Reverted Mean and Asymptotic Stationary Distribution for Timber Price Using a Mean-Reverting Stochastic Model

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Abstract: We use a mean-reverting stochastic model to demonstrate a procedure for calculating the reverted mean and asymptotic stationary distribution of sawlog price data. The data used for this study comes from a local auction market in the Fukuoka Prefecture, Japan where 4m long sugi (*Cryptomeria japonica*) and hinoki (*Chamercypress obtusa*) are commonly traded. Parameter estimation is completed using a quasi-maximum likelihood method based on a local linearization scheme. The stationary distribution for our stochastic model is numerically constructed using a generalized Pearson system.

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1. Introduction

Many researchers have acknowledged the importance of considering the stochastic nature of management decisions. Among them are Reed (1984), Brazee and Mendelsohn (1988), Clarke and Reed (1989), Clarke and Reed (1990), Teeter and Caulfield (1991), Zinkhan (1991), Thomson (1992), Reed and Ye (1994), Yin and Newman (1995), Yin and Newman (1996), Yoshimoto and Shoji (1998), Willassen (1998), Plantinga (1998), Morck et al. (1989), Thorsen (1999), Brazee and Bulte (2000), Fina et al. (2000), Hughes (2000), Yoshimoto (2002), Sødal (2003), Insley (2002), Insley and Rollins (2005) and Penttinen (2006). Geometric Brownian motion is widely used in stochastic modeling because it is easily applied in this context. Clarke and Reed (1989), Thomson (1992), Yoshimoto and Shoji (1998), and Yoshimoto (2002) have supported the use of geometric Brownian motion for price dynamics, whereas Haight and Holmes (1991) applied a non-stationary random walk to a log-transformed price process.

Recognizing that the long-run marginal cost of production reflects the mean market price (a microeconomic concept), Insley (2002) and Insley and Rollins (2005) employed a mean-reverting process in their models. Yoshimoto (2008) employed four variants of a mean-reverting stochastic model to analyze reverted mean price using local market data from 1970 to 2006, as well as national monthly averages. Yoshimoto (2009) constructed a stochastic dynamic programming model that applied mean-reverting processes to seek the minimum harvest age and threshold price for sustainable forest management.

The reverted mean price plays an important role in making decisions about future management activities. Likewise, information about the future variance or distribution of price dynamics is needed when analyzing the risk that forest owners will abandon active management. The objective of this paper is to demonstrate use of a mean-reverting

stochastic model to calculate the reverted mean and asymptotic stationary distribution of sawlog price data. We use market-based price data for 4m sugi (*Cryptomeria japonica*) and hinoki (*Chamercypress obtusa*) sawlogs taken from the Fukuoka Prefecture, Japan. The remainder of this paper is organized as follows. In the next section we describe the mean-reverting stochastic model and parameter estimation method. In the third section, we derive the reverted mean and asymptotic stationary distribution of price dynamics of our model. Whereas in the fourth section we compute the reverted mean and asymptotic stationary distribution of the model estimated from the data. The final section presents some concluding remarks.

2. Mean-Reverting Processes and Parameter Estimation

In this paper, we use the following mean-reverting stochastic model for log price dynamics to estimate the reverted mean of the price over time and derive its asymptotic stationary distribution:

$$[1] \quad dx_t = (\alpha - \beta x_t)dt + x_t \sigma dB_t$$

where x_t is the log price at time t , and B_t is a standard Brownian motion with the following characteristics:

1. $B_0 = 0$
2. $\{B_t, t \geq 0\}$ has independent increments with Gaussian distribution $B_t - B_s \sim N(0, t - s)$

The set of parameters (α, β, σ) are assumed to be positive so that equation is strictly constrained to be mean-reverting.

Parameter estimation for the model presented above is achieved via a pseudo-likelihood approach based on a local linearization method (Ozaki, 1985a, 1992, 1993, Shoji and Ozaki, 1997, and Shoji, 1998). A number of comparative studies among different estimation methods

have shown that in practical situations the pseudo-likelihood estimators based on the local linearization method outperform other for simplicity, computational efficiency and negligible bias (see Shoji and Ozaki ,1997, Durham and Gallant, 2002, Singer 2002, Hurn *et al.*, 2007). In brief, the method begins by converting a nonlinear stochastic differential equation into a stochastic differential equation with a constant volatility term. Then, over a small time lapse, the nonlinear drift term of the stochastic differential equation is locally approximated by a linear function of the state and time. The resulting linear stochastic differential equation can be solved analytically, which enables derivation of the corresponding likelihood function for parameter estimation. A detailed description of the method can be found in Yoshimoto and Shoji (2002).

