Fuzzy Time-Delay Systems and Their Stability

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Abstract: In this paper, we consider new generalized delay-dependent stability conditions of Takagi-Sugeno fuzzy time-delay systems. In the literature, both delay-independent stability conditions and delay-dependent stability conditions for fuzzy time-delay systems have already been obtained. However, those conditions are rather conservative and do not guarantee wide stability regions. This is also true in case of the robust stability for uncertain fuzzy time-delay systems. Here we choose a generalized Lyapunov functional and introduce more free matrices to it, in order to obtain generalized delay-dependent stability conditions. These techniques lead to generalized and less conservative stability conditions. In fact, delay-dependent stability conditions thus obtained are given in linear matrix inequalities(LMIs) and guarantee a wide stability region. We take a simple example to illustrate our result. Comparison with other stability conditions in the literature shows our conditions are the most powerful ones to guarantee the widest stability region. Moreover, we consider the robust stability of fuzzy time-delay systems with uncertain parameters. Applying the same techniques made on the stability conditions, we obtain delay-dependent sufficient conditions for the robust stability of uncertain fuzzy systems.

Keywords: Takagi-Sugeno Fuzzy Systems, Time-Delay Systems, Stability, Uncertain Systems, Robust Stability

1 Introduction

Takagi-Sugeno fuzzy models [7], [8] are nonlinear systems described by a set of if-then rules which gives a local linear representation of an underlining system. It is well-known that such models can describe or approximate a wide class of nonlinear systems. Hence it is important to study their stability and the synthesis of stabilizing controllers. Since the work by Tanaka and Sugeno [8] on stability analysis and state feedback stabilization there has been much effort developing system theory for such systems.

On the other hand, time-delay systems often appear in industrial systems and information networks. Thus, it is also important to analyze time-delay systems and design controllers for them. However, time-delay systems are, in general, infinite dimensional systems, which make the analysis and synthesis complicated. Recent research has investigated stability conditions based on linear matrix inequalities(LMIs). Both delay-independent and delay-dependent stability conditions for time-delay systems to be stable have been obtained in [3]-[6] and [9]. Moreover, robust stability analysis for uncertain systems has also been investigated. The result has been extended to uncertain fuzzy systems in [12].

In this paper we consider the stability and robust stability of Takagi-Sugeno fuzzy time-delay systems. We have already given both delay-independent and delay-dependent sufficient conditions for the stability for a fuzzy time-delay system in [1], [11], [13], [14] and [15]. However, these conditions still do not guarantee a wide stability region. Here we give new generalized delay-dependent sufficient conditions for the stability of fuzzy time-delay systems. The key technique to obtain such generalized conditions is to select an appropriate Lyapunov functional and to introduce more free matrices to it. In fact, this leads to a generalized stability conditions. We then compare our conditions with others in the literature. A simple example is given to illustrate our result. It is easy to see that our stability conditions guarantee wider stability region than other conditions. Moreover, we consider the robust stability of fuzzy time-delay systems with uncertain parameters.

2 Time-Delay Systems

In this section, we introduce Takagi-Sugeno fuzzy time-delay systems. Consider the Takagi-Sugeno fuzzy model with time-delay, described by the following IF-THEN rules:

\[
\begin{align*}
\text{IF} & \quad \xi_1 \text{ is } M_{i1} \text{ and } \cdots \xi_p \text{ is } M_{ip}, \\
\text{THEN} & \quad \dot{x}(t) = (A_i + \Delta A_i(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t - \tau),
\end{align*}
\]

where \( \tau \) is a time-delay and \( x(t) \in \mathbb{R}^n \) is the state. The matrices \( A_i \) and \( A_{di} \) are of appropriate dimensions. \( r \) is the number of IF-THEN rules. \( M_{ij} \) are fuzzy sets and \( \xi_1, \cdots, \xi_p \) are premise variables. We set \( \xi = [\xi_1 \cdots \xi_p]^T \) and \( \xi(t) \) is assumed to be given or to be a measurable function. The uncertain matrices are of the form

\[
[\Delta A_i(t) \quad \Delta A_{di}(t)] = H_i F_i(t) \begin{bmatrix} E_1 & E_2 \end{bmatrix} \quad \forall i = 1, \cdots, r
\]

where \( H_i, E_1 \) and \( E_2 \) are of appropriate dimensions, and \( F_i(t) \) is an unknown real time varying matrices
satisfying
\[ F_i^T(t)F_i(T) \leq I. \]

