - \beta - 1, \beta' - 1, \gamma' - \beta' - 1, -\alpha \notin \mathbf{Z}_{<0}$$

and

$$\gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha - 1 \notin \mathbf{Z}_{\leq 0},$$

then the integrals $I_j^{(2)}(x, y) = \int_{\text{reg } D_j^{(2)}} u_{D_j^{(2)}}^{(2)}(x, y; t) dt$ for $1 \leq j \leq 4$ constitute a fundamental set of solutions of E_2 .

2.2. E_3 case

Let (x, y) be a point of

$$X^{(3)} = \mathbf{C}^2 \setminus \{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\} \cup \{xy - x - y = 0\} \subset (\mathbf{P}^1)^2,$$

and $u^{(3)}(x, y; t)$ a function

$$u^{(3)}(x, y; t) = t_1^{\beta-1} (1 - xt_1)^{-\alpha} t_2^{\beta'-1} (1 - yt_2)^{-\alpha'} (1 - t_1 - t_2)^{\gamma-\beta-\beta'-1}$$

on $T_{(x,y)}^{(3)} = \mathbf{C}^2 \setminus \bigcup_{i, \lambda_i \notin \mathbf{Z}_{\geq 0}} \{f_i^{(3)}(t) = 0\}$, where $f_1^{(3)}(t) = t_1, f_2^{(3)}(t) = 1 - xt_1, \dots, f_5^{(3)}(t) = 1 - t_1 - t_2$. When a pair of complex variables (x, y) are real and sufficiently large positive numbers, assign the name $D_j^{(3)}$ for $1 \leq j \leq 4$ to each domain of the real manifold $T_{\mathbf{R}}^{(3)}$ (the real locus of $T^{(3)} = T_{(x,y)}^{(3)}$) as in Figure 4:

$$D_1^{(3)} = \{0 < t_1 < 1/x, 0 < t_2 < 1/y\},$$

$$D_2^{(3)} = \{1/x < t_1, 1/y < t_2, t_1 + t_2 < 1\},$$

$$D_3^{(3)} = \{0 < t_2 < 1/y, 1/x < t_1, t_1 + t_2 < 1\},$$

$$D_4^{(3)} = \{0 < t_1 < 1/x, 1/y < t_2, t_1 + t_2 < 1\},$$

where the orientation of $D_j^{(3)}$ is fixed to be natural one induced from $T_{\mathbf{R}}^{(3)}$.

Fig. 4

For a domain $D_j^{(3)}$, we let $u_{D_j^{(3)}}^{(3)}(x, y; t)$ denote the function

$$u_{D_j^{(3)}}^{(3)}(x, y; t) = \prod_i (\varepsilon_i f_i^{(3)}(t))^{\lambda_i},$$

where

$$\lambda_1 = \beta - 1, \quad \lambda_2 = -\alpha, \quad \lambda_3 = \beta' - 1, \quad \lambda_4 = -\alpha', \quad \lambda_5 = \gamma - \beta - \beta' - 1,$$

and $\varepsilon_i = \pm$ is so determined that each $\varepsilon_i f_i(t)$ is positive and the argument of $\varepsilon_i f_i(t)$ is assigned to be zero on $D_j^{(3)}$.

Then we have the following.

Theorem 2.4. *If*

$$\beta - 1, -\alpha, \beta' - 1, -\alpha', \gamma - \beta - \beta' - 1 \notin \mathbf{Z}_{<0},$$

then each of the integrals $\int_{\text{reg } D_j^{(3)}} u_{D_j^{(3)}}^{(3)}(x, y; t) dt$ for $1 \leq j \leq 4$ satisfies the system of differential equations

$$(E_3) \quad \begin{cases} \left[x(1-x) \frac{\partial^2}{\partial x^2} + y \frac{\partial^2}{\partial x \partial y} + \{\gamma - (\alpha + \beta + 1)x\} \frac{\partial}{\partial x} - \alpha\beta \right] F = 0, \\ \left[y(1-y) \frac{\partial^2}{\partial y^2} + x \frac{\partial^2}{\partial x \partial y} + \{\gamma - (\alpha' + \beta' + 1)y\} \frac{\partial}{\partial y} - \alpha'\beta' \right] F = 0. \end{cases}$$

Proof. Theorem 1.1 implies

$$[\theta_{z_1}(\theta_{z_1} - \lambda_{12} - 1) - z_1(\theta_{z_1} - \lambda_2)(\theta_{z_1} + \theta_{z_2} - 2 - \lambda_{12345})] \int_{\text{reg } D_j} u_{D_j}(t) dt = 0,$$

$$[\theta_{z_2}(\theta_{z_2} - \lambda_{34} - 1) - z_2(\theta_{z_2} - \lambda_4)(\theta_{z_2} + \theta_{z_4} - 2 - \lambda_{12345})] \int_{\text{reg } D_j} u_{D_j}(t) dt = 0,$$

for $\lambda_1, \dots, \lambda_5 \notin \mathbf{Z}_{<0}$. The change of variables $z_1 = 1/x$, $z_2 = 1/y$ implies $\theta_{z_1} = -\theta_x$, $\theta_{z_2} = -\theta_y$ and

$$\begin{aligned} & \int_{\text{reg } D_j} u_{D_j}(t) dt \\ &= \int_{\text{reg } D_j} (\varepsilon_1 t_1)^{\lambda_1} \left(\varepsilon_2 \left(t_1 - \frac{1}{x} \right) \right)^{\lambda_2} (\varepsilon_3 t_2)^{\lambda_3} \left(\varepsilon_4 \left(t_2 - \frac{1}{y} \right) \right)^{\lambda_4} (\varepsilon_5 (t_1 + t_2 - 1))^{\lambda_5} dt \\ &= x^{-\lambda_2} y^{-\lambda_4} \int_{\text{reg } D_j^{(3)}} (\varepsilon_1 t_1)^{\lambda_1} (\varepsilon_2 (x t_1 - 1))^{\lambda_2} (\varepsilon_3 t_2)^{\lambda_3} (\varepsilon_4 (y t_2 - 1))^{\lambda_4} \\ & \quad \times (\varepsilon_5 (t_1 + t_2 - 1))^{\lambda_5} dt \\ &= x^{-\lambda_2} y^{-\lambda_4} \int_{\text{reg } D_j^{(3)}} u_{D_j^{(3)}}^{(3)}(x, y; t) dt. \end{aligned}$$

On the other hand, as operators, the relations

$$\theta_x x^{-\lambda_2} = x^{-\lambda_2} (\theta_x - \lambda_2) \quad \text{and} \quad \theta_y y^{-\lambda_4} = y^{-\lambda_4} (\theta_y - \lambda_4)$$

imply

$$\begin{aligned} & [x\theta_x(\theta_x + \lambda_{12} + 1) - (\theta_x + \lambda_2)(\theta_x + \theta_y + 2 + \lambda_{12345})] x^{-\lambda_2} y^{-\lambda_4} \\ &= x^{-\lambda_2} y^{-\lambda_4} [x(\theta_x - \lambda_2)(\theta_x + \lambda_1 + 1) - \theta_x(\theta_x + \theta_y + 2 + \lambda_{135})] \end{aligned}$$

and

$$\begin{aligned} & [x\theta_y(\theta_y + \lambda_{34} + 1) - (\theta_y + \lambda_4)(\theta_x + \theta_y + 2 + \lambda_{12345})] x^{-\lambda_2} y^{-\lambda_4} \\ &= x^{-\lambda_2} y^{-\lambda_4} [y(\theta_y - \lambda_4)(\theta_y + \lambda_3 + 1) - \theta_y(\theta_x + \theta_y + 2 + \lambda_{135})]. \end{aligned}$$

Therefore, the integral

$$\int_{\text{reg } D_j^{(3)}} u_{D_j^{(3)}}^{(3)}(x, y; t) dt$$

satisfies the system of differential equations

$$\begin{cases} [x(\theta_x - \lambda_2)(\theta_x + \lambda_1 + 1) - \theta_x(\theta_x + \theta_y + 2 + \lambda_{135})] F = 0, \\ [y(\theta_y - \lambda_4)(\theta_y + \lambda_3 + 1) - \theta_y(\theta_x + \theta_y + 2 + \lambda_{135})] F = 0. \end{cases}$$

Under the condition

$$\lambda_1 = \beta - 1, \quad \lambda_2 = -\alpha, \quad \lambda_3 = \beta' - 1, \quad \lambda_4 = -\alpha', \quad \lambda_5 = \gamma - \beta - \beta' - 1,$$

it follows that $\int_{\text{reg } D_j^{(3)}} u_{D_j^{(3)}}^{(3)}(x, y; t) dt$ satisfies the system of differential equations

$$\begin{cases} [x(\theta_x + \alpha)(\theta_x + \beta) - \theta_x(\theta_x + \theta_y + \gamma - 1)]F = 0, \\ [y(\theta_y + \alpha')(\theta_y + \beta') - \theta_y(\theta_x + \theta_y + \gamma - 1)]F = 0, \end{cases}$$

which is equivalent to E_3 . This completes the proof. $\square$

Theorem 2.5. *If*

$$\beta - 1, -\alpha, \beta' - 1, -\alpha', \gamma - \beta - \beta' - 1 \notin \mathbf{Z}_{<0}$$

and

$$\gamma - \alpha - \beta', \gamma - \alpha' - \beta, \gamma - \alpha - \alpha' - 1 \notin \mathbf{Z}_{\leq 0},$$

then the integrals $\int_{\text{reg } D_j^{(3)}} u_{D_j^{(3)}}^{(3)}(x, y; t) dt$ for $1 \leq j \leq 4$ are linearly independent.

