Chapter 3  COMPRESSIVE STRENGTH TESTS AT HIGH TEMPERATURES

3.1 Overview of the chapter

We have performed high-temperature compressive strength tests for plain concrete and concrete bound with hoops (confined concrete). The stress-strain relations for plain concrete and confined concrete obtained from the tests were used to derive the plastic flow rule. Furthermore, a discussion of the failure criterion has been provided based on the results of the compression tests for confined concrete. On the basis of the plastic flow rule and the failure criterion as derived from the experimental results, the stress-strain relation for confined concrete has been calculated using finite element analysis, and the validity of the derived constitutive laws was verified.

3.2 Compressive strength and stress-strain relation

A list of the results of the compressive strength test is provided in Table 3.1. The Young modulus and the Poisson ratio for 1/3 of the maximal load are given in Table 3.2. The stress-strain relation is provided in Figures 3.2 to 3.16. A list of the loading rates for the specimens is given in Table 3.3.

The compressive strength was about 120 N/mm$^2$ for plain concrete with water-binder ratio of 22.5% at normal temperature, 93 N/mm$^2$ for a water-binder ratio of 30% and 53 N/mm$^2$ for a water-binder ratio of 50%. From Table 3.2, it is clear that the lower the water-binder ratio, the lower the Young modulus becomes at higher temperatures. The dependency of the Poisson ratio on the temperature is shown in Figure 3.1. Although the Poisson ratio drops once at 200°C for water-binder ratios of 30% and 50%, above that it grows continuously together with the temperature. However, since the lateral displacement was negligible at 200°C during the tests, the obtained curve for the stress-strain ratio was not smooth, and the resolution was low. There is still room for improvement with respect to this issue.

In the uniaxial compressive strength test, there was a match between the strain in the direction of the hoops and the radial strain. Given this fact, it was considered that the hoop strain for the highest strength of the concrete is equal to the measured lateral strain, and by using the hoop stress (mean of two values) based on Figure 3.17, the lateral pressure was calculated from Equation 2.1. In the same figure, the hoop stress values which were used are marked with O. Most of the hoops yield when the specimen is at maximal load. In addition, friction force is generated between the end faces of the steel cylinder and the specimen since the specimen is in direct contact with the steel cylinder, which entails the generation of lateral stress due to the confinement of the lateral strain in the vicinity of the upper and the lower end faces. However, lateral stress due to friction has not been considered here.
The target values for the lateral stress mentioned in 2.2.2 (4.5 N/mm$^2$ at normal temperature, 4 N/mm$^2$ at 200°C, 2.3 N/mm$^2$ at 400°C, 0.3 N/mm$^2$ at 600°C and 0.1 N/mm$^2$ at 800°C) were met.

The increase in strength due to the confined effect for the specimen with hoops was notable for temperatures up to 600°C. At 800°C, since the number of hoops is low and the high-temperature strength of the hoops is low, which entails low lateral stress, the increase in strength due to the confined effect was virtually unobservable.

As visible from Figures 3.2 to 3.16, the longitudinal strain and the lateral strain at maximal load increase together with the temperature of the concrete, and the ratio of the lateral strain to the longitudinal strain becomes larger as the temperature increases. At 800°C, the lateral strain almost overtook the longitudinal strain.

In the Guidebook for Fire-Resistive Performance of Structural materials[1] published by the Architectural Institute of Japan (AIJ), the calculation results for the compressive strength of concrete at high temperatures and the Young modulus are collected and organized, and the formulated compressive strength and Young modulus are presented as shown in Table 3.4. According to Table 3.4, the water-binder ratio affects the residual compressive strength ratio but not the Young modulus. However, since there are few case studies involving high-temperature compressive strength tests, the data in Table 3.4 are based almost exclusively on hard sandstone, with scarce examples of data based on breccia, riverstone and limestone.