The state equation is defined as follows:
\[
\dot{x}(t) = \sum_{i=1}^{r} \lambda_i(\xi(t)) \{(A_i + \Delta A_i(t))x(t) + (A_{di} + \Delta A_{di}(t))x(t - \tau)\}
\]

where
\[
\lambda_i(\xi) = \frac{\beta_i(\xi)}{\sum_{i=1}^{\eta} \beta_i(\xi)}, \quad \beta_i(\xi) = \prod_{j=1}^{\eta} M_{ij}(\xi)
\]

and \(M_{ij}(\cdot)\) is the grade of the membership function of \(M_{ij}\). We assume
\[
\beta_i(\xi(t)) \geq 0, \quad i = 1, \ldots, r, \quad \sum_{i=1}^{r} \beta_i(\xi(t)) > 0
\]

for any \(\xi(t)\). Hence \(\lambda_i(\xi(t))\) satisfy
\[
\lambda_i(\xi(t)) \geq 0, \quad i = 1, \ldots, r, \quad \sum_{i=1}^{r} \lambda_i(\xi(t)) = 1
\]

for any \(\xi(t)\). When we consider the stability of fuzzy time-delay systems, we set
\[
\Delta A_i = 0, \quad \Delta A_{di} = 0 \quad \forall i = 1, \ldots, r.
\]

Then the nominal system is given by
\[
\dot{x}(t) = \sum_{i=1}^{r} \lambda_i(\xi(t)) \{(A_i x(t) + A_{di} x(t - \tau))\}. \tag{2}
\]

3 Stability Analysis and Synthesis

This section states several conventional stability conditions on Takagi-Sugeno fuzzy time-delay systems as well as a new stability condition. It also gives a design method of stabilizing controllers for such systems.

3.1 Stability

First we give several stability conditions in the literature for Takagi-Sugeno fuzzy time-delay system (2). Theorem 3.1 gives delay-independent stability conditions in [1] and [11] while Theorems 3.2, 3.3 and 3.4 provide delay-dependent stability conditions in [13], [14] and [15], respectively.

Theorem 3.1 ([1],[11]) If there exist common matrices \(X > 0, \quad Q > 0\) such that
\[
\begin{bmatrix}
A_i^T X + X A_i + Q & X A_{di} \\
A_{di}^T X & -Q
\end{bmatrix} < 0 \quad \forall i = 1, \ldots, r, \tag{3}
\]

then (2) is stable.

The following theorem is a slightly modified version of Theorem 3.1 in [13].

Theorem 3.2 If there exists common matrices \(X > 0\) and some number \(0 < \beta < 1\) such that
\[
\begin{bmatrix}
G_i & \tau X A_{di} & \tau A_i^T \\
0 & -Q & 0 & \tau A_{di}^T \\
\tau A_i & 0 & 0 & -\beta \tau I \\
0 & \tau A_{di} & 0 & -\tau (1 - \beta) I
\end{bmatrix} < 0
\]

where
\[
G_i = X(A_i + A_{di}) + (A_i + A_{di})^T X + Q,
\]

then (2) is stable.

Theorem 3.3 ([14]) If there exists common matrices \(X > 0, \quad Q > 0, \quad R_1 > 0, \quad R_2 > 0\) and some matrices \(M_i\) such that
\[
\begin{bmatrix}
G_i & \tau M_i & \tau M_i^T & XA_{di} - M_i \\
\tau M_i^T & -\tau R_1 & 0 & 0 \\
\tau M_i & 0 & -\tau R_2 & 0 \\
A_{di}^T X - M_i^T & 0 & 0 & G_i
\end{bmatrix} < 0 \quad \forall i = 1, \ldots, r. \tag{5}
\]

where
\[
G_i = X(A_i + A_{di}) + M_i + M_i^T + Q + \tau A_{di}^T R_1 A_i, \\
G_i = -Q + \tau A_{di}^T R_2 A_{di},
\]

then (2) is stable.