Proof. Theorem 1.2 with

$$\lambda_1 = \beta - 1, \quad \lambda_2 = -\alpha, \quad \lambda_3 = \beta' - 1, \quad \lambda_4 = -\alpha', \quad \lambda_5 = \gamma - \beta - \beta' - 1,$$

and

$$\lambda_{125} = \gamma - \alpha - \beta' - 2, \quad \lambda_{345} = \gamma - \alpha' - \beta - 2, \quad \lambda_{12345} = \gamma - \alpha - \alpha' - 4,$$

implies the result. $\square$

Therefore, combination of Theorem 2.4 with Theorem 2.5 gives the following.

Corollary 2.6. *If*

$$\beta - 1, -\alpha, \beta' - 1, -\alpha', \gamma - \beta - \beta' - 1 \notin \mathbf{Z}_{<0}$$

and

$$\gamma - \alpha - \beta', \gamma - \alpha' - \beta, \gamma - \alpha - \alpha' - 1 \notin \mathbf{Z}_{\leq 0},$$

then the integrals $I_j^{(3)}(x, y) = \int_{\text{reg } D_j^{(3)}} U_{D_j^{(3)}}^{(3)}(x, y; t) dt$ for $1 \leq j \leq 4$ constitute a fundamental set of solutions of E_3 .

3. A monodromy representation associated with the auxiliary integrals

Fix a base point $z^0 = (z_1^0, z_2^0) \in Z$ to be real and a set of small positive numbers as in Figure 5. In the sequel, we consider the monodromy representation

$$\rho : \pi_1(Z, z^0) \rightarrow GL(4, \mathbf{C})$$

Fig. 5. The base point z^0 and the singular locus

Fig. 6. Paths in z_1 -space

Fig. 7. Paths in z_2 -space

on the space

$$V = \sum_{1 \leq j \leq 4} C I_j(z),$$

where $I_j(z) = \int_{\text{reg } D_j} u_{D_j}(t) dt$. The action $\gamma^* I_j(z)$ of $\gamma \in \pi_1(Z, z^0)$ is given by the consequence of the analytic continuation of $I_j(z)$ on the path $\gamma \in \pi_1(Z, z^0)$.

The paths $\gamma_{z_1, \{0\}}, \gamma_{z_1, \{1-z_2^0\}}, \gamma_{z_1, \{1-z_2^0, 1\}}, \gamma_{z_2, \{0\}}, \gamma_{z_1, \{1-z_1^0, 1\}}$ depicted in Figure 6 and Figure 7 give a set of generators of $\pi_1(X, z^0)$.

Then we have the following.

Theorem 3.1. Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \notin \mathbf{Z}_{<0}$. The matrices $\rho(\gamma)$ of the monodromy representation $\gamma^* \Xi(z) = \rho(\gamma) \Xi(z)$ with respect to

$$\Xi(z) = {}^t \left(I_1(z), e\left(\frac{\lambda_{24}}{2}\right) I_2(z), e\left(\frac{\lambda_2}{2}\right) I_3(z), e\left(\frac{\lambda_4}{2}\right) I_4(z) \right),$$

where $I_j(z) = \int_{\text{reg } D_j} u_{D_j}(t) dt$, are given by

$$\begin{aligned} \rho(\gamma_{z_1, \{0\}}) &= \begin{bmatrix} e_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & e_2(1-e_1) \\ e_2(1-e_1) & 0 & 1 & 0 \\ 0 & 0 & 0 & e_{12} \end{bmatrix}, \\ \rho(\gamma_{z_1, \{1-z_2^0, 1\}}) &= \begin{bmatrix} 1 & 0 & 1-e_5 & 0 \\ 0 & e_{2345} & e_{245}(e_3-1) & 0 \\ 0 & 0 & e_{25} & 0 \\ 0 & 1-e_{345} & e_{45}(1-e_3) & 1 \end{bmatrix}, \\ \rho(\gamma_{z_1, \{1-z_2^0\}}) &= \begin{bmatrix} 1 & e_5-1 & 0 & 0 \\ 0 & e_{245} & 0 & 0 \\ 0 & 1-e_{25} & 1 & 0 \\ 0 & 1-e_{45} & 0 & 1 \end{bmatrix}, \\ \rho(\gamma_{z_2, \{0\}}) &= \begin{bmatrix} e_{34} & 0 & 0 & 0 \\ 0 & 1 & e_4(1-e_3) & 0 \\ 0 & 0 & e_{34} & 0 \\ e_4(1-e_3) & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

and

$$\rho(\gamma_{z_2, \{1-z_1^0, 1\}}) = \begin{bmatrix} 1 & 0 & 0 & 1-e_5 \\ 0 & e_{2345} & 0 & e_{245}(e_1-1) \\ 0 & 1-e_{125} & 1 & e_{25}(1-e_1) \\ 0 & 0 & 0 & e_{45} \end{bmatrix}.$$

Proof. By the symmetry

$$\begin{aligned} \gamma_{z_1, \{0\}} &\leftrightarrow \gamma_{z_2, \{0\}}, \\ \gamma_{z_1, \{1-z_2^0, 1\}} &\leftrightarrow \gamma_{z_2, \{1-z_1^0, 1\}}, \\ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) &\leftrightarrow (\lambda_3, \lambda_4, \lambda_1, \lambda_2, \lambda_5), \\ (D_1, D_2, D_3, D_4) &\leftrightarrow (D_1, D_2, D_4, D_3), \end{aligned}$$

the two matrices $\rho(\gamma_{z_2, \{0\}})$, $\rho(\gamma_{z_2, \{1-z_1^0, 1\}})$ are obtained from $\rho(\gamma_{z_1, \{0\}})$, $\rho(\gamma_{z_1, \{1-z_2^0, 1\}})$. Hence it is enough to calculate the three matrices

$$\rho(\gamma_{z_1, \{0\}}), \rho(\gamma_{z_1, \{1-z_2^0, 1\}}), \rho(\gamma_{z_1, \{1-z_2^0\}}),$$

defined by

$$\gamma^* \int_{S_i} u_{D_i}(t) dt = \sum_{j=1}^4 \rho(\gamma)_{ij} \int_{S_j} u_{D_j}(t) dt,$$

where

$$S_1 = \text{reg } D_1, \quad S_2 = e_2^{1/2} \text{reg } D_2, \quad S_3 = e_2^{1/2} \text{reg } D_3, \quad S_4 = e_4^{1/2} \text{reg } D_4.$$

In what follows, we write only the chain C to express the function $\int_C u(t) dt$ for brevity. For instance, $\text{reg } D_j$ designates $\int_{\text{reg } D_j} u_{D_j}(t) dt$.

(1) First, we consider the action of $\gamma_{z_1, \{0\}}$:

$$\begin{aligned} \gamma_{z_1, \{0\}}^*(\text{reg } D_1) &= \gamma_{z_1, \{0\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \xrightarrow{\quad} z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \xrightarrow{\quad} z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \right\} \\ &= e_{12} \text{reg } D_1, \\ \gamma_{z_1, \{0\}}^*(\text{reg } D_2) &= \gamma_{z_1, \{0\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \xrightarrow{\quad} \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \xrightarrow{\quad} \quad \circ \end{array} \right\} \right\} \\ &= \left\{ \begin{array}{c} t_1\text{-space} \\ \text{loop at } 0 \text{ with arrow to } z_1 \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{loop at } z_2^0 \text{ with arrow to } 1 - z_1 \end{array} \right\} \\ &= \text{reg } D_2 + e_2 \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \xrightarrow{\quad} \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \xrightarrow{\quad} \quad \circ \end{array} \right\} \right\} \\ &\quad + e_1 \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \xleftarrow{\quad} \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \xleftarrow{\quad} \quad \circ \end{array} \right\} \right\} \\ &= \text{reg } D_2 + e_2^{1/2} (1 - e_1) \text{reg } D_4. \end{aligned}$$

Here and in what follows, the vertical arrow in each picture indicates the point at which the argument of each factor of the integrand is fixed to be zero, while we omit such an arrow when the chain in the picture is an interval of T_R . We return to proceed our calculations.

$$\begin{aligned}
\gamma_{z_1, \{0\}}^*(\text{reg } D_3) &= \gamma_{z_1, \{0\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \right\} \\
&+ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \\
&= \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \\
&+ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \\
&= \text{reg } D_3 + e_2^{1/2}(1 - e_1) \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \\
&= \text{reg } D_3 + e_2^{1/2}(1 - e_1) \text{reg } D_1,
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{z_1, \{0\}}^*(\text{reg } D_4) &= \gamma_{z_1, \{0\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \right\} \\
&= e_{12} \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \quad \quad \circ \end{array} \right\} = e_{12} \text{reg } D_4.
\end{aligned}$$

Therefore, we obtain

$$\gamma_{z_1, \{0\}}^* \begin{bmatrix} \text{reg } D_1 \\ \text{reg } D_2 \\ \text{reg } D_3 \\ \text{reg } D_4 \end{bmatrix} = \begin{bmatrix} e_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & e_2^{1/2}(1 - e_1) \\ e_2^{1/2}(1 - e_1) & 0 & 1 & 0 \\ 0 & 0 & 0 & e_{12} \end{bmatrix} \begin{bmatrix} \text{reg } D_1 \\ \text{reg } D_2 \\ \text{reg } D_3 \\ \text{reg } D_4 \end{bmatrix},$$

which leads to

$$\gamma_{z_1, \{0\}}^* \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} e_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & e_2(1-e_1) \\ e_2(1-e_1) & 0 & 1 & 0 \\ 0 & 0 & 0 & e_{12} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}.$$

(2) Second, we consider the action of $\gamma_{z_1, \{1-z_2^0, 1\}}$:

$$\begin{aligned} & \gamma_{z_1, \{1-z_2^0, 1\}}^* (\text{reg } D_1) \\ &= \gamma_{z_1, \{1-z_2^0, 1\}}^* \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \xrightarrow{\quad} z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \xrightarrow{\quad} z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \\ &= \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \xrightarrow{\quad} z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \xrightarrow{\quad} z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \\ &= \text{reg } D_1 + e_2 \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \xrightarrow{\quad} z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \xrightarrow{\quad} z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \\ &+ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \xrightarrow{\quad} z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \xrightarrow{\quad} z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \\ &+ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \xrightarrow{\quad} z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \xrightarrow{\quad} z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} e_5 \\ &+ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \xrightarrow{\quad} z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \xrightarrow{\quad} z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} e_5 \\ &= \text{reg } D_1 + e_2^{1/2} (1 - e_5) \text{reg } D_3. \end{aligned}$$