A comparison between the values proposed by AIJ (Table 3.4) and the experimental results is provided in Figure 3.18. According to the same figure, for temperatures between 200°C and 600°C, the values for the high-temperature strength obtained from the present tests for all water-binder ratios are lower than the proposed values. The discrepancies are larger for lower water-binder ratios, and the values proposed by AIJ are exceeded for temperatures above 600°C. The residual strength ratio around 400°C is lower for concrete made with andesite as compared to concrete made with hard sandstone as the coarse aggregate, however, it becomes higher for temperatures above 600°C. Furthermore, except in the case of the data for water-binder ratio of 22.5 % at 400°C, the Young modulus matched the values proposed by AIJ.

Regarding the failure modes after the experiments had been terminated, the samples with lower number of hoops broke in half, while the ones with higher number of hoops displayed a slip surface. However, these failure modes are the same for ambient temperature, and unique high-temperature failure modes were not confirmed.
### Table 3.1 List of the results from the strength tests

<table>
<thead>
<tr>
<th>Water/binder ratio</th>
<th>temperature °C</th>
<th>pw=0%</th>
<th>pw=0.3%</th>
<th>pw=0.6%</th>
<th>pw=1.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Strength N/mm²</td>
<td>Longitudinal strain x10⁻⁶</td>
<td>Lateral strain N/mm²</td>
<td>Longitudinal strain x10⁻⁶</td>
<td>Lateral strain N/mm²</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>W/B = 22.5%</td>
<td>20</td>
<td>119</td>
<td>-3266</td>
<td>886</td>
<td>120</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>116</td>
<td>-3250</td>
<td>1549</td>
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<tr>
<td></td>
<td>20</td>
<td>121</td>
<td>-3125</td>
<td>1536</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>79.4</td>
<td>-3677</td>
<td>1790</td>
<td></td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>54.3</td>
<td>-7775</td>
<td>6040</td>
<td></td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>41.4</td>
<td>-8457</td>
<td>14760</td>
<td>42.2</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>43.6</td>
<td>-22438</td>
<td>28390</td>
<td>38.8</td>
</tr>
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<td>W/B = 30%</td>
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<td>2109</td>
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<td></td>
<td>20</td>
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<td>1900</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>92.9</td>
<td>-2710</td>
<td>1117</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>63.4</td>
<td>-3285</td>
<td>1480</td>
<td></td>
</tr>
<tr>
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<td>400</td>
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<td>24400</td>
<td>33.5</td>
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<td>W/C = 50%</td>
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<td>1895</td>
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<td></td>
<td>20</td>
<td>52.3</td>
<td>-2200</td>
<td>1755</td>
<td></td>
</tr>
<tr>
<td></td>
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<td>835</td>
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<td>7235</td>
<td>16.7</td>
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</table>

### Table 3.2 Comparisons of the values of the Young modulus and the Poisson ratios for plain concrete

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<thead>
<tr>
<th>Water/binder ratio</th>
<th>Item</th>
<th>20°C</th>
<th>20°C</th>
<th>20°C</th>
<th>200°C</th>
<th>400°C</th>
<th>600°C</th>
<th>800°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>W/B = 22.5%</td>
<td>Young modulus</td>
<td>4.25</td>
<td>4.17</td>
<td>4.30</td>
<td>2.48</td>
<td>0.81</td>
<td>0.94</td>
<td>0.42</td>
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<td>Poisson ratio</td>
<td>0.22</td>
<td>0.22</td>
<td>0.20</td>
<td>0.31</td>
<td>0.25</td>
<td>0.28</td>
<td>0.29</td>
</tr>
<tr>
<td>W/B = 30%</td>
<td>Young modulus</td>
<td>4.05</td>
<td>4.18</td>
<td>3.97</td>
<td>2.63</td>
<td>1.58</td>
<td>0.99</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>Poisson ratio</td>
<td>0.19</td>
<td>0.23</td>
<td>0.22</td>
<td>0.14</td>
<td>0.30</td>
<td>0.35</td>
<td>0.26</td>
</tr>
<tr>
<td>W/C = 50%</td>
<td>Young modulus</td>
<td>3.70</td>
<td>3.53</td>
<td>3.65</td>
<td>2.42</td>
<td>1.45</td>
<td>1.08</td>
<td>0.87</td>
</tr>
<tr>
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<td>Poisson ratio</td>
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<td>0.20</td>
<td>0.21</td>
<td>0.03</td>
<td>0.19</td>
<td>0.20</td>
<td>0.29</td>
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</table>
### Table 3.3 Loading rates of the specimens