Following Yoshimoto and Shoji (2002), the quasi log-likelihood function for the data set, $\{x_t\}$, is derived as

$$[2] \quad \begin{aligned} \log(p(x_{t_0}, x_{t_1}, \dots, x_{t_N})) = \\ -\frac{1}{2} \sum_{i=1}^N \left\{ \frac{(y_{t_i} - E_{t_{i-1}})^2}{V_{t_{i-1}}} + \log(2\pi V_{t_{i-1}}) \right\} + \log(p(y_{t_0})) \end{aligned}$$

where

$$[3] \quad E_{t_i} = y_{t_i} + \frac{h(y_{t_i})}{L_{t_i}}(e^{L_{t_i}\Delta t} - 1) + \frac{M_{t_i}}{L_{t_i}^2}(e^{L_{t_i}\Delta t} - 1 - L_{t_i}\Delta t)$$

$$[4] \quad V_{t_i} = \frac{\sigma^2 \cdot (\exp(2L_{t_i}\Delta t) - 1)}{2L_{t_i}}$$

$$[5] \quad y_{t_i} = \phi(x_{t_i}) = \ln(x_{t_i})$$

$$[6] \quad h(y_{t_i}) = \frac{1}{x_{t_i}} \cdot (\alpha - \beta x_{t_i}) - \frac{1}{2}\sigma^2$$

$$[7] \quad L_{t_i} = -\frac{\alpha}{x_{t_i}}$$

$$[8] \quad M_{t_i} = \frac{\alpha\sigma^2}{2x_{t_i}}$$

Note that t_i is the time of the i^{th} observation and y_{t_i} is the corresponding log-transformed data of x_{t_i} . To strictly constrain parameters (α, β, σ) to positive values, the following exponential transformation is applied

$$[9] \quad \alpha = e^{\theta_1}, \beta = e^{\theta_2}, \sigma = e^{\theta_3}$$

where the new parameters $(\theta_1, \theta_2, \theta_3)$ are unrestricted in the range of $(-\infty, \infty)$. All parameters are estimated by maximizing the above log-likelihood function.

3. Reverted Mean Values and Asymptotic Stationary Distribution

The reverted mean is derived by first converting the stochastic differential equation into a form that includes a constant volatility term, using the following transformation:

$$[10] \quad y_t = \ln(x_t)$$

With this conversion, we have the following constant volatility process:

$$[11] \quad dy_t = \left\{ \alpha e^{-y_t} - \beta - \frac{1}{2} \sigma^2 \right\} dt + \sigma dB_t$$

Setting the drift term equal to zero, we get the reverted mean, x_r , as a function of all parameter estimates:

$$[12] \quad x_r = \frac{2\alpha}{2\beta + \sigma^2}$$

To derive the asymptotic stationary distribution, we first consider the transition density function, P , of the diffusion process, which is governed by the following general form of a stochastic differential equation:

$$[13] \quad dx = f(x)dt + g(x)dw$$

with $x(t_0) = x_0$. This density function P satisfies the so-called Kolmogorov Forward Equation (KFE)

$$[14] \quad \frac{dP(t, x)}{dt} = -\frac{\partial}{\partial x}(f(x)P(t, x)) + \frac{1}{2} \frac{\partial^2}{\partial x^2}(g^2(x)P(t, x))$$

for all $t > t_0$. When t goes to infinity, the stationary distribution P_∞ of the diffusion process x is then defined by the following generalized Pearson system:

$$[15] \quad \frac{dP_\infty(x)}{dx} = \frac{2f(x) - g(x)g'(x)}{g^2(x)}P_\infty(x)$$

which results in the KFE by setting $\frac{dP(t, x)}{dt} = 0$ (see Ozaki, 1985b). Thus, for the mean-reverting stochastic model used here, the stationary distribution of x is governed by

$$[16] \quad \frac{dP_\infty(x)}{dx} = \frac{2(\alpha - \beta x) - \sigma^2 x}{\sigma^2 x^2}P_\infty(x)$$

This equation is numerically integrated using a classical order-2 local linearization method for ordinary differential equations (Jimenez *et al.*, 2005), defined by the recursive formula

$$[17] \quad P_{n+1} = P_n + Le^{A_n h_n} r$$

where $P_n = P_\infty(x_n)$ and $x_n = nh_n$ for all $n = 0, \dots, N$, and $P_0 = 0$. Here, N denotes the maximum number of iterations, and h_n the step-size at each iteration $n+1$.