Theorem 3.4 ([15]) If there exist common matrices \(X > 0, \quad R > 0, \quad S > 0\) and some matrices \(M_i\) such that
\[
\begin{bmatrix}
S_{11i} & S_{12i} & S_{13i} \\
S_{12i}^T & -\frac{1}{\tau} R & 0 \\
S_{13i}^T & 0 & -S
\end{bmatrix} < 0 \quad \forall i = 1, \ldots, r \tag{6}
\]

where
\[
S_{11i} = \begin{bmatrix} A_i & A_{di} \end{bmatrix} X + X \begin{bmatrix} A_i & A_{di} \end{bmatrix}^T + \begin{bmatrix} 0 & M_i \end{bmatrix} + \begin{bmatrix} 0 & M_i^T \end{bmatrix} + S + \tau \begin{bmatrix} A_{di} & 0 \end{bmatrix}^T R \begin{bmatrix} A_{di}^T & 0 \end{bmatrix},
\]
\[
S_{12i} = M_i^T, \\
S_{13i} = \begin{bmatrix} A_{di} & -A_{di} \end{bmatrix} X,
\]

then (2) is stable.

Remark 3.1 Theorem 3.1 gives delay-independent conditions. There conditions can be applied to fuzzy systems with unknown time-delay. However, they are
known to be rather conservative. The other theo-
rem 3.2 guarantee a stability region that is not covered
by the one in Theorem 3.1. But, there conditions in
Theorems 3.1 and 3.2 are rather complementary. The
conditions in Theorem 3.3 exactly cover the stability
regions guaranteed by the ones in Theorem 3.1 and
3.2. The conditions in Theorem 3.4 are generalized
delay-dependent conditions that reduce the conserva-
tiveness of the stability of the system (2). The com-
parison of these conditions shall be made in Section 5.

Now we give new and generalized delay-dependent
stability conditions for the system (2).

**Theorem 3.5** If there exist common matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $W > 0$ and some matrices
$N_i$, $i = 1, 2, 3$ and $T_i$, $i = 1, 2, 3$ such that

$$
\Phi_i = \begin{bmatrix}
\Phi_{11i} & \Phi_{12i} & -\tau N_1 \\
\Phi_{21i} & \Phi_{22i} & -\tau N_2 \\
-\tau N_1^T & -\tau N_2^T & -\tau W
\end{bmatrix} < 0
$$

where

$$
\Phi_{11i} = Q_1 + N_i + N_i^T - T_i A_i - A_i^T T_i^T,
\Phi_{12i} = P + N_2^2 - A_i^T T_i^2 + T_i,
\Phi_{21i} = N_1^2 - N_i - A_i^T T_i^3 - T_i A_d_i,
\Phi_{22i} = P_2 + \tau W + T_2 + T_2^T,
\Phi_{23i} = -N_2 + T_3 - T_3 A_d_i,
\Phi_{33i} = -Q_1 - N_3 - N_3^2 - T_3 A_d_i - A_d_i^T T_3^T,
$$

then (2) is stable.

**Outline of Proof:** We consider the following Lyap-
unov functional

$$
V(x) = x^T(t) P x(t)
+ \int_0^t \left( x^T(s) Q_1 x(s) + \dot{x}^T(s) Q_2 \dot{x}(s) \right) ds
+ \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}^T(s) W \dot{x}(s) ds d\theta
$$

where $P, Q_1, Q_2$ and $W$ are positive definite ma-
trices to be determined.