<table>
<thead>
<tr>
<th>Water-binder ratio</th>
<th>Temperature °C</th>
<th>(p_w = 0%)</th>
<th>Strain rate x10^-6/sec</th>
<th>(p_w = 0.3%)</th>
<th>Strain rate x10^-6/sec</th>
<th>(p_w = 0.6%)</th>
<th>Strain rate x10^-6/sec</th>
<th>(p_w = 1.5%)</th>
<th>Strain rate x10^-6/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>W/B = 22.5</td>
<td>200</td>
<td>0.10</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>0.13</td>
<td>8</td>
<td>0.10</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0.09</td>
<td>13</td>
<td>-</td>
<td>-</td>
<td>0.10</td>
<td>14</td>
<td>0.12</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>0.09</td>
<td>19</td>
<td>0.09</td>
<td>22</td>
<td>0.09</td>
<td>23</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>0.10</td>
<td>52</td>
<td>0.09</td>
<td>61</td>
<td>0.09</td>
<td>62</td>
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<td>-</td>
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<tr>
<td>W/B = 30</td>
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<td>0.13</td>
<td>7</td>
<td>-</td>
<td>-</td>
<td>0.11</td>
<td>7</td>
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<td></td>
<td>400</td>
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<td>10</td>
<td>-</td>
<td>-</td>
<td>0.12</td>
<td>18</td>
<td>0.10</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>0.13</td>
<td>24</td>
<td>0.10</td>
<td>24</td>
<td>0.09</td>
<td>31</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>0.10</td>
<td>53</td>
<td>0.11</td>
<td>60</td>
<td>0.12</td>
<td>62</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>W/C = 50</td>
<td>200</td>
<td>0.11</td>
<td>7</td>
<td>0.12</td>
<td>10</td>
<td>0.09</td>
<td>7</td>
<td>0.08</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0.09</td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>0.08</td>
<td>14</td>
<td>0.10</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>0.07</td>
<td>18</td>
<td>0.09</td>
<td>29</td>
<td>0.08</td>
<td>30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>0.06</td>
<td>21</td>
<td>0.07</td>
<td>26</td>
<td>0.07</td>
<td>37</td>
<td>-</td>
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</tr>
</tbody>
</table>

**Figure 3.1 Poisson ratios**

**Figure 3.2 Stress-strain relations (W/B=22.5%, ambient temperature)**
Figure 3.3 Stress-strain relations (W/B=22.5%, 200°C)

Figure 3.4 Stress-strain relations (W/B=22.5%, 400°C)

Figure 3.5 Stress-strain relations (W/B=22.5%, 600°C)
Figure 3.6 Stress-strain relations (W/B=22.5%, 800°C)

Figure 3.7 Stress-strain relations (W/B=30%, ambient temperature)

Figure 3.8 Stress-strain relations (W/B=30%, 200°C)
Figure 3.9 Stress-strain relations (W/B=30%, 400°C)

Figure 3.10 Stress-strain relations (W/B=30%, 600°C)

Figure 3.11 Stress-strain relations (W/B=30%, 800°C)
Figure 3.12 Stress-strain relations (W/B=50%, ambient temperature)

Figure 3.13 Stress-strain relations (W/B=50%, 200°C)

Figure 3.14 Stress-strain relations (W/B=50%, 400°C)
Figure 3.15 Stress-strain relations (W/B=50%, 600°C)

Figure 3.16 Stress-strain relations (W/B=50%, 800°C)

Figure 3.17 Hoop stress at the time of failure of the specimens
Table 3.4 Formulation of the high-temperature strength and the Young modulus [1]

