Ground States for Semi-Relativistic Schrödinger-Poisson-Slater Energy

By

Jacopo Bellazzini, Tohru Ozawa and Nicola Visciglia

(Università di Sassari, Italy, Waseda University, Japan and Università di Pisa, Italy)

Abstract. We prove the existence of ground states for the semi-relativistic Schrödinger-Poisson-Slater energy

\[ I_{\alpha, \beta}(r) = \inf_{u \in H^{1/2}(\mathbb{R}^3)} \left( \frac{1}{2} \| u \|_{H^{1/2}(\mathbb{R}^3)}^2 + \alpha \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|} \, dx \, dy - \beta \int_{\mathbb{R}^3} |u|^{8/3} \, dx \right) \]

\( \alpha, \beta > 0 \) and \( r > 0 \) is small enough. The minimization problem is \( L^2 \) critical and in order to characterize the values \( \alpha, \beta > 0 \) such that \( I_{\alpha, \beta}(r) > 0 \) for every \( r > 0 \), we prove a new lower bound on the Coulomb energy involving the kinetic energy and the exchange energy. We prove the existence of a constant \( S > 0 \) such that

\[ \frac{1}{S} \left( \int_{\mathbb{R}^3} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x - y|} \, dx \, dy \right)^{1/8} \leq \left( \int_{\mathbb{R}^3} |\varphi(x)|^2 \, dx \right)^{1/2} \]

for all \( \varphi \in C_0^\infty(\mathbb{R}^3) \). Besides, we show that similar compactness property fails if we replace the inhomogeneous Sobolev norm \( \| u \|_{H^{1/2}(\mathbb{R}^3)} \) by the homogeneous one \( \| u \|_{H^{1/2}(\mathbb{R}^3)} \) in the energy above.

Key Words and Phrases. Semi-relativistic Schrödinger equation, Ground states, Concentration-compactness.

2010 Mathematics Subject Classification Numbers. 35J60, 35Q55, 35Q40.

1. Introduction

The aim of this paper is to prove the existence of ground states for the following minimization problem:

\[ I^{x, \beta}(\rho) = \inf_{u \in S(\rho)} \mathcal{E}^{x, \beta}(u) \]

where

\[ \mathcal{E}^{x, \beta}(u) = \frac{1}{2} \| u \|_{H^{1/2}(\mathbb{R}^3)}^2 + \alpha \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x - y|} \, dx \, dy - \beta \int_{\mathbb{R}^3} |u|^{8/3} \, dx, \]

and

\[ S(\rho) = \left\{ u \in H^{1/2}(\mathbb{R}^3) \mid \text{s.t. } \int_{\mathbb{R}^3} |u|^2 \, dx = \rho \right\} \]
for $\alpha, \beta > 0$ and $\rho > 0$ and $H^s(\mathbb{R}^3)$ denotes for general $s \in \mathbb{R}$ the usual Sobolev spaces endowed with the norm:

$$
\|u\|^2_{H^s(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1 + |2\pi \xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi
$$

with $\hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-2\pi i x \cdot \xi} u(x) \, dx$.

The aforementioned energy functional is the semi-relativistic version of the Hartree-Fock energy functional proposed by Slater [18] for a system of electrons interacting with each other via the Coulomb law. In the Hartree-Fock model proposed by Slater [18] the focusing term $\|u\|^8_{L^8(\mathbb{R}^3)}$ is the exchange energy due to the Pauli principle and $\iint |u(x)|^2 |u(y)|^2 |x - y| \, dx \, dy$ describes the repulsive Coulomb interaction while the kinetic energy is given by $(1/2)\|V u\|^2_{L^2(\mathbb{R}^3)}$. The quantity $\rho$ measures the total number of electrons.

In this paper we treat the semi-relativistic case, i.e. considering the kinetic term given by $\|u\|^2_{H^{1/2}(\mathbb{R}^3)}$ instead of the classical term $\|u\|^2_{H^1(\mathbb{R}^3)}$ proposed by Slater [18]. For this reason we call (1.2) the semi-relativistic Schrödinger-Poisson-Slater energy. The main question addressed in this paper is the role of $\alpha$, $\beta$ and $\rho$ in the minimization problem.

It is important to underline the main difficulties to prove the existence of minimizers for the semi-relativistic Schrödinger-Poisson-Slater energy:

- it is not simple to show that $I^{\alpha, \beta}(\rho) > -\infty$ for all $\rho$: the problem is $L^2$ critical due to the fractional Gagliardo-Nirenberg inequality
  $$
  \|\varphi\|^8_{L^{8/3}(\mathbb{R}^3)} \leq C \|\varphi\|^{2/3}_{L^2(\mathbb{R}^3)} \|\varphi\|^2_{H^{1/2}(\mathbb{R}^3)};
  $$

- it is not clear if one can choose a sequence of radially symmetric functions as a minimizing sequence due to the competition between the Coulomb energy term and the kinetic one;

- in $H^{1/2}(\mathbb{R}^3)$ without symmetry informations it not straightforward to prove that a bounded minimizing sequence with some additional assumption has a non vanishing weak limit;

- it is not elementary to avoid dichotomy, i.e. to prove that the weak limit belongs to $S(\rho)$, due to the presence of three terms in the energy functional.

Recall that a general strategy to attack constrained minimization problems is the celebrated concentration-compactness principle of P. L. Lions, see [16]. The main point is that in general if $u_n$ is a bounded minimizing sequence for (1.1) then up to translations two possible bad scenarios can occur (that can be shortly summarized as follows):

- (vanishing) $u_n \rightharpoonup 0$;
- (dichotomy) $u_n \rightharpoonup \bar{u} \neq 0$ and $0 < \|\bar{u}\|_{L^2(\mathbb{R}^3)} < \rho$. 

The vanishing can be excluded typically by proving that any minimizing sequence weakly converges, up to translation, to a function \( \bar{u} \) different from zero (actually it can be accomplished in general by a suitable localized Gagliardo-Nirenberg inequality in conjunction with the Rellich compactness theorem). Concerning the dichotomy, the classical way to rule out this case is to establish the following strong subadditivity inequality

\[
I^{\alpha, \beta}(\rho) < I^{\alpha, \beta}(\mu) + I^{\alpha, \beta}(\rho - \mu) \quad \forall 0 < \mu < \rho.
\]

Although the following weak version of (1.4)

\[
I^{\alpha, \beta}(\rho) \leq I^{\alpha, \beta}(\mu) + I^{\alpha, \beta}(\rho - \mu) \quad \text{for all } 0 < \mu < \rho.
\]

can be easily proved, in general the proof of (1.4) requires some extra arguments which heavily depend on the structure of the functional we are looking at.

