An Axiomatization for the formal system $\Sigma$ of subjective epistemic reasoning

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Abstract: D.G. Schwartz introduced the formal system $\Sigma$ of subjective epistemic reasoning in which belief was measured along a series of linguistic degrees, e.g., unequivocally believes, strongly believes, fairly confidently believes, somewhat believes, neither believes nor disbelieves, somewhat disbelieves, fairly confidently disbelieves, strongly disbelieves, unequivocally disbelieves. The purpose of this paper is to present an axiomatization of D.G. Schwarz's system, by using the matrix of many-valued logic. Then we prove the completeness theorem as well as the soundness theorem.

1. Introduction
In [4] and [5], D.G. Schwartz introduced the formal system $\Sigma$ of subjective epistemic reasoning in which belief was measured along a series of linguistic degrees, e.g., unequivocally believes, strongly believes, fairly confidently believes, somewhat believes, neither believes nor disbelieves, somewhat disbelieves, fairly confidently disbelieves, strongly disbelieves, unequivocally disbelieves. In his system we can express the following reasoning such as “if the agent fairly confidently believes that John will visit Europe during the summer, and the agent strongly disbelieves that Tom will finish his degree by spring, then the agent fairly confidently believes at least one of these will occur”. (see lemma 4.4.(3)) His system employs a dual-leveled language [3] in which the lower-level follows fuzzy logic and the upper-level follows classical propositional logic. The main purpose of this paper is to present an axiomatization of D.G. Schwartz's system, by using the matrix of many-valued logic. Then we prove the completeness theorem as well as the soundness theorem.

2. Matrices
2.1. Truth values
We take 0,1,2,...,8 as truth values. Let $T=\{0,1,2,...,8\}$ be the set of all truth values. Elements of $T$ are denoted by $m,n,...$. Intuitively “0” stands for “false” and “8” stands for “true”.

2.2. Primitive symbols
(1) Propositional variables: $p,q,r,...$
(2) Propositional connectives: $\neg,\lor,\land,\rightarrow,\forall,\exists$
(3) Belief operators: $B_m(\cdot)$ (m=0,1,...,8).
(4) Auxiliary symbols: $\cdot$.

Intuitively “$B_0(P)$” means “$P$ is unequivocally disbelieved”, “$B_1(P)$” means “$P$ is strongly disbelieved”, “$B_2(P)$” means “$P$ is fairly confidently disbelieved”, “$B_3(P)$” means “$P$ is somewhat disbelieved”, “$B_4(P)$” means “$P$ is neither believed nor disbelieved”, “$B_5(P)$” means “$P$ is somewhat believed”, “$B_6(P)$” means “$P$ is fairly confidently believed”, “$B_7(P)$” means “$P$ is strongly believed” and “$B_8(P)$” means “$P$ is unequivocally believed”.

2.3. Definition of a lower-level formula and an upper-level formula
1. A propositional variable is a lower-level formula.
2. If P and Q are lower-level formulas, then \( \neg P, P \lor Q \) and \( P \land Q \) are lower-level formulas.
3. If A is a lower-level formula, then \( B_m(A) \) is an upper-level formula.
4. If A and B are upper-level formulas, then \( \neg P, P \lor Q \) and \( P \land Q \) are upper-level formulas.

We denote a set of lower-level formulas by \( F \) and a set of upper-level formulas by \( G \).

A Gentzen’s sequent \( A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n \) means intuitively that some formula of \( A_1, \ldots, A_m \) is false or some formula of \( B_1, \ldots, B_n \) is true. The truth value “0” corresponds to the succedent and the truth value “8” corresponds to the antecedent. We extend the notion of a sequent to a many-valued logic case.

2.4. Definition of a matrix
A valued formula is a pair consisting of a formula (a lower-level formula or an upper-level formula) and a truth value. We call the following finite set of valued formula a matrix: \( \{(A_1, m_1), \ldots, (A_k, m_k)\} \), where for any \( A_k \in G \) \( m_k = 0 \) or \( m_k = 8 \). We call \( A_1 \) or \( \ldots \) or \( A_k \) the \( m_1 \)-part of this matrix or \( \ldots \) or \( m_k \)-part of this matrix, respectively. Intuitively the matrix \( \{(A_1, m_1), \ldots, (A_k, m_k)\} \) means that \( A_j \) has the truth value \( m_j \) for some \( j = 1, \ldots, k \).