<table>
<thead>
<tr>
<th>Temperature °C</th>
<th>Residual ratio</th>
<th>Hot compressive strength</th>
<th>Young modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>100</td>
<td>0.80</td>
<td>0.80</td>
<td>0.80</td>
</tr>
<tr>
<td>200</td>
<td>0.334×W/B+0.764</td>
<td>0.68</td>
<td>0.68</td>
</tr>
<tr>
<td>300</td>
<td>0.364×W/B+0.711</td>
<td>0.57</td>
<td>0.57</td>
</tr>
<tr>
<td>400</td>
<td>0.451×W/B+0.561</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>500</td>
<td>0.385×W/B+0.411</td>
<td>0.35</td>
<td>0.35</td>
</tr>
<tr>
<td>600</td>
<td>0.471×W/B+0.200</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>700</td>
<td>0.436×W/B+0.114</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>800</td>
<td>0.15</td>
<td>0.075</td>
<td>0.075</td>
</tr>
<tr>
<td>900</td>
<td></td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Figure 3.18 High-temperature residual ratios of the compressive strengths and the Young modulus

3.3 Stress-strain relation

The stress-volumetric strain relations ($\varepsilon_V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$) and the high-temperature stress-volumetric strain relations for the plain concrete samples are given in Figures 3.19 to 3.23.

When compressive loading is applied to concrete at ambient temperature, the volume contraction is linear for relatively low stress. However, it is known that over a certain stress level the direction of the volume changes reverses and proceeds as expansion. This tendency is confirmed by Figures 3.19 to 3.21, which present the stress-volumetric strain relations for ambient to high temperature.

As shown, at ambient temperature, when the water-binder ratio is lower (i.e. the strength is higher), the stress for which the volumetric strain is lowest contains a higher proportion of the compressive strength, and the volume expansion is smaller at maximal loading.
However, at high temperatures, regardless of the water-binder ratio, the volumetric strain at maximal loading becomes higher as the temperature increases whereas the portion of the stress for the lowest strain corresponding to the compressive strength decreases. Therefore, at high temperatures, the volumetric strain becomes greater when the water-binder ratio is low, and the portion of the stress for the minimal strain corresponding to the compressive strength decreases. In the case of water-binder ratios under 30%, at 800°C and high-temperature compressive strength of about 40%, the direction of the change in the volumetric strain reversed, becoming over 20000 μ at maximal load. Hence, in contrast to the ambient temperature scenario, there is a notable increase in volume at high temperatures, and it can be said that the volume expansion is dependent on the temperature.

The volumetric expansion of concrete, which generates compression force, is induced by the emergence and growth of micro cracks, and represents a visible increase in volume. If the applied loading exceeds the stress where the volumetric strain is lowest, micro cracks begin to grow even if the stress is maintained constant, which results in a failure after a certain amount of time has passed\[2\], in other words, a creep failure occurs.

The relations of the compressive strength and the stress at minimal volumetric strain with respect to the temperature are given in Figures 3.22 to 3.24. The relation between the stress at minimal volumetric strain and the high-temperature compressive strength with respect to the temperature is given in Figure 3.25.

According to Figures 3.22 and 3.23, the stress at minimal volumetric strain for concrete with water-binder ratio of 22.5% and 30% decreases to 20 N/mm\(^2\) almost linearly for temperatures up to 400°C, while the decrease is small for temperatures above 400°C. On the other hand, the stress at minimal volumetric strain for concrete with water-binder ratio of 50% is lower than the value of the high-temperature compressive strength by about 10 N/mm\(^2\), which confirms that there is a difference with regards to the water-binder ratio. This means that concrete whose strength is relatively lower does not experience creep failures when the stress is below 50% of the high-temperature strength and the temperature is lower than 800°C, while concrete whose compressive strength at ambient temperature exceeds 90 N/mm\(^2\) is prone to creep failures if the applied compressive stress is above 20 N/mm\(^2\) and the temperature rises above 400°C.