The existence of minimizers for semi-relativistic energies is not a novelty, see e.g. [10], [13], [14] for the case \( \beta = 0 \) and \( \alpha < 0 \). We should underline however that when \( \beta = 0 \) and \( \alpha < 0 \), the Boson star minimization problem, we can choose a minimizing sequence consisting of radially symmetric functions. On the other hand, when \( \alpha > 0 \) and \( \beta > 0 \), the existence of ground states was known only for the classical (i.e., non relativistic) Schrödinger-Poisson-Slater energy (1.1), with \( \|u\|^2_{H^{1/2}} \) replaced by \( \|u\|^2_{H^{1}} \). In the classical case with \( \alpha, \beta > 0 \), the existence of ground states for small \( \rho \) is proved in [17], and also in [3] and [4] for the case of the focusing term \( \int |u|^{8/3} \, dx \) replaced by \( \int |u|^p \, dx \) with \( 3 < p < 10/3 \) and \( 2 < p < 3 \) respectively (see also [7] for a review on this subject). We further mention [6] where the classical energy with focusing term \( \int ([|u|^{10/3} - |u|^{8/3}) \, dx \) is studied. It is worth noting that the minimization problems for the classical Schrödinger-Poisson-Slater energies mentioned above are all \( L^2 \) subcritical, namely that the energy is bounded from below and that the minimizing sequence is bounded.

2. Main results

Now we are ready to state our main results. The first result concerns the characterization of the values \( \alpha, \beta > 0 \) yielding \( I^{\alpha, \beta}(\rho) > -\infty \) for every \( \rho > 0 \).

In order to state the first result (Theorem 2.1 below), we introduce the constant \( S \) defined as follows:

\[
S := \sup_{\phi \in C_0^\infty(\mathbb{R}^3)} \frac{\|\phi\|_{L^{8/3}(\mathbb{R}^3)}^{2/3}}{\|\phi\|_{H^{1/2}(\mathbb{R}^3)}^{1/2}} \left( \int_{\mathbb{R}^3} \left( \frac{\|\phi(x)\|^{8/3}_{L^8(\mathbb{R}^3)}}{|x-y|} \right) \, dx \right)^{1/8}.
\]
where \( \|\varphi\|_2^{2} = \int_{\mathbb{R}^{3}} |2\pi \xi | |\hat{\varphi}(\xi)|^{2} d\xi \). In the next section we shall prove that \( S < +\infty \) by proving the following inequality: for some \( C > 0 \)

\[
(2.1) \quad \|\varphi\|_{L^{\infty}(\mathbb{R}^{3})} \leq C\|\varphi\|_{H^{1/2}(\mathbb{R}^{3})}^{1/2} \left( \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{|\varphi(x)|^{2} |\varphi(y)|^{2}}{|x - y|} dxdy \right)^{1/8}.
\]

Our first result is:

**Theorem 2.1.** Let \( \alpha, \beta > 0 \) be fixed. Then the following facts are equivalent:

(i) \( \exists \rho > 0 \) s.t. \( I^{\alpha, \beta}(\rho) = -\infty \);  
(ii) \( (27\alpha/\beta)^{1/8} < \sqrt{2}S \).

The second result concerns the existence of minimizers for the semi-relativistic Schrödinger-Poisson-Slater energy.

**Theorem 2.2.** For every \( \alpha, \beta > 0 \) there exists \( r \) with \( 0 < r < r_{1} \) such that \( I^{\alpha, \beta}(r) = -\infty \). Moreover for every sequence \( \{u_{n}\} \) which satisfy:

\( u_{n} \in S(r) \) and \( E^{\alpha, \beta}(u_{n}) \to I^{\alpha, \beta}(r) \), with \( 0 < r < r_{1} \)

there exists \( \tau_{n} \in \mathbb{R}^{3} \) such that, up to subsequence, we have

\( u_{n}(\cdot + \tau_{n}) \) has a strong limit in \( H^{1/2}(\mathbb{R}^{3}) \).

In particular the set of minimizers for \( I^{\alpha, \beta}(r) \) is not empty for \( 0 < r < r_{1} \).

In our opinion Theorem 2.2 is quite surprising in view of the next non-existence result. First we introduce the following minimization problems

\[
\tilde{I}^{\alpha, \beta}(\rho) = \inf_{u \in S(\rho)} \tilde{E}^{\alpha, \beta}(u)
\]

where

\[
(2.2) \quad \tilde{E}^{\alpha, \beta}(u) = \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^{3})}^{2} + \alpha \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{|u(x)|^{2} |u(y)|^{2}}{|x - y|} dxdy - \beta \int_{\mathbb{R}^{3}} |u|^{8/3} dx.
\]

and

\[
\|u\|_{2}^{2} = \int_{\mathbb{R}^{3}} |2\pi \xi | |\hat{u}(\xi)|^{2} d\xi
\]

(here \( \hat{u}(\xi) \) denotes the Fourier transform of \( u \) and \( S(\rho) \) is defined in (1.3)). We note that the only difference between \( E^{\alpha, \beta} \) and \( \tilde{E}^{\alpha, \beta} \) is that the kinetic energy term (quadratic term) of the latter is homogeneous and that of the former inhomogeneous.
Theorem 2.3. For every \( \alpha, \beta > 0 \) there exists \( \tilde{\rho} = \tilde{\rho}(\alpha, \beta) > 0 \) such that:

- \( I^{\alpha, \beta}(\rho) > -\infty \) \( \forall \rho \in (0, \tilde{\rho}) \);
- \( \forall \rho \in (0, \tilde{\rho}) \) and \( \forall v \in S(\rho) \) we have \( \tilde{\mathcal{I}}^{\alpha, \beta}(v) > I^{\alpha, \beta}(\rho) \) (i.e. there are not minimizers for \( I^{\alpha, \beta}(\rho) \) with \( \rho \) small).

Remark 2.1. The statement of Theorem 2.2 holds true, if we replace the exchange energy by \( k_u k_p L_p(R^3) \) for \( 2 < p < 8/3 \). In this case the minimization problem is \( L^2 \) subcritical such that the energy is bounded from below for all \( \alpha, \beta > 0 \). The existence of ground states for \( 2 < p < 8/3 \) can be proved in much the same way as it is for \( p = 8/3 \) (Theorem 2.2).

We conclude the introduction with discussing the connection between minimizers for (1.1) and steady states of a suitable semi-relativistic nonlinear Schrödinger Equation.

By using the well-known property \( ||w||^2_{H^{1/2}(R^3)} \leq ||w||^2_{H^{1/2}(R^3)} \), where equality occurs if and only if there exists \( \theta \in R \) such that \( e^{i\theta}w \) is real-valued (see for instance [11]), one can deduce that if \( v(x) \) is a minimizer for (1.1) then there exists \( \theta \in R \) such that \( e^{i\theta}v \) is real-valued. In particular, defining \( \sqrt{I - \Delta} \) as the inverse Fourier transform of \( \sqrt{1 + |2\pi \xi|^2} \) (see e.g. [11]), any minimizer \( v \) to (1.1) solves the following equation:

\[
\sqrt{1 - \Delta} v + 4\alpha(|x|^{-1} * |v|^2)v - \frac{8}{3}\beta|v|^{2/3}v = \omega v \quad \text{in } R^3
\]

for a suitable Lagrange multiplier \( \omega \in R \). Moreover the corresponding time-dependent function

\[
\psi(x, t) = e^{-\omega t}v(x)
\]

is a solution of the time-dependent Nonlinear Schrödinger Equation

\[
i\psi_t = \sqrt{1 - \Delta} \psi + 4\alpha(|x|^{-1} * |\psi|^2)\psi - \frac{8}{3}\beta|\psi|^{2/3}\psi \quad \text{in } R^3.
\]

As far as we know this evolutionary problem has not been studied in the literature. In this context we quote the paper [16] where it is studied the following Cauchy problem:

\[
i\psi_t = \sqrt{1 - \Delta} \psi - (|x|^{-1} * |\psi|^2)\psi \quad \text{in } R^3.
\]

In this case the main advantage is the smoothing effect associated with the Hartree nonlinearity which allows us to solve the Cauchy problem by using the classical energy estimates. On the contrary the nonlinearity in (2.5) does not enjoy the same smoothness and it makes more complicated the analysis of the corresponding Cauchy problem.
Acknowledgement. The authors would like to thank T. Cazenave, V. Georgiev and L. Vega for fruitful conversations.

3. Proof of Theorem 2.1

This section is devoted to proving Theorem 2.1.

Proposition 3.1 ([2]). There exists $C > 0$ such that

$$
\|\varphi\|_{L^{3/1}(\mathbb{R}^3)} \leq C \|\varphi\|_{H^{1/2}(\mathbb{R}^3)}^{1/2} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x - y|} \, dx \, dy \right)^{1/8}
$$

It is worth noting here that inequality (2.1) has been recently proved in [2] to involve the general $L^p$-norm, $H^s$-norm and potentials of the form $|x|^{-\lambda}$ with appropriate combinations of indices $p, s$ and $\lambda$ (see Proposition 2.1 in [2]). The proof of the Theorem 2.1 follows from (2.1) and the subsequent Propositions 3.2 and 3.3. In the sequel the energy $\tilde{\mathcal{E}}^{x, \beta}$ is the one defined in (2.2).

Proposition 3.2. The following facts are equivalent: for any $\rho > 0$

(i) $I^{x, \beta}(\rho) = -\infty$

(ii) We have $\tilde{\mathcal{E}}^{x, \beta}(\varphi) < 0$, for some $\varphi \in S(\rho)$

Proof. If $I^{x, \beta}(\rho) = -\infty$ then there exists $\varphi \in S(\rho)$ such that $\mathcal{E}^{x, \beta}(\varphi) < 0$ and hence $\tilde{\mathcal{E}}^{x, \beta}(\varphi) < 0$.

Next we prove the opposite implication. We introduce $\varphi_\theta(x) = \theta^{3/2} \varphi(\theta x)$ then by a scaling argument

$$
\tilde{\mathcal{E}}^{x, \beta}(\varphi_\theta) = \theta \tilde{\mathcal{E}}^{x, \beta}(\varphi).
$$

Noticing that

$$
\|\varphi_\theta\|_{H^{1/2}}^2 - \|\varphi_\theta\|_{H^{1/2}}^2 = \int \sqrt{1 + \frac{|2\pi \xi|^2}{\theta^3}} \left| \varphi (\frac{\xi}{\theta}) \right|^2 \, d\xi - \int \frac{|2\pi \xi| |\varphi(\xi)|^2}{\theta^3} \, d\xi
$$

$$
= \int \left( \sqrt{1 + \theta^2 |2\pi \xi|^2} - \theta |2\pi \xi| \right) |\varphi(\xi)|^2 \, d\xi
$$

$$
= \int \frac{1}{\sqrt{1 + \theta^2 |2\pi \xi|^2 + \theta |2\pi \xi|}} |\varphi(\xi)|^2 \, d\xi = o(1),
$$

as $\theta \to \infty$, so that we have

$$
\mathcal{E}^{x, \beta}(\varphi_\theta) = \tilde{\mathcal{E}}^{x, \beta}(\varphi_\theta) + \frac{1}{2} (\|\varphi_\theta\|_{H^{1/2}}^2 - \|\varphi_\theta\|_{H^{1/2}}^2)
$$

$$
= \theta \tilde{\mathcal{E}}^{x, \beta}(\varphi) + o(1)
$$
and hence

$$I_{\alpha, \beta}^{\ast} (\rho) \leq \lim_{\theta \to \infty} \tilde{\mathcal{E}}_{\alpha, \beta}^{\ast} (\varphi_\theta) = -\infty.$$  \hfill \Box

**Proposition 3.3.** The following facts are equivalent:

(i) \( \tilde{\mathcal{E}}_{\alpha, \beta}^{\ast} (\varphi) \geq 0 \ \forall \varphi \in H^{1/2}, \)

(ii) \( (27 \alpha / \beta^2)^{1/8} \geq \sqrt{2} S. \)

**Proof.** Let \( \varphi_\theta (x) = \varphi(x/\theta) \) for \( \theta \in (0, \infty). \) Then we have

\[
\inf_{\varphi \in H^{1/2}(\mathbb{R}^3)} \tilde{\mathcal{E}}_{\alpha, \beta}^{\ast} (\varphi) \geq 0
\]

if and only if

\[
\inf_{\theta \in (0, \infty)} \tilde{\mathcal{E}}_{\alpha, \beta}^{\ast} (\varphi_\theta) \geq 0 \quad \text{for any } \varphi \in H^{1/2}.
\]

By explicit computation this is equivalent to

\[
\frac{1}{2} \theta^2 \| \varphi \|^2_{H^{1/2}} + \alpha \theta^3 \iint |\varphi(x)|^2 |\varphi(y)|^2 \frac{dxdy}{|x-y|} - \beta \theta^3 \| \varphi \|_{8/3}^8 \geq 0
\]

\( \forall \varphi \in H^{1/2}, \ \theta \in (0, \infty). \)

Hence the condition \( \tilde{\mathcal{E}}_{\alpha, \beta}^{\ast} (\varphi) \geq 0 \ \forall \varphi \in H^{1/2} \) can be rewritten as follows:

\[
(3.1) \quad \inf_{\theta \in (0, \infty)} J_{\varphi}^{\alpha, \beta} (\theta) \geq 0 \quad \forall \varphi \in H^{1/2}, \ \theta \in (0, \infty)
\]

where

\[
J_{\varphi}^{\alpha, \beta} (\theta) = \frac{1}{2} \| \varphi \|^2_{H^{1/2}(\mathbb{R}^3)} + \alpha \theta^3 \iint |\varphi(x)|^2 |\varphi(y)|^2 \frac{dxdy}{|x-y|} - \beta \theta \| \varphi \|_{8/3}^{8/3}(\mathbb{R}^3)
\]

By elementary computation we get

\[
\inf_{(0, \infty)} J_{\varphi}^{\alpha, \beta} (\theta) = \tilde{J}_{\varphi}^{\alpha, \beta} \left( \| \varphi \|_{8/3}^{4/3} \sqrt{\frac{\beta \iint |\varphi(x)|^2 |\varphi(y)|^2 \frac{dxdy}{|x-y|}}{3\alpha}} \right)^{-1}
\]

\[
= \frac{1}{2} \| \varphi \|^2_{H^{1/2}(\mathbb{R}^3)} - \left( \frac{\alpha}{3\alpha} \right)^{3/2} \beta \sqrt{3\alpha} \sqrt{\iint |\varphi(x)|^2 |\varphi(y)|^2 \frac{dxdy}{|x-y|}}
\]

\[
= \frac{1}{2} \| \varphi \|^2_{H^{1/2}(\mathbb{R}^3)} - \frac{2}{3} \beta \sqrt{\frac{3\alpha}{\iint |\varphi(x)|^2 |\varphi(y)|^2 \frac{dxdy}{|x-y|}}}.
\]
Hence the condition (3.1) becomes
\[ \frac{4}{27z}\|\phi\|_{L^{3}\beta(R^{3})}^4 \leq \|\phi\|_{H^{1/2}(R^{3})}^2 \int \int \frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|} \, dx \, dy \quad \forall \phi \in H^{1/2}(R^{3}) \]
and we can conclude since by definition \( S \) is the best constant in the inequality
\[ \|\phi\|_{L^{3}\beta(R^{3})} \leq S\|\phi\|_{H^{1/2}(R^{3})}^{1/2} \left( \int \int \frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|} \, dx \, dy \right)^{1/8} \quad \forall \phi \in H^{1/2}(R^{3}). \]

4. Proof of Theorem 2.2

First we quote a recent result of [2] to avoid the vanishing. It is a generalization of the Lieb Translation Lemma which holds in \( H^1 \), see [12].

**Lemma 4.1** (Lieb Translation Lemma in \( \mathcal{H}_s \), Lemma 2.1 in [2]). Let \( s > 0, 1 < p < \infty \) and \( u_n \in \mathcal{H}_s(R^d) \cap L^p(R^d) \) be a sequence with
\[ \sup_n (\|u_n\|_{H^s} + \|u_n\|_{L^p}) < \infty \]
and, for some \( \eta > 0 \), (with \( |\cdot| \) denoting Lebesgue measure)
\[ \inf_n \{|\{u_n\} > \eta\} > 0. \]

Then there is a sequence \( (x_n) \subset R^d \) such that a subsequence of \( u_n(\cdot + x_n) \) has a weak limit \( u \neq 0 \) in \( \mathcal{H}_s(R^d) \cap L^p(R^d) \).

Now we state four propositions that are important in the sequel.

**Proposition 4.1.** For every \( \alpha, \beta > 0 \) there exists \( \rho_0 = \rho_0(\beta) > 0 \) such that
\[ I^{\alpha,\beta}(\rho) \geq 0 \quad \forall 0 < \rho < \rho_0. \]
Moreover if \( u_n \in S(\rho) \) is a minimizing sequence for \( I^{\alpha,\beta}(\rho) \) with \( 0 < \rho < \rho_0 \) then
\[ \sup_n \|u_n\|_{H^{1/2}(R^3)} < \infty. \]

**Proof.** It follows from the following estimate
\[ e^{\alpha,\beta}(\varphi) \geq \frac{1}{2} \|\varphi\|_{H^{1/2}(R^3)}^2 - C\|\varphi\|_{L^3(R^3)}^{2/3}\|\varphi\|_{H^{1/2}(R^3)}^2 \]
where we have used the Hölder inequality in conjunction with the Sobolev embedding \( H^{1/2}(R^3) \subset L^3(R^3) \). \( \square \)
Proposition 4.2. Let $\alpha, \beta > 0$ be fixed and $\bar{\rho} = \bar{\rho}(\alpha, \beta) > 0$ be such that $I^{\alpha, \beta}(\rho) > -\infty$ for $\rho \in (0, \bar{\rho})$. Then the function

$$(0, \bar{\rho}) \ni \rho \to I^{\alpha, \beta}(\rho) \in R$$

is continuous.

Proof. Assume it is not continuous, then there exists a sequence $\rho_n$ and $\varepsilon > 0$ such that $\lim_{n \to \infty} \rho_n = \rho^* > 0$ and $|I^{\alpha, \beta}(\rho_n) - I^{\alpha, \beta}(\rho^*)| \geq \varepsilon > 0$. In particular up to subsequence we can assume that either

$$I^{\alpha, \beta}(\rho_n) - I^{\alpha, \beta}(\rho^*) \geq \varepsilon \quad \text{(4.3)}$$

or

$$I^{\alpha, \beta}(\rho^*) - I^{\alpha, \beta}(\rho_n) \geq \varepsilon \quad \text{(4.4)}$$

First we shall prove that (4.3) cannot occur. We fix $w \in H^{1/2}$ such that

$$w \in S(\rho^*) \quad \text{and} \quad \delta^{\alpha, \beta}(w) - I^{\alpha, \beta}(\rho^*) \leq \frac{\varepsilon}{2} \quad \text{(4.5)}$$

and we introduce

$$w_n = \sqrt{\frac{\rho_n}{\rho^*}} w.$$ 

Notice that

$$w_n \in S(\rho_n) \quad \text{and} \quad \lim_{n \to \infty} \delta^{\alpha, \beta}(w_n) = \delta^{\alpha, \beta}(w). \quad \text{(4.6)}$$

By combining (4.6) with (4.5) we get the existence of $\bar{n} \in N$ such that

$$I^{\alpha, \beta}(\rho_n) \leq \delta^{\alpha, \beta}(w_n) \leq \delta^{\alpha, \beta}(w) + \frac{\varepsilon}{4} \leq I^{\alpha, \beta}(\rho^*) + \frac{3}{4} \varepsilon \quad \forall n > \bar{n} \quad \text{(4.7)}$$

which is contrary to (4.3).

In order to deny the case of (4.4) we argue as follows. Let $v_n \in H^{1/2}$ such that

$$v_n \in S(\rho_n) \quad \text{and} \quad \delta^{\alpha, \beta}(v_n) - I^{\alpha, \beta}(\rho_n) \leq \frac{\varepsilon}{2} \quad \text{(4.8)}$$

We can prove the following:

Claim. We can choose a sequence $v_n$ that satisfies (4.8) and moreover

$$\sup_n \|v_n\|_{H^{1/2}(R^3)} < \infty.$$
Assuming this claim for now, we proceed. It is easy to prove that

\[(4.9) \quad \lim_{n \to \infty} (\mathcal{E}^{\alpha,\beta}(v_n) - \mathcal{E}^{\alpha,\beta}(u_n)) = 0.\]

where

\[u_n = \sqrt{\frac{\rho^*}{p_n}} v_n \in S(\rho^*).\]

By combining (4.8) with (4.9) we can fined \(n \in N\) such that

\[I^{\alpha,\beta}(\rho^*) \leq \mathcal{E}^{\alpha,\beta}(u_n) \leq \mathcal{E}^{\alpha,\beta}(v_n) + \frac{\varepsilon}{4} \leq I^{\alpha,\beta}(\rho_n) + \frac{3}{4}\varepsilon \quad \forall n > n\]

hence contradicting (4.4).

Next we shall prove the claim. Notice that if (4.4) is true then

\[K = \sup_n I^{\alpha,\beta}(\rho_n) < \infty\]

and we deduce that \(v_n\) can be choosen in such a way that:

\[K + 1 \geq \mathcal{E}^{\alpha,\beta}(v_n) \geq h_{\rho_n}(\|v_n\|_{H^{1/2}})\]

where \(h_{\rho_n}(t) = t^2/2 - C p_n^{1/3} t^2\). It is now easy to deduce the claim since for every \(M > 0\) there exists \(R > 0\) such that

\[h_{\rho_n}(t) \geq M \quad \forall t \geq R \quad \forall n \in N.\]

**Proposition 4.3.** For every \(\alpha, \beta > 0\) there exists \(\rho_1 = \rho_1(\alpha, \beta) > 0\) such that

\[(4.10) \quad \frac{I^{\alpha,\beta}(\rho)}{\rho} < \frac{1}{2} \quad \forall 0 < \rho < \rho_1.\]

Moreover

\[(4.11) \quad \lim_{\rho \to 0} \frac{I^{\alpha,\beta}(\rho)}{\rho} = \frac{1}{2}.

Proof of (4.10). Now, defining the symbol \(\langle \xi \rangle = \sqrt{1 + |2\pi \xi|^2}\), we introduce the functional

\[(4.12) \quad \mathcal{F}^{\alpha,\beta}(u) = \mathcal{E}^{\alpha,\beta}(u) - \frac{1}{2}\|u\|_2^2 = \mathcal{E}^{\alpha,\beta}(u) - \frac{1}{2}\|\hat{u}\|_2^2 = \frac{1}{2} \int \frac{|2\pi \xi|^2}{1 + \langle \xi \rangle} |\hat{u}|^2 d\xi + \alpha \int \frac{|u(x)|^2 |u(y)|^2}{|x - y|} dxdy - \beta \|u\|_{L^{8/3}(\mathbb{R}^n)}^{8/3}\]
where we have used the Plancharel identity. Notice that (4.10) is equivalent to show that

\[(4.13) \quad \inf_{u \in S(\rho)} \mathcal{F}^{x, \beta}(u) < 0 \quad \forall 0 < \rho < \rho_1\]

with \(\rho_1\) small enough. In order to prove (4.13) we fix \(\varphi \in C^\infty_0(R^3)\) such that \(\varphi \in S(1)\) and we introduce \(\varphi_0 = \varphi(\theta x)\) where \(\gamma\) will be chosen later. Notice that by looking at the expression of \(\mathcal{F}^{x, \beta}\) in (4.12) we get

\[
\inf_{u \in S(\theta^{2\gamma-3})} \mathcal{F}^{x, \beta}(u) \leq \mathcal{F}^{x, \beta}(\varphi_0) \leq \frac{1}{2} \int |2\pi \xi|^2 |\varphi_0|^2 d\xi
\]

\[+ \, \frac{2}{\theta} \int \int \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} dx dy - \beta \|\varphi_0\|_{L^{8/3}(R^3)}^{8/3}]
\]

\[= \frac{1}{2} \theta^{2\gamma-1} \|\varphi_0\|_{H^1(R^3)}^2 + \, \theta^{4\gamma-5} \int \int \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} dx dy
\]

\[- \, \beta \theta^{(8/3)\gamma-3} \|\varphi_0\|_{L^{8/3}(R^3)}^{8/3}.
\]

We note that the last term of the formula above is negative for \(0 < \gamma < \gamma_0\), if we can choose \(\gamma\) such that

\[2\gamma - 1 > 0, \quad 4\gamma - 5 > 0, \quad \frac{8}{3} \gamma - 3 > 0
\]

\[\frac{8}{3} \gamma - 3 < 2\gamma - 1, \quad \frac{8}{3} \gamma - 3 < 4\gamma - 5.
\]

Indeed, these conditions above are satisfied for any \(\gamma \in (3/2, 3)\). Notice that \(\varphi_0 \in S(\theta^{2\gamma-3})\), and that \(\theta^{2\gamma-3} \to 0\) when \(\theta \to 0\) if \(\gamma > 3/2\).

**Proof of (4.11).** Due to (4.10) it is sufficient to prove that

\[\liminf_{\rho \to 0} \frac{I^{x, \beta}(\rho)}{\rho} \geq \frac{1}{2}.
\]

For every \(\rho > 0\) we take a minimizing sequence \(u_n \in S(\rho)\) for \(I^{x, \beta}(\rho)\) and fix it, so that we have

\[
\frac{I^{x, \beta}(\rho)}{\rho} \geq \limsup_{n \to \infty} \left( \frac{1}{2} \rho^{-1} \|u_n\|_{H^{1/2}(R^3)}^2 - \beta \rho^{-1} \|u_n\|_{L^{8/3}(R^3)}^{8/3} \right).
\]

Notice that \(\limsup_{n \to \infty} (1/2) \rho^{-1} \|u_n\|_{H^{1/2}(R^3)}^2 \geq \limsup_{n \to \infty} (1/2) \rho^{-1} \|u_n\|_{L^2(R^3)}^2 = 1/2\) hence it is sufficient to prove that

\[\limsup_{\rho \to 0} \left( \limsup_{n \to \infty} \rho^{-1} \|u_n\|_{L^{8/3}(R^3)}^{8/3} \right) = 0.
\]
This fact will follow by combining next claim with the usual Sobolev embedding \( H^{1/2} \subset L^{8/3} \).

Claim.

(4.14) \( \exists \bar{\rho} > 0, C > 0 \text{ s.t. } \limsup_{n \to \infty} ||u_n||_{H^{1/2}(\mathbb{R}^3)} < C\sqrt{\bar{\rho}} \quad \forall \rho < \bar{\rho}. \)

By Hölder inequality, Sobolev embedding theorem and (4.10), we get:

(4.15) \( \frac{1}{2}\rho > I^{x,\beta}(\rho) = \lim_{n \to \infty} \delta^{x,\beta}(u_n) \geq \limsup_{n \to \infty} h_{\rho}(||u_n||_{H^{1/2}(\mathbb{R}^3)}) \quad \forall 0 < \rho < \rho_1 \)

where \( h_{\rho}(t) = t^2/2 - C_\rho^{1/3}t^2 \geq t^2/4 \) (for \( \rho \) small enough) and \( u_n \in S(\rho) \) is a minimizing sequence for \( I^{x,\beta}(\rho) \). This implies that

\[ \limsup_{n \to \infty} ||u_n||_{H^{1/2}(\mathbb{R}^3)} \leq C\sqrt{\rho} \]

for \( 0 < \rho < \rho_1 \) with \( \rho_1 \) suitable small number.

**Proposition 4.4** [Concentration-Compactness]. Let \( x, \beta > 0 \) be fixed. Let \( \rho > 0 \) be such that \( I^{x,\beta}(\rho) > -\infty \). Assume moreover

(4.16) \( \rho I^{x,\beta}(\rho') < \rho' I^{x,\beta}(\rho) \quad \text{for any } \rho' \in (0, \rho). \)

Then for every minimizing sequence \( u_n \in S(\rho) \) for \( I^{x,\beta}(\rho) \), there exists \( \tau_n \in \mathbb{R}^3 \), such that, for some subsequences, \( u_n(\cdot + \tau_n) \) converge strongly to \( \bar{u} \) in \( H^{1/2} \).

**Proof of Proposition 4.4.** Recall that by Proposition 4.1 we can assume \( \sup_n ||u_n||_{H^{1/2}(\mathbb{R}^3)} < \infty. \)

**First step: no-vanishing**

First we prove the following

Claim. \( \exists \varepsilon_0 > 0 \text{ s.t. } ||u_n||_{L^{8/3}(\mathbb{R}^3)} \geq \varepsilon_0 \)

Assume it is not true then \( \lim_{n \to \infty} ||u_n||_{L^{8/3}(\mathbb{R}^3)} = 0 \) and in particular

\[ I^{x,\beta}(\rho) = \lim_{n \to \infty} \frac{1}{2} ||u_n||_{H^{1/2}(\mathbb{R}^3)}^2 + z \left\{ \int \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} \, dx dy - \beta ||u_n||_{L^{8/3}(\mathbb{R}^3)}^{8/3} \right\} \]

\[ \geq \lim_{n \to \infty} \frac{1}{2} ||u_n||_{L^2(\mathbb{R}^3)}^2 = \frac{1}{2} \rho \]

which contradicts (4.10).
By the well known PQR Lemma (see [15]), this claim together with 
\[ \sup_{n \in \mathbb{N}} (\|u_n\|_{L^2} + \infty, \|u_n\|_{L^3}) < \infty, \] 
implies that there exists of \( \eta > 0 \) such that 
\[ \inf_n \{x \in \mathbb{R}^3 | |u_n| > \eta\} > 0. \]
Thus Lieb’s Translation Lemma in \( H^{1/2}(\mathbb{R}^3) \) gives us what we want.

Second step: \( v_n \) converges strongly to \( \bar{v} \) in \( H^{1/2}(\mathbb{R}^3) \)

It is sufficient to prove that \( v_n \) converges strongly to \( \bar{v} \) in \( L^2 \) (then the strong convergence in \( H^{1/2}(\mathbb{R}^3) \) follows by the fact that \( v_n \) is a minimizing sequence for \( I^{\alpha,\beta}(\rho) \)). In particular it is sufficient to prove that \( \|\bar{v}\|^2_{L^2(\mathbb{R}^3)} = \rho \). Suppose the contrary that \( \|\bar{v}\|^2_{L^2(\mathbb{R}^3)} = \delta \in (0, \rho) \), then since \( L^2 \) and \( H^{1/2} \) are Hilbert spaces we get:

(4.17) \[ \|v_n - \bar{v}\|^2_{L^2(\mathbb{R}^3)} = \rho - \delta + o(1) \]

and also

(4.18) \[ \|v_n - \bar{v}\|^2_{H^{1/2}(\mathbb{R}^3)} = \|v_n\|^2_{H^{1/2}(\mathbb{R}^3)} - \|\bar{v}\|^2_{H^{1/2}(\mathbb{R}^3)} + o(1). \]

Moreover, up to subsequence, we can assume that

\[ v_n(x) \rightarrow \bar{v}(x) \quad \text{a.e. } x \in \mathbb{R}^3. \]

Hence via the Brézis-Lieb Lemma (see [5]) we get

(4.19) \[ \|v_n - \bar{v}\|^p_{L^p(\mathbb{R}^3)} = \|v_n\|^p_{L^p(\mathbb{R}^3)} - \|\bar{v}\|^p_{L^p(\mathbb{R}^3)} + o(1) \]

and by a “non local version” of Brezis-Lieb Lemma (see Lemma 2.2 of [2]) we have

(4.20) \[ \int \int \frac{|(v_n - \bar{v})(x)|^2 |(v_n - \bar{v})(y)|^2}{|x - y|} \, dx \, dy \]

\[ = \int \int \frac{|v_n(x)|^2 |v_n(y)|^2}{|x - y|} \, dx \, dy - \int \int \frac{|\bar{v}(x)|^2 |\bar{v}(y)|^2}{|x - y|} \, dx \, dy + o(1). \]

By combining (4.17), (4.18), (4.19), (4.20) and the fact that \( v_n \) is a minimizing sequence for \( I^{\alpha,\beta}(\rho) \) we get

\[ I^{\alpha,\beta}(\rho) = \mathcal{E}^{\alpha,\beta}(v_n) + o(1) = \mathcal{E}^{\alpha,\beta}(v_n - \bar{v}) + \mathcal{E}^{\alpha,\beta}(\bar{v}) + o(1) \]

\[ \geq I^{\alpha,\beta}(\rho - \delta + o(1)) + I^{\alpha,\beta}(\delta) + o(1) \]

and in particular by the continuity of the function \( I^{\alpha,\beta}(\rho) \) (see Proposition 4.2) we get

(4.21) \[ I^{\alpha,\beta}(\rho) \geq I^{\alpha,\beta}(\rho - \delta) + I^{\alpha,\beta}(\delta). \]
On the other hand, we have by (4.16) that for \(0 < \delta < \rho\)

\[
I^{x,\beta}(\rho - \delta) > \frac{\rho - \delta}{\rho} I^{x,\beta}(\rho)
\text{ and } I^{x,\beta}(\delta) > \frac{\delta}{\rho} I^{x,\beta}(\rho)
\]

which imply

\[
I^{x,\beta}(\rho - \delta) + I^{x,\beta}(\delta) > I^{x,\beta}(\rho)
\]

hence contradicting (4.21).

Proof of Theorem 2.2. We shall prove the existence of \(\rho_1\) sufficiently small with \(0 < \rho_1 \leq \rho(x,\beta)\), such that there exist ground states for all \(0 < \rho < \rho_1\).

First we prove the existence of a sequence of ground states for \(I^{x,\beta}(\rho_n)\)

Claim. \(\exists\) a sequence \(\rho_n \to 0\), and \(u_n \in S(\rho_n)\) s.t. \(I^{x,\beta}(\rho_n) = E^{x,\beta}(u_n)\).

The proof of the claim follows from a continuity argument. Fix \(\varepsilon > 0\), and define

\[
\rho_\varepsilon := \inf \left\{ \rho > 0, \text{ s.t. } \frac{I^{x,\beta}(\rho)}{\rho} = \frac{1}{2} - \varepsilon \right\}.
\]

By Proposition 4.3 and Proposition 4.2, \(\rho_\varepsilon > 0\) and

\[
(4.22) \quad \frac{I^{x,\beta}(\rho_\varepsilon)}{\rho_\varepsilon} < \frac{I^{x,\beta}(\rho)}{\rho} \quad \forall 0 < \rho < \rho_\varepsilon.
\]

The existence of a ground state for \(I^{x,\beta}(\rho_\varepsilon)\) follows from Proposition 4.4, since (4.22) is exactly the same condition as (4.16). Sending \(\varepsilon \to 0\) we get the claim.

By Proposition 4.4 it is sufficient to prove the monotonicity of \(I^{x,\beta}(\rho_\varepsilon)/\rho\) for all \(0 < \rho < \rho_1\). Fix \(\rho > 0\), and define \(c = \min_{[0,\rho]} I^{x,\beta}(s)/s < 1/2\) and

\[
\rho_0 := \inf \left\{ s \in (0,\rho], \text{ s.t. } \frac{I^{x,\beta}(s)}{s} = c \right\}.
\]

We have to prove that \(\rho_0 = \rho\).

Suppose the contrary that \(\rho_0 < \rho\). By the claim above, we have a ground state \(u_{\rho_0}\) for \(I^{x,\beta}(\rho_0)\). If the monotonicity of \(I^{x,\beta}(s)/s\) broke at \(\rho_0\), the following would hold:

\[
(4.23) \quad \frac{E^{x,\beta}(u_{\rho_0})}{\rho_0} = \frac{I^{x,\beta}(\rho_0)}{\rho_0} \leq \frac{I^{x,\beta}(\rho_0)}{\rho_0} \leq \frac{E^{x,\beta}(\theta u_{\rho_0})}{\theta^2 \rho_0}
\]

for all \(0 < \theta < 1\) and for a sequence \(\theta_n \to 1\) with \(\lim_{n \to \infty} \theta_n = 1\) (we shall consider only a sequence because \(I^{x,\beta}(\rho)/\rho\) can be oscillating for \(\rho > \rho_0\)). Inequality (4.23) implies that

\[
\frac{d}{d\theta} \left( \frac{E^{x,\beta}(\rho_0)}{\rho_0} - \frac{E^{x,\beta}(\theta u_{\rho_0})}{\theta^2 \rho_0} \right)_{\theta=1} = 0
\]
which is equivalent to
\[
2a \int \left( \frac{|u_{\rho_0}(x)|^2 |u_{\rho_0}(y)|^2}{|x-y|} \right) dxdy - \frac{2}{3} \beta \|u_{\rho_0}\|_{L^{8/3}(\mathbb{R}^3)}^8 = 0.
\]

To conclude the proof it suffices to apply the Hardy-Littlewood-Sobolev inequality and the interpolation inequality to get
\[
\|u_{\rho_0}\|_{L^{8/3}(\mathbb{R}^3)}^8 = \frac{3}{(2b)k^{8/3}} \leq C \|u_{\rho_0}\|_{L^{12/3}(\mathbb{R}^3)}^4 \leq C \rho_0^{2/3} \|u_{\rho_0}\|_{L^{8/3}(\mathbb{R}^3)}^8
\]
which cannot hold if \(\rho_0\) and hence \(\rho\) and \(\rho_1\) are sufficiently small.

\(\square\)

5. Proof of Theorem 2.3

We shall need the following lemma.

Lemma 5.1. For each \(\rho > 0\), the following dichotomy occurs:

\[
\text{either } \tilde{I}^{\alpha,\beta}(\rho) = 0 \text{ or } I^{\alpha,\beta}(\rho) = -\infty.
\]

Moreover there exists \(\bar{\rho} > 0\) such that

\[
\text{\forall } \rho \in (0, \bar{\rho})
\]

\[
\tilde{I}^{\alpha,\beta}(\rho) = 0.
\]

Proof. First step: \(\tilde{I}^{\alpha,\beta}(\rho) \leq 0\)

We choose any \(\varphi \in C_0^\infty(\mathbb{R}^3)\) such that \(\|\varphi\|_{L^2(\mathbb{R}^3)}^2 = \rho\), and put \(\varphi_0 = \theta^{3/2}\varphi(\theta x)\). Then \(\|\varphi\|_{L^2(\mathbb{R}^3)}^2 = \rho\).

By direct computations, we have
- \(\|\varphi\|_{L^{8/3}(\mathbb{R}^3)}^8 = \theta\|\varphi\|_{L^{8/3}(\mathbb{R}^3)}^8\)
- \(\int \|\varphi_0(x)\|^2 |\varphi_0(y)|^2 /|x-y|dxdy = \theta \int \|\varphi(x)\|^2 |\varphi(y)|^2 /|x-y|dxdy\)
- \(\|\varphi\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2 = \theta \|\varphi\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2\)

In particular we get
\[
\tilde{\varphi}^{\alpha,\beta}(\varphi_0) = \theta \tilde{\varphi}^{\alpha,\beta}(\varphi)
\]
which implies
\[
\tilde{I}^{\alpha,\beta}(\rho) \leq \lim_{\theta \to 0} \tilde{\varphi}^{\alpha,\beta}(\varphi_0) = 0.
\]

Second step: if \(\tilde{I}^{\alpha,\beta}(\rho) < 0\) then \(\tilde{I}^{\alpha,\beta}(\rho) = -\infty\)

Let \(\varphi \in S(\rho)\) be such that \(\tilde{\varphi}^{\alpha,\beta}(\rho) < 0\). Then arguing as above, we have
\[
\tilde{\varphi}^{\alpha,\beta}(\varphi_0) = \theta \tilde{\varphi}^{\alpha,\beta}(\varphi)
\]
where \( \varphi_\theta = \theta^{3/2}\varphi(\theta x) \). Hence
\[
\tilde{I}^{x,\beta}(\rho) \leq \lim_{\theta \to \infty} \tilde{E}^{x,\beta}(\varphi_\theta) = \lim_{\theta \to \infty} \theta \tilde{E}^{x,\beta}(\varphi) = -\infty.
\]

The proof of (5.1) follows easily.

Next we shall prove (5.2). By Hölder inequality and the Sobolev embedding theorem \( H^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) \), we have
\[
\tilde{E}^{x,\beta}(u) \geq \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^3)}^2 - \beta \|u\|_{L^3(\mathbb{R}^3)}^2 \|u\|_{L^3(\mathbb{R}^3)}^{2/3} \geq \frac{1}{2} \|u\|_{H^{1/2}(\mathbb{R}^3)}^2 - C \|u\|_{H^{1/2}(\mathbb{R}^3)}^2 \rho_\beta^{1/3} \quad \forall u \in S(\rho).
\]

In particular if \( \rho \) is sufficiently small, then \( \tilde{E}^{x,\beta}(u) \geq 0 \) for any \( u \in S(\rho) \) and hence
\[
\tilde{I}^{x,\beta}(\rho) \geq 0.
\]

By combining this fact with (5.1) we deduce (5.2).

**Proof of Theorem 2.3.** Let \( \rho_* > 0 \) be such that
\[
\frac{1}{2} - C \rho_*^{4/3} > 0
\]
where \( C \) is the universal constant that appears in (5.3). Let \( \tilde{\rho} \) be as in Lemma 5.1. Then by using Lemma 5.1 \( \tilde{I}^{x,\beta}(\rho) = 0 \) for every \( \rho < \min\{\tilde{\rho}, \rho_*\} \). By combining this fact with (5.3) we deduce that there is no \( \varphi \in S(c) \) such that \( \tilde{E}^{x,\beta}(\varphi) = \tilde{I}^{x,\beta} = 0 \).

\[\square\]

**References**


Catto, I., Dolbeaut, J., Sanchez, O. and Soler, J., Existence of steady states for the Maxwell-
Schrödinger-Poisson system: exploring the applicability of the concentration-compactness

Christ, F. M. and Weinstein, M. I., Dispersion of small amplitude solutions of the


Lenzmann, E. and Lewin, M., Minimizers for the Hartree-Fock-Bogoliubov theory of


Lieb, E. H., On the lowest eigenvalue of the Laplacian for the intersection of two domains,

Lieb, E. and Yau, H. T., The Chandrasekhar theory of stellar collapse as the limit of

Frohlich, J., Jonsson, B. L. G. and Lenzmann, E., Boson stars as solitary waves, Comm.


Lions, P. L., The concentration-compactness principle in the Calculus of Variation. The

Sanchez, O. and Soler, J., Long time dynamics of the Schrödinger-Poisson-Slater system,
Journal of Statistical Physics, 114 (2004), 179–204.


nuna adreso:
Jacopo Bellazzini
Università di Sassari
Via Piandanna 4, 07100, Sassari
Italy
E-mail: jbellazzini@uniss.it

Tohru Ozawa
Department of Applied Physics
Waseda University
Tokyo, 169-8555
Japan
E-mail: txozawa@waseda.jp

Nicola Visciglia
Dipartimento di Matematica
Università di Pisa
Largo B. Pontecorvo 5, 56100, Pisa
Italy
E-mail: viscigli@dm.unipi.it

(Ricevita la 2-an de junio, 2015)
(Revizita la 11-an de marco, 2016)