Abstract

A fuzzy (linguistic) truth value is a truth value specified by a fuzzy set over a closed interval \([0, 1]\). Logical operations, defined according to the extension principle, do not satisfy all identities to be lattice because there are subnormal truth values and non-convex truth values that violate the absorption laws. In 2000, Brzozowski proposed de Morgan bisemilattice, which is generalized algebra of de Morgan lattice in order for applications in multi-valued simulations of digital circuits. This paper studies a notion of fuzzy-interval equivalent relation defined for fuzzy functions taking linguistic truth value.

1 Introduction

A linguistic truth value, proposed by L. A. Zadeh in [1], is of fuzzy set defined over a closed interval \([0, 1]\). An ordinal numerical truth value, which is commonly used in fuzzy theory, is less expressive if we wish to explicitly represent semantics of truth values; unknown or contradict truth. The notion of linguistic truth value allows us to take the limitation of numerical truth value away and to carry explicit semantics of truth as representative values labeled by “very true”, “almost true” or “more or less false.” The labels are useful because there are infinite number of fuzzy sets over \([0, 1]\) possibly defined as linguistic truth values. The cardinality of \(\mathcal{F}_1\) is too much (notice that fuzzy set defined over \([0, 1]\) is equivalent to a mapping from \([0, 1]\) to \([0, 1]\).) Moreover, the properties of linguistic truth values are not clear except some results studied in very earlier time of fuzzy logic.

Mizumoto, et al. showed that the set of convex and normal linguistic truth values with logical operations \(\land, \lor, \neg\) satisfy a de Morgan algebra in [2]. Mukaidono pointed out that \(\alpha\)-cut of any convex linguistic truth value results in a closed interval on \([0, 1]\) and clarified some fundamental properties of linguistic truth values [4]. Kawaguchi, et al., examined algebraic structure of linguistic truth values satisfying continuous property [5]. Emoto and Mukaidono noticed an interesting fact that not all subsets of convex and normal linguistic truth value are closed, i.e., there is a subset \(S\) in which element \(a\) and \(b\) in \(S\) yield \(a \land b\) or \(a \lor b\) that does not belong to \(S\). The typical example includes the triangular-shaped truth value in which a conjunction or disjunction of two triangular-shaped fuzzy sets would be a four-sided polygon [8]. In order to deal with the inconsistency, they adopt a new definition of logical operation which does not follow the extension principle.

In 2000, Brzozowski proposed de Morgan bisemilattice, which is generalized algebra of de Morgan lattice in order for applications in multi-valued simulations of digital circuits. A bisemilattice is algebra with two binary operations \(\sqcup\) and \(\land\) defined over a set \(S\) such that \((S, \sqcup)\) and \((S, \land)\) are semilattices in which the absorption laws do not hold.

The problem what we would like to solve in this paper is given set of fuzzy truth values[1], to find a clear condition for which the set satisfies the distributive laws. A fuzzy truth value is a fuzzy set defined over \([0, 1]\), allowing us to have an expressive degree of truth than the conventional numerical truth. However, in the cost of representation, the logic of fuzzy
truth value may lose some useful algebraic properties including the complementary law and the absorption laws. In particular, we focus to the distributive laws in this paper.

Emoto and Mukaidono studied the problem and proved some useful properties of fuzzy truth value\[8\]. They showed a set of fuzzy truth value satisfies for all \(x \in [0, 1]\),

\[
\begin{align*}
&\text{(1) } A \text{ is convex at } x, \\
&\text{(2a) } A \text{ is concave at } x, \mu_A(x) \geq \mu_B(x) \lor \mu_C(x) \text{ for all } B, C, \text{ or} \\
&\text{(2b) } \mu_A(x) \geq \max_B \land \max_C, \text{ where } \max_A = \sup_x \mu_A(x).
\end{align*}
\]

The disadvantage of the above property is that the condition is too strong to characterize a proper set of truth value in which distributive laws hold. For instance, by letting \(A = \{0, 2\}, B = \{2\} \text{ and } C = \{0, 1\}, \) we can show \(A \lor BC = (A \lor B)(A \lor C) \) but both of the above conditions (2a) and (2b) fail here.  