The reason for the stress at minimal volumetric strain to decrease at high temperatures and for the volumetric strain to increase at maximal loading is considered to be that a multitude of micro cracks emerge at the border of the mortar and the aggregate when they are exposed to heat since the mortar and the aggregate have different thermal expansion coefficients. Furthermore, both the free water and the bonded water in the concrete dissipate, leaving an increasing number of empty gaps.
Figure 3.19 Relation between the stress and the volumetric strain (W/B=22.5%)

Figure 3.20 Relation between the stress and the volumetric strain (W/B=30%)

Figure 3.21 Relation between the stress and the volumetric strain (W/B=50%)
Figure 3.22 Compressive strength and stress at minimal volumetric strain ($W/B=22.5\%$)

Figure 3.23 Compressive strength and stress at minimal volumetric strain ($W/B=30\%$)

Figure 3.24 Compressive strength and stress at minimal volumetric strain ($W/B=50\%$)

Figure 3.25 Stress/high-temperature strength at minimal volumetric strain
3.4 Plastic flow rule

As mentioned in the preceding section, when compressive loading is applied to concrete at high temperatures, volume contraction occurs for relatively low stress, however, volumetric expansion becomes prominent as the stress increases, and considerable volumetric expansion occurs as compared to the ambient temperature scenario.

In order to obtain a quantitative understanding of the volumetric changes of concrete under compressive stress, we will discuss the strain behavior by looking at the stress-strain relations obtained in section 3.2.

3.4.1 Deriving the plastic flow rule for plain concrete

The strain for inelastic concrete can be decomposed into elastic strain components and plastic strain components, as shown in Figure 3.26. In order to determine the extent of the lateral strain for concrete under compressive stress, it is necessary to understand the emergence ratio of the longitudinal and the lateral plastic strain.

\[
\text{Figure 3.26 Elastic strain and plastic strain}
\]

In order to define the plastic strain increment, we introduce the concept of plastic potential, which is used in the plastic flow Equation 3.1.

\[
d\varepsilon_{ij}^{pl} = d\lambda \frac{\partial G}{\partial \sigma_{ij}}
\]  

(3.1)

Here, \(d\varepsilon_{ij}^{pl}\): Plastic strain increment in the \(ij\) direction
\(d\lambda\): A positive proportionality coefficient
\(G\): Plastic potential function
\(\sigma_{ij}\): Stress in the \(ij\) direction (expansion is considered positive, contraction is negative)

Equation 3.1 is proportional to the stress gradient of the plastic potential function \(G\), and indicates that there is a plastic potential increment.

Since there is a change in volume when compressive stress is applied to concrete, as shown in Figures 3.19 to 3.21, it is necessary to use a function capable of taking into account the changes in the volumetric strain with respect to the plastic potential
function $G$. Here, we use Equation 3.2 as a function capable of taking into account the changes in volumetric strain.

$$
G = t - p \cdot \tan \psi
$$

(3.2)

$t$: Mises equivalent stress ($\sigma_n$ is the principal stress)

$$
t = \sqrt{\frac{1}{2} \left( (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right)}
$$

$p$: Hydrostatic stress, $p = -\frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33})$

$\psi$: Dilatancy angle

The positive proportionality coefficient $d\lambda$ in Equation 3.1 and the dilatancy angle $\psi$ in Equation 3.2 can be derived from the relation between the longitudinal plastic strain increment $\varepsilon_{11}^{pl}$ and the lateral plastic strain increment $\varepsilon_{22}^{pl}$ from the stress-strain relation obtained during the experiments. The method for deriving the positive proportionality coefficient $d\lambda$ and the dilatancy angle $\psi$ is explained below.

Substituting Equation 3.2 in Equation 3.1 with respect to the plastic strain increment, we obtain

$$
de_{y}^{pl} = d\lambda \cdot \frac{\partial (t - p \cdot \tan \psi)}{\partial \sigma_y} 
$$

$$
de_{y}^{pl} = d\lambda \cdot \frac{\partial \left( \sqrt{\frac{1}{2} \left( (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right)} + \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \cdot \tan \psi \right)}{\partial \sigma_y}
$$

Upon differentiating the plastic potential function $G$ in Equation 3.1 with respect to the compressive stress $\sigma_{11}$, we obtain the following equation:

$$
\delta \left( \sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11} \cdot \sigma_{22} - \sigma_{22} \cdot \sigma_{33} - \sigma_{33} \cdot \sigma_{11}} + \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \cdot \tan \psi \right)
$$

(3.3)

