Analytic and Summable Solutions of Inhomogeneous Moment Partial Differential Equations

By

Sławomir MICHALIK

(Cardinal Stefan Wyszyński University, Poland)

Abstract. We study the Cauchy problem for a general inhomogeneous linear moment partial differential equation of two complex variables with constant coefficients, where the inhomogeneity is given by the formal power series. We state sufficient conditions for the convergence, analytic continuation and summability of formal power series solutions in terms of properties of the inhomogeneity. We consider both the summability in one variable t (with coefficients belonging to some Banach space of Gevrey series with respect to the second variable z) and the summability in two variables (t, z) .

Key Words and Phrases. Linear PDEs with constant coefficients, Formal power series, Moment functions, Moment-PDEs, Gevrey order, Borel summability, Multi-summability.

2010 *Mathematics Subject Classification Numbers.* 35C10, 35C15, 35E15, 40G10.

1. Introduction

The formal m -moment differentiation $\partial_{m,z}$ is a linear operator on the space of formal power series defined by

$$\partial_{m,z} \left(\sum_{j=0}^{\infty} \frac{u_j z^j}{m(j)} \right) := \sum_{j=0}^{\infty} \frac{u_{j+1} z^j}{m(j)},$$

where $m(u)$ is a moment function constructed in the Balser theory of moment summability [1, Section 5.5]. In the special case $m(u) = \Gamma(1+u)$, the operator $\partial_{m,z}$ coincides with the usual differentiation ∂_z . More generally, for $p \in \mathbb{N}$ and $m(u) = \Gamma(1+u/p)$ the operator $\partial_{m,z}$ is closely related to the $1/p$ -fractional differentiation $\partial_z^{1/p}$.

The concept of moment differentiation was introduced by Balser and Yoshino [6], who studied the Gevrey order of formal solutions of inhomogeneous moment partial differential equations with constant coefficients.

The theory of moment partial differential equations was developed by the author [10, 11]. In [10] formal, analytic and summable solutions of homogeneous moment partial differential equations were studied. In the same paper

the author also calculated normalised formal solutions in the inhomogeneous case and he determined their Gevrey orders. Next, in [11] the author extended the results of [10] to homogeneous equations with divergent Cauchy data. Moreover, Lastra, Malek and Sanz [7] generalised the results of [10] about homogeneous moment equations to the case when moment functions are determined by given strongly regular moment sequences.

The present paper is a natural generalisation of [10, 11] to the inhomogeneous case with the inhomogeneity given by the formal power series. More precisely, we consider the initial value problem for a general inhomogeneous linear moment partial differential equation of two complex variables (t, z) with constant coefficients

$$(1) \quad P(\partial_{m_1, t}, \partial_{m_2, z})\hat{u} = \hat{f}(t, z), \quad \partial_{m_1, t}^j \hat{u}(0, z) = 0 \quad \text{for } j = 0, \dots, n-1,$$

where $P(\lambda, \zeta)$ is a polynomial of two variables of degree n with respect to λ and the inhomogeneity $\hat{f}(t, z)$ is a formal power series in both variables of Gevrey order $(\tilde{s}_1, \tilde{s}_2) \in \mathbf{R}^2$, and m_1, m_2 are moment functions of orders s_1, s_2 respectively.

Similarly to [11], we consider the wider class of moment functions than in [6] or [10], which creates the group with respect to multiplication. This extension allows us to study some integro-differential operators as moment differential operators.

As in the previous papers [10, 11] we use the moment pseudodifferential operators $\lambda_i(\partial_{m_2, z})$ to factorise the operator P as

$$P(\partial_{m_1, t}, \partial_{m_2, z}) = P_0(\partial_{m_2, z})(\partial_{m_1, t} - \lambda_1(\partial_{m_2, z}))^{n_1} \dots (\partial_{m_1, t} - \lambda_l(\partial_{m_2, z}))^{n_l},$$

where $P_0(\zeta)$ is a polynomial and $\lambda_1(\zeta), \dots, \lambda_l(\zeta)$ are the roots of the characteristic equation $P(\lambda, \zeta) = 0$ of (1) with multiplicities $n_1, \dots, n_l$ ($n_1 + \dots + n_l = n$) respectively.

In general, a formal solution of (1) is not unique, but it is uniquely determined by every formal power series $\hat{g}$ satisfying $P_0(\partial_{m_2, z})\hat{g} = \hat{f}$, since there is exactly one formal solution of (1) satisfying also the moment pseudodifferential equation

$$(\partial_{m_1, t} - \lambda_1(\partial_{m_2, z}))^{n_1} \dots (\partial_{m_1, t} - \lambda_l(\partial_{m_2, z}))^{n_l} \hat{u} = \hat{g}.$$

If $\hat{u}$ is a formal solution of (1) determined by $\hat{g}$ then $\hat{u} = \sum_{\alpha=1}^l \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$, where $\hat{u}_{\alpha\beta}$ satisfies the simple moment pseudodifferential equation

$$(2) \quad (\partial_{m_1, t} - \lambda_\alpha(\partial_{m_2, z}))^\beta \hat{u}_{\alpha\beta} = \hat{g}_{\alpha\beta}, \quad \partial_{m_1, t}^j \hat{u}_{\alpha\beta}(0, z) = 0 \quad \text{for } j = 0, \dots, \beta-1,$$

for some formal series $\hat{g}_{\alpha\beta}$ constructed from $\hat{g}$.

In this way we reduce the problem (1) to (2). We immediately calculate the Gevrey order of $\hat{u}_{\alpha\beta}$, which depends on a pole order of $\lambda_\alpha(\zeta)$, the Gevrey order of $\hat{g}$ and orders of moment functions m_1 and m_2 (Theorem 1).

If the Gevrey order of $\hat{u}_{\alpha\beta}$ is equal to zero, we study the analytic continuation properties of $u_{\alpha\beta}$. To this end we observe that by the general theory of moment summability, without loss of generality we may assume that $m_1(u) = \Gamma(1 + s_1 u)$ and $m_2(u) = \Gamma(1 + s_2 u)$. In this case we construct the integral representation of $u_{\alpha\beta}$, which allows us to find the connection between analytic continuation properties of solution $u_{\alpha\beta}$ and inhomogeneity $g_{\alpha\beta}$ (Theorem 2).

If the Gevrey order of $\hat{u}_{\alpha\beta}$ is greater than zero, then applying appropriate moment Borel transforms $\mathcal{B}_{m'_1, t}$ and $\mathcal{B}_{m'_2, z}$ to (2) we transform the formal solution $\hat{u}_{\alpha\beta}$ of (2) with divergent inhomogeneity $\hat{g}_{\alpha\beta}$ into the analytic solution $v_{\alpha\beta} = \mathcal{B}_{m'_1, t} \mathcal{B}_{m'_2, z} \hat{u}_{\alpha\beta}$ of the equation

$$(\partial_{m_1 m'_1, t} - \lambda_\alpha(\partial_{m_2 m'_2, z}))^\beta v_{\alpha\beta} = \mathcal{B}_{m'_1, t} \mathcal{B}_{m'_2, z} \hat{g}_{\alpha\beta}$$

with the convergent inhomogeneity $\mathcal{B}_{m'_1, t} \mathcal{B}_{m'_2, z} \hat{g}_{\alpha\beta}$. In this way we obtain the sufficient conditions for the summability of $\hat{u}_{\alpha\beta}$ (both in one variable t and in two variables (t, z)) in terms of the analytic continuation properties of the inhomogeneity (Theorem 3).

Finally, returning to the general equation (1), we get the sufficient condition for the multisummability of $\hat{u}$ in terms of the inhomogeneity $\hat{g}$ (Theorem 5).

In the last section we define the Newton polygon for moment partial differential operators with constant coefficients and we use it to describe the multisummable solutions of (1) in the case when inhomogeneity f is analytic in some complex neighbourhood of the origin.

Since in the special case $m_1(u) = m_2(u) = \Gamma(1 + u)$, (1) is the Cauchy problem for an inhomogeneous linear partial differential equation with constant coefficients and with a divergent inhomogeneity, the paper provides new sufficient conditions for the summability of formal solutions of such initial value problem. In this way the results of this paper extends our knowledge about summability of formal solutions of inhomogeneous linear partial differential equations, which was studied earlier by such authors as Balser [3], Balser and Loday-Richaud [5], Balser, Duval and Malek [4], Michalik [8, 9] and Tahara and Yamazawa [13].

2. Notation

Throughout this paper we use the following notation. The complex disc in $\mathbb{C}^n$ with centre at the origin and radius $r > 0$ is denoted by $D_r^n :=$

$\{z \in \mathbf{C}^n : |z| < r\}$. To simplify notation, we write D_r instead of D_r^1 . If the radius r is not essential, then we denote it briefly by D^n (resp. D).

A sector in a direction $d \in \mathbf{R}$ with an opening $\varepsilon > 0$ in the universal covering space $\mathbf{C} \setminus \{0\}$ of $\mathbf{C} \setminus \{0\}$ is defined by

$$S_d(\varepsilon) := \{z \in \widetilde{\mathbf{C} \setminus \{0\}} : z = re^{i\theta}, d - \varepsilon/2 < \theta < d + \varepsilon/2, r > 0\}.$$

Moreover, if the value of opening angle ε is not essential, then we denote it briefly by S_d .

Analogously, by a disc-sector in a direction $d \in \mathbf{R}$ with an opening $\varepsilon > 0$ and radius $r > 0$ we mean a domain $\hat{S}_d(\varepsilon; r) := S_d(\varepsilon) \cup D_r$. If the values of ε and r are not essential, we write it $\hat{S}_d$ for brevity (i.e. $\hat{S}_d = S_d \cup D$).

By $\mathcal{O}(G)$ we understand the space of holomorphic functions on a domain $G \subseteq \mathbf{C}^n$. Analogously, the space of analytic functions of the variables $z_1^{1/\kappa_1}, \dots, z_n^{1/\kappa_n}$ ($(\kappa_1, \dots, \kappa_n) \in \mathbf{N}^n$) on G is denoted by $\mathcal{O}_{1/\kappa_1, \dots, 1/\kappa_n}(G)$. More generally, if $\mathbf{E}$ denotes a Banach space with a norm $\|\cdot\|_{\mathbf{E}}$, then by $\mathcal{O}(G, \mathbf{E})$ (resp. $\mathcal{O}_{1/\kappa_1, \dots, 1/\kappa_n}(G, \mathbf{E})$) we shall denote the set of all $\mathbf{E}$ -valued holomorphic functions (resp. holomorphic functions of the variables $z_1^{1/\kappa_1}, \dots, z_n^{1/\kappa_n}$) on a domain $G \subseteq \mathbf{C}^n$. For more information about functions with values in Banach spaces we refer the reader to [1, Appendix B]. In the paper, as a Banach space $\mathbf{E}$ we will take the space of complex numbers $\mathbf{C}$ (we abbreviate $\mathcal{O}(G, \mathbf{C})$ to $\mathcal{O}(G)$ and $\mathcal{O}_{1/\kappa_1, \dots, 1/\kappa_n}(G, \mathbf{C})$ to $\mathcal{O}_{1/\kappa_1, \dots, 1/\kappa_n}(G)$) or the space of Gevrey series $G_{s, 1/\kappa}(r)$ (see Definition 7).

Definition 1. A function $u \in \mathcal{O}_{1/\kappa}(\hat{S}_d(\varepsilon; r), \mathbf{E})$ is of *exponential growth of order at most $K \in \mathbf{R}$ as $x \rightarrow \infty$ in $\hat{S}_d(\varepsilon; r)$* if for any $\tilde{\varepsilon} \in (0, \varepsilon)$ and $\tilde{r} \in (0, r)$ there exist $A, B < \infty$ such that

$$\|u(x)\|_{\mathbf{E}} < Ae^{B|x|^K} \quad \text{for every } x \in \hat{S}_d(\tilde{\varepsilon}; \tilde{r}).$$

The space of such functions is denoted by $\mathcal{O}_{1/\kappa}^K(\hat{S}_d(\varepsilon; r), \mathbf{E})$.

Analogously, a function $u \in \mathcal{O}_{1/\kappa_1, 1/\kappa_2}(\hat{S}_{d_1}(\varepsilon_1; r_1) \times \hat{S}_{d_2}(\varepsilon_2; r_2))$ is of *exponential growth of order at most $(K_1, K_2) \in \mathbf{R}^2$ as $(t, z) \rightarrow \infty$ in $\hat{S}_{d_1}(\varepsilon_1; r_1) \times \hat{S}_{d_2}(\varepsilon_2; r_2)$* if for any $\tilde{\varepsilon}_i \in (0, \varepsilon_i)$ and any $\tilde{r}_i \in (0, r_i)$ ($i = 1, 2$) there exist $A, B_1, B_2 < \infty$ such that

$$|u(t, z)| < Ae^{B_1|t|^{K_1}} e^{B_2|z|^{K_2}} \quad \text{for every } (t, z) \in \hat{S}_{d_1}(\tilde{\varepsilon}_1; \tilde{r}_1) \times \hat{S}_{d_2}(\tilde{\varepsilon}_2; \tilde{r}_2).$$

The space of such functions is denoted by $\mathcal{O}_{1/\kappa_1, 1/\kappa_2}^{K_1, K_2}(\hat{S}_{d_1}(\varepsilon_1; r_1) \times \hat{S}_{d_2}(\varepsilon_2; r_2))$.