Here the last equality follows from

$$\left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} 0 & \xrightarrow{1-t_1} & z_2^0 \\ \uparrow & \circlearrowleft & \end{array} \end{array} \right\} = \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} \circ & \xrightarrow{\quad} & \circ \\ 0 & 1-t_1 & z_2^0 \end{array} \end{array} \right\} + e_5^{1/2} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} \circ & \xrightarrow{\quad} & \circ \\ 0 & 1-t_1 & z_2^0 \end{array} \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} \circ & \xrightarrow{\quad} & \circ \\ 0 & 1-t_1 & z_2^0 \end{array} \end{array} \right\} = \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} \circ & \xrightarrow{\quad} & \circ \\ 0 & 1-t_1 & z_2^0 \end{array} \end{array} \right\} + e_5^{-1/2} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} \circ & \xrightarrow{\quad} & \circ \\ 0 & 1-t_1 & z_2^0 \end{array} \end{array} \right\}.$$

Similarly, we have

$$\begin{aligned} & \gamma_{z_1, \{1-z_2^0, 1\}}^*(\text{reg } D_2) \\ &= \gamma_{z_1, \{1-z_2^0, 1\}}^* \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} 0 & z_1 & 1-z_2^0 \quad 1 \\ \circ & \xrightarrow{\quad} & \circ \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} & 1-t_1 & \\ \circ & \circlearrowleft & \circ \\ 0 & z_2^0 & 1-z_1 \quad 1 \end{array} \end{array} \right\} \\ &= e_2 \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} & & \\ \circ & z_1 & \begin{array}{c} \circlearrowleft \\ \uparrow \end{array} \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} & 1-t_1 & \\ \circ & \begin{array}{c} \circlearrowleft \\ \downarrow \end{array} & \circ \\ 0 & z_2^0 & 1-z_1 \quad 1 \end{array} \end{array} \right\} \\ &= e_2 \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} & & \\ \circ & z_1 & \begin{array}{c} \circlearrowleft \\ \downarrow \end{array} \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} & 1-t_1 & \\ \circ & \begin{array}{c} \circlearrowleft \\ \downarrow \end{array} & \circ \\ 0 & z_2^0 & 1-z_1 \quad 1 \end{array} \end{array} \right\} \\ &= -e_2 e_{45}^{1/2} \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} 0 & z_1 & 1-z_2^0 \quad 1 \\ \circ & \circ & \xrightarrow{\quad} \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} \circ & \circlearrowleft & \circ \\ 0 & 1-t_1 & z_2^0 \quad 1-z_1 \quad 1 \end{array} \end{array} \right\} \right. \\ &+ \left. \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} 0 & z_1 & 1-z_2^0 \quad 1 \\ \circ & \circ & \xleftarrow{\quad} \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccc} \circ & \circlearrowleft & \circ \\ 0 & \begin{array}{c} \downarrow \\ 1-t_1 \end{array} & z_2^0 \quad 1-z_1 \quad 1 \end{array} \end{array} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
& + e_5^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{Diagram: } 0 \text{ to } z_2^0 \text{ to } 1 - z_1 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \\
& = -e_2 e_{45}^{1/2} \left\{ e_5^{1/2} (1 - e_3) \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{Diagram: } 0 \text{ to } z_2^0 \text{ to } 1 - z_1 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \right\} \\
& + e_5^{1/2} (1 - e_3) \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{Diagram: } 0 \text{ to } z_2^0 \text{ to } 1 - z_1 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \\
& + e_3 e_{45}^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{Diagram: } 0 \text{ to } z_2^0 \text{ to } 1 - z_1 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \\
& = -e_4^{1/2} e_{25} (1 - e_3) \text{reg } D_3 + e_{2345} \text{reg } D_2, \\
& \gamma_{z_1, \{1 - z_2^0, 1\}}^* (\text{reg } D_3) \\
& = \gamma_{z_1, \{1 - z_2^0, 1\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{Diagram: } 0 \text{ to } z_2^0 \text{ to } 1 - z_1 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \right\} \\
& + \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{Diagram: } 0 \text{ to } z_2^0 \text{ to } 1 - z_1 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \\
& = e_2 \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ \text{Diagram: } 0 \text{ to } z_1 \text{ to } 1 - z_2^0 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{Diagram: } 0 \text{ to } z_2^0 \text{ to } 1 - z_1 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \right\} \\
& + \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \text{Diagram: } 0 \text{ to } z_2^0 \text{ to } 1 - z_1 \text{ to } 1, \text{ with } 1 - t_1 \text{ at bottom} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad 1 \\ \circ \quad \circ \quad \circ \quad \circ \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
&= e_2 \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 1 \\ \circ \quad \circ \quad \circ \leftarrow \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 1 - t_1 \\ \circ \uparrow \circ \rightarrow z_2^0 \quad 1 - z_1 \quad \circ 1 \\ 0 \end{array} \right\} \right. \\
&+ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 1 \\ \circ \quad \circ \quad \circ \rightarrow \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \downarrow \circ \rightarrow z_2^0 \quad 1 - z_1 \quad \circ 1 \\ 1 - t_1 \end{array} \right\} e_5 \\
&+ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 1 \\ \circ \quad \circ \rightarrow \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \rightarrow z_2^0 \quad \cdots \circ \quad 1 - z_1 \quad \circ 1 \\ 1 - t_1 \end{array} \right\} e_5 \\
&+ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 1 \\ \circ \quad \circ \quad \circ \rightarrow \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 1 - t_1 \\ \circ \leftarrow \cdots \circ \rightarrow z_2^0 \quad 1 - z_1 \quad \circ 1 \\ 0 \end{array} \right\} \Big\} \\
&= e_{25} \operatorname{reg} D_3,
\end{aligned}$$

and

$$\begin{aligned}
&\gamma_{z_1, \{1 - z_2^0, 1\}}^*(\operatorname{reg} D_4) \\
&= \gamma_{z_1, \{1 - z_2^0, 1\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 1 \\ \circ \rightarrow \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \circ \quad \downarrow \circ \rightarrow 1 - t_1 \\ 0 \quad z_2^0 \quad 1 - z_1 \quad \circ 1 \end{array} \right\} \right\} \\
&= \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \rightarrow z_1 \quad \circ \quad 1 - z_2^0 1 \\ \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \circ \quad \downarrow \circ \rightarrow 1 \\ \circ \quad z_2^0 \quad 1 - z_1 \quad \circ \end{array} \right\} \\
&= \operatorname{reg} D_4 + e_2^{1/2} \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 1 \\ \circ \quad \circ \rightarrow \circ \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1 - z_1 \quad \circ 1 \\ \circ \rightarrow \cdots \circ \quad 1 - t_1 \end{array} \right\} \right. \\
&+ e_{45}^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1 - z_2^0 1 \\ \circ \quad \circ \quad \circ \rightarrow \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \cdots \circ \rightarrow \circ \quad \circ \quad \circ \\ 0 \quad 1 - t_1 \quad z_2^0 \quad 1 - z_1 \quad \circ 1 \end{array} \right\} \Big\}
\end{aligned}$$

$$\begin{aligned}
& + e_{45}^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} 0 & z_1 & 1 - z_2^0 1 \\ \circ & \circ & \circ \leftarrow \circ \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{c} \text{Diagram: } \begin{array}{c} \circ \cdots \circ \downarrow \circ \\ \circ \quad \quad \quad \circ \\ 0 \quad \quad \quad z_2^0 1 - z_1 1 \\ \quad \quad \quad 1 - t_1 \end{array} \end{array} \right\} \\
& + e_5 e_4^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} 0 & z_1 & 1 - z_2^0 1 \\ \circ & \circ & \circ \leftarrow \circ \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{c} \text{Diagram: } \begin{array}{c} \circ \cdots \circ \downarrow \circ \\ \circ \quad \quad \quad \circ \\ 0 \quad \quad \quad z_2^0 1 - z_1 1 \\ \quad \quad \quad 1 - t_1 \end{array} \end{array} \right\} \\
& = \text{reg } D_4 + e_2^{1/2} \left\{ (1 - e_{345}) \text{reg } D_2 \right. \\
& + e_5 e_4^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} 0 & z_1 & 1 - z_2^0 1 \\ \circ & \circ & \circ \leftarrow \circ \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{c} \text{Diagram: } \begin{array}{c} \circ \cdots \circ \downarrow \circ \\ \circ \quad \quad \quad \circ \\ 0 \quad \quad \quad 1 - t_1 \\ \quad \quad \quad z_2^0 1 - z_1 1 \end{array} \end{array} \right\} \\
& + e_5 e_4^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccc} 0 & z_1 & 1 - z_2^0 1 \\ \circ & \circ & \circ \leftarrow \circ \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{c} \text{Diagram: } \begin{array}{c} \circ \cdots \circ \downarrow \circ \\ \circ \quad \quad \quad \circ \\ 0 \quad \quad \quad 1 - t_1 \\ \quad \quad \quad z_2^0 1 - z_1 1 \end{array} \end{array} \right\} \\
& = \text{reg } D_4 + e_2^{1/2} \left\{ (1 - e_{345}) \text{reg } D_2 + e_5 e_4^{1/2} (1 - e_3) \text{reg } D_3 \right\} \\
& = \text{reg } D_4 + e_2^{1/2} (1 - e_{345}) \text{reg } D_2 + e_5 e_{24}^{1/2} (1 - e_3) \text{reg } D_3.
\end{aligned}$$