Letting $f(x) = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11} \cdot \sigma_{22} - \sigma_{22} \cdot \sigma_{33} - \sigma_{33} \cdot \sigma_{11}$ and using the functions $y = f(x)^n$, $y' = nf(x)^{n-1} f'(x)$, Equation 3.3 can be solved as follows, giving the positive proportionality coefficient:

$$
de_{11}^{pl} = d\lambda \left( 0.5 \cdot \frac{2 \sigma_{11} - \sigma_{22} - \sigma_{33}}{\sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11} \cdot \sigma_{22} - \sigma_{22} \cdot \sigma_{33} - \sigma_{33} \cdot \sigma_{11}}} + \frac{1}{3} \tan \psi \right)
$$

(3.4)

$$
d\lambda = \frac{de_{11}^{pl}}{0.5 \cdot \frac{2 \sigma_{11} - \sigma_{22} - \sigma_{33}}{\sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11} \cdot \sigma_{22} - \sigma_{22} \cdot \sigma_{33} - \sigma_{33} \cdot \sigma_{11}}} + \frac{1}{3} \tan \psi}
$$

(3.5)
Next, regarding the lateral plastic strain increment $\varepsilon_{22}^{pl}$, we obtain

\[ \frac{d\varepsilon_{22}^{pl}}{d\lambda} = -\frac{\partial}{\partial \sigma_{22}} \left( \sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11} \cdot \sigma_{22} - \sigma_{22} \cdot \sigma_{33} - \sigma_{33} \cdot \sigma_{11}} + \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \cdot \tan \psi \right) \]

Letting $f(x) = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11} \cdot \sigma_{22} - \sigma_{22} \cdot \sigma_{33} - \sigma_{33} \cdot \sigma_{11}$ and solving the above equation in a similar way, we arrive at

\[ d\varepsilon_{22}^{pl} = d\lambda \left( \frac{2\sigma_{22} - \sigma_{11} - \sigma_{33}}{\sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11} \cdot \sigma_{22} - \sigma_{22} \cdot \sigma_{33} - \sigma_{33} \cdot \sigma_{11}}} + \frac{1}{3} \tan \psi \right) \]

(3.6)

Substituting Equation 3.5 into Equation 3.6 and eliminating the positive proportionality coefficient $d\lambda$, we obtain Equation 3.7:

\[ \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} = \frac{0.5(2\sigma_{22} - \sigma_{11} - \sigma_{33})}{0.5(2\sigma_{11} - \sigma_{22} - \sigma_{33})} + \frac{1}{3} \tan \psi \]

(3.7)

Extracting the longitudinal plastic strain increment $\varepsilon_{11}^{pl}$ and the lateral plastic strain increment $\varepsilon_{22}^{pl}$ from the principal stress during the experiment and the $\sigma - \varepsilon$ relation, and substituting them into Equation 3.7, we can derive the $\sigma_{11}$, $\sigma_{22}$, $\sigma_{33}$ dilatancy angle $\Psi$.

Here, in the case of the cylinder compression test, the stress in the direction of the hoop is equal to the radial stress, i.e. $\sigma_{22} = \sigma_{33}$, and Equation 3.7 can be written as follows:

\[ \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} = \frac{0.5(\sigma_{22} - \sigma_{11}) + \frac{1}{3} \tan \psi}{0.5(2\sigma_{11} - 2\sigma_{22})} + \frac{1}{3} \tan \psi \]

(3.8)

$\sigma_{11}$ and $\sigma_{22}$ can be eliminated from the above equation, and the relation between the plastic strain increment ratio, corresponding to the axial direction, and the dilatancy angle can be expressed with the following equation:

\[ \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} = \frac{1 + \frac{1}{3} \tan \psi}{2 + \frac{1}{3} \tan \psi} \]

(3.9)

The relation between the plastic strain increment ratio and the dilatancy angle is shown in Figure 3.27. This figure shows that when the angle is $0^\circ$, then $d\varepsilon_{22}^{pl} = -0.5d\varepsilon_{11}^{pl}$ and there is no dilatancy ($d\varepsilon_{\nu} = d\varepsilon_{11}^{pl} + d\varepsilon_{22}^{pl} + d\varepsilon_{33}^{pl} = d\varepsilon_{11