Differentiation of $V(x(t))$ with respect to $t$ along the
solution of (2) and addition of the zero quantities in
terms of Leibniz-Newton formula and the state equa-
tion (2) give

$$
dt V(x(t)) = \frac{1}{\tau} \sum_{i=1}^\tau \lambda_i(\zeta(t)) \int_{t-i\tau}^t \zeta^T(t,s) \Phi_i \zeta(t,s) ds
$$

where $\zeta(t,s) = [x^T(t) \dot{x}^T(t) \dot{x}^T(t-\tau) \dot{x}^T(s)]^T$ and $\Phi_i$
is defined in (7). If $\Phi_i < 0$, then $\dot{V}(x) \leq -\epsilon \|x(t)\|^2$
for sufficiently small $\epsilon > 0$. Thus, the system (2) is
asymptotically stable if the conditions (7) holds.

## 4 Robust Stability

Here we consider the robust stability of fuzzy time-
delay systems with uncertain parameters (1). Theo-
rem 3.5 can be extended to a class of uncertain fuzzy
time-delay systems.

**Theorem 4.1** If there exist common matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $W > 0$ and some matrices
$N_i$, $i = 1, 2, 3$ and $T_i$, $i = 1, 2, 3$ such that

$$
\Pi_i < 0 \quad \forall i = 1, \cdots, r
$$

where

$$
\Pi_i = \begin{bmatrix}
\Pi_{11i} & \Pi_{12i} & -\tau N_1 \\
\Pi_{21i} & \Pi_{22i} & -\tau N_2 \\
-\tau N_1^T & -\tau N_2^T & -\tau W
\end{bmatrix} < 0
$$

then (1) is robustly stable.

**Outline of Proof:** Replacing $A_i$ and $A_d_i$ in (7) by
$A_i + H_i F_i(t) E_1$ and $A_d + H_i F_i(t) E_2$, respec-
tively, we find that (7) for the system (1) is equivalent to

$$
\dot{H}_i = \begin{bmatrix}
-H_i^T T_1^T & -H_i^T T_2^T & -H_i^T T_3^T \\
E_1 & 0 & E_2
\end{bmatrix},
$$

where

$$
H_i = \sum_{i=1}^r \Pi_i,
\bar{E} = \begin{bmatrix}
0 & E_2
\end{bmatrix}.
$$

It follows from the technique in [10] that the above
inequalities hold if

$$
\Phi_i + \bar{H}_i \dot{H}_i + \bar{E}^T \bar{E} < 0 \quad \forall i = 1, \cdots, r.
$$

Application of Schur complement leads to (8).

## 5 Example

Consider a scalar example with the following IF-
THEN rules

**IF** $x$ is $M_i$, **THEN** $\dot{x}(t) = ax(t) + a_i x(t-1), i = 1, 2$
where

$$a = \alpha, \quad a_d = \beta, \quad a_d = \beta + 0.5.$$

The fuzzy system is hence given by

$$
\dot{x}(t) = \sum_{i=1}^2 (ax(t) + a_i x(t-1)).$$
We attempt to obtain the stability regions corresponding to the conditions (3), (4), (5), (6) and (7). These stability regions are depicted in Figure 1. The thin line indicates the boundary of the stability region obtained from (3), the dotted line the one from (4), the thick solid line the one from (5), the thickest solid line the one from (6) and the dot-dash line the one from (7). For all the lines, the stability regions are left sides of their boundaries.

Figure 1: The stability regions

It is easy to see that the stability regions derived from (3) and (4) are rather complementary. The stability region in (5) exactly contains those of (3) and (4). Furthermore, the stability region obtained from (6) does not only includes those of (3), (4) and (5) but also give a wider stability region. (7) gives even wider stability region. These show that the conditions (7) guarantee the stability of the widest class of fuzzy time-delay systems.

6 Conclusion

We have obtained the generalized and relaxed delay-dependent stability conditions for Takagi-Sugeno fuzzy time-delay systems. The generalized conditions obtained here guarantee a wider stability region of fuzzy time-delay systems than other conditions in the literature. Finally, a simple example has given to illustrate our results.

References