In this paper, we study the problem in the model of a partial order\[9\], which is a generalized algebra of lattice. A bisemilattice is an algebra with two binary operations \(\lor\) and \(\land\) defined over a set \(S\) such that \((S, \lor)\) and \((S, \land)\) are semilattices in which the absorption laws do not hold.

2 Preliminary

2.1 Linguistic Truth Values

Let \(V = [0, 1]\) be a set of (numerical) truth value. A linguistic truth value or fuzzy truth value is a truth value specified by a fuzzy subset \(V_L\) over \(V\). The whole set of linguistic truth value is denoted by \(V_L\), namely, \(V_L = \{A \mid \mu_A : V \to V\}\).

Let \(A\) and \(B\) be linguistic truth values. According to the extension principle, logical operations, \(A \lor B, A \land B \) and \(\neg A\) are defined as follows. For any \(x \in V\),

\[
\begin{align*}
\mu_{A \lor B}(z) &= \sup_{z = \max(a, b), a, b \in V} \min(\mu_A(a), \mu_B(b)), \\
\mu_{A \land B}(z) &= \sup_{z = \min(a, b), a, b \in V} \min(\mu_A(a), \mu_B(b)), \\
\mu_{\neg A}(x) &= \mu_A(1 - x).
\end{align*}
\]

Note that the unusual notations for logical operation are used here for a convenience in later discussion.

A subset \(V_A\) of \(V_L\) is closed in terms of the extension principle if and only if logical operations \(A \lor B\) and \(\neg A\) are in \(V_A\) for any elements \(A\) and \(B\) in \(V_L\).

A linguistic truth value \(A\) is normal iff \(\mu_A(x) = 1\) for some \(x \in V\). A linguistic truth value is convex iff for any \(x \leq y \leq z \in V\), \(\mu_A(y) \geq \max(\mu_A(x), \mu_A(z))\) holds. A linguistic truth value \(A\) is regular iff for any element \(x, y \in V\), \(x \geq y\) implies \(\mu_A(x) \geq \mu_B(y)\). We denote the whole sets of normal, convex, and regular linguistic truth values by \(V_N, V_C\) and \(V_R\).

An interval truth value is a linguistic truth value \(A\) such that

\[
\mu_A(x) = \begin{cases} 
1 & \text{if } n \leq x \leq p, \\
0 & \text{otherwise,}
\end{cases}
\]

for some \(n \leq p \in V\). We denote an interval truth value by \(A = [n, p]\), and the whole set of interval truth value by \(V_I\).

A numerical truth value is characterized by \(i \in V\) as \(\mu_A(x) = 1\) if \(x = i\); \(0\) otherwise. Explicitly, we write a numerical truth value by \([i]\). A linguistic truth value with \(\mu_A(x) = 0\) for all \(a \in V\) is called empty set and written by \(\emptyset\).

2.2 Bisemilattice

Let \(S\) be a subset of \(V_L\). A partial order \(\subseteq\) is defined by \(A \subseteq B\) iff \(A = A \lor B\). A partially ordered set \((S, \subseteq)\) is an upper semilattice if every pair of elements of \(S\) has a least upper bound \(\text{hub}_{\subseteq}[7]\). A partially ordered set is bounded if it has a unite \(1_{\subseteq}\) and a zero \(0_{\subseteq}\), i.e., \(A \lor 1_{\subseteq} = 1_{\subseteq}\) for any \(A\) in \(S\), and \(A \lor 0_{\subseteq} = A\) for any \(A\) in \(S\). An upper semilattice is complete if every subset of \(S\) has a least upper bound. Note all
complete upper semilattice is bounded. We define a new operation induced from \( \sqcup \) by

\[
A \sqcap B = \text{lub}_{\leq} \{A, B\}.
\]

Similarly, we define a partial order \( \leq \) over \( S \) by \( A \leq B \) iff \( A = A \sqcap B \). A partially ordered set \((S, \leq)\) is a lower semilattice if every pair of elements of \( S \) has a least upper bound \( \text{lub}_{\leq} \). With the partial order, we define a dual operation as

\[
A \land B = \text{lub}_{\leq} \{A, B\}.
\]

A partial ordered set \((S, \sqcup, \land)\) is bisemilattice if \((S, \sqcup)\) and \((S, \land)\) are semilattices. A bisemilattice satisfies L1-L3 and L1’-L3’ in Table 1.