The space of formal power series $\hat{u}(x) = \sum_{j=0}^{\infty} u_j x^{j/\kappa}$ with $u_j \in \mathbf{E}$ is denoted by $\mathbf{E}[[x^{1/\kappa}]]$. Analogously, the space of formal power series $\hat{u}(t, z) = \sum_{j, n=0}^{\infty} u_{jn} t^{j/\kappa_1} z^{n/\kappa_2}$ with $u_{jn} \in \mathbf{E}$ is denoted by $\mathbf{E}[[t^{1/\kappa_1}, z^{1/\kappa_2}]]$.

We use the “hat” notation ($\hat{u}$, $\hat{v}$, $\hat{\phi}$, $\hat{f}$, $\hat{g}$) to denote the formal power series. If the formal power series $\hat{u}$ (resp. $\hat{v}$, $\hat{\phi}$, $\hat{f}$, $\hat{g}$) is convergent, we denote its sum by u (resp. v , ϕ , f , g).

3. Moment functions

In this section we recall the notion of moment methods introduced by Balser [1].

Definition 2 (see [1, Section 5.5]). A pair of functions e_m and E_m is said to be *kernel functions of order k* ($k > 1/2$) if they have the following properties:

1. $e_m \in \mathcal{O}(S_0(\pi/k))$, $e_m(z)/z$ is integrable at the origin, $e_m(x) \in \mathbf{R}_+$ for $x \in \mathbf{R}_+$ and e_m is exponentially flat of order k as $z \rightarrow \infty$ in $S_0(\pi/k)$ (i.e. $\forall \varepsilon > 0 \exists A, B > 0$ such that $|e_m(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S_0(\pi/k - \varepsilon)$).
2. $E_m \in \mathcal{O}^k(\mathbf{C})$ and $E_m(1/z)/z$ is integrable at the origin in $S_\pi(2\pi - \pi/k)$.
3. The connection between e_m and E_m is given by the *corresponding moment function m of order $1/k$* as follows. The function m is defined by the Mellin transform of e_m

$$(3) \quad m(u) := \int_0^\infty x^{u-1} e_m(x) dx \quad \text{for } \operatorname{Re} u \geq 0$$

and the kernel function E_m has the power series expansion

$$(4) \quad E_m(z) = \sum_{n=0}^{\infty} \frac{z^n}{m(n)} \quad \text{for } z \in \mathbf{C}.$$

Remark 1. Observe that by the inverse Mellin transform and by (4), the moment function m uniquely determines the kernel functions e_m and E_m .

In case $k \leq 1/2$ the set $S_\pi(2\pi - \pi/k)$ is not defined, so the second property in Definition 2 can not be satisfied. It means that we must define the kernel functions of order $k \leq 1/2$ and the corresponding moment functions in another way.

Definition 3 (see [1, Section 5.6]). A function e_m is called a *kernel function of order $k > 0$* if we can find a pair of kernel functions $e_{\bar{m}}$ and $E_{\bar{m}}$ of order $pk > 1/2$ (for some $p \in \mathbf{N}$) so that

$$e_m(z) = e_{\bar{m}}(z^{1/p})/p \quad \text{for } z \in S_0(\pi/k).$$

For a given kernel function e_m of order $k > 0$ we define the *corresponding moment function m of order $1/k > 0$* by (3) and the *kernel function E_m of order $k > 0$* by (4).

Remark 2. Observe that by Definitions 2 and 3 we have

$$m(u) = \tilde{m}(pu) \quad \text{and} \quad E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} = \sum_{j=0}^{\infty} \frac{z^j}{\tilde{m}(jp)}.$$

As in [11], we extend the notion of moment functions to real orders.

Definition 4. We say that m is a *moment function of order* $1/k < 0$ if $1/m$ is a moment function of order $-1/k > 0$.

We say that m is a *moment function of order 0* if there exist moment functions m_1 and m_2 of the same order $1/k > 0$ such that $m = m_1/m_2$.

By Definition 4 and by [1, Theorems 31 and 32] we have

Proposition 1. Let m_1, m_2 be moment functions of orders $s_1, s_2 \in \mathbf{R}$ respectively. Then

- $m_1 m_2$ is a moment function of order $s_1 + s_2$,
- m_1/m_2 is a moment function of order $s_1 - s_2$.

Example 1. For any $a \geq 0$, $b \geq 1$ and $k > 0$ we can construct the following examples of kernel functions e_m and E_m of orders $k > 0$ with the corresponding moment function m of order $1/k$ satisfying Definition 2 or 3:

- $e_m(z) = akz^{bk}e^{-z^k}$,
- $m(u) = a\Gamma(b + u/k)$,
- $E_m(z) = \frac{1}{a} \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(b + j/k)}$.

In particular for $a = b = 1$ we get the kernel functions and the corresponding moment function, which are used in the classical theory of k -summability.

- $e_m(z) = kz^k e^{-z^k}$,
- $m(u) = \Gamma(1 + u/k)$,
- $E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1 + j/k)} =: \mathbf{E}_{1/k}(z)$, where $\mathbf{E}_{1/k}$ is the Mittag-Leffler function of index $1/k$.

Example 2. For any $s \in \mathbf{R}$ we will denote by Γ_s the function

$$\Gamma_s(u) := \begin{cases} \Gamma(1 + su) & \text{for } s \geq 0 \\ 1/\Gamma(1 - su) & \text{for } s < 0. \end{cases}$$

Observe that by Example 1 and Definition 4, Γ_s is an example of a moment function of order $s \in \mathbf{R}$.

The moment functions Γ_s will be extensively used in the paper, since every moment function m of order s has the same growth as Γ_s . Precisely speaking, we have

Proposition 2 (see [1, Section 5.5]). *If m is a moment function of order $s \in \mathbf{R}$ then there exist constants $c, C > 0$ such that*

$$c^n \Gamma_s(n) \leq m(n) \leq C^n \Gamma_s(n) \quad \text{for every } n \in \mathbf{N}.$$

4. Moment Borel transforms, Gevrey order and Borel summability

We use the moment function to define moment Borel transforms, the Gevrey order and the Borel summability. We first introduce

Definition 5. Let $\kappa \in \mathbf{N}$ and m be a moment function. Then the linear operator $\mathcal{B}_{m, x^{1/\kappa}} : E[[x^{1/\kappa}]] \rightarrow E[[x^{1/\kappa}]]$ defined by

$$\mathcal{B}_{m, x^{1/\kappa}} \left(\sum_{j=0}^{\infty} u_j x^{j/\kappa} \right) := \sum_{j=0}^{\infty} \frac{u_j}{m(j/\kappa)} x^{j/\kappa}$$

is called an m -moment Borel transform with respect to $x^{1/\kappa}$.

We define the Gevrey order of formal power series as follows

Definition 6. Let $\kappa \in \mathbf{N}$ and $s \in \mathbf{R}$. Then $\hat{u} \in E[[x^{1/\kappa}]]$ is called a *formal power series of Gevrey order s* if there exists a disc $D \subset \mathbf{C}$ with centre at the origin such that $\mathcal{B}_{\Gamma_s, x^{1/\kappa}} \hat{u} \in \mathcal{O}_{1/\kappa}(D, \mathbf{E})$. The space of formal power series of Gevrey order s is denoted by $E[[x^{1/\kappa}]]_s$.

Analogously, if $\kappa_1, \kappa_2 \in \mathbf{N}$ and $s_1, s_2 \in \mathbf{R}$ then $\hat{u} \in E[[t^{1/\kappa_1}, z^{1/\kappa_2}]]$ is called a *formal power series of Gevrey order (s_1, s_2)* if there exists a disc $D^2 \subset \mathbf{C}^2$ with centre at the origin such that $\mathcal{B}_{\Gamma_{s_1}, t^{1/\kappa_1}} \mathcal{B}_{\Gamma_{s_2}, z^{1/\kappa_2}} \hat{u} \in \mathcal{O}_{1/\kappa_1, 1/\kappa_2}(D^2, \mathbf{E})$. The space of formal power series of Gevrey order (s_1, s_2) is denoted by $E[[t^{1/\kappa_1}, z^{1/\kappa_2}]]_{s_1, s_2}$.

Remark 3. By Proposition 2, we may replace Γ_s (resp. Γ_{s_1} and Γ_{s_2}) in Definition 6 by any moment function m of order s (resp. by any moment functions m_1 and m_2 of orders s_1 and s_2).

Remark 4. If $\hat{u} \in E[[x^{1/\kappa}]]_s$ and $s \leq 0$ then the formal series $\hat{u}$ is convergent, so its sum u is well defined. Moreover, $\hat{u} \in E[[x^{1/\kappa}]]_0 \Leftrightarrow u \in \mathcal{O}_{1/\kappa}(D, \mathbf{E})$ and $\hat{u} \in E[[x^{1/\kappa}]]_s \Leftrightarrow u \in \mathcal{O}_{1/\kappa}^{-1/s}(\mathbf{C}, \mathbf{E})$ for $s < 0$.

By Definitions 5 and 6 we obtain

Proposition 3. *For every $\hat{u} \in E[[x^{1/\kappa}]]$ the following properties of moment Borel transforms are satisfied:*

- $\mathcal{B}_{m_1, x^{1/\kappa}} \mathcal{B}_{m_2, x^{1/\kappa}} \hat{u} = \mathcal{B}_{m_1 m_2, x^{1/\kappa}} \hat{u}$ for every moment functions m_1 and m_2 .
- $\mathcal{B}_{m, x^{1/\kappa}} \mathcal{B}_{1/m, x^{1/\kappa}} \hat{u} = \mathcal{B}_{1/m, x^{1/\kappa}} \mathcal{B}_{m, x^{1/\kappa}} \hat{u} = \mathcal{B}_{1, x^{1/\kappa}} \hat{u} = \hat{u}$ for every moment function m .
- $\hat{u} \in E[[x^{1/\kappa}]]_{s_1} \Leftrightarrow \mathcal{B}_{m, x^{1/\kappa}} \hat{u} \in E[[x^{1/\kappa}]]_{s_1 - s}$ for every $s, s_1 \in \mathbf{R}$ and for every moment function m of order s .

As a Banach space $\mathbf{E}$ we will take the space of complex numbers $\mathbf{C}$ or the space of Gevrey series $G_{s,1/\kappa}(r)$ defined below.

Definition 7. Fix $\kappa \in \mathbf{N}$, $r > 0$ and $s \in \mathbf{R}$. By $G_{s,1/\kappa}(r)$ we denote a Banach space of Gevrey series

$$G_{s,1/\kappa}(r) := \{\hat{\phi} \in \mathbf{C}[[z^{1/\kappa}]]_s : \mathcal{B}_{\Gamma_s, z^{1/\kappa}} \hat{\phi} \in \mathcal{O}_{1/\kappa}(D_r) \cap C(\overline{D_r})\}$$

equipped with the norm

$$\|\hat{\phi}\|_{G_{s,1/\kappa}(r)} := \max_{|z| \leq r} |\mathcal{B}_{\Gamma_s, z^{1/\kappa}} \hat{\phi}(z)|.$$

We also set $G_{s,1/\kappa} := \lim_{r \rightarrow 0} G_{s,1/\kappa}(r)$. Analogously, we define $\mathcal{O}_{1/\kappa}(G, G_{s,1/\kappa}) := \lim_{r > 0} \mathcal{O}_{1/\kappa}(G, G_{s,1/\kappa}(r))$ and $\mathcal{O}_{1/\kappa}^K(G, G_{s,1/\kappa}) := \lim_{r > 0} \mathcal{O}_{1/\kappa}^K(G, G_{s,1/\kappa}(r))$.

Moreover, we denote by $G_{s_2,1/\kappa}[[t]]_{s_1}$ the space of formal power series $\hat{u}(t, z) = \sum_{j=0}^{\infty} \hat{u}_j(z) t^j$ of Gevrey order s_1 with coefficients $\hat{u}_j(z) \in G_{s_2,1/\kappa}$.

By Definitions 6, 7, Remark 3 and Proposition 3 we conclude

Proposition 4. For every $\kappa \in \mathbf{N}$, $s, \bar{s} \in \mathbf{R}$ (resp. $s_1, s_2, \bar{s} \in \mathbf{R}$) and for every moment function m of order $\bar{s}$ the following conditions are equivalent:

- $\hat{u} \in \mathbf{C}[[x^{1/\kappa}]]_s$ (resp. $\hat{u} \in \mathbf{C}[[t, z^{1/\kappa}]]_{s_1, s_2}$),
- $\mathcal{B}_{\Gamma_s, x^{1/\kappa}} \hat{u} \in \mathcal{O}_{1/\kappa}(D)$ (resp. $\mathcal{B}_{\Gamma_{s_1}, t} \mathcal{B}_{\Gamma_{s_2}, z^{1/\kappa}} \hat{u} \in \mathcal{O}_{1,1/\kappa}(D^2)$),
- there exists $r > 0$ such that $\hat{u} \in G_{s,1/\kappa}(r)$ (resp. $\hat{u} \in G_{s_2,1/\kappa}(r)[[t]]_{s_1}$),
- $\hat{u} \in G_{s,1/\kappa}$ (resp. $\hat{u} \in G_{s_2,1/\kappa}[[t]]_{s_1}$),
- $\mathcal{B}_{m, x^{1/\kappa}} \hat{u} \in \mathbf{C}[[x^{1/\kappa}]]_{s-\bar{s}}$ (resp. $\mathcal{B}_{m, z^{1/\kappa}} \hat{u} \in G_{s_2-\bar{s}, 1/\kappa}[[t]]_{s_1}$).