Therefore, we obtain

$$\gamma_{z_1, \{1-z_2^0, 1\}}^* \begin{bmatrix} \text{reg } D_1 \\ \text{reg } D_2 \\ \text{reg } D_3 \\ \text{reg } D_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & e_2^{1/2}(1 - e_5) & 0 \\ 0 & e_{2345} & e_{25} e_4^{1/2}(e_3 - 1) & 0 \\ 0 & 0 & e_{25} & 0 \\ 0 & e_2^{1/2}(1 - e_{345}) & e_5 e_{24}^{1/2}(1 - e_3) & 1 \end{bmatrix} \begin{bmatrix} \text{reg } D_1 \\ \text{reg } D_2 \\ \text{reg } D_3 \\ \text{reg } D_4 \end{bmatrix},$$

which leads to

$$\gamma_{z_1, \{1-z_2^0, 1\}}^* \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 - e_5 & 0 \\ 0 & e_{2345} & e_{245}(e_3 - 1) & 0 \\ 0 & 0 & e_{25} & 0 \\ 0 & 1 - e_{345} & e_{45}(1 - e_3) & 1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}.$$

(3) Third, we consider the action of $\gamma_{z_1, \{1-z_2^0\}}$:

$$\begin{aligned}
& \gamma_{z_1, \{1-z_2^0\}}^* (\text{reg } D_1) \\
&= \gamma_{z_1, \{1-z_2^0\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \xrightarrow{\quad} z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \xrightarrow{\quad} z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \right\} \\
&= \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \\
&= \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \\
&+ e_2^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \\
&+ e_2^{1/2} \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\}
\end{aligned}$$

$$= \text{reg } D_1 - e_{24}^{1/2} (1 - e_5) \text{reg } D_2,$$

$$\begin{aligned}
& \gamma_{z_1, \{1-z_2^0\}}^* \text{reg } D_2 \\
&= \gamma_{z_1, \{1-z_2^0\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \right\} \\
&= e_{245} \text{reg } D_2,
\end{aligned}$$

$$\begin{aligned}
& \gamma_{z_1, \{1-z_2^0\}}^* \text{reg } D_3 \\
&= \gamma_{z_1, \{1-z_2^0\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ 0 \quad z_1 \quad 1-z_2^0 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ 0 \quad z_2^0 \quad 1-z_1 \quad 1 \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \right\} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \begin{array}{ccc} t_1\text{-space} & & \\ 0 & z_1 & 1 - z_2^0 \quad 1 \\ \circ & \circ & \circ \longrightarrow \circ \end{array} \right\} \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 1 - t_1 & & \\ \circ \longleftarrow \circ \cdots \circ & \circ & \circ \\ 0 & z_2^0 & 1 - z_1 \quad 1 \end{array} \right\} \\
& = e_2 \left\{ \begin{array}{ccc} t_1\text{-space} & & \\ 0 & z_1 & 1 - z_2^0 \quad 1 \\ \circ & \circ & \circ \longrightarrow \circ \end{array} \right\} \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 0 & z_2^0 & 1 - z_1 \quad 1 \\ \circ & \circ & \circ \\ \uparrow & \text{loop} & \circ \end{array} \right\} \\
& + \left\{ \begin{array}{ccc} t_1\text{-space} & & \\ 0 & z_1 & 1 - z_2^0 \quad 1 \\ \circ & \circ & \circ \\ \text{loop} & & \downarrow \circ \end{array} \right\} \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 0 & z_2^0 & 1 - z_1 \quad 1 \\ \circ & \circ & \circ \\ \text{loop} & & \circ \end{array} \right\} \\
& = e_2 \left\{ \begin{array}{ccc} t_1\text{-space} & & \\ 0 & z_1 & 1 - z_2^0 \quad 1 \\ \circ & \circ & \circ \longrightarrow \circ \end{array} \right\} \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 0 & z_2^0 & 1 - z_1 \quad 1 \\ \circ & \circ & \circ \\ \longrightarrow & & \circ \end{array} \right\} \\
& + e_4^{1/2} (1 - e_5) \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 0 & z_2^0 & 1 - z_1 \quad 1 \\ \circ & \circ & \circ \\ \text{loop} & & \circ \end{array} \right\} \\
& + \left\{ \begin{array}{ccc} t_1\text{-space} & & \\ 0 & z_1 & 1 - z_2^0 \quad 1 \\ \circ & \circ & \circ \longrightarrow \circ \end{array} \right\} \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 1 - t_1 & & \\ \circ \longleftarrow \circ \cdots \circ & \circ & \circ \\ 0 & z_2^0 & 1 - z_1 \quad 1 \end{array} \right\} \\
& + \left\{ \begin{array}{ccc} t_1\text{-space} & & \\ 0 & z_1 & 1 - z_2^0 \quad 1 \\ \circ & \circ & \circ \longrightarrow \circ \end{array} \right\} \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 0 & z_2^0 & 1 - z_1 \quad 1 \\ \circ & \circ & \circ \\ \uparrow & \text{loop} & \circ \end{array} \right\} \\
& + \left\{ \begin{array}{ccc} t_1\text{-space} & & \\ 0 & z_1 & 1 - z_2^0 \quad 1 \\ \circ & \circ & \circ \longrightarrow \circ \end{array} \right\} \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 0 & z_2^0 & 1 - z_1 \quad 1 \\ \circ & \circ & \circ \\ \uparrow & \text{loop} & \circ \end{array} \right\} \\
& = \left\{ \begin{array}{ccc} t_1\text{-space} & & \\ 0 & z_1 & 1 - z_2^0 \quad 1 \\ \circ & \circ & \circ \longrightarrow \circ \end{array} \right\} \left\{ \begin{array}{ccc} t_2\text{-space} & & \\ 0 & z_2^0 & 1 - z_1 \quad 1 \\ \circ & \circ & \circ \\ \longrightarrow & & \circ \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
& + e_4^{1/2}(1 - e_{25}) \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccccc} 0 & z_2^0 & 1 - z_1 & 1 & \\ \circ & \circ \cdots \circ & & & \\ & 1 - t_1 & & & \end{array} \end{array} \right\} \\
& + \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccccc} 0 & z_1 & 1 - z_2^0 & 1 & \\ \circ & \circ & \circ \rightarrow \circ & & \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccccc} 1 - t_1 & & & & \\ \circ \cdots \circ & z_2^0 & 1 - z_1 & 1 & \\ 0 & & & & \end{array} \end{array} \right\} \\
& = \text{reg } D_3 + e_4^{1/2}(1 - e_{25}) \text{reg } D_2,
\end{aligned}$$

and

$$\begin{aligned}
& \gamma_{z_1, \{1 - z_2^0\}}^*(\text{reg } D_4) \\
& = \gamma_{z_1, \{1 - z_2^0\}}^* \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccccc} 0 & z_1 & 1 - z_2^0 & 1 & \\ \circ & \circ \rightarrow \circ & & & \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccccc} & & 1 - t_1 & & \\ \circ & \circ \cdots \circ & & & \\ 0 & z_2^0 & 1 - z_1 & 1 & \end{array} \end{array} \right\} \right\} \\
& = \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccccc} 0 & z_1 & 1 - z_2^0 & 1 & \\ \circ & \circ \rightarrow \circ & & & \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccccc} & & 1 - t_1 & & \\ \circ & \circ \cdots \circ & & & \\ 0 & z_2^0 & 1 - z_1 & 1 & \end{array} \end{array} \right\} \\
& = \text{reg } D_4 + e_2 \left\{ \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccccc} 0 & z_1 & 1 - z_2^0 & 1 & \\ \circ & \circ \rightarrow \circ & & & \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccccc} & & 1 - t_1 & & \\ \circ & \circ \cdots \circ & & & \\ 0 & z_2^0 & 1 - z_1 & 1 & \end{array} \end{array} \right\} \right\} \\
& + e_{45} \left\{ \begin{array}{c} t_1\text{-space} \\ \begin{array}{ccccc} 0 & z_1 & 1 - z_2^0 & 1 & \\ \circ & \circ \leftarrow \circ & & & \end{array} \end{array} \right\} \left\{ \begin{array}{c} t_2\text{-space} \\ \begin{array}{ccccc} & & 1 - t_1 & & \\ \circ & \circ \cdots \circ & & & \\ 0 & z_2^0 & 1 - z_1 & 1 & \end{array} \end{array} \right\} \\
& = \text{reg } D_4 + e_2^{1/2}(1 - e_{45}) \text{reg } D_2.
\end{aligned}$$

Therefore, we obtain

$$\gamma_{z_1, \{1 - z_2^0\}}^* \begin{bmatrix} \text{reg } D_1 \\ \text{reg } D_2 \\ \text{reg } D_3 \\ \text{reg } D_4 \end{bmatrix} = \begin{bmatrix} 1 & e_{24}^{1/2}(e_5 - 1) & 0 & 0 \\ 0 & e_{245} & 0 & 0 \\ 0 & e_4^{1/2}(1 - e_{25}) & 1 & 0 \\ 0 & e_2^{1/2}(1 - e_{45}) & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{reg } D_1 \\ \text{reg } D_2 \\ \text{reg } D_3 \\ \text{reg } D_4 \end{bmatrix},$$

and thus

$$\gamma_{z_1, \{1-z_2^0\}}^* \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} = \begin{bmatrix} 1 & e_5 - 1 & 0 & 0 \\ 0 & e_{245} & 0 & 0 \\ 0 & 1 - e_{25} & 1 & 0 \\ 0 & 1 - e_{45} & 0 & 1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}.$$

It completes the proof of Theorem 3.1. $\square$

4. Monodromy representations associated with F_2 and F_3

4.1. F_2 case

Next we consider the monodromy representation for F_2 . Fix a base point (x^0, y^0) to be a pair of real and sufficiently small positive numbers as in Figure 8.