2.5. Abbreviations
In the following, \( K, L, \ldots \) denote matrices. \( \Gamma, \Delta, \ldots \) denote finite sets of formulas, \( A, B, \ldots \) denote formulas. Let \( S \subseteq T \). The matrix \( \{(A, m) : A \in \Gamma, m \in S\} \) is abbreviated as \( (\Gamma, S) \), \( (A \cup \Gamma, S) \) as \( (\Gamma, m) \), \( (A \cup \Gamma, m) \) as \( (A, \Gamma, m) \) and \( K \cup \{(A, m)\} \) as \( K \cup (A, m) \), respectively.
We define \( K \subseteq L \), if and only if for all \( m \in T \) every formula that occurs in the \( m \)-part of \( K \) also occurs in the \( m \)-part of \( L \).

3. Model
3.1. Definition of a model
A likelihood mapping is a function \( \mu : F \rightarrow [0,1] \) satisfying the following conditions: for any propositional variable \( p \), \( \mu(p) \in [0,1] \), for any lower-level formulas \( P, Q \),
1. \( \mu(\neg P)=1 - \mu(P) \),
2. \( \mu(P \lor Q) = \max\{\mu(P), \mu(Q)\} \),
3. \( \mu(P \land Q) = \min\{\mu(P), \mu(Q)\} \).

Let \( t_0 = [0,0], t_1 = (\frac{1}{7}, \frac{2}{7}), t_2 = (\frac{2}{7}, \frac{3}{7}), t_3 = (\frac{3}{7}, \frac{4}{7}), t_4 = [\frac{4}{7}, \frac{5}{7}), t_5 = \frac{5}{7}, t_6 = \frac{6}{7}, t_7 = [1,1] \).

A truth valuation is a function \( v : G \rightarrow \{0,8\} \) satisfying the following conditions:
1. for any lower-level formula \( P \) and any upper-level formulas \( A \) and \( B \),
2. \( v(B_m(P)) = 8 \) if and only if \( \mu(P) \in t_m \) \( (m = 0, 1, \ldots, 8) \),
3. \( v(\neg A) = 8 \) if and only if \( v(A) = 0 \),
4. \( v(A \lor B) = 8 \) if and only if \( v(A) = 8 \) or \( v(B) = 8 \),
5. \( v(A \land B) = 8 \) if and only if \( v(A) = v(B) = 8 \).

A matrix \( L \) is to be valid, if for any likelihood mapping \( \mu \) and truth valuation \( v \), either there exists a lower-level formula \( P \) such that \( (P, m) \in L \) and \( \mu(P) \in t_m \) or there exists an upper-level formula \( B \) such that \( (B, v(B)) \in L \).

4. Formal system
We introduce the formal system. The formal system is constituted by its axioms and its inference rules.
4.1. Axiom:
(beginnning matrix) for any lower-level formul
4.2. Inference rules

1. Weakening
\[ \frac{L}{K} \] , where \( L \subseteq K \).

2. Cut
For any formula \( A \)
\[ \frac{K \cup (A, k), L \cup (A, m)}{K \cup L} \] , where \( k \neq m \).

3. Inference for lower-level propositional connectives
For any lower-level formulas \( P \) and \( Q \)
\[ \frac{L \cup (A, m)}{L \cup (\neg A, 8 - m)} \] (\( \neg \))
\[ \frac{L \cup (P, m) \cup (Q, n)}{L \cup (P \lor Q, \max(m, n))} \] (\( \lor \))
\[ \frac{L \cup (P, m) \cup (Q, n)}{L \cup (P \land Q, \min(m, n))} \] (\( \land \))

4. Inferences for belief operators
For any lower-level formula \( P \),
\[ \frac{L \cup (P, m)}{L \cup (B_m(P), 8)} \] (\( B_m(P) \))
\[ \frac{L \cup (P, n)}{L \cup (B_m(P), 0)} \] (\( \neg B_m(P) \))
\[ \frac{L \cup (A, m)}{L \cup (A \lor B, 8)} \] (\( A \lor B \))
\[ \frac{L \cup (A, 8)}{L \cup (A \land B, 8)} \] (\( A \land B \))

5. Inference for upper-level propositional connectives
For any upper-level formulas \( A, B \)
\[ \frac{L \cup (A, m)}{L \cup (\neg A, 8 - m)} \] (\( \neg A \))
\[ \frac{L \cup (A, 8)}{L \cup (A \lor B, 8)} \] (\( A \lor B \))
\[ \frac{L \cup (A, m) \cup (B, n)}{L \cup (A \lor B, \max(m, n))} \] (\( A \lor B \))

4.3. Definition of provable matrices

A matrix \( L \) is called provable if it is obtained from axioms by a finite number of application of the above inference rules.