Now we are ready to define the summability of formal power series in one variable (see Balser [1])

Definition 8. Let $\kappa \in \mathbf{N}$, $K > 0$ and $d \in \mathbf{R}$. Then $\hat{u} \in \mathbf{E}[[x^{1/\kappa}]]$ is called K -summable in a direction d if there exists a disc-sector $\hat{S}_d$ in a direction d such that $\mathcal{B}_{\Gamma_{1/K}, x^{1/\kappa}} \hat{u} \in \mathcal{O}_{1/K}^K(\hat{S}_d, \mathbf{E})$.

Remark 5. By Definitions 7 and 8, $\hat{u} \in G_{s,1/\kappa}[[t]]$ is K -summable in a direction d if and only if $\mathcal{B}_{\Gamma_{1/K}, t} \mathcal{B}_{\Gamma_s, z^{1/\kappa}} \hat{u} \in \mathcal{O}_{1,1/\kappa}^K(\hat{S}_d \times D)$.

We can now define the multisummability in a multidirection.

Definition 9. Let $k_1 > \dots > k_n > 0$. We say that a real vector $(d_1, \dots, d_n) \in \mathbf{R}^n$ is an *admissible multidirection* if

$$|d_j - d_{j-1}| \leq \pi(1/k_j - 1/k_{j-1})/2 \quad \text{for } j = 2, \dots, n.$$

Let $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{R}_+^n$ and let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbf{R}^n$ be an admissible multidirection. We say that a formal power series $\hat{u} \in E[[x]]$ is $\mathbf{k}$ -multisummable in the multidirection $\mathbf{d}$ if $\hat{u} = \hat{u}_1 + \dots + \hat{u}_n$, where $\hat{u}_j \in E[[x]]$ is K_j -summable in the direction d_j for $j = 1, \dots, n$.

Following Sanz [12] we extend the notion of summability to two variables

Definition 10. For $\kappa_1, \kappa_2 \in \mathbf{N}$, $K_1, K_2 > 0$ and $d_1, d_2 \in \mathbf{R}$ the formal power series $\hat{u} \in C[[t^{1/\kappa_1}, z^{1/\kappa_2}]]$ is called (K_1, K_2) -summable in the direction (d_1, d_2) if there exist disc-sectors $\hat{S}_{d_1}$ and $\hat{S}_{d_2}$ such that $\mathcal{B}_{\Gamma_{1/K_1}, t^{1/\kappa_1}} \mathcal{B}_{\Gamma_{1/K_2}, z^{1/\kappa_2}} \hat{u} \in \mathcal{O}_{1/\kappa_1, 1/\kappa_2}^{K_1, K_2}(\hat{S}_{d_1} \times \hat{S}_{d_2})$.

Remark 6. By the general theory of moment summability (see [1, Section 6.5 and Theorem 38]), we may replace $\Gamma_{1/K}$ in Definition 8 (resp. Γ_{1/K_1} and Γ_{1/K_2} in Definition 10) by any moment function m of order $1/K$ (resp. by any moment functions m_1 of order $1/K_1$ and m_2 of order $1/K_2$).

Using moment functions of order 0 we can formulate the above remark as

Proposition 5. We assume that $\kappa_1, \kappa_2 \in \mathbf{N}$, $K_1, K_2 > 0$, $d_1, d_2 \in \mathbf{R}$, $\bar{m}_1, \bar{m}_2$ are moment functions of order 0, $\hat{u} \in C[[t^{1/\kappa_1}, z^{1/\kappa_2}]]$ and $v \in \mathcal{O}_{1/\kappa_1, 1/\kappa_2}(D^2)$. Then

- $\hat{u}$ is K_1 -summable in the direction d_1 (with respect to t^{1/κ_1}) if and only if $\mathcal{B}_{\bar{m}_1, t^{1/\kappa_1}} \mathcal{B}_{\bar{m}_2, z^{1/\kappa_2}} \hat{u}$ is K_1 -summable in the direction d_1 (with respect to t^{1/κ_1}).
- $\hat{u}$ is (K_1, K_2) -summable in the direction (d_1, d_2) if and only if $\mathcal{B}_{\bar{m}_1, t^{1/\kappa_1}} \mathcal{B}_{\bar{m}_2, z^{1/\kappa_2}} \hat{u}$ is (K_1, K_2) -summable in the direction (d_1, d_2) .
- $v \in \mathcal{O}_{1/\kappa_1, 1/\kappa_2}^{K_1}(\hat{S}_{d_1} \times D)$ if and only if $\mathcal{B}_{\bar{m}_1, t^{1/\kappa_1}} \mathcal{B}_{\bar{m}_2, z^{1/\kappa_2}} v \in \mathcal{O}_{1/\kappa_1, 1/\kappa_2}^{K_1}(\hat{S}_{d_1} \times D)$.
- $v \in \mathcal{O}_{1/\kappa_1, 1/\kappa_2}^{K_1, K_2}(\hat{S}_{d_1} \times \hat{S}_{d_2})$ if and only if $\mathcal{B}_{\bar{m}_1, t^{1/\kappa_1}} \mathcal{B}_{\bar{m}_2, z^{1/\kappa_2}} v \in \mathcal{O}_{1/\kappa_1, 1/\kappa_2}^{K_1, K_2}(\hat{S}_{d_1} \times \hat{S}_{d_2})$.

5. Moment operators

In this section we recall the notion of moment differential operators constructed by Balser and Yoshino [6] and the concept of moment pseudo-differential operators introduced in our previous papers [10, 11].

Definition 11. Let m be a moment function. Then the linear operator $\partial_{m,x} : E[[x]] \rightarrow E[[x]]$ defined by

$$\partial_{m,x} \left(\sum_{j=0}^{\infty} \frac{u_j}{m(j)} x^j \right) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{m(j)} x^j$$

is called the m -moment differential operator $\partial_{m,x}$.

More generally, if $\kappa \in \mathbb{N}$ then the linear operator $\partial_{m,x^{1/\kappa}} : \mathbf{E}[[x^{1/\kappa}]] \rightarrow \mathbf{E}[[x^{1/\kappa}]]$ defined by

$$\partial_{m,x^{1/\kappa}} \left(\sum_{j=0}^{\infty} \frac{u_j}{m(j/\kappa)} x^{j/\kappa} \right) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{m(j/\kappa)} x^{j/\kappa}$$

is called the *m-moment $1/\kappa$ -fractional differential operator* $\partial_{m,x^{1/\kappa}}$.

Moreover, the right-inversion operator $\partial_{m,x^{1/\kappa}}^{-1} : \mathbf{E}[[x^{1/\kappa}]] \rightarrow \mathbf{E}[[x^{1/\kappa}]]$ given by

$$\partial_{m,x^{1/\kappa}}^{-1} \left(\sum_{j=0}^{\infty} \frac{u_j}{m(j/\kappa)} x^{j/\kappa} \right) := \sum_{j=1}^{\infty} \frac{u_{j-1}}{m(j/\kappa)} x^{j/\kappa}$$

is called the *m-moment $1/\kappa$ -fractional integration operator* $\partial_{m,x^{1/\kappa}}^{-1}$.

Below we present most important examples of moment differential operators. Other examples, including also integro-differential operators, can be found in [11, Example 3].

Example 3. If $m(u) = \Gamma_1(u)$ then the operator $\partial_{m,x}$ coincides with the usual differentiation ∂_x . More generally, if $s > 0$ and $m(u) = \Gamma_s(u)$ then the operator $\partial_{m,x}$ satisfies

$$(\partial_{m,x} \hat{u})(x^s) = \partial_x^s (\hat{u}(x^s)),$$

where ∂_x^s denotes the Caputo fractional derivative of order s defined by

$$\partial_x^s \left(\sum_{j=0}^{\infty} \frac{u_j}{\Gamma_s(j)} x^{sj} \right) := \sum_{j=0}^{\infty} \frac{u_{j+1}}{\Gamma_s(j)} x^{sj}.$$

The moment differential operator $\partial_{m,z}$ is well-defined for every $\varphi \in \mathcal{O}(D)$. In addition, we have the following integral representation of $\partial_{m,z}^n \varphi$.

Proposition 6 (see [10, Proposition 3]). *Let $\varphi \in \mathcal{O}(D_r)$ and m be a moment function of order $1/k > 0$. Then for every $|z| < \varepsilon < r$ and $n \in \mathbb{N}$ we have*

$$\partial_{m,z}^n \varphi(z) = \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_0^{\infty(\theta)} \zeta^n E_m(z\zeta) \frac{e_m(w\zeta)}{w\zeta} d\zeta dw,$$

where $\theta \in (-\arg w - \pi/(2k), -\arg w + \pi/(2k))$.

Using the above formula, we have defined in [10, Definition 8] a moment pseudodifferential operator $\lambda(\partial_{m,z}) : \mathcal{O}(D) \rightarrow \mathcal{O}(D)$ as an operator satisfying

$$\lambda(\partial_{m,z}) E_m(\zeta z) := \lambda(\zeta) E_m(\zeta z) \quad \text{for } |\zeta| \geq r_0.$$

Namely, if $\lambda(\zeta)$ is an analytic function for $|\zeta| \geq r_0$ then $\lambda(\partial_{m,z})$ is defined by

$$\lambda(\partial_{m,z})\varphi(z) := \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \varphi(w) \int_{r_0 e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_m(\zeta z) \frac{e_m(\zeta w)}{\zeta w} d\zeta dw$$

for every $\varphi \in \mathcal{O}(D_r)$ and $|z| < \varepsilon < r$, where $\theta \in (-\arg w - \pi/(2k), -\arg w + \pi/(2k))$.

In [11] we have extended this definition to the case where $\lambda(\zeta)$ is an analytic function of the variable $\zeta = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbb{N}$ and $r_0 > 0$). Since $(\partial_{m,z}\varphi)(z^\kappa) = \partial_{\tilde{m},z}^\kappa(\varphi(z^\kappa))$ for every $\varphi \in \mathcal{O}(D)$, where $\tilde{m}(u) := m(u/\kappa)$ (see [10, Lemma 3]), the operator $\lambda(\partial_{m,z})$ should satisfy the formula

$$(5) \quad (\lambda(\partial_{m,z})\varphi)(z^\kappa) = \lambda(\partial_{\tilde{m},z}^\kappa)(\varphi(z^\kappa)) \quad \text{for every } \varphi \in \mathcal{O}_{1/\kappa}(D).$$

For this reason we have

Definition 12 ([11, Definition 13]). Let m be a moment function of order $1/k > 0$ and $\lambda(\zeta)$ be an analytic function of the variable $\zeta = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbb{N}$ and $r_0 > 0$) of polynomial growth at infinity. A *moment pseudodifferential operator* $\lambda(\partial_{m,z}) : \mathcal{O}_{1/\kappa}(D) \rightarrow \mathcal{O}_{1/\kappa}(D)$ (or, more generally, $\lambda(\partial_{m,z}) : E[[z^{1/\kappa}]]_0 \rightarrow E[[z^{1/\kappa}]]_0$) is defined by

$$(6) \quad \lambda(\partial_{m,z})\varphi(z) := \frac{1}{2\kappa\pi i} \oint_{|w|=\varepsilon}^\kappa \varphi(w) \int_{r_0 e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_{\tilde{m}}(\zeta^{1/\kappa} z^{1/\kappa}) \frac{e_m(\zeta w)}{\zeta w} d\zeta dw$$

for every $\varphi \in \mathcal{O}_{1/\kappa}(D_r)$ and $|z| < \varepsilon < r$, where $\tilde{m}(u) := m(u/\kappa)$, $E_{\tilde{m}}(\zeta^{1/\kappa} z^{1/\kappa}) = \sum_{n=0}^{\infty} \zeta^{n/\kappa} z^{n/\kappa} / \tilde{m}(n)$, $\theta \in (-\arg w - \pi/(2k), -\arg w + \pi/(2k))$ and $\oint_{|w|=\varepsilon}^\kappa$ means that we integrate κ times along the positively oriented circle of radius ε . Here the integration in the inner integral is taken over a ray $\{re^{i\theta} : r \geq r_0\}$.

Observe that

$$\begin{aligned} (\lambda(\partial_{m,z})\varphi)(z^\kappa) &= \frac{1}{2\kappa\pi i} \oint_{|w|=\varepsilon}^\kappa \varphi(w) \int_{r_0 e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_{\tilde{m}}(\zeta^{1/\kappa} z) \frac{e_m(\zeta w)}{\zeta w} d\zeta dw \\ &= \frac{1}{2\pi i} \oint_{|w^\kappa|=\varepsilon} \varphi(w^\kappa) \int_{r_0^{1/\kappa} e^{i\theta/\kappa}}^{\infty(\theta/\kappa)} \lambda(\zeta^\kappa) E_{\tilde{m}}(\zeta z) \frac{e_{\tilde{m}}(\zeta w)}{\zeta w} d\zeta dw \\ &= \lambda(\partial_{\tilde{m},z}^\kappa)(\varphi(z^\kappa)), \end{aligned}$$

so (5) holds for the operators $\lambda(\partial_{m,z})$ defined by (6).

Immediately by the definition, we obtain the following connection between the moment Borel transform and the moment differentiation.

Proposition 7. *Let m and m' be moment functions of positive orders. Then the operators $\mathcal{B}_{m',x}, \partial_{m,x} : \mathbf{E}[[x]] \rightarrow \mathbf{E}[[x]]$ satisfy the following commutation formulas for every $\hat{u} \in \mathbf{E}[[x]]$ and for $\bar{m} = mm'$:*

- i) $\mathcal{B}_{m',x} \partial_{m,x} \hat{u} = \partial_{\bar{m},x} \mathcal{B}_{m',x} \hat{u},$
- ii) $\mathcal{B}_{m',x} P(\partial_{m,x}) \hat{u} = P(\partial_{\bar{m},x}) \mathcal{B}_{m',x} \hat{u}$ for any polynomial P with constant coefficients.

The same commutation formula holds if we replace $P(\partial_{m,x})$ by $\lambda(\partial_{m,x})$. Namely, we have

Proposition 8 (see [11, Proposition 8]). *Let m and m' be moment functions of positive orders and $\lambda(\zeta)$ be an analytic function of the variable $\zeta = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbf{N}$ and $r_0 > 0$) of polynomial growth at infinity. Then the operators $\mathcal{B}_{m',x^{1/\kappa}}, \lambda(\partial_{m,x}) : \mathbf{E}[[x^{1/\kappa}]]_0 \rightarrow \mathbf{E}[[x^{1/\kappa}]]_0$ satisfy the commutation formula*

$$\mathcal{B}_{m',x^{1/\kappa}} \lambda(\partial_{m,x}) \hat{u} = \lambda(\partial_{\bar{m},x}) \mathcal{B}_{m',x^{1/\kappa}} \hat{u}$$

for every $\hat{u} \in \mathbf{E}[[x^{1/\kappa}]]_0$ and for $\bar{m} = mm'$.

Using Proposition 8 we are able to extend Definition 12 to the formal power series and to the moment functions of real orders.

Definition 13 ([11, Definition 14]). Let $s \in \mathbf{R}$, m be a moment function of order $\tilde{s} \in \mathbf{R}$ and $\lambda(\zeta)$ be an analytic function of the variable $\zeta = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ of polynomial growth at infinity. A moment pseudodifferential operator $\lambda(\partial_{m,z})$ for the formal power series $\hat{\phi} \in \mathbf{E}[[z^{1/\kappa}]]_s$ is defined by

$$(7) \quad \lambda(\partial_{m,z}) \hat{\phi}(z) := \mathcal{B}_{\Gamma_{\tilde{s}, z^{1/\kappa}}} \lambda(\partial_{\bar{m},z}) \mathcal{B}_{\Gamma_{\tilde{s}, z^{1/\kappa}}} \hat{\phi}(z),$$

where $\bar{m} := m\Gamma_{\tilde{s}}$, $\tilde{s} := \max\{s, 1 - \tilde{s}\}$ and the operator $\lambda(\partial_{\bar{m},z})$ is constructed in Definition 12.

Remark 7. Note that the choice of $\tilde{s} := \max\{s, 1 - \tilde{s}\}$ guarantees that $\mathcal{B}_{\Gamma_{\tilde{s}, z^{1/\kappa}}} \hat{\phi}(z) \in \mathbf{E}[[z^{1/\kappa}]]_0$ and that $\bar{m}$ is a moment function of positive order $\tilde{s} + \tilde{s} \geq 1$. Thanks to that, the operator $\lambda(\partial_{m,z})$ given by (7) is well-defined.

Definition 14 ([10, Definition 9]). We define a pole order $q \in \mathbf{Q}$ and a leading term $\lambda_0 \in \mathbf{C} \setminus \{0\}$ of $\lambda(\zeta)$ as the numbers satisfying the formula $\lim_{\zeta \rightarrow \infty} \lambda(\zeta)/\zeta^q = \lambda_0$. We write it also $\lambda(\zeta) \sim \lambda_0 \zeta^q$.

At the end of the section we recall the estimate given in [11]

Lemma 1 ([11, Lemma 2]). *Let $\hat{\phi} \in \mathbf{C}[[z^{1/\kappa}]]_s$, $s \leq 0$, m be a moment function of order $1/k > 0$ and $\lambda(\partial_{m,z})$ be a moment pseudodifferential oper-*

ator with $\lambda(\zeta) \sim \lambda_0 \zeta^q$ and $q \in \mathcal{Q}$. Then there exist $r > 0$ and $A, B < \infty$ such that

$$\sup_{|z| < r} |\lambda^j(\partial_{m,z})\varphi(z)| \leq |\lambda_0|^j AB^j \Gamma_{q^+(s+1/k)}(j) \quad \text{for } j = 0, 1, \dots,$$

where $q^+ := \max\{0, q\}$.

6. Formal solutions and Gevrey estimates

In a similar way to [10] we generalise the results of [11] to inhomogeneous case. Without loss of generality we may assume that the initial data vanish. So it is sufficient to consider the following Cauchy problem for general inhomogeneous linear moment partial differential equation with constant coefficients

$$(8) \quad \begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})u = \hat{f} \in \mathcal{C}[[t, z]]_{\tilde{s}_1, \tilde{s}_2} \\ \partial_{m_1,t}^j u(0, z) = 0 \quad \text{for } j = 0, \dots, n-1, \end{cases}$$

where m_1, m_2 are moment functions of orders $s_1, s_2 \in \mathbf{R}$ respectively, $(\tilde{s}_1, \tilde{s}_2) \in \mathbf{R}^2$ is a Gevrey order of the inhomogeneity $\hat{f}$ and $P(\lambda, \zeta)$ is a general polynomial of two variables, which is of order n with respect to λ . It means that

$$P(\lambda, \zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^n P_j(\zeta)\lambda^{n-j} = P_0(\zeta)\tilde{P}(\lambda, \zeta) = P_0(\zeta) \prod_{\alpha=1}^l (\lambda - \lambda_\alpha(\zeta))^{n_\alpha},$$

where $\lambda_1(\zeta), \dots, \lambda_l(\zeta)$ are the roots of the characteristic equation $P(\lambda, \zeta) = 0$ with multiplicities $n_1, \dots, n_l$ ($n_1 + \dots + n_l = n$) respectively. Since $\lambda_\alpha(\zeta)$ are algebraic functions, we may assume that there exist $\kappa \in \mathbf{N}$ and $r_0 < \infty$ such that $\lambda_\alpha(\zeta)$ are holomorphic functions of the variable $\xi = \zeta^{1/\kappa}$ (for $|\zeta| \geq r_0$) and, moreover, there exist $\lambda_\alpha \in \mathbf{C} \setminus \{0\}$ and $q_\alpha = \mu_\alpha / \nu_\alpha$ (for some relatively prime numbers $\mu_\alpha \in \mathbf{Z}$ and $\nu_\alpha \in \mathbf{N}$) such that $\lambda_\alpha(\zeta) \sim \lambda_\alpha \zeta^{q_\alpha}$ for $\alpha = 1, \dots, l$.

Hence $\lambda_\alpha(\partial_{m_2,z})$ are well-defined moment pseudodifferential operators and consequently also the operator

$$\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z}) = (\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1} \dots (\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l}$$

is well-defined.

If $P_0(\zeta) \neq \text{const.}$ then the formal solution of (8) is not uniquely determined. For this reason we choose a formal power series $\hat{g} \in \mathcal{C}[[t, z]]_{\tilde{s}_1, \tilde{s}_2}$ satisfying the equation $P_0(\partial_{m_2,z})\hat{g} = \hat{f}$. For such $\hat{g}$ we may construct the uniquely determined solution $\hat{u}$ of

$$\begin{cases} \tilde{P}(\partial_{m_1,t}, \partial_{m_2,z})u = \hat{g} \in \mathcal{C}[[t, z]]_{\tilde{s}_1, \tilde{s}_2} \\ \partial_{m_1,t}^j u(0, z) = 0 \quad \text{for } j = 0, \dots, n-1, \end{cases}$$

which is also a formal solution of (8) and is called the *formal solution of (8) determined by $\hat{g}$* .

In a similar way to [11, Theorem 1], we generalise the results for the analytic Cauchy data given in [10, Theorems 7 and 8] as follows

Theorem 1. *Let $\hat{u}$ be a formal solution of (8) determined by $\hat{g} \in \mathcal{C}[[t, z]]_{\tilde{s}_1, \tilde{s}_2}$. Then $\hat{u} = \sum_{\alpha=1}^l \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$ with $\hat{u}_{\alpha\beta}$ being a formal solution of simple inhomogeneous pseudodifferential equation*

$$(9) \quad \begin{cases} (\partial_{\tilde{m}_1, t} - \lambda_\alpha(\partial_{\tilde{m}_2, z}))^\beta u_{\alpha\beta} = \hat{g}_{\alpha\beta} \\ \partial_{\tilde{m}_1, t}^j u_{\alpha\beta}(0, z) = 0 \quad \text{for } j = 0, \dots, \beta - 1 \end{cases}$$

where $\hat{g}_{\alpha\beta}(t, z) := d_{\alpha\beta}(\partial_{\tilde{m}_2, z})\hat{g}(t, z) \in \mathcal{C}[[t, z^{1/\kappa}]]_{\tilde{s}_1, \tilde{s}_2}$ and $d_{\alpha\beta}(\zeta)$ is a holomorphic function of the variable $\zeta = \zeta^{1/\kappa}$ and of polynomial growth.

Moreover, if q_α is a pole order of $\lambda_\alpha(\zeta)$ for some $\alpha \in \{1, \dots, l\}$ and $q_\alpha^+ := \max\{0, q_\alpha\}$, then a formal solution $\hat{u}_{\alpha\beta}$ is a Gevrey series of order $Q_\alpha := \max\{q_\alpha^+(s_2 + \tilde{s}_2) - s_1, \tilde{s}_1\}$ with respect to t . More precisely, $\hat{u}_{\alpha\beta} \in \mathcal{C}[[t, z^{1/\kappa}]]_{Q_\alpha, \tilde{s}_2}$ or, equivalently, $\hat{u}_{\alpha\beta} \in G_{\tilde{s}_2, 1/\kappa}[[t]]_{Q_\alpha}$.

Proof. For fixed $\tilde{s}_2 > \max\{\tilde{s}_2, -s_2\}$ we define $\hat{v} := \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{u}$. By Proposition 7, $\hat{v}$ is a formal solution of

$$\begin{cases} P(\partial_{\tilde{m}_1, t}, \partial_{\tilde{m}_2, z})v = \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{f} \\ \partial_{\tilde{m}_1, t}^j v(0, z) = 0 \quad \text{for } j = 0, \dots, n - 1, \end{cases}$$

determined by $\hat{h} = \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{g}$, where $\tilde{m}_1 := m_1 \Gamma_{\tilde{s}_1}$ and $\tilde{m}_2 := m_2 \Gamma_{\tilde{s}_2}$. Since $\tilde{m}_2$ is a moment function of order $\tilde{s}_2 + s_2 > 0$ and $\hat{h} \in \mathcal{C}[[t, z]]_{0, \tilde{s}_2 - \tilde{s}_2}$ with $\tilde{s}_2 - \tilde{s}_2 < 0$, repeating the proof of [10, Theorem 7] we conclude that $\hat{v} = \sum_{\alpha=1}^l \sum_{\beta=1}^{n_\alpha} \hat{v}_{\alpha\beta}$ with $\hat{v}_{\alpha\beta}$ being a formal solution of

$$\begin{cases} (\partial_{\tilde{m}_1, t} - \lambda_\alpha(\partial_{\tilde{m}_2, z}))^\beta v_{\alpha\beta} = \hat{h}_{\alpha\beta}(t, z) \\ \partial_{\tilde{m}_1, t}^j v_{\alpha\beta}(0, z) = 0 \quad \text{for } j = 0, \dots, \beta - 1, \end{cases}$$

where $\hat{h}_{\alpha\beta}(t, z) = d_{\alpha\beta}(\partial_{\tilde{m}_2, z})\hat{h}(t, z) \in \mathcal{C}[[t, z^{1/\kappa}]]_{0, \tilde{s}_2 - \tilde{s}_2}$ and $d_{\alpha\beta}(\zeta)$ is a holomorphic function of the variable $\zeta = \zeta^{1/\kappa}$ and of polynomial growth. Hence, by Definition 13, $\hat{u} = \sum_{\alpha=1}^l \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$ where $\hat{u}_{\alpha\beta} = \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z^{1/\kappa}} \hat{v}_{\alpha\beta}$ satisfies (9) with

$$\begin{aligned} \hat{g}_{\alpha\beta}(t, z) &= \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z^{1/\kappa}} \hat{h}_{\alpha\beta}(t, z) = \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z^{1/\kappa}} d_{\alpha\beta}(\partial_{\tilde{m}_2, z}) \hat{h}(t, z) \\ &= d_{\alpha\beta}(\partial_{m_2, z}) \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{h}(t, z) = d_{\alpha\beta}(\partial_{m_2, z}) \hat{g}(t, z) \end{aligned}$$

for $\beta = 1, \dots, n_\alpha$ and $\alpha = 1, \dots, l$.

To find the Gevrey order of $\hat{v}_{\alpha\beta}(t, z) = \sum_{j=0}^{\infty} (v_{\alpha\beta j}(z)/\tilde{m}_1(j))t^j$ with respect to t , observe that by [10, Lemma 8]

$$(10) \quad \hat{v}_{\alpha\beta}(t, z) = \sum_{k=\beta-1}^{\infty} \binom{k}{\beta-1} (\partial_{\tilde{m}_1, t}^{-1})^{k+1} \lambda_{\alpha\beta}^{k-\beta+1} (\partial_{\tilde{m}_2, z}) \hat{h}_{\alpha\beta}(t, z).$$

Since

$$(11) \quad \hat{h}_{\alpha\beta}(t, z) = \sum_{j=0}^{\infty} \frac{h_{\alpha\beta, j}(z)}{\tilde{m}_1(j)} t^j$$

is a Gevrey series of order 0 with respect to t and $\tilde{m}_1$ is a moment function of order $s_1 + \tilde{s}_1$, there exist $r > 0$ and $A, B < \infty$ such that

$$\sup_{|z| < r} |h_{\alpha\beta, j}(z)| \leq AB^j \Gamma_{s_1 + \tilde{s}_1}(j) \quad \text{for } j = 0, 1, \dots$$

Substituting (11) into (10) we obtain

$$\hat{v}_{\alpha\beta}(t, z) = \sum_{j=\beta}^{\infty} \frac{t^j}{\tilde{m}_1(j)} \sum_{k=\beta-1}^{j-1} \binom{k}{\beta-1} \lambda_{\alpha\beta}^{k-\beta+1} (\partial_{\tilde{m}_2, z}) h_{\alpha\beta, j-k-1}(z).$$

Hence, by Lemma 1, there exist $r > 0$ and $C, D < \infty$ such that for every $z \in D_r$

$$\begin{aligned} |v_{\alpha\beta j}(z)| &\leq \sum_{k=\beta-1}^{j-1} \binom{k}{\beta-1} |\lambda_{\alpha\beta}^{k-\beta+1} (\partial_{\tilde{m}_2, z}) h_{\alpha\beta, j-k-1}(z)| \\ &\leq CD^j \sum_{k=\beta-1}^{j-1} \Gamma_{q_{\alpha}^+(\tilde{s}_2 - \tilde{s}_2 + s_2 + \tilde{s}_2)}(k) \Gamma_{s_1 + \tilde{s}_1}(j - k - 1). \end{aligned}$$

Hence there exist $A, B < \infty$ such that

$$\sup_{|z| < r} |v_{\alpha\beta j}(z)| \leq AB^j \Gamma_{Q_{\alpha} + s_1}(j) \quad \text{for } j = 0, 1, \dots,$$

where $Q_{\alpha} = \max\{q_{\alpha}^+(s_2 + \tilde{s}_2) - s_1, \tilde{s}_1\}$.

It means that $\hat{v}_{\alpha\beta}(t, z) = \sum_{j=0}^{\infty} (v_{\alpha\beta j}(z)/\tilde{m}_1(j))t^j \in G_{\tilde{s}_2 - \tilde{s}_2, 1/\kappa}[[t]]_{Q_{\alpha} - \tilde{s}_1}$. Finally, by Proposition 4 we conclude that $\hat{u}_{\alpha\beta} = \mathcal{B}_{\Gamma_{-\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{-\tilde{s}_2}, z^{1/\kappa}} \hat{v}_{\alpha\beta} \in G_{\tilde{s}_2, 1/\kappa}[[t]]_{Q_{\alpha}}$ or, equivalently, $\hat{u}_{\alpha\beta} \in \mathcal{C}[[t, z^{1/\kappa}]]_{Q_{\alpha}, \tilde{s}_2}$. $\square$

7. Integral representation and analytic solution

By Proposition 5, to study analytic continuation or K -summability of $\hat{u}$ satisfying (9) we may replace m_1 and m_2 by Γ_{s_1} and Γ_{s_2} , where s_1 and s_2 are

orders of m_1 and m_2 respectively. So by Propositions 7 and 8, we may assume that $\hat{u}$ is a formal solution of

$$(12) \quad \begin{cases} (\partial_{\Gamma_{s_1}, t} - \lambda(\partial_{\Gamma_{s_2}, z}))^\beta u = \hat{g} \\ \partial_{\Gamma_{s_1}, t}^j u(0, z) = 0 \quad \text{for } j = 0, \dots, \beta - 1, \end{cases}$$

where $\lambda(\zeta)$ is a root of the characteristic equation of (8). It means that $\lambda(\zeta)$ is an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ and $\lambda(\zeta) \sim \lambda_0 \zeta^q$.

By [10, Lemma 8] the formal solution of (12) has the power series representation

$$(13) \quad \hat{u}(t, z) = \sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} (\partial_{\Gamma_{s_1}, t}^{-1})^{j+1} \lambda^{j-\beta+1} (\partial_{\Gamma_{s_2}, z}) \hat{g}(t, z).$$

To find the integral representation of $\hat{u}$, we first show

Lemma 2. *If $\hat{\phi}(x) \in E[[x]]$, $k \in \mathbb{N}$ and $s > 0$ then*

$$(14) \quad (\partial_{\Gamma_{s,x}}^{-1})^k \hat{\phi}(x) = - \int_0^{x^{1/s}} \hat{\phi}(y^s) \partial_y \frac{(x^{1/s} - y)^{ks}}{\Gamma_s(k)} dy.$$

Proof. After the change of variables $t := y/x^{1/s}$, the right-hand side of (14) is equal to

$$\text{RHS} = \frac{x^k}{\Gamma(ks)} \int_0^1 (1-t)^{ks-1} \hat{\phi}(t^s x) dt.$$

Since $\hat{\phi}(x) = \sum_{n=0}^{\infty} \frac{\varphi_n}{\Gamma_s(n)} x^n$, we have

$$\text{RHS} = \sum_{n=0}^{\infty} \frac{\varphi_n x^{n+k}}{\Gamma(ks) \Gamma(1+ns)} \int_0^1 (1-t)^{ks-1} t^{ns} dt.$$

Moreover, using the beta integral formula we conclude

$$\int_0^1 (1-t)^{ks-1} t^{ns} dt = B(ks, 1+ns) = \frac{\Gamma(ks) \Gamma(1+ns)}{\Gamma(1+(k+n)s)}.$$

Hence

$$\text{RHS} = \sum_{n=0}^{\infty} \frac{\varphi_n}{\Gamma(1+(k+n)s)} x^{n+k} = \sum_{n=k}^{\infty} \frac{\varphi_{n-k}}{\Gamma(1+ns)} x^n = \text{LHS}. \quad \square$$

We also prove

Lemma 3. *Let $s > 0$ and $\beta \in \mathbb{N}$. Then the function*

$$(15) \quad e_{s,\beta}(x) := \sum_{j=\beta}^{\infty} \binom{j-1}{\beta-1} \frac{x^j}{\Gamma_s(j)}$$

possesses the following properties

- (a) $e_{s,\beta}(x) \in \mathcal{O}^{1/s}(\mathbb{C})$,
- (b) *if $s < 2$ and $\arg x \in (s\pi/2, 2\pi - s\pi/2)$ then $e_{s,\beta}(x) \rightarrow 0$ as $x \rightarrow \infty$.*

Proof. Observe that by [1, Lemma 6] the properties (a) and (b) are satisfied by the Mittag-Leffler function $\mathbf{E}_s(x) = \sum_{j=0}^{\infty} x^j / \Gamma_s(j)$. Moreover, we can express the function $e_{s,\beta}(x)$ in terms of $\mathbf{E}_s(x)$ as

$$e_{s,\beta}(x) = \frac{1}{(\beta-1)!} x^\beta \left(\frac{\mathbf{E}_s(x) - 1}{x} \right)^{(\beta-1)}.$$

Hence $e_{s,\beta}$ also satisfies (a) and (b). $\square$

Using the definition of moment pseudodifferential operators and the power series representation of $\hat{u}$ we can find the integral representation of solution u

Proposition 9. *Let $\lambda(\zeta) \sim \lambda_0 \zeta^q$ be an analytic function of the variable $\zeta = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$. We also assume that $g \in \mathcal{O}_{1,1/\kappa}(D^2)$, $s_1, s_2 > 0$ and $s_1 \geq qs_2$. Then the solution u of (12) belongs to the space $\mathcal{O}_{1,1/\kappa}(D^2)$ and has the integral representation*

$$u(t, z) = \frac{-1}{2\kappa\pi i} \int_0^{t^{1/s_1}} \oint_{|w|=\varepsilon} g(\tau^{s_1}, w) \partial_\tau k(t, \tau, z, w) dw d\tau,$$

where

$$k(t, \tau, z, w) := \int_{r_0 e^{i\theta}}^{\infty(\theta)} \lambda^{-\beta}(\zeta) e_{s_1,\beta}((t^{1/s_1} - \tau)^{s_1} \lambda(\zeta)) E_{\Gamma_{s_2/\kappa}}(\zeta^{1/\kappa} z^{1/\kappa}) \frac{e_{\Gamma_{s_2}}(\zeta w)}{\zeta w} d\zeta$$

and $e_{s_1,\beta}(x)$ is given by (15).

Moreover, if $s_1 > qs_2$ and $g \in \mathcal{O}_{1,1/\kappa}^{1/(s_1-qs_2)}(\mathbb{C} \times D)$ then also $u \in \mathcal{O}_{1,1/\kappa}^{1/(s_1-qs_2)}(\mathbb{C} \times D)$.

Proof. Since $g \in \mathcal{O}_{1,1/\kappa}(D^2)$ and $s_1 \geq qs_2$, by Theorem 1 we get $u \in \mathcal{O}_{1,1/\kappa}(D^2)$. To show that the integral given in the definition of $k(t, \tau, z, w)$ is convergent, observe that by Lemma 3, Definitions 2 and 3 there exist constants A_i and b_i ($i = 1, 2, 3$) such that

- $|e_{s_1,\beta}((t^{1/s_1} - \tau)^{s_1} \lambda(\zeta))| \leq A_1 e^{b_1 |t^{1/s_1} - \tau| |\zeta|^{q/s_1}},$
- $|E_{\Gamma_{s_2/\kappa}}(\zeta^{1/\kappa} z^{1/\kappa})| \leq A_2 e^{b_2 |\zeta|^{1/s_2} |z|^{1/s_2}},$
- $|e_{\Gamma_{s_2}}(\zeta w)| \leq A_3 e^{-b_3 |\zeta|^{1/s_2} |w|^{1/s_2}}.$

Hence, for fixed $w \in \mathbf{C} \setminus \{0\}$ such that $|z|$ is small relative to $|w|$ and for $|t| < a|w|^q$ with some fixed $a > 0$ and for $\tau \in [0, t^{1/s_1}]$, we have

$$|k(t, \tau, z, w)| \leq \int_{r_0}^{\infty} \tilde{A} e^{-\tilde{b} x^{1/s_2} |w|^{1/s_2}} dx < \infty.$$

It means that the integral representation of the solution u is a well-defined holomorphic function in some complex neighbourhood of the origin. To show that this integral representation holds, observe that by (6), (13) and (14) we have

$$\begin{aligned} u(t, z) &= \sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} (\partial_{\Gamma_{s_1}, t}^{-1})^{j+1} \lambda^{j-\beta+1} (\partial_{\Gamma_{s_2}, z}) g(t, z) \\ &= - \sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} \int_0^{t^{1/s_1}} \lambda^{j-\beta+1} (\partial_{\Gamma_{s_2}, z}) g(\tau^{s_1}, z) \partial_{\tau} \frac{(t^{1/s_1} - \tau)^{(j+1)s_1}}{\Gamma_{s_1}(j+1)} d\tau \\ &= \frac{-1}{2\kappa\pi i} \int_0^{t^{1/s_1}} \oint_{|w|=\varepsilon}^{\kappa} g(\tau^{s_1}, w) \partial_{\tau} \int_{r_0 e^{i\theta}}^{\infty(\theta)} \lambda^{-\beta}(\zeta) \\ &\quad \times \sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} \frac{(t^{1/s_1} - \tau)^{s_1(j+1)} \lambda^{j+1}(\zeta)}{\Gamma_{s_1}(j+1)} E_{\Gamma_{s_2/\kappa}}(\zeta^{1/\kappa} z^{1/\kappa}) \frac{e_{\Gamma_{s_2}}(\zeta w)}{\zeta w} d\zeta dw d\tau \\ &= \frac{-1}{2\kappa\pi i} \int_0^{t^{1/s_1}} \oint_{|w|=\varepsilon}^{\kappa} g(\tau^{s_1}, w) \partial_{\tau} k(t, \tau, z, w) dw d\tau. \end{aligned}$$

The second part of the proposition is given by Theorem 1 and by the observation that $g, u \in \mathcal{O}_{1, 1/\kappa}^{1/(s_1 - qs_2)}(\mathbf{C} \times D)$ if and only if $\hat{g}, \hat{u} \in \mathbf{C}[[t, z^{1/\kappa}]]_{qs_2 - s_1, 0}$. $\square$

Using the above integral representation of solution of (12) we conclude that

Theorem 2. *Let m_1, m_2 be moment functions of orders $s_1, s_2 > 0$ respectively and let $\lambda(\zeta) \sim \lambda_0 \zeta^q$ be an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$. We also assume that $s_1 = qs_2$, $K > 0$, $d \in \mathbf{R}$ and u is a solution of*

$$\begin{cases} (\partial_{m_1, t} - \lambda(\partial_{m_2, z}))^\beta u = g \in \mathcal{O}_{1, 1/\kappa}(D^2) \\ \partial_{m_1, t}^j u(0, z) = 0 \quad \text{for } j = 0, \dots, \beta - 1. \end{cases}$$

Then $u(t, z) \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ for $k \in \mathbf{N}$ if and only if $g(t, z) \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ for $k \in \mathbf{N}$.

Proof. Let $U := \mathcal{B}_{\bar{m}_1, t} \mathcal{B}_{\bar{m}_2, z^{1/\kappa}} u$ and $G := \mathcal{B}_{\bar{m}_1, t} \mathcal{B}_{\bar{m}_2, z^{1/\kappa}} g$, where $\bar{m}_1$ and $\bar{m}_2$ are moment functions of order 0 defined by $\bar{m}_1(u) := \Gamma_{s_1}(u)/m_1(u)$ and $\bar{m}_2(u) := \Gamma_{s_2}(u)/m_2(u)$. By Propositions 7 and 8, we conclude that U is a solution of

$$\begin{cases} (\partial_{\Gamma_{s_1}, t} - \lambda(\partial_{\Gamma_{s_2}, z}))^\beta U = G \in \mathcal{O}_{1, 1/\kappa}(D^2) \\ \partial_{\Gamma_{s_1}, t}^j U(0, z) = 0 \quad \text{for } j = 0, \dots, \beta - 1. \end{cases}$$

To finish the proof, by Proposition 5 it is sufficient to show the equivalence $U(t, z) \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0+2k\pi)/q})$ for $k \in \mathbb{N} \Leftrightarrow G(t, z) \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0+2k\pi)/q})$ for $k \in \mathbb{N}$.

($\Rightarrow$) Since $G(t, z) = (\partial_{\Gamma_{s_1}, t} - \lambda(\partial_{\Gamma_{s_2}, z}))U(t, z)$ then $G(t, z)$ belongs to the same spaces $\mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0+2k\pi)/q})$ for $k \in \mathbb{N}$ as $U(t, z)$.

($\Leftarrow$) Deforming the path of integration with respect to w in the integral representation of solution given in Proposition 9 and repeating the proof of [11, Lemma 4] we conclude the assertion. $\square$

8. Summable solution

Let $\hat{u}$ be a formal solution of

$$(16) \quad \begin{cases} (\partial_{m_1, t} - \lambda(\partial_{m_2, z}))^\beta u = \hat{g} \in \mathcal{C}[[t, z^{1/\kappa}]]_{\tilde{s}_1, \tilde{s}_2} \\ \partial_{m_1, t}^j u(0, z) = 0 \quad \text{for } j = 0, \dots, \beta - 1, \end{cases}$$

where m_1, m_2 are moment functions of orders s_1, s_2 respectively and let $\lambda(\zeta) \sim \lambda_0 \zeta^q$ be an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$.

By the Gevrey estimates $\hat{u} \in \mathcal{C}[[t, z^{1/\kappa}]]_{\max\{q(s_2+\tilde{s}_2)-s_1, \tilde{s}_1\}, \tilde{s}_2}$. In the case when the Gevrey order is positive, it is natural to ask about summability of $\hat{u}$. Our aim is a characterisation of summable solutions $\hat{u}$ in terms of the inhomogeneity $\hat{g}$. Using Theorem 2 we prove

Theorem 3. *Let $d \in \mathbb{R}$ and $\hat{u}$ be a formal solution of (16) with $q > 0$. We consider two cases.*

I. $q(s_2 + \tilde{s}_2) - s_1 \geq \tilde{s}_1$, $q(s_2 + \tilde{s}_2) - s_1 > 0$ and $s_2 + \tilde{s}_2 > 0$. In this case we assume that $G(t, z) = \mathcal{B}_{\Gamma_{q(s_2+\tilde{s}_2)-s_1}, t} \mathcal{B}_{\Gamma_{s_2}, z^{1/\kappa}} \hat{g}(t, z)$ and $K = 1/(q(s_2 + \tilde{s}_2) - s_1)$. Then the following statements hold:

- If $G \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0+2k\pi)/q})$ ($k \in \mathbb{N}$) then $\hat{u}$ is K -summable in a direction d with respect to t .
- If $G \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0+2k\pi)/q})$ ($k \in \mathbb{N}$), $s_1 \leq qs_2$ and $\tilde{s}_2 > 0$ then $\hat{u}$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_0 + 2k\pi)/q)$ ($k \in \mathbb{N}$).
- If $s_1 = qs_2$ and $\tilde{s}_2 > 0$ then $\hat{u}$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_0 + 2k\pi)/q)$ ($k \in \mathbb{N}$) if and only if $\hat{g}$ is $(K, 1/\tilde{s}_2)$ -summable in the same directions.

Moreover, if $\tilde{s}_1 \leq 0$ then we may replace the condition $G \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0+2k\pi)/q})$ ($k \in \mathbb{N}$) by $G \in \mathcal{O}_{1, 1/\kappa}^{qK}(D \times \hat{S}_{(d+\arg \lambda_0+2k\pi)/q})$ ($k \in \mathbb{N}$).

II. $q(s_2 + \tilde{s}_2) - s_1 \leq \tilde{s}_1$, $\tilde{s}_1 > 0$ and $s_1 + \tilde{s}_1 > 0$. In this case we assume that $G(t, z) = \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{(s_1+\tilde{s}_1)/q-s_2}, z^{1/\kappa}} \hat{g}(t, z)$ and $K = 1/\tilde{s}_1$. If $G \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_0+2k\pi)/q})$ ($k \in \mathbb{N}$) then $\hat{u}$ is K -summable in a direction d with respect to t .

Moreover, if we additionally assume that $s_1 \geq q(s_2 + \tilde{s}_2)$ then $\hat{u}$ is K -summable in a direction d if and only if $\hat{g}$ has the same property.

Proof. I. We assume that $q(s_2 + \tilde{s}_2) - s_1 \geq \tilde{s}_1$, $q(s_2 + \tilde{s}_2) - s_1 > 0$ and $s_2 + \tilde{s}_2 > 0$. Applying the moment Borel transforms $\mathcal{B}_{\Gamma_{q(s_2 + \tilde{s}_2) - s_1}, t}$ and $\mathcal{B}_{\Gamma_{\tilde{s}_2}, z^{1/\kappa}}$ to (16) we obtain

$$\begin{cases} (\partial_{m_1} \Gamma_{q(s_2 + \tilde{s}_2) - s_1}, t - \lambda(\partial_{m_2} \Gamma_{\tilde{s}_2}, z))^\beta U = G \in \mathcal{O}_{1, 1/\kappa}(D^2), \\ \partial_{m_1}^j \Gamma_{q(s_2 + \tilde{s}_2) - s_1}, t U(0, z) = 0 \quad \text{for } j = 0, \dots, \beta - 1, \end{cases}$$

where $U(t, z) := \mathcal{B}_{\Gamma_{q(s_2 + \tilde{s}_2) - s_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z^{1/\kappa}} \hat{u}(t, z)$ and $G(t, z) := \mathcal{B}_{\Gamma_{q(s_2 + \tilde{s}_2) - s_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z^{1/\kappa}} \hat{g}(t, z)$.

If $G \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$), then by Theorem 2 we conclude that also $U \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$). Hence by Definition 8, Remark 5 and Proposition 5, $\hat{u} \in G_{\tilde{s}_2, 1/\kappa}[[t]]$ is K -summable in a direction d .

If we additionally assume that $s_1 \leq qs_2$ and $\tilde{s}_2 > 0$ then $qK \leq 1/\tilde{s}_2$ and consequently $U \in \mathcal{O}_{1, 1/\kappa}^{K, 1/\tilde{s}_2}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$). It means by Definition 10 and Proposition 5 that $\hat{u} \in \mathcal{C}[[t, z^{1/\kappa}]]$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_0 + 2k\pi)/q)$ ($k \in \mathbb{N}$).

If $qK = 1/\tilde{s}_2$ then by Theorem 2 we conclude that $U(t, z) \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$) if and only if $G(t, z) \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$). By Definition 10 and Proposition 5 it means that $\hat{u} \in \mathcal{C}[[t, z^{1/\kappa}]]$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_0 + 2k\pi)/q)$ ($k \in \mathbb{N}$) if and only if $\hat{g} \in \mathcal{C}[[t, z^{1/\kappa}]]$ is $(K, 1/\tilde{s}_2)$ -summable in the same directions.

Finally, observe that $\hat{G} \in \mathcal{C}[[t, z^{1/\kappa}]]_{s_1 + \tilde{s}_1 - q(s_2 + \tilde{s}_2), 0}$, it means by Remark 4 that $G \in \mathcal{O}_{1, 1/\kappa}^{1/(q(s_2 + \tilde{s}_2) - s_1 - \tilde{s}_1)}(\mathcal{C} \times D)$ for $q(s_2 + \tilde{s}_2) > \tilde{s}_1 + s_1$. Hence, if $\tilde{s}_1 \leq 0$ then $1/(q(s_2 + \tilde{s}_2) - s_1 - \tilde{s}_1) \leq K$ and consequently $G \in \mathcal{O}_{1, 1/\kappa}^K(\hat{S}_d \times D)$. So, in this case the conditions $G \in \mathcal{O}_{1, 1/\kappa}^{qK}(D \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$) and $G \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$) are equivalent.

II. Now we assume that $q(s_2 + \tilde{s}_2) - s_1 \leq \tilde{s}_1$, $\tilde{s}_1 > 0$ and $s_1 + \tilde{s}_1 > 0$. As in the previous case we obtain that $\hat{u}$ is K -summable in a direction d with respect to t under condition that $G \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$).

Similarly to the first part of the proof, observe that in our case $\hat{G} \in \mathcal{C}[[t, z^{1/\kappa}]]_{0, s_2 + \tilde{s}_2 - (s_1 + \tilde{s}_1)/q}$ and consequently $G \in \mathcal{O}_{1, 1/\kappa}^{q/(s_1 + \tilde{s}_1 - q(s_2 + \tilde{s}_2))}(D \times \mathcal{C})$ for $q(s_2 + \tilde{s}_2) < \tilde{s}_1 + s_1$. So, if $s_1 \geq q(s_2 + \tilde{s}_2)$ then also $G \in \mathcal{O}_{1, 1/\kappa}^{qK}(D \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$). It means that if $\hat{g}$ is K -summable in a direction d with respect to t , or equivalently $G \in \mathcal{O}_{1, 1/\kappa}^K(\hat{S}_d \times D)$, then $G \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_0 + 2k\pi)/q})$ ($k \in \mathbb{N}$) and consequently also $\hat{u}$ is K -summable in a direction d with respect to t . Conversely, if $\hat{u}$ is K -summable in a direction d with respect to t then of course also $\hat{g} = (\partial_{m_1, t} - \lambda(\partial_{m_2, z}))^\beta \hat{u}$ has the same properties. $\square$

Example 4. To illustrate the above theorem, let us discuss the summability of the formal solution $\hat{u}$ of the Cauchy problem

$$\begin{cases} (\partial_t - \partial_z^q)u = \hat{g} \in \mathcal{O}[[t]]_s, \\ \partial_t u(0, z) = 0, \end{cases}$$

where $q \in \mathbf{N}$ and $s \in \mathbf{R}$. Observe that $\hat{u}$ satisfies (16) with $m_1 = m_2 = \Gamma_1$, $s_1 = s_2 = 1$, $\lambda(\zeta) = \zeta^q$, $\beta = 1$, $\tilde{s}_1 = s$ and $\tilde{s}_2 = 0$. By Theorem 3 we conclude that

- If $q > 1$, $q - 1 \geq s$ and $\mathcal{B}_{\Gamma_{q-1}, t} \hat{g}(t, z) \in \mathcal{O}^{1/(q-1), q/(q-1)}(S_d \times S_{(d+2k\pi)/q})$ for $k = 0, \dots, q - 1$ then $\hat{u}$ is $1/(q - 1)$ -summable in a direction d with respect to t .
- If $q > 1$, $s \leq 0$ and $\mathcal{B}_{\Gamma_{q-1}, t} \hat{g}(t, z) \in \mathcal{O}^{q/(q-1)}(D \times S_{(d+2k\pi)/q})$ for $k = 0, \dots, q - 1$ then $\hat{u}$ is $1/(q - 1)$ -summable in a direction d with respect to t .
- If $q - 1 \leq s$, $s > 0$ and $\mathcal{B}_{\Gamma_s, t} \hat{g}(t, z) \in \mathcal{O}^{1/s, q/s}(S_d \times S_{(d+2k\pi)/q})$ for $k = 0, \dots, q - 1$ then $\hat{u}$ is $1/s$ -summable in a direction d with respect to t .
- If $q = 1$ and $s > 0$ then $\hat{u}$ is $1/s$ -summable in a direction d with respect to t if and only if $\hat{g}$ has the same property.

Now, let us return to the Cauchy problem (8) with the additional condition that all $\hat{u}_{\alpha\beta}$ constructed in Theorem 1 have the same Gevrey order with respect to t . In this case we have

Theorem 4. *Let $d \in \mathbf{R}$ and $\hat{u}$ be a formal solution of (8) determined by $\hat{g} \in \mathcal{C}[[t, z]]_{\tilde{s}_1, \tilde{s}_2}$. We consider two cases.*

I. $s_2 + \tilde{s}_2 > 0$ and there exist $q \in \mathbf{Q}_+$ being the common pole order of $\lambda_\alpha(\zeta)$ (i.e. $\lambda_\alpha(\zeta) \sim \lambda_\alpha \zeta^q$) for every $\alpha = 1, \dots, l$ and satisfying $q(s_2 + \tilde{s}_2) \geq s_1 + \tilde{s}_1$ and $q(s_2 + \tilde{s}_2) > s_1$. In this case we assume that $G(t, z) = \mathcal{B}_{\Gamma_{q(s_2 + \tilde{s}_2) - s_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{g}(t, z)$ and $K = 1/(q(s_2 + \tilde{s}_2) - s_1)$. Then the following statements hold:

- If $G \in \mathcal{O}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_\alpha + 2k\pi)/q})$ (for $\alpha = 1, \dots, l$ and $k \in \mathbf{N}$) then $\hat{u}$ is K -summable in a direction d with respect to t .
- If $G \in \mathcal{O}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_\alpha + 2k\pi)/q})$ (for $\alpha = 1, \dots, l$ and $k \in \mathbf{N}$), $s_1 \leq qs_2$ and $\tilde{s}_2 > 0$ then $\hat{u}$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_\alpha + 2k\pi)/q)$ (for $\alpha = 1, \dots, l$ and $k \in \mathbf{N}$).
- If $s_1 = qs_2$ and $\tilde{s}_2 > 0$ then $\hat{u}$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_\alpha + 2k\pi)/q)$ (for $\alpha = 1, \dots, l$ and $k \in \mathbf{N}$) if and only if $\hat{g}$ is $(K, 1/\tilde{s}_2)$ -summable in the same directions.

Moreover, if $\tilde{s}_1 \leq 0$ then we may replace the condition $G \in \mathcal{O}^{K, qK}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_\alpha + 2k\pi)/q})$ by $G \in \mathcal{O}^{qK}(D \times \hat{S}_{(d + \arg \lambda_\alpha + 2k\pi)/q})$ (for $\alpha = 1, \dots, l$ and $k \in \mathbf{N}$).

II. $\tilde{s}_1 > 0$, $s_1 + \tilde{s}_1 > 0$ and $q_\alpha(s_2 + \tilde{s}_2) - s_1 \leq \tilde{s}_1$ for $\alpha = 1, \dots, l$, where $q_\alpha \in \mathbf{Q}_+$ is a pole order of $\lambda_\alpha(\zeta)$ (i.e. $\lambda_\alpha(\zeta) \sim \lambda_\alpha \zeta^{q_\alpha}$) for every $\alpha = 1, \dots, l$. In this case we assume that $G_\alpha(t, z) = \mathcal{B}_{\Gamma_{\tilde{s}_1}, t} \mathcal{B}_{\Gamma_{(s_1 + \tilde{s}_1)/q_\alpha - s_2}, z} \hat{g}(t, z)$ and $K = 1/\tilde{s}_1$. If $G_\alpha \in \mathcal{O}^{K, q_\alpha K}(\hat{S}_d \times \hat{S}_{(d + \arg \lambda_\alpha + 2k\pi)/q})$ (for $\alpha = 1, \dots, l$ and $k \in \mathbf{N}$) then $\hat{u}$ is K -summable in a direction d with respect to t .

Moreover, if we additionally assume that $s_1 \geq q_\alpha(s_2 + \tilde{s}_2)$ (for $\alpha = 1, \dots, l$) then $\hat{u}$ is K -summable in a direction d if and only if $\hat{g}$ has the same property.

Proof. By Theorem 1 we conclude that $\hat{u} = \sum_{\alpha=1}^l \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$ with $\hat{u}_{\alpha\beta}$ being a formal solution of simple inhomogeneous pseudodifferential equation

$$\begin{cases} (\partial_{m_1, t} - \lambda_\alpha(\partial_{m_2, z}))^\beta u_{\alpha\beta} = \hat{g}_{\alpha\beta} \\ \partial_{m_1, t}^j u_{\alpha\beta}(0, z) = 0 \quad \text{for } j = 0, \dots, \beta - 1 \end{cases}$$

where $\hat{g}_{\alpha\beta}(t, z) := d_{\alpha\beta}(\partial_{m_2, z})\hat{g}(t, z) \in \mathcal{C}[[t, z^{1/\kappa}]]_{\tilde{s}_1, \tilde{s}_2}$ and $d_{\alpha\beta}(\zeta)$ is a holomorphic function of the variable $\zeta = \zeta^{1/\kappa}$ and of polynomial growth.

In the first case, if $G_{\alpha\beta} := \mathcal{B}_{\Gamma_{q(s_2+\tilde{s}_2)-s_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{g}_{\alpha\beta}$ then by Proposition 8

$$G_{\alpha\beta} = d_{\alpha\beta}(\partial_{m_2} \Gamma_{\tilde{s}_2}, z) \mathcal{B}_{\Gamma_{q(s_2+\tilde{s}_2)-s_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{g} = d_{\alpha\beta}(\partial_{m_2} \Gamma_{\tilde{s}_2}, z) G$$

for $\alpha = 1, \dots, l$ and $\beta = 1, \dots, n_\alpha$. Hence if $G \in \mathcal{O}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_\alpha+2k\pi)/q})$ (for $\alpha = 1, \dots, l$ and $k \in \mathbb{N}$) then for every fixed $\alpha \in \{1, \dots, l\}$ and $\beta \in \{1, \dots, n_\alpha\}$ we conclude that $G_{\alpha\beta} \in \mathcal{O}_{1, 1/\kappa}^{K, qK}(\hat{S}_d \times \hat{S}_{(d+\arg \lambda_\alpha+2k\pi)/q})$ (for $k \in \mathbb{N}$).

By Theorem 3 we see that $\hat{u}_{\alpha\beta}$ is K -summable in a direction d with respect to t for every $\alpha \in \{1, \dots, l\}$ and $\beta \in \{1, \dots, n_\alpha\}$. Consequently also $\hat{u} = \sum_{\alpha=1}^l \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$ is K -summable in a direction d with respect to t .

If additionally we assume that $s_1 \leq q s_2$ and $\tilde{s}_2 > 0$ then by Theorem 3 we conclude that $\hat{u}_{\alpha\beta}$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_{\tilde{\alpha}} + 2k\pi)/q)$ for every $\alpha, \tilde{\alpha} \in \{1, \dots, l\}$ and $\beta \in \{1, \dots, n_\alpha\}$, so $\hat{u}$ also is $(K, 1/\tilde{s}_2)$ -summable in the same directions.

Finally, we assume that $s_1 = q s_2$ and $\tilde{s}_2 > 0$. If $\hat{u}$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_{\tilde{\alpha}} + 2k\pi)/q)$ (for $\tilde{\alpha} = 1, \dots, l$ and $k \in \mathbb{N}$) then also $\hat{g} = \hat{P}(\partial_{m_1, t}, \partial_{m_2, z})\hat{u}$ is $(K, 1/\tilde{s}_2)$ -summable in the same directions. For the converse, if $\hat{g}$ is $(K, 1/\tilde{s}_2)$ -summable in directions $(d, (d + \arg \lambda_{\tilde{\alpha}} + 2k\pi)/q)$ (for $\tilde{\alpha} = 1, \dots, l$ and $k \in \mathbb{N}$) then also $\hat{g}_{\alpha\beta} = d_{\alpha\beta}(\partial_{m_2, z})\hat{g}(t, z)$ and, by Theorem 3, $\hat{u}_{\alpha\beta}$ are $(K, 1/\tilde{s}_2)$ -summable in the same directions for every $\alpha = 1, \dots, l$ and $\beta = 1, \dots, n_\alpha$. It means that $\hat{u} = \sum_{\alpha=1}^l \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$ is $(K, 1/\tilde{s}_2)$ -summable in the same directions, too.

Using the same arguments as in the proof of Theorem 3, we finish the proof of the first case. In the same way we prove the second part of the theorem. $\square$

9. Multisummable solution

Now we return to the general Cauchy problem (8) with $s_1, s_2, \tilde{s}_1, \tilde{s}_2 \in \mathbb{R}$ satisfying $s_1 > 0$, $s_2 > 0$ and $s_2 + \tilde{s}_2 > 0$. For convenience we assume that

$$(17) \quad P(\lambda, \zeta) = P_0(\zeta) \tilde{P}(\lambda, \zeta) = P_0(\zeta) \prod_{\alpha=1}^{\tilde{n}} \prod_{\beta=1}^{l_\alpha} (\lambda - \lambda_{\alpha\beta}(\zeta))^{n_{\alpha\beta}},$$

where $\lambda_{\alpha\beta}(\zeta) \sim \lambda_{\alpha\beta}\zeta^{q_\alpha}$ is the root of the characteristic equation with $q_\alpha \in \mathbf{Q}$ and $\lambda_{\alpha\beta} \in \mathbf{C} \setminus \{0\}$ for $\alpha = 1, \dots, \tilde{n}$ and $\beta = 1, \dots, l_\alpha$. Without loss of generality we may assume that there exist exactly N positive pole orders of the roots of the characteristic equation, which are greater than $s_1/(s_2 + \tilde{s}_2)$ and $(s_1 + \tilde{s}_1)/(s_2 + \tilde{s}_2)$, say $\max\{s_1/(s_2 + \tilde{s}_2), (s_1 + \tilde{s}_1)/(s_2 + \tilde{s}_2)\} < q_N < \dots < q_1 < \infty$. Let $K_\alpha := (q_\alpha(s_2 + \tilde{s}_2) - s_1)^{-1}$ for $\alpha = 1, \dots, N$.

By Theorem 1, the formal solution $\hat{u}$ of (8) determined by $\hat{g} \in C[[t, z]]_{\tilde{s}_1, \tilde{s}_2}$ is given by

$$(18) \quad \hat{u} = \sum_{\alpha=1}^{\tilde{n}} \sum_{\beta=1}^{l_\alpha} \sum_{\gamma=1}^{n_{\alpha\beta}} \hat{u}_{\alpha\beta\gamma}$$

with $\hat{u}_{\alpha\beta\gamma}$ satisfying

$$\begin{cases} (\partial_{m_1, t} - \lambda_{\alpha\beta}(\partial_{m_2, z}))^\gamma u_{\alpha\beta\gamma} = \hat{g}_{\alpha\beta\gamma} \\ \partial_{m_1, t}^j u_{\alpha\beta\gamma}(0, z) = 0 \quad \text{for } j = 0, \dots, \gamma - 1, \end{cases}$$

where $\hat{g}_{\alpha\beta\gamma}(t, z) = d_{\alpha\beta\gamma}(\partial_{m_2, z})\hat{g}(t, z) \in C[[t, z^{1/\kappa}]]_{\tilde{s}_1, \tilde{s}_2}$ and $d_{\alpha\beta\gamma}(\zeta)$ is a holomorphic function of the variable $\zeta = \zeta^{1/\kappa}$ and of polynomial growth at infinity. Moreover $\hat{u}_{\alpha\beta\gamma} \in C[[t, z^{1/\kappa}]]_{Q_\alpha, \tilde{s}_2}$, where $Q_\alpha = \max\{q_\alpha^+(s_2 + \tilde{s}_2) - s_1, \tilde{s}_1\}$ for $\alpha = 1, \dots, \tilde{n}$. It means that $Q_\alpha = q_\alpha(s_2 + \tilde{s}_2) - s_1$ for $\alpha = 1, \dots, N$. If $N = \tilde{n}$ then we may ask about $(K_N, \dots, K_1)$ -summable solution $\hat{u}$. Similarly, if we assume that $\tilde{s}_1 \leq 0$ then $Q_\alpha \leq 0$ for $\alpha = N + 1, \dots, \tilde{n}$, so also in this case we may try to find $(K_N, \dots, K_1)$ -summable solution $\hat{u}$. On the other hand if $\tilde{s}_1 > 0$ then $Q_\alpha = \tilde{s}_1$ for $\alpha = N + 1, \dots, \tilde{n}$ and we can ask about $(\tilde{K}, K_N, \dots, K_1)$ -summable solution $\hat{u}$ with $\tilde{K} = 1/\tilde{s}_1$.

So, under the above conditions, using Theorem 3 and similar arguments as in the proof of Theorem 4 we obtain

Theorem 5. *Case I: $\tilde{s}_1 \leq 0$ or $N = \tilde{n}$.*

We assume that $(d_N, \dots, d_1) \in \mathbf{R}^N$ is an admissible multidirection with respect to $(K_N, \dots, K_1)$ and $G_\alpha(t, z) := \mathcal{B}_{\Gamma_{q_\alpha(s_2 + \tilde{s}_2) - s_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{g}(t, z)$ satisfies

$$G_\alpha(t, z) \in \mathcal{O}^{K_\alpha, q_\alpha K_\alpha}(\hat{S}_{d_\alpha} \times \hat{S}_{(d_\alpha + \arg \lambda_{\alpha\beta} + 2k\pi)/q_\alpha})$$

for every $\alpha = 1, \dots, N$, $\beta = 1, \dots, l_\alpha$ and $k \in \mathbf{N}$. Then the formal solution $\hat{u}$ of (8) determined by $\hat{g}$ is $(K_N, \dots, K_1)$ -summable in the multidirection $(d_N, \dots, d_1)$.

Case II: $\tilde{s}_1 > 0$ and $N < \tilde{n}$.

We assume that $(\tilde{d}, d_N, \dots, d_1) \in \mathbf{R}^{N+1}$ is an admissible multidirection with respect to $(\tilde{K}, K_N, \dots, K_1)$ and $G_\alpha(t, z) := \mathcal{B}_{\Gamma_{q_\alpha(s_2 + \tilde{s}_2) - s_1}, t} \mathcal{B}_{\Gamma_{\tilde{s}_2}, z} \hat{g}(t, z)$ satisfies

$$G_\alpha(t, z) \in \mathcal{O}^{K_\alpha, q_\alpha K_\alpha}(\hat{S}_{d_\alpha} \times \hat{S}_{(d_\alpha + \arg \lambda_{\alpha\beta} + 2k\pi)/q_\alpha})$$

for every $\alpha = 1, \dots, N$, $\beta = 1, \dots, l_\alpha$ and $k \in \mathbf{N}$. Additionally we assume that $q_\alpha > 0$ and $G_0(t, z) = \mathcal{B}_{\Gamma_{\tilde{s}_1, t}} \mathcal{B}_{\Gamma_{\tilde{s}_2, z}} \hat{g}(t, z)$ satisfies

$$G_0(t, z) \in \mathcal{O}^{1/\tilde{s}_1, q_\alpha/\tilde{s}_1}(\hat{\mathcal{S}}_{\tilde{d}} \times \hat{\mathcal{S}}_{(\tilde{d} + \arg \lambda_{\alpha\beta} + 2k\pi)/q_\alpha})$$

for every $\alpha = N + 1, \dots, \tilde{n}$, $\beta = 1, \dots, l_\alpha$ and $k \in \mathbf{N}$. Then the formal solution $\hat{u}$ of (8) determined by $\hat{g}$ is $(\tilde{K}, K_N, \dots, K_1)$ -summable in the multidirection $(\tilde{d}, d_N, \dots, d_1)$.

10. Newton polygon

In this section we introduce the notion of the Newton polygon for the moment partial differential operators with constant coefficients. This concept is a generalisation of the Newton polygon introduced by Yonemura [14] for partial differential operators. The similar considerations for the partial differential operators with constant coefficients can be found in [2, Section 3].

We will define the Newton polygon for the operator $P(\partial_{m_1, t}, \partial_{m_2, z})$, where

$$P(\lambda, \zeta) = \sum_{i=0}^n \sum_{j=0}^k a_{ij} \lambda^i \zeta^j.$$

To this end we denote by $\mathcal{A}$ the set of indices $\mathcal{A} := \{(i, j) \in \mathbf{N}_0^2 : a_{ij} \neq 0\}$. So, we may write the polynomial P as

$$P(\lambda, \zeta) = \sum_{(i, j) \in \mathcal{A}} a_{ij} \lambda^i \zeta^j.$$

We have

Definition 15. The *Newton polygon* for the operator $P(\partial_{m_1, t}, \partial_{m_2, z})$ with $s_1, s_2 > 0$ is defined by the convex hull of the union of sets $C(is_1 + js_2, -i)$ $((i, j) \in \mathcal{A})$, that is

$$N(P, s_1, s_2) = \text{conv} \left\{ \bigcup_{(i, j) \in \mathcal{A}} C(is_1 + js_2, -i) \right\},$$

where $C(a, b) := \{(x, y) \in \mathbf{R}^2 : x \leq a, y \geq b\}$ for any $(a, b) \in \mathbf{R}^2$.

Remark 8. Observe that if $(i, j) \in \mathcal{A}$ then the point $(is_1 + js_2, -i)$ in the Newton polygon $N(P, s_1, s_2)$ is determined by the operator $a_{ij} \partial_{m_1, t}^i \partial_{m_2, z}^j$.

Remark 9. We may extend this definition to the case of pseudodifferential operators

$$\tilde{P}(\lambda, \zeta) = \sum_{i=0}^n a_i \lambda^i \tilde{P}_i(\zeta) \quad \text{with } \tilde{P}_i(\zeta) \sim \zeta^{q_i} \text{ and } q_i \in \mathcal{Q}$$

putting

$$N(\tilde{P}, s_1, s_2) = \text{conv} \left\{ \bigcup_{i \in \tilde{A}} C(is_1 + q_i s_2, -i) \right\},$$

where $\tilde{A} = \{i \in N_0 : a_i \neq 0\}$. Analogously as in the previous case, if $i \in \tilde{A}$ then the point $(is_1 + q_i s_2, -i)$ is determined by the pseudodifferential operator $a_i \partial_{m_1, t}^i \tilde{P}_i(\partial_{m_2, z})$.

The Newton polygon allows us to describe the multisummable solutions of (8) in case of $\tilde{s}_1 = 0$. First observe that the application of the moment Borel transform $\mathcal{B}_{\tilde{m}_2, z}$, where $\tilde{m}_2$ is a moment function of order $\tilde{s}_2$, gives us for any operator P the formula

$$N(\mathcal{B}_{\tilde{m}_2, z} P, s_1, s_2) = N(P, s_1, s_2 + \tilde{s}_2).$$

Hence, it is sufficient to consider the solutions of (8) in case of $\tilde{s}_2 = 0$.

We find the shape of the Newton polygon for the operators $P(\partial_{m_1, t}, \partial_{m_2, z})$ and $\tilde{P}(\partial_{m_1, t}, \partial_{m_2, z})$ given by (17) with moment functions m_1 and m_2 of positive orders s_1 and s_2 respectively.

So we assume that $P(\lambda, \zeta) = P_0(\zeta) \tilde{P}(\lambda, \zeta)$ and $\deg P_0(\zeta) = k \in N_0$. By the definitions of the Newton polygons for differential and pseudodifferential operators we conclude that the Newton polygon $N(P, s_1, s_2)$ is created from $N(\tilde{P}, s_1, s_2)$ by the translation on the vector $v = (ks_2, 0)$, i.e.

$$(19) \quad N(P, s_1, s_2) = N(\tilde{P}, s_1, s_2) + (ks_2, 0).$$

Now we describe the shape of the Newton polygon for the operator

$$\tilde{P}(\partial_{m_1, t}, \partial_{m_2, z}) = \prod_{\alpha=1}^{\tilde{n}} \prod_{\beta=1}^{l_\alpha} (\partial_{m_1, t} - \lambda_{\alpha\beta}(\partial_{m_2, z}))^{n_{\alpha\beta}},$$

where $\lambda_{\alpha\beta}(\zeta) \sim \lambda_{\alpha\beta} \zeta^{q_\alpha}$ with a pole order $q_\alpha \in \mathcal{Q}$ and a leading term $\lambda_{\alpha\beta} \in \mathbb{C} \setminus \{0\}$ for $\alpha = 1, \dots, \tilde{n}$ and $\beta = 1, \dots, l_\alpha$. As in the previous section we may assume that there exist exactly N positive pole orders which are greater than s_1/s_2 , say $s_1/s_2 < q_N < \dots < q_1 < \infty$.

The set of vertices of $N(\tilde{P}, s_1, s_2)$ consists of $N + 1$ points $v_0, v_1, \dots, v_N$ such that $v_0 = (ns_1, -n)$ and

$$v_\alpha = ((n - n_1 - \dots - n_\alpha)s_1 + (n_1 q_1 + \dots + n_\alpha q_\alpha)s_2, n_1 + \dots + n_\alpha - n)$$

for $\alpha = 1, \dots, N$, where $n_\alpha = \sum_{\beta=1}^{l_\alpha} n_{\alpha\beta}$ is a number of formal solutions of the equation $\tilde{P}(\partial_{m_1, t}, \partial_{m_2, z})u = 0$, which are of Gevrey order $q_\alpha s_2 - s_1$ (see Theorem 1).

Moreover, by (19), $n_\alpha q_\alpha$ is an integer for $\alpha = 1, \dots, N$. Observe also that the vertex v_0 is determined by $\partial_{m_1, t}^n$ and v_α ($\alpha = 1, \dots, N$) by the operator

$$\partial_{m_1, t}^{n-n_1-\dots-n_\alpha} \prod_{\gamma=1}^{\alpha} \prod_{\beta=1}^{l_\gamma} \lambda_{\gamma\beta}^{n_{\gamma\beta}}(\partial_{m_2, z}).$$

It means that the boundary of $N(\tilde{P}, s_1, s_2)$ consists of a horizontal half line Γ_0 , N -segments $\Gamma_1, \dots, \Gamma_N$ and a vertical half line Γ_{N+1} . Let us denote by K_α the slope of Γ_α . Observe that $K_0 = 0$, $K_{N+1} = \infty$ and

$$K_\alpha = \frac{n_\alpha}{s_2 n_\alpha q_\alpha - s_1 n_\alpha} = \frac{1}{q_\alpha s_2 - s_1} \quad \text{for } \alpha = 1, \dots, N.$$

So $K_0 = 0 < K_1 < \dots < K_N < K_{N+1} = \infty$.

Fix $\alpha \in \{1, \dots, N\}$. The set of points of segment Γ_α (with ends $v_{\alpha-1}$ and v_α) is determined by the operator

$$(20) \quad \left(\partial_{m_1, t}^{n-n_1-\dots-n_\alpha} \prod_{\gamma=1}^{\alpha-1} \prod_{\beta=1}^{l_\gamma} \lambda_{\gamma\beta}^{n_{\gamma\beta}}(\partial_{m_2, z}) \right) \prod_{\beta=1}^{l_\alpha} (\partial_{m_1, t} - \lambda_{\alpha\beta}(\partial_{m_2, z}))^{n_{\alpha\beta}}.$$

Moreover the product $\partial_{m_1, t}^{n-n_1-\dots-n_\alpha} \prod_{\gamma=1}^{\alpha-1} \prod_{\beta=1}^{l_\gamma} \lambda_{\gamma\beta}^{n_{\gamma\beta}}(\partial_{m_2, z})$ from (20) is a common part of all the operators determining the points of segment Γ_α . So it is sufficient to consider the rest of the operator (20), i.e. the operator

$$\tilde{P}_{\Gamma_\alpha}(\partial_{m_1, t}, \partial_{m_2, z}) := \prod_{\beta=1}^{l_\alpha} (\partial_{m_1, t} - \lambda_{\alpha\beta}(\partial_{m_2, z}))^{n_{\alpha\beta}}.$$

Let $\hat{u}$ be a formal solution of (8) with $\tilde{s}_1 = \tilde{s}_2 = 0$ determined by $g \in \mathcal{O}(D^2)$ and satisfying the decomposition (18). As in the previous section we conclude that $\hat{u}_{\alpha\beta\gamma}$ is a formal series of Gevrey order $1/K_\alpha$ with respect to t , which satisfies $\tilde{P}_{\Gamma_\alpha}(\partial_{m_1, t}, \partial_{m_2, z}) u_{\alpha\beta\gamma} = g_{\alpha\beta\gamma} \in \mathcal{O}_{1, 1/\kappa}(D^2)$ (for every $\beta \in \{1, \dots, l_\alpha\}$ and $\gamma \in \{1, \dots, n_{\alpha\beta}\}$).

So, the Gevrey order $1/K_\alpha$ of $\hat{u}_{\alpha\beta\gamma}$ is determined by the slope of Γ_α and K_α -summability is determined by the operator $\tilde{P}_{\Gamma_\alpha}(\partial_{m_1, t}, \partial_{m_2, z})$.

References

- [1] Balser, W., *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer-Verlag, New York, 2000.
- [2] Balser, W., Multisummability of formal power series solutions of partial differential equations with constant coefficients, *J. Differential Equations*, **201** (2004), 63–74.
- [3] Balser, W., Power series solutions of the inhomogeneous heat equation, in *Recent trends in Microlocal Analysis*, *Kôkyûroku RIMS*, **1412** (2005), 151–159.

- [4] Balser, W., Duval, A. and Malek, S., Summability of formal solutions for abstract Cauchy problems and related convolution equations, *Ulmer Seminare*, **11** (2006), 29–44.
- [5] Balser, W. and Loday-Richaud, M., Summability of solutions of the heat equation with inhomogeneous thermal conductivity in two variables, *Adv. Dyn. Syst. Appl.*, **4** (2009), 159–177.
- [6] Balser, W. and Yoshino, M., Gevrey order of formal power series solutions of inhomogeneous partial differential equations with constant coefficients, *Funkcial. Ekvac.*, **53** (2010), 411–434.
- [7] Lastra, A., Malek, S. and Sanz, J., Summability in general Carleman ultraholomorphic classes, *J. Math. Anal. Appl.*, **430** (2015), 1175–1206.
- [8] Michalik, S., Summability of formal solutions to the n -dimensional inhomogeneous heat equation, *J. Math. Anal. Appl.*, **347** (2008), 323–332.
- [9] Michalik, S., Multisummability of formal solutions of inhomogeneous linear partial differential equations with constant coefficients, *J. Dyn. Control Syst.*, **18** (2012), 103–133.
- [10] Michalik, S., Analytic solutions of moment partial differential equations with constant coefficients, *Funkcial. Ekvac.*, **56** (2013), 19–50.
- [11] Michalik, S., Summability of formal solutions of linear partial differential equations with divergent initial data, *J. Math. Anal. Appl.*, **406** (2013), 243–260.
- [12] Sanz, J., Summability in a direction of formal power series in several variables, *Asymptot. Anal.*, **29** (2002), 115–141.
- [13] Tahara, H. and Yamazawa, H., Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations, *J. Differential Equations*, **255** (2013), 3592–3637.
- [14] Yonemura, A., Newton Polygons and Formal Gevrey Classes, *Publ. RIMS Kyoto Univ.*, **26** (1990), 197–204.

nuna adreso:

Faculty of Mathematics and Natural Sciences
 College of Science
 Cardinal Stefan Wyszyński University
 Wóycickiego 1/3, 01-938 Warszawa
 Poland
 E-mail: s.michalik@uksw.edu.pl

(Ricevita la 26-an de majo, 2015)

(Revizita la 14-an de januaro, 2016)