A bisemilattice is a lattice if additionally the absorbtion laws holds:

\[
L8 \quad A \sqcup (A \land B) = A, \\
L8' \quad A \land (A \sqcup B) = A.
\]

Then, lattice satisfies \( A \sqcap B = A \sqcup B \).

A bisemilattice \((S, \sqcup, \land)\) is complete if \((S, \sqcup)\) and \((S, \land)\) are complete, i.e., for any subset \( A \) of \( S \), \( \text{lub}_{\sqcup}(A) \) and \( \text{lub}_{\land}(A) \) are in \( V \).

A bisemilattice \((S, \sqcup, \land)\) is bilattice if \((S, \sqcup)\) and \((S, \land)\) are lattices. A complete bisemilattice is always bilattice.

A bisemilattice \((S, \sqcup, \land)\) is consistently bounded if \((S, \sqcup)\) and \((S, \land)\) are bounded and \( 0 \sqcup = 0 \land \) and \( 1 \sqcup = 1 \land \). With unified unite 1 and zero 0, laws L4 and L5 are satisfied for consistently bounded bisemilattice.

3 Classes of Linguistic Truth Values

3.1 Common Properties

First, we show any closed subset of \( V_L \) is bisemilattice, i.e., the extension principle follows that logical operations \( \sqcup \) and \( \land \) satisfies L1, L2 and L3. Then, we consider de Morgan’s law.

Lemma 3.1 Let \( S \) be a subset of \( V_L \). If \( S \) is closed with regards to \( \sqcup \) and \( \land \), then L1: \( A \sqcup A = A \) and \( A \land A = A \) for any \( A \) in \( S \).

Lemma 3.2 Let \( S \) be a closed subset of \( V_L \) with regards to \( \sqcup \) and \( \land \).

\[
L2: \quad A \sqcup B = B \sqcup A, \quad A \land B = B \land A \\
L3: \quad A \sqcup (B \land C) = (A \sqcup B) \land C, \\
A \land (B \sqcup C) = (A \land B) \sqcup C.
\]

Note that the above identities hold even if \( S \) contains \( \emptyset \) since an operation with \( \emptyset \) always yields \( \emptyset \).

Theorem 3.1 Let \( S \) be a subset of \( V_L \). If \( S \) is closed with regards to \( \sqcup \) and \( \land \), then \((S, \sqcup, \land)\) is a bisemilattice \((S, \sqcup, \land)\).

The absorption law, L8: \( A \sqcup (A \land B) = A \), does not hold for \( \sqcup \) and \( \land \), but note that it locally holds for each of \( \sqcup \) and \( \land \), i.e., \( A \sqcup (A \sqcap B) = A \) and \( A \land (A \sqcup B) = A \). In the sense, \((S, \sqcup, \land)\) can be said bilattice.

The involution law (L6) and the de Morgan’s laws (L7) are always satisfies with the unary operation \( \neg \).