We can easily prove the following lemma.

Lemma 4.4. The following matrices are provable,
1. for any upper-level formula \( A \), \( (A, 0) \cup (A, 8) \),
2. for any lower-level formulas \( P \) and \( Q \),
\[ (B_m(P) \land B_m(Q), 0) \cup (B_{\min(m, n)}(P \land Q), 8) \],
3. for any lower-level formulas \( P \) and \( Q \),
\[ (B_m(P) \land B_m(Q), 0) \cup (B_{\max(m, n)}(P \lor Q), 8) \],
4. for any lower-level formula \( P \),
\[ (B_m(\neg P), 0) \cup (B_{8-m}(P), 8) \],
5. for any lower-level formula \( P \),
\[ (B_{8-m}(P), 0) \cup (B_m(\neg P), 8) \],
6. for any lower-level formula \( P \),
\[ (\neg B_m(P), 0) \cup (\lor_{k \neq m} B_k(P), 8) \],
7. for any lower-level formula \( P \),
\[ (\lor_k B_k(P), 0) \cup (\neg B_m(P), 8) \],
8. for any upper-level formula \( A \),
\[ (A, 0) \cup (\neg A, 8) \],
9. for any upper-level formula \( A \),
\[ (A, 8) \cup (\neg A, 8) \],
10. for any upper-level formulas \( A \) and \( B \),
\[ (A, 0) \cup (A \lor B, 8) \].

Theorem 4.5. (Soundness theorem) If a matrix is provable, then it is valid.

Proof. It can easily be proved by induction on the construction of a proof of the given matrix.
5. Completeness theorem

5.1. Abbreviation

Let the matrix K be fixed. \( \Omega \) denotes a set of all subformulas of formula occurring in the matrix K. We denote a set of formulas occurring in the m-part of L by \( L_m \). \( L_m \cap \ldots \cap L_k \) is denoted by \( L_{m,k} \) and \( \bigcap_{k=m}^{\infty} L_k \) is denoted by \( L_m^* \).

By \( (A,m) \) we mean \( (A, T\setminus\{m\}) \).

5.2. Definition of complete

If the matrix L is unprovable and for any \( \Omega \in A \), there exists an \( m \in T \) such that \( A \in L_m^* \), we call the matrix L complete. We denote the set of all complete matrices by \( C(K) \).

Lemma 5.3. For any \( L \in C(K) \),

1. for any upper-level formula A, \( m=0,8 \), \( A \in L_m^* \) if and only if \( L \cup \{(A,m)\} \) is provable,
2. for any lower-level formula P, if \( P \in L_m^* \) then \( \neg P \in L_{m-1}^* \),
3. for any lower-level formulas P and Q, if \( P \in L_m^* \) and \( Q \in L_n^* \) then \( P \lor Q \in L_{\max(m,n)}^* \),
4. for any lower-level formulas P and Q, if \( P \in L_m^* \) and \( Q \in L_n^* \) then \( P \land Q \in L_{\min(m,n)}^* \),
5. for any lower-level formula P, if \( B_m(P) \in L_0 \) then \( P \in L_m^* \),
6. for any lower-level formula P, if \( B_m(P) \in L_8 \) then \( P \in L_m^* \),
7. for any upper-level formula A, \( \neg A \in L_0 \) if and only if \( A \in L_8 \),
8. for any upper-level formula A, \( \neg A \in L_8 \) if and only if \( A \in L_0 \),
9. for any upper-level formulas A and B, \( A \lor B \in L_0 \) if and only if either \( A \in L_0 \) or \( B \in L_0 \).

5.4. Definition of a canonical model

we prove the completeness theorem by a powerful method of a canonical model, see [1] and [2]. Let a complete matrix K be fixed. We define the function \( \mu_K : F \to [0,1] \) and the function \( \nu_K : G \to \{0,8\} \) as follows:

\[
\mu_K(P) = \begin{cases} 1 & (P \in K_8^*) \\ \frac{2^{m-1}}{14} & (\text{there exists } m \text{ such that } P \in K_m^* \land m \neq 0,8) \\ 0 & (P \in K_0^*) \end{cases}
\]

\[
\nu_K(A) = \begin{cases} 0 & (A \in K_8^*) \\ 8 & (A \in K_0^*) \end{cases}
\]

We can prove the following theorem.

Theorem 5.5. Completeness Theorem

If a finite matrix is valid, then it is provable.

Reference