Fig. 8. The base point and singular locus of E_2

We consider the monodromy representation

$$\rho : \pi_1(X^{(2)}, (x^0, y^0)) \rightarrow GL(4, \mathbb{C})$$

on the solution space

$$V = \sum_{1 \leq j \leq 4} \mathbf{C} I_j^{(2)}(x, y)$$

of E_2 , where $I_j^{(2)} = I_j^{(2)}(x, y) = \int_{\text{reg}} D_j^{(2)} u_{D_j^{(2)}}^{(2)}(x, y; t) dt$. The action $\gamma^* I^{(2)}(x, y)$ of $\gamma \in \pi_1(X^{(2)}, (x^0, y^0))$ is given by the consequence of the analytic continuation of $I^{(2)}(x, y)$ on the path $\gamma \in \pi_1(X^{(2)}, (x^0, y^0))$.

The paths $\gamma_0^{(2)}, \gamma_1^{(2)}, \gamma_2^{(2)}, \gamma_0'^{(2)}, \gamma_1'^{(2)}$ depicted in Figure 9 and Figure 10 give a set of generators of $\pi_1(X^{(2)}, (x^0, y^0))$.

Fig. 9. Paths in x -space

Fig. 10. Paths in y -space

Then we have the following.

Theorem 4.1. *Suppose that*

$$\beta - 1, \gamma - \beta - 1, \beta' - 1, \gamma' - \beta' - 1, -\alpha \notin \mathbf{Z}_{<0}.$$

Then the matrices $\rho(\gamma)$ of the monodromy representation $\gamma^ \Xi(x, y) = \rho(\gamma) \Xi(x, y)$ for $\gamma \in \pi_1(X^{(2)}, (x^0, y^0))$ in the basis*

$$\Xi(x, y) = {}^t \left(I_1^{(2)}, e \left(\frac{\gamma + \gamma' - \beta - \beta'}{2} \right) I_2^{(2)}, e \left(\frac{\gamma - \beta - 1}{2} \right) I_3^{(2)}, e \left(\frac{\gamma' - \beta' - 1}{2} \right) I_4^{(2)} \right),$$

where $I_j^{(2)} = \int_{\text{reg } D_j^{(2)}} u_{D_j^{(2)}}^{(2)}(x, y; t) dt$, are given by

$$\rho(\gamma_0^{(2)}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e(-\gamma) & 0 & e(-\beta) - 1 \\ e(-\beta) - 1 & 0 & e(-\gamma) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\rho(\gamma_1^{(2)}) = \begin{bmatrix} 1 & 0 & 1 - e(-\alpha) & 0 \\ 0 & e(-\alpha - \beta + \gamma + \gamma') & e(-\alpha - \beta - \beta' + \gamma + \gamma')(e(\beta') - 1) & 0 \\ 0 & 0 & e(-\alpha - \beta + \gamma) & 0 \\ 0 & 1 - e(-\alpha + \gamma') & e(-\alpha - \beta' + \gamma')(1 - e(\beta')) & 1 \end{bmatrix},$$

$$\rho(\gamma_2^{(2)}) = \begin{bmatrix} 1 & e(-\alpha) - 1 & 0 & 0 \\ 0 & e(-\alpha - \beta - \beta' + \gamma + \gamma') & 0 & 0 \\ 0 & 1 - e(-\alpha - \beta + \gamma) & 1 & 0 \\ 0 & 1 - e(-\alpha - \beta' + \gamma') & 0 & 1 \end{bmatrix},$$

$$\rho(\gamma_0'^{(2)}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e(-\gamma') & e(-\beta') - 1 & 0 \\ 0 & 0 & 1 & 0 \\ e(-\beta') - 1 & 0 & 0 & e(-\gamma') \end{bmatrix},$$

and

$$\rho(\gamma_1'^{(2)}) = \begin{bmatrix} 1 & 0 & 0 & 1 - e(-\alpha) \\ 0 & e(-\alpha - \beta' + \gamma + \gamma') & 0 & e(-\alpha - \beta - \beta' + \gamma + \gamma')(e(\beta) - 1) \\ 0 & 1 - e(-\alpha + \gamma) & 1 & e(-\alpha - \beta + \gamma)(1 - e(\beta)) \\ 0 & 0 & 0 & e(-\alpha - \beta' + \gamma'). \end{bmatrix}.$$

Proof. Recall that

$$u(x, y; t) = t_1^{\lambda_1} (t_1 - x)^{\lambda_2} t_2^{\lambda_3} (t_2 - y)^{\lambda_4} (t_1 + t_2 - 1)^{\lambda_5},$$

$$u^{(2)}(x, y; t) = t_1^{\lambda_1} (t_1 - 1)^{\lambda_2} t_2^{\lambda_3} (t_2 - 1)^{\lambda_4} (1 - xt_1 - yt_2)^{\lambda_5}.$$

Then the change of integration variables $t_1 \rightarrow t_1/x$ and $t_2 \rightarrow t_2/y$ induces

$$(4.1) \quad \int_{\text{reg } D_i^{(2)}} u_{D_i^{(2)}}^{(2)}(x, y; t) dt = x^{-\lambda_{12}-1} y^{-\lambda_{34}-1} \int_{\text{reg } D_i^{(0)}} u_{D_i^{(0)}}(x, y; t) dt,$$

where

$$D_1^{(2)} = \{0 < t_1 < 1, 0 < t_2 < 1\},$$

$$D_2^{(2)} = \{1 < t_1, 1 < t_2, xt_1 + yt_2 < 1\},$$

$$D_3^{(2)} = \{0 < t_2 < 1, 1 < t_1, xt_1 + yt_2 < 1\},$$

$$D_4^{(2)} = \{0 < t_1 < 1, 1 < t_2, xt_1 + yt_2 < 1\}$$

and

$$D_1^{(0)} = \{0 < t_1 < x, 0 < t_2 < y\},$$

$$D_2^{(0)} = \{x < t_1, y < t_2, t_1 + t_2 < 1\},$$

$$D_3^{(0)} = \{0 < t_2 < y, x < t_1, t_1 + t_2 < 1\},$$

$$D_4^{(0)} = \{0 < t_1 < x, y < t_2, t_1 + t_2 < 1\}.$$

For $\gamma^{(2)} \in \pi_1(X^{(2)}, (x^0, y^0))$, we define the matrix elements $\rho(\gamma^{(2)})_{ij}$ to be

$$(4.2) \quad \gamma^{(2)*} \int_{S_i^{(2)}} u_{D_i^{(2)}}^{(2)}(x, y; t) dt = \sum_j \rho(\gamma^{(2)})_{ij} \int_{S_j^{(2)}} u_{D_j^{(2)}}^{(2)}(x, y; t) dt.$$

Here and in what follows, we use the symbols

$$\begin{aligned} S_1^{(k)} &= \text{reg } D_1^{(k)}, & S_2^{(k)} &= e_{24}^{1/2} \text{reg } D_2^{(k)}, \\ S_3^{(k)} &= e_2^{1/2} \text{reg } D_3^{(k)}, & S_4^{(k)} &= e_4^{1/2} \text{reg } D_4^{(k)} \end{aligned}$$

for $k = 0, 2$.

Substituting (4.1) into (4.2) leads to

$$\begin{aligned} & \gamma^{(2)*} \left\{ x^{-\lambda_{12}-1} y^{-\lambda_{34}-1} \int_{S_i^{(0)}} u_{D_i^{(0)}}(x, y; t) dt \right\} \\ &= \sum_j \rho(\gamma^{(2)})_{ij} \left\{ x^{-\lambda_{12}-1} y^{-\lambda_{34}-1} \int_{S_j^{(0)}} u_{D_j^{(0)}}(x, y; t) dt \right\}, \end{aligned}$$

which implies

$$\begin{aligned} & \gamma^{(2)*} \left\{ \int_{S_i^{(0)}} u_{D_i^{(0)}}(x, y; t) dt \right\} \\ &= \sum_j \rho(\gamma^{(2)})_{ij} \frac{x^{-\lambda_{12}-1} y^{-\lambda_{34}-1}}{\gamma^{(2)*} \{x^{-\lambda_{12}-1} y^{-\lambda_{34}-1}\}} \left\{ \int_{S_j^{(0)}} u_{D_j^{(0)}}(x, y; t) dt \right\}. \end{aligned}$$

The change of the variables $x = z_1$ and $y = z_2$ with $(x^0, y^0) = (z_1^0, z_2^0)$ induces the map

$$\psi : \pi_1(X^{(2)}, (x^0, y^0)) \ni \gamma^{(2)} \rightarrow \psi(\gamma^{(2)}) \in \pi_1(Z, (z_1^0, z_2^0))$$

and

$$\begin{aligned} & \psi(\gamma^{(2)})^* \left\{ \int_{S_i} u_{D_i}(z_1, z_2; t) dt \right\} \\ &= \sum_j \rho(\gamma^{(2)})_{ij} \frac{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}}{\psi(\gamma^{(2)})^* \{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}\}} \left\{ \int_{S_j} u_{D_j}(z_1, z_2; t) dt \right\}. \end{aligned}$$

This means

$$\rho(\psi(\gamma^{(2)}))_{ij} = \rho(\gamma^{(2)})_{ij} \frac{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}}{\psi(\gamma^{(2)})^* \{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}\}},$$

which is equivalent to

$$\rho(\gamma^{(2)})_{ij} = \rho(\psi(\gamma^{(2)}))_{ij} \frac{\psi(\gamma^{(2)})^* \{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}\}}{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}}.$$