Lemma 3.3 Let \( S \) be a closed subset of \( V_L \). For any \( A \) and \( B \) in \( S \),

\[
L6 \quad \neg \neg A = A \\
L7 \quad \neg (A \sqcup B) = (\neg A) \land (\neg B) \\
L7' \quad \neg (A \land B) = (\neg A) \sqcup (\neg B)
\]

Consequently, any closed subset of linguistic truth value satisfies L1 - L7. A bisemilattice is called de Morgan bisemilattice if it is consistently bounded and satisfies L1-L7. Unfortunately, \( S \) may have an \( \emptyset \), which is a unite for \( \sqcup \) and a zero for \( \land \) and thus spoils the consistency of two semilattices. Hence, we should carefully examine subsets if it is de Morgan bisemilattice or not.
3.2 Conditions for de Morgan bisemilattice

**Lemma 3.4** Let $S$ be a subset of $V_L$, and $\sqcup, \land, \neg$ be logical operations defined according to the extension principle. A de Morgan bisemilattice $(S, \sqcup, \land, \neg, 0, 1)$ does not have $\emptyset$.

Note that the exclusion of $\emptyset$ from $S$ does not immediately mean the de Morgan bisemilattice. Consider a subset $S_3$ of $V_L$ consisting of $\{1\}, \{.5, 1\}, \{0, .5, 1\}$ and $\{0, 1\}$, for which $0 \sqcup = \{0, 1\} \neq 0 = \{0, .5, 1\}$ as shown in Figure 1. Even without $\emptyset$, a bisemilattice can be consistently bounded.

Instead, let us examine the condition of bisemilattice to be consistently bounded.

**Definition 3.1** A subset $S$ of $V_L$ is closed with regard to $\neg$ if and only if for any element $a$ of $S$, $\neg a$ belongs to $S$.

**Lemma 3.5** Let $S$ be a closed subset of $V_L$ with regard to $\neg$.

\[ \neg 1_\sqcup = 0_\sqcup \implies 0_\sqcup = 0_\land, 1_\sqcup = 1_\land \]

**Lemma 3.6** Let $(S, \sqcup, \land, \neg)$ be a bisemilattice such that $L6$ and $L7$ ($L7'$) hold. Then,

\[ \neg 1_\sqcup = 0_\land. \]

**Theorem 3.2** Let $S$ be a subset of $V_L$. $(S, \sqcup, \land, \neg)$ is a de Morgan bisemilattice with $1 = 1_\sqcup = 1_\land$ and $0 = 0_\sqcup = 0_\land$ if and only if

1. $S$ is closed with regard to $\neg$,
2. $\neg 1_\sqcup = 0_\sqcup (\neg 1_\land = 0_\land)$.

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4 Some Properties of Bisemilattice of Fuzzy Truth Value

4.1 A Closed Subset of $V_L$

Consider a subset of ternary truth value $S_7 = \{0, 1, 2, 01, 12, 012, 02\}$, letting $0 = \{0, 0\}$, $1 = \{1, 1\}$, $2 = \{.5, .5\}$, $12 = 1 \sqcup 2 = \{.5, .5, 1, 1\}$ and so on. $S_7$ has the non-convex element $02 = \{0, 0, [1, 1]\}$ for which the absorption law $(L8, L8')$

\[ 02 \sqcup (02 \land 1) = 12 \neq 02 \]

fails. Figure 2 shows the inconsistent operations defined over $S_7$.

4.2 Distributive Laws in Bilattice

A bilattice $(S, \sqcup, \land)$ is distributive\(^2\) if it satisfies

\[ L9 \quad A \sqcup (B \land C) = (A \sqcup B) \land (A \sqcup C), \]

\[ L9' \quad A \land (B \sqcup C) = (A \land B) \sqcup (A \land C). \]

For the simplicity, we concentrate to consider a multiple-interval truth value in this section because it is the simplest closed subset of fuzzy truth value and we can easily extend a property of multiple-interval

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\(^2\)There is a locally distributive bilattice if $L9$ and $L9'$ hold for each of lattice. We explicitly say a mutually distributive bilattice if necessary.
truth value to that of more general fuzzy truth value through the extension principle.

The distributive laws are generalized as follows.

**Theorem 4.1**

\[
A(B \sqcup C) = A(AB \sqcup AC)
\]

\[
A \sqcup BC = A \sqcup (A \sqcup B)(A \sqcup C)
\]

### 4.3 Subset of Bilattice of Fuzzy Truth Value

Let \( f : S_7^n \rightarrow S_7 \) be a \( S_7 \)-logic function if \( f \) is represented with \( n \)-variable logic formula consisting of logical operation \( \sqcup \), \( \wedge \), and \( \neg \).