To compute P_n in our example, we set $N = 10^7$, $h_n = 0.01$ for all n , and the matrices

$$A_n = \begin{bmatrix} F(x_n) & \frac{d}{dx_n}F(x_n)P_n & F(x_n)P_n \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and

$$r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with

$$F(x) = \frac{2(\alpha - \beta x) - \sigma^2 x}{\sigma^2 x^2}$$

The normalized density function p_∞ is finally computed as

$$[18] \quad p_\infty(x_i) = \frac{P_\infty(x_i)}{\sum_{i=0}^N P_\infty(x_i) h_i}$$

The mean M , variance Std^2 , skewness Skw , and kurtosis Kur of p_∞ are defined as follows:

$$M = \sum_{i=0}^N x_i p_\infty(x_i) h_i$$

$$Std^2 = \sum_{i=0}^N (x_i - M)^2 p_\infty(x_i) h_i$$

$$Skw = \frac{1}{Std^3} \sum_{i=0}^N (x_i - M)^3 p_\infty(x_i) h_i$$

$$Kur = \frac{1}{Std^4} \sum_{i=0}^N (x_i - M)^4 p_\infty(x_i) h_i$$

4. Demonstrative Examples

For demonstrative purposes, we use time series sawlog price data for two different species, sugi (*Cryptomeria japonica*) and hinoki (*Chamaecypress obtusa*). This market-based data, which includes 1,075 sugi and 1,074 hinoki data points, was taken from Fukuoka prefecture log market (Ukiha log auction). The sawlogs were 4m in length and diameter

ranged from 14–18cm. Figure 1 shows the price dynamics of these data sets from April 7, 1970 to December 18, 2009.

Figure 1. Price dynamics of 4m sawlogs.

a) sugi (*Cryptomeria japonica*), b) hinoki (*Chamaecypress obtusa*)

Estimated parameter values, the derived reverted mean, and statistics for the asymptotic stationary distribution of the two data sets are provided in Table 1. Comparing the estimates between sugi and hinoki in terms of the value of σ , the price of sugi showed slightly less volatility than that of hinoki. From Eq.[12], the reverted mean of the price for sugi was 17,413 Yen/m³, while it was 35,321 Yen/m³ for hinoki. Note that the mean of the asymptotic stationary distribution is not the same as the reverted mean because the distribution is not Gaussian. Thus, the most likely value or mode differs from the mean of the distribution (Figure 2).

Table 1. Parameter estimates, reverted mean, and asymptotic stationary distribution.

		Sugi	Hinoki
Parameter	α	3029.345	10688.812
	β	0.132952	0.2524661
	σ^2	0.082041	0.1003003
Reverted Mean*	x_r	17,413	35,321
Asymptotic Stationary Distribution	Mean M	29,840	47,865
	Standard Dev Std	20,993.53	27,482.69
	Skewness Skw	5.7025	3.8273
	Kurtosis Kur	1.5103	0.9422

*Note: The reverted mean is also the distribution's mode, or most frequently observed value.

5. Conclusions

In this paper we used a mean-reverting stochastic model to demonstrate a procedure for deriving the reverted mean price and asymptotic stationary distribution for sawlog prices. The reverted mean was derived by first converting the stochastic differential equation into a form that includes a constant volatility term, then setting the drift term equal to zero. The asymptotic stationary distribution was derived in terms of the general Pearson system, which was numerically integrated with the classical order-2 local linearization method.

Our analysis showed that the reverted mean price for sugi sawlogs was much less than that for hinoki. On the other hand, the variance of the asymptotic stationary distribution for sugi was less than that for hinoki, suggesting sugi sawlog prices may be more stable than hinoki prices (based on the market data used for this analysis). Many researchers have suggested use of a stochastic dynamic programming model to identify the minimum threshold price level at which forest owners will maintain management practices. Using this information in

Figure 2. Asymptotic stationary distribution.

conjunction with the asymptotic stationary distribution of price demonstrated here, it is possible to estimate the level of risk or probability that forest owners will abandon active management. If sustainable forest management is desired in a certain area, it may be possible to issue an insurance policy that is based on this derived probability to avoid the abandonment of management practices.

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