On the other hand, we have

$$\begin{aligned} \psi(\gamma_0^{(2)}) &= \gamma_{z_1, \{0\}}, & \psi(\gamma_1^{(2)}) &= \gamma_{z_1, \{1-z_2^0, 1\}}, & \psi(\gamma_2^{(2)}) &= \gamma_{z_1, \{1-z_2^0\}}, \\ \psi(\gamma_0'^{(2)}) &= \gamma_{z_2, \{0\}}, & \psi(\gamma_1'^{(2)}) &= \gamma_{z_2, \{1-z_1^0, 1\}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho(\gamma_0^{(2)}) &= \rho(\psi(\gamma_0^{(2)})) \frac{\psi(\gamma_0^{(2)})^* \{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}\}}{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}} \\ &= e(-\lambda_{12}) \rho(\gamma_{z_1, \{0\}}) = e_{12}^{-1} \begin{bmatrix} e_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & e_2(1-e_1) \\ e_2(1-e_1) & 0 & 1 & 0 \\ 0 & 0 & 0 & e_{12} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{12}^{-1} & 0 & e_1^{-1}-1 \\ e_1^{-1}-1 & 0 & e_{12}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \rho(\gamma_1^{(2)}) &= \rho(\psi(\gamma_1^{(2)})) \frac{\rho(\psi(\gamma_1^{(2)}))^* \{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}\}}{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}} \\ &= \rho(\gamma_{z_1, \{1-z_2^0, 1\}}) = \begin{bmatrix} 1 & 0 & 1-e_5 & 0 \\ 0 & e_{2345} & e_{245}(e_3-1) & 0 \\ 0 & 0 & e_{25} & 0 \\ 0 & 1-e_{345} & e_{45}(1-e_3) & 1 \end{bmatrix}, \\ \rho(\gamma_2^{(2)}) &= \rho(\psi(\gamma_2^{(2)})) \frac{\psi(\gamma_2^{(2)})^* \{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}\}}{z_1^{-\lambda_{12}-1} z_2^{-\lambda_{34}-1}} \\ &= \rho(\gamma_{z_1, \{1-z_2^0\}}) = \begin{bmatrix} 1 & e_5-1 & 0 & 0 \\ 0 & e_{245} & 0 & 0 \\ 0 & 1-e_{25} & 1 & 0 \\ 0 & 1-e_{45} & 0 & 1 \end{bmatrix}. \end{aligned}$$

When the parameters are given by

$$\lambda_1 = \beta - 1, \quad \lambda_2 = \gamma - \beta - 1, \quad \lambda_3 = \beta' - 1, \quad \lambda_4 = \gamma' - \beta' - 1, \quad \lambda_5 = -\alpha,$$

it follows that

$$\begin{aligned}\lambda_{125} &= \gamma - \alpha - 2, & \lambda_{345} &= \gamma' - \alpha - 2, & \lambda_{12345} &= \gamma + \gamma' - \alpha - 4, \\ \lambda_{12} &= \gamma - 2, & \lambda_{2345} &= \gamma + \gamma' - \alpha - \beta - 3, & \lambda_{245} &= \gamma + \gamma' - \alpha - \beta - \beta' - 2, \\ \lambda_{45} &= \gamma' - \alpha - \beta' - 1.\end{aligned}$$

Therefore we reach the result. $\square$

4.2. F_3 case

Next we consider the monodromy representation for F_3 . Fix a base point (x^0, y^0) to be a pair of real and sufficiently large positive numbers as in Figure 11.

Fig. 11. Singular locus of E_3

We consider the monodromy representation

$$\rho : \pi_1(X^{(3)}, (x^0, y^0)) \rightarrow GL(4, \mathbb{C})$$

on the solution space

$$V = \sum_{1 \leq j \leq 4} \mathbf{C} I_j^{(3)}(x, y)$$

of E_3 , where $I_j^{(3)} = I_j^{(3)}(x, y) = \int_{\text{reg } D_j^{(3)}} u_{D_j^{(3)}}^{(3)}(x, y; t) dt$. The action $\gamma^* I^{(3)}(x, y)$ of $\gamma \in \pi_1(X^{(3)}, (x^0, y^0))$ is given by the consequence of the analytic continuation of $I^{(3)}(x, y)$ on the path $\gamma \in \pi_1(X^{(3)}, (x^0, y^0))$.

The paths $\gamma_0^{(3)}, \gamma_1^{(3)}, \gamma_2^{(3)}, \gamma_0'^{(3)}, \gamma_1'^{(3)}$ depicted in Figure 12 and Figure 13 give a set of generators of $\pi_1(X^{(3)}, (x^0, y^0))$.

Fig. 12. Paths in x -space Fig. 13. Paths in y -space

Then we have the following.

Theorem 4.2. *Suppose that*

$$\beta - 1, -\alpha, \beta' - 1, -\alpha', \gamma - \beta - \beta' - 1 \notin \mathbf{Z}_{<0}.$$

Then the matrices $\rho(\gamma)$ of the monodromy representation $\gamma^ \Xi(x, y) = \rho(\gamma) \Xi(x, y)$ for $\gamma \in \pi_1(X^{(3)}, (x^0, y^0))$ in the basis*

$$\Xi(x, y) = {}^t \left(I_1^{(3)}, e\left(\frac{-\alpha - \alpha'}{2}\right) I_2^{(3)}, e\left(\frac{-\alpha}{2}\right) I_3^{(3)}, e\left(\frac{-\alpha'}{2}\right) I_4^{(3)} \right),$$

where $I_j^{(3)} = \int_{\text{reg } D_j^{(3)}} u_{D_j^{(3)}}^{(3)}(x, y; t) dt$, are given by

$$\rho(\gamma_0^{(3)}) = \begin{bmatrix} e(-\beta) & 0 & 0 & 0 \\ 0 & e(-\alpha) & 0 & e(-\alpha)[1 - e(-\beta)] \\ e(-\alpha)[1 - e(-\beta)] & 0 & e(-\alpha) & 0 \\ 0 & 0 & 0 & e(-\beta) \end{bmatrix},$$

$$\rho(\gamma_1^{(3)}) = \begin{bmatrix} 1 & 0 & 1 - e(-\beta - \beta' + \gamma) & 0 \\ 0 & e(-\alpha - \alpha' - \beta + \gamma) & e(-\alpha - \alpha' - \beta - \beta' + \gamma)(e(\beta') - 1) & 0 \\ 0 & 0 & e(-\alpha - \beta - \beta' + \gamma) & 0 \\ 0 & 1 - e(-\alpha' - \beta + \gamma) & e(-\alpha' - \beta - \beta' + \gamma)(1 - e(\beta')) & 1 \end{bmatrix},$$

$$\rho(\gamma_2^{(3)}) = \begin{bmatrix} 1 & e(-\beta - \beta' + \gamma) - 1 & 0 & 0 \\ 0 & e(-\alpha - \alpha' - \beta - \beta' + \gamma) & 0 & 0 \\ 0 & 1 - e(-\alpha - \beta - \beta' + \gamma) & 1 & 0 \\ 0 & 1 - e(-\alpha' - \beta - \beta' + \gamma) & 0 & 1 \end{bmatrix},$$

$$\rho(\gamma_0'^{(3)}) = \begin{bmatrix} e(-\beta') & 0 & 0 & 0 \\ 0 & e(-\alpha') & e(-\alpha')[1 - e(-\beta')] & 0 \\ 0 & 0 & e(-\beta') & 0 \\ e(-\alpha')[1 - e(-\beta')] & 0 & 0 & e(-\alpha') \end{bmatrix},$$

and

$$\rho(\gamma_1'^{(3)}) = \begin{bmatrix} 1 & 0 & 0 & 1 - e(-\beta - \beta' + \gamma) \\ 0 & e(-\alpha - \alpha' - \beta' + \gamma) & 0 & e(-\alpha - \alpha' - \beta - \beta' + \gamma)(e(\beta) - 1) \\ 0 & 1 - e(-\alpha - \beta' + \gamma) & 1 & e(-\alpha - \beta - \beta' + \gamma)(1 - e(\beta)) \\ 0 & 0 & 0 & e(-\alpha' - \beta - \beta' + \gamma) \end{bmatrix}.$$

Proof. Recall that

$$u(x, y; t) = t_1^{\lambda_1} (t_1 - x)^{\lambda_2} t_2^{\lambda_3} (t_2 - y)^{\lambda_4} (t_1 + t_2 - 1)^{\lambda_5},$$

$$u^{(3)}(x, y; t) = t_1^{\lambda_1} (1 - xt_1)^{\lambda_2} t_2^{\lambda_3} (1 - yt_2)^{\lambda_4} (1 - t_1 - t_2)^{\lambda_5}.$$

Then the change of the variables $x = 1/z_1$ and $y = 1/z_2$ with $(x^0, y^0) = (1/z_1^0, 1/z_2^0)$ implies

$$(4.3) \quad \int_{\text{reg } D_i^{(3)}} u_{D_i^{(3)}}^{(3)}(x, y; t) dt = z_1^{-\lambda_2} z_2^{-\lambda_4} \int_{\text{reg } D_i} u_{D_i}(z_1, z_2; t) dt$$

and induces the map

$$\psi : \pi_1(X^{(3)}, (x^0, y^0)) \ni \gamma^{(3)} \rightarrow \psi(\gamma^{(3)}) \in \pi_1(Z, (z_1^0, z_2^0)).$$

For $\gamma^{(3)} \in \pi_1(X^{(3)}, (x^0, y^0))$, we define the matrix elements $\rho(\gamma^{(3)})_{ij}$ to be

$$(4.4) \quad \gamma^{(3)*} \int_{S_i^{(3)}} u_{D_i^{(3)}}^{(3)}(x, y; t) dt = \sum_j \rho(\gamma^{(3)})_{ij} \int_{S_j^{(3)}} u_{D_j^{(3)}}^{(3)}(x, y; t) dt.$$

Here and in what follows, we use the symbols

$$S_1^{(3)} = \text{reg } D_1^{(3)}, \quad S_2^{(3)} = e_{24}^{1/2} \text{reg } D_2^{(3)},$$

$$S_3^{(3)} = e_2^{1/2} \text{reg } D_3^{(3)}, \quad S_4^{(3)} = e_4^{1/2} \text{reg } D_4^{(3)}.$$