Table ?? illustrates two \( S_7 \)-functions, \( f_A \) and \( f_B \). \( f_A \) is a valid \( S_7 \)-logic function since formula \( x \sqcup \neg x \) implements \( f_A \), while no formula can represent function \( f_B \).

<table>
<thead>
<tr>
<th>Table 2: Example ( S_7 )-logic function</th>
</tr>
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<tbody>
<tr>
<td>( f_A )</td>
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<tr>
<td>( f_B )</td>
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</table>

Let \( a \succeq b \) be a partially relation defined by Hasse diagram in Figure 3, where \( a \) and \( b \) are element of \( S_7 \). Note that the relation is naturally defined by extending set implication over \( \{0, 1, 2\} \).

\[
\begin{array}{c|c|c|c|c|c|c|c}
0 & 1 & 2 & 01 & 02 & 12 & 012 & 02 \\
\hline
f_A & 2 & 1 & 2 & 12 & 12 & 012 & 02 \\
\hline
f_B & 0 & 0 & 1 & 2 & 2 & 01 & 02 \\
\end{array}
\]

Figure 3: Partially Relation on \( S_7 \)

**Theorem 4.2** Let \( f \) be \( S_7 \)-logic function. Then,

\[
f(a_1, \ldots, a_n) \succeq f(b_1, \ldots, b_n)
\]

for any \( (a_1, \ldots, a_n), (b_1, \ldots, b_n) \) of \( S_7^n \) such that \( (a_1, \ldots, a_n) \succeq (b_1, \ldots, b_n) \)

**Theorem 4.3** [11] Let \( f \) be \( S_7 \)-logic function and \( A \) be a subset of \( S_7 \). Then,

\[
f(A^n) \in A
\]

if \( A \) is any of

1. \( \{0, 2\} \),
2. \( \{0, 1, 2\} \),
3. \( \{01, 12\} \),
4. \( \{1, 01, 12, 012\} \),
5. \( \{1, 01, 12, 012\} \),
6. \( \{0, 2, 01, 02, 12, 012\} \),
7. \( \{0, 02\} \),
8. \( \{2, 02\} \),
9. \( \{0, 2, 02\} \),
10. \( \{0, 2, 01, 12\} \).

There is an essential subset of \( S_7 \) for which \( (A, \sqcup, \wedge, \neg) \) satisfies De Morgan bilattice.

**Theorem 4.4** Let \( f \) be \( S_7 \)-logic function and \( S_5 \) be a \( \{01, 12, 1, 012, 02\} \) subset of \( S_7 \). Then,

\[
f(S_5^n) \in S_5.
\]

**Theorem 4.5** Let \( f \) and \( g \) be \( S_7 \)-logic function. For any element \( A \) of \( S_5^n \), satisfying \( f(A) = g(A) \) implies \( f(A) = g(A) \) for any all elements of \( S_7^n \).

In \( S_7 \), the conventional absorption laws do not hold any more. Instead, the following weak absorption laws hold.

**Theorem 4.6**

\[
A \sqcup B \sqcup AB = A \sqcup B;
\]

\[
AB(A \sqcup B) = AB
\]
For example, the followings are identical.

\[ F = ABC \]

\[ = AB \sqcup AC \]

\[ = AB \sqcup AC \sqcup ABAC \]

\[ = AB \sqcup AC \sqcup ABC \]

Figure 4 illustrates 8 distinct \( S_7 \)-logic functions that are all identical if the range is restricted with \( \{1, 01, 12, 012\} \).

![Figure 4: \( S_4 \)-Equilalent Sets of \( S_7 \)-logic functions](image)

5 Conclusion

We have clarified a necessary and sufficient condition for distributive bilattice of fuzzy truth value. The future study is to find a new alternative condition which can replaces the second equation of Theorem.

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