Substituting (4.3) into (4.4) leads to

$$\psi(\gamma^{(3)})^* \left\{ z_1^{-\lambda_2} z_2^{-\lambda_4} \int_{S_i} u_{D_i}(t) dt \right\} = \sum_j \rho(\gamma^{(3)})_{ij} \left\{ z_1^{-\lambda_2} z_2^{-\lambda_4} \int_{S_j} u_{D_j}(t) dt \right\},$$

which implies

$$\psi(\gamma^{(3)})^* \left\{ \int_{S_i} u_{D_i}(t) dt \right\} = \sum \rho(\gamma^{(3)})_{ij} \frac{z_1^{-\lambda_2} z_2^{-\lambda_4}}{\psi(\gamma^{(3)})^* \{z_1^{-\lambda_2} z_2^{-\lambda_4}\}} \int_{S_j} u_{D_j}(t) dt.$$

This means

$$\rho(\psi(\gamma^{(3)}))_{ij} = \rho(\gamma^{(3)})_{ij} \frac{z_1^{-\lambda_2} z_2^{-\lambda_4}}{\psi(\gamma^{(3)})^* \{z_1^{-\lambda_2} z_2^{-\lambda_4}\}},$$

which is equivalent to

$$\rho(\gamma^{(3)})_{ij} = \rho(\psi(\gamma^{(3)}))_{ij} \frac{\psi(\gamma^{(3)})^* \{z_1^{-\lambda_2} z_2^{-\lambda_4}\}}{z_1^{-\lambda_2} z_2^{-\lambda_4}}.$$

On the other hand, we have

$$\begin{aligned} \psi(\gamma_0^{(3)}) &= \gamma_{z_1, \{0\}}^{-1}, & \psi(\gamma_1^{(3)}) &= \gamma_{z_1, \{1-z_2^0, 1\}}, & \psi(\gamma_2^{(3)}) &= \gamma_{z_1, \{1-z_2^0\}}, \\ \psi(\gamma_0'^{(3)}) &= \gamma_{z_2, \{0\}}^{-1}, & \psi(\gamma_1'^{(3)}) &= \gamma_{z_2, \{1-z_1^0, 1\}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \rho(\gamma_0^{(3)}) &= \rho(\gamma_{z_1, \{0\}}^{-1}) \frac{(\gamma_{z_1, \{0\}}^{-1})^* \{z_1^{-\lambda_2} z_2^{-\lambda_4}\}}{z_1^{-\lambda_2} z_2^{-\lambda_4}} \\ &= e(\lambda_2) \rho(\gamma_{z_1, \{0\}})^{-1} \\ &= e_2 \begin{bmatrix} e_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & e_2(1-e_1) \\ e_2(1-e_1) & 0 & 1 & 0 \\ 0 & 0 & 0 & e_{12} \end{bmatrix}^{-1} = e_2 \begin{bmatrix} e_{12}^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1-e_1^{-1} \\ 1-e_1^{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & e_{12}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} e_1^{-1} & 0 & 0 & 0 \\ 0 & e_2 & 0 & e_2(1-e_1^{-1}) \\ e_2(1-e_1^{-1}) & 0 & e_2 & 0 \\ 0 & 0 & 0 & e_1^{-1} \end{bmatrix}, \\ \rho(\gamma_1^{(3)}) &= \rho(\gamma_{z_1, \{1-z_2^0, 1\}}) \frac{\gamma_{z_1, \{1-z_2^0, 1\}}^* \{z_1^{-\lambda_2} z_2^{-\lambda_4}\}}{z_1^{-\lambda_2} z_2^{-\lambda_4}} \\ &= \rho(\gamma_{z_1, \{1-z_2^0, 1\}}) = \begin{bmatrix} 1 & 0 & 1-e_5 & 0 \\ 0 & e_{2345} & e_{245}(e_3-1) & 0 \\ 0 & 0 & e_{25} & 0 \\ 0 & 1-e_{345} & e_{45}(1-e_3) & 1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
\rho(\gamma_2^{(3)}) &= \rho(\gamma_{z_1, \{1-z_2^0\}}) \frac{\gamma_{z_1, \{1-z_2^0\}}^* \{z_1^{-\lambda_2} z_2^{-\lambda_4}\}}{z_1^{-\lambda_2} z_2^{-\lambda_4}} \\
&= \rho(\gamma_{z_1, \{1-z_2^0\}}) = \begin{bmatrix} 1 & e_5 - 1 & 0 & 0 \\ 0 & e_{245} & 0 & 0 \\ 0 & 1 - e_{25} & 1 & 0 \\ 0 & 1 - e_{45} & 0 & 1 \end{bmatrix}.
\end{aligned}$$

When the parameters are given by

$$\lambda_1 = \beta - 1, \quad \lambda_2 = -\alpha, \quad \lambda_3 = \beta' - 1, \quad \lambda_4 = -\alpha', \quad \lambda_5 = \gamma - \beta - \beta' - 1,$$

it follows that

$$\begin{aligned}
\lambda_{125} &= \gamma - \alpha - \beta' - 2, & \lambda_{345} &= \gamma - \alpha' - \beta - 2, & \lambda_{12345} &= \gamma - \alpha - \alpha' - 4, \\
\lambda_{245} &= \gamma - \alpha - \alpha' - \beta - \beta' - 1, & \lambda_{2345} &= -\alpha - \alpha' - \beta - \gamma - 2.
\end{aligned}$$

Therefore we reach the result. $\square$

5. Irreducibility

By using the expression of the matrix elements in Theorem 3.1, we derive a sufficient condition for the monodromy representation associated with the auxiliary integrals being irreducible.

Theorem 5.1. *If none of $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_{125}, \lambda_{345}$ and λ_{12345} are integers, the monodromy representation ρ in Section 3 is irreducible.*

Proof. We simply denote

$$\gamma_{z_1, \{0\}}, \gamma_{z_1, \{1-z_2^0, 1\}}, \gamma_{z_1, \{1-z_2^0\}}, \gamma_{z_2, \{0\}}, \gamma_{z_2, \{1-z_1^0, 1\}}$$

by

$$\gamma_0, \gamma_1, \gamma_2, \gamma'_0, \gamma'_1.$$

We show that a non-zero invariant subspace $W \subset V = \sum_{j=1}^4 \mathbf{C}S_j$ should be the total space V .

First, choose an arbitrary non-zero element $w = c_1 S_1 + c_2 S_2 + c_3 S_3 + c_4 S_4 \in W$, where $S_1 = I_1$, $S_2 = e_{24}^{1/2} I_2$, $S_3 = e_2^{1/2} I_3$, $S_4 = e_4^{1/2} I_4$. Then the system of equations

$$\begin{aligned}
(5.1) \quad (\rho(\gamma_0) - 1)(w) &= \{c_1(e_{12} - 1) + c_3 e_2(1 - e_1)\} S_1 \\
&\quad + \{c_2 e_2(1 - e_1) + c_4(e_{12} - 1)\} S_4,
\end{aligned}$$

$$\begin{aligned}
(5.2) \quad (\rho(\gamma_1) - 1)(w) &= \{c_2(e_{2345} - 1) + c_4(1 - e_{345})\}S_2 \\
&\quad + \{c_1(1 - e_5) + c_2e_{245}(e_3 - 1) \\
&\quad + c_3(e_{25} - 1) + c_4e_{45}(1 - e_3)\}S_3
\end{aligned}$$

implies that there exists an element $w_1 = c'_1S_1 + c'_4S_4 \in W$ with $(c'_1, c'_4) \neq (0, 0)$ or an element $w_2 = c'_2S_2 + c'_3S_3 \in W$ with $(c'_2, c'_3) \neq (0, 0)$ under the condition $\lambda_2, \lambda_{125}, \lambda_{12345} \notin \mathbf{Z}$. The reason is that, if all the coefficients of $S_1, \dots, S_4$ in (5.1) and (5.2) are zero, we have

$$M \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where

$$M = \begin{bmatrix} e_{12} - 1 & 0 & e_2(1 - e_1) & 0 \\ 0 & e_2(1 - e_1) & 0 & e_{12} - 1 \\ 0 & e_{2345} - 1 & 0 & 1 - e_{345} \\ 1 - e_5 & e_{245}(e_3 - 1) & e_{25} - 1 & e_{45}(1 - e_3) \end{bmatrix}$$

and

$$\det(M) = (e_2 - 1)^2(e_{125} - 1)(e_{12345} - 1),$$

which implies the contradiction $(c_1, \dots, c_4) = (0, \dots, 0)$.

Second, the equality

$$(\rho(\gamma_0) - 1)(w_2) = e_2(1 - e_1)(c'_2S_4 + c'_3S_1)$$

for $w_2 = c'_2S_2 + c'_3S_3$ implies that, if $w_2 \in W$, there exists an element $w_1 = c'_1S_1 + c'_4S_4 \in W$ with $(c'_1, c'_4) \neq (0, 0)$ under the condition $\lambda_1 \notin \mathbf{Z}$.

Third, the system of equations

$$\begin{aligned}
(\rho(\gamma'_0) - 1)(w_1) &= \{(e_{34} - 1)c'_1 + e_4(1 - e_3)c'_4\}S_1, \\
(\rho(\gamma'_1) - 1)(w_1) &= \{(1 - e_5)c'_1 + (e_{45} - 1)c'_4\}S_4
\end{aligned}$$

for $w_1 = c'_1S_1 + c'_4S_4$ with

$$\begin{bmatrix} e_{34} - 1 & e_4(1 - e_3) \\ 1 - e_5 & e_{45} - 1 \end{bmatrix} \begin{bmatrix} c'_1 \\ c'_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\det \begin{bmatrix} e_{34} - 1 & e_4(1 - e_3) \\ 1 - e_5 & e_{45} - 1 \end{bmatrix} = (e_4 - 1)(e_{345} - 1)$$

implies that if $w_1 \in W$ then $S_1 \in W$ or $S_4 \in W$ under the condition $\lambda_4, \lambda_{345} \notin \mathbf{Z}$.

Fourth, the relation $(\rho(\gamma'_0) - 1)(S_4) = e_4(1 - e_3)S_1$ implies that if $S_4 \in W$ then $S_1 \in W$ under the condition $\lambda_3 \notin \mathbf{Z}$.

Fifth, the relations

$$(\rho(\gamma_2) - 1)(S_1) = (e_5 - 1)S_2,$$

$$(\rho(\gamma_1) - 1)(S_1) = (1 - e_5)S_3,$$

$$(\rho(\gamma'_1) - 1)(S_1) = (1 - e_5)S_4$$

imply that if $S_1 \in W$ then $S_2, S_3, S_4 \in W$ under the condition $\lambda_5 \notin \mathbf{Z}$.

Therefore, combining the five steps above, we arrive at the result. $\square$

As corollaries of Theorem 5.1, we obtain the following.

Theorem 5.2. (1) *If none of*

$$\alpha, \beta, \beta', \gamma - \beta, \gamma' - \beta', \gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha$$

is an integer, then the monodromy representation for E_2 is irreducible.

(2) *If none of*

$$\alpha, \alpha', \beta, \beta', \gamma - \beta - \beta', \gamma - \alpha - \beta', \gamma - \alpha' - \beta, \gamma - \alpha - \alpha'$$

is an integer, then the monodromy representation for E_3 is irreducible.

In [MS4], the following is proved.

Theorem 5.3. (1) *If at least one of*

$$\alpha, \beta, \beta', \gamma - \beta, \gamma' - \beta', \gamma - \alpha, \gamma' - \alpha, \gamma + \gamma' - \alpha$$

is an integer, then the monodromy representation of the system E_2 is reducible.

(2) *If at least one of*

$$\alpha, \alpha', \beta, \beta', \gamma - \beta - \beta', \gamma - \alpha - \beta', \gamma - \alpha' - \beta, \gamma - \alpha - \alpha'$$

is an integer, then the monodromy representation of the system E_3 is reducible.

Combining Theorem 5.2 and Theorem 5.3, we arrive at the necessary and sufficient conditions for the systems E_2 and E_3 being irreducible obtained by Bod [Bo] and Kato [Ka].

6. Appendix

Let $\mathcal{L}_z^\vee$ be the local system (locally constant sheaf) determined by the function $u(z; t)$ in Section 1: the sheaf consisting of the local solutions of $dL = L\omega$ for $\omega = du(t)/u(t)$.

If we assume the condition for the exponents λ_i ($1 \leq i \leq 5$)

$$(6.1) \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_{125}, \lambda_{345}, \lambda_{12345} \notin \mathbf{Z},$$

the twisted homology groups $H_j(T_z, \mathcal{L}_z^\vee)$ and $H_j^{\text{lf}}(T_z, \mathcal{L}_z^\vee)$ vanish for $j \neq 2$, $\dim H_2(T_z, \mathcal{L}_z^\vee) = \dim H_2^{\text{lf}}(T_z, \mathcal{L}_z^\vee) = 4$, and the natural map $\iota : H_2(T_z, \mathcal{L}_z^\vee) \rightarrow H_2^{\text{lf}}(T_z, \mathcal{L}_z^\vee)$ is an isomorphism (See [C]). The condition (6.1) guarantees that the loaded chains D_j ($1 \leq j \leq 4$) constitute a basis of $H_2^{\text{lf}}(T_z, \mathcal{L}_z^\vee)$, because the determinant of the intersection matrix for the loaded cycles ([KY][MY1][MY2]) turns out to be

$$\begin{aligned} & \det(D_i \bullet D_j) \\ &= \det \begin{pmatrix} \frac{-d_{12} - d_{34}}{d_1 d_2} \frac{-d_{34}}{d_3 d_4}, & \frac{e_2}{d_2} \frac{e_4}{d_4}, & \frac{e_2}{d_2} \frac{-d_{34}}{d_3 d_4}, & \frac{e_4}{d_4} \frac{-d_{12}}{d_1 d_2} \\ \frac{e_2}{d_2} \frac{e_4}{d_4}, & \frac{d_{123}}{d_1 d_2 d_3}, & \frac{e_4}{d_4} \frac{-d_{25}}{d_2 d_5}, & \frac{e_2}{d_2} \frac{-d_{45}}{d_4 d_5} \\ \frac{e_2}{d_2} \frac{-d_{34}}{d_3 d_4}, & \frac{e_4}{d_4} \frac{-d_{25}}{d_2 d_5}, & \frac{-d_{25}}{d_2 d_5} \frac{-d_{34}}{d_3 d_4}, & \frac{e_2}{d_2} \frac{e_4}{d_4} \\ \frac{e_4}{d_4} \frac{-d_{12}}{d_1 d_2}, & \frac{e_2}{d_2} \frac{-d_{45}}{d_4 d_5}, & \frac{e_2}{d_2} \frac{e_4}{d_4}, & \frac{-d_{12}}{d_1 d_2} \frac{-d_{45}}{d_4 d_5} \end{pmatrix} \\ &= \frac{\langle e_{125} \rangle \langle e_{345} \rangle \langle e_{12345} \rangle}{\langle e_1 \rangle^2 \langle e_2 \rangle^2 \langle e_3 \rangle^2 \langle e_4 \rangle^2 \langle e_5 \rangle^3}, \end{aligned}$$

where $d_{ij\dots k} = e_{ij\dots k} - 1$ and $\langle A \rangle = A - A^{-1}$. Therefore, our representation ρ can be considered as the irreducible representation on $H_2(T_z, \mathcal{L}_z^\vee)$ or $H_2^{\text{lf}}(T_z, \mathcal{L}_z^\vee)$ if (6.1) is satisfied.

Acknowledgement

The author is grateful to Professors Kyoichi Takano, Takeshi Sasaki, Masaaki Yoshida and Masatoshi Noumi for valuable comments, and also to the referee for a careful reading of the manuscript.

References

- [Ao1] Aomoto, K., On vanishing of cohomology attached to certain many valued meromorphic functions, J. Math. Soc. Japan, **27** (1975), 248–255.

- [Ao2] Aomoto, K., On the structure of integrals of power product of linear functions, Sci. Papers College Gen. Ed. Univ. Tokyo, **27** (1977), 49–61.
- [AK] Appell, P. and Kempé de Fériet, J., *Fonctions hypergéométriques et hypersphériques; Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.
- [Ba] Bailey, W. N., *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 32, Stechert-Hafner, Inc., New York, 1964.
- [Be] Beukers, F., Irreducibility of A -hypergeometric systems, Indag. Math. (N.S.), **21** (2011), 30–39.
- [Bo] Bod, E., Algebraicity of the Appell-Lauricella and Horn hypergeometric functions, J. Differential Equations, **252** (2012), 541–566.
- [C] Cho, K., A generalization of Kita and Noumi's vanishing theorems of cohomology groups of local system, Nagoya Math. J., **147** (1997), 63–69.
- [E] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., *Higher Transcendental Functions*, vol. 1, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.
- [HK] Hattori, A. and Kimura, T., On the Euler integral representations of hypergeometric functions in several variables, J. Math. Soc. Japan, **26** (1974), 1–16.
- [Ka] Kato, M., Connection formulas and irreducibility conditions for Appell's F_2 , Kyushu J. Math., **66** (2012), 325–363.
- [Kim] Kimura, T., *Hypergeometric Functions of Two Variables*, Lecture Notes, University of Tokyo, 1973.
- [Kit] Kita, M., On hypergeometric functions in several variables, I, New integral representations of Euler type, Japan. J. Math. (N.S.), **18** (1992), 25–74.
- [KY] Kita, M. and Yoshida, M., Intersection theory for twisted cycles, Math. Nachr., **166** (1994), 287–304.
- [Mi1] Mimachi, K., Homological representations of the Iwahori-Hecke algebra associated with a Selberg type integral, Int. Math. Res. Not., **2005** (2005), 2031–2057.
- [Mi2] Mimachi, K., Connection matrices associated with the generalized hypergeometric function ${}_3F_2$, Funkcial. Ekvac., **51** (2008), 107–133.
- [MS1] Mimachi, K. and Sasaki, T., Monodromy representations associated with the Gauss hypergeometric function using integrals of a multivalued function, Kyushu J. Math., **66** (2012), 35–60.
- [MS2] Mimachi, K. and Sasaki, T., Irreducibility and reducibility of Lauricella's system of differential equations E_D and the Jordan-Pochhammer differential equation E_{JP} , Kyushu J. Math., **66** (2012), 61–87.
- [MS3] Mimachi, K. and Sasaki, T., Monodromy representations associated with Appell's hypergeometric function F_1 using integrals of a multivalued function, Kyushu J. Math., **66** (2012), 89–114.
- [MS4] Mimachi, K. and Sasaki, T., Reducibility of the systems of differential equations satisfied by Appell's F_2 , F_3 and F_4 , Kyushu J. Math., **69** (2015), 429–435.
- [MY1] Mimachi, K. and Yoshida, M., Intersection numbers of twisted cycles and the correlation functions of the conformal field theory, Comm. Math. Phys., **234** (2003), 339–358.
- [MY2] Mimachi, K. and Yoshida, M., Intersection numbers of twisted cycles associated with the Selberg integral and an application to the conformal field theory, Comm. Math. Phys., **250** (2004), 23–45.
- [Na] Nakagiri, E., Monodromy representations of hypergeometric differential equations of two variables (in Japanese), Master's thesis of Kobe Univ., 1979.

- [ST] Sasaki, T. and Takano, K., On Appell's systems of hypergeometric differential equations, Ann. of Math. Stud., 100, Princeton Univ. Press, Princeton, N.J., 1981, pp. 345–356.

nuna adreso:

Department of Pure and Applied Mathematics
Graduate School of Information Science and
Technology

Osaka University

1-5 Yamadaoka, Suita-shi

Osaka 565-0871

Japan

E-mail: mimachi@math.sci.osaka-u.ac.jp

(Ricevita la 11-an de aprilo, 2015)

(Reviziita la 6-an de decembro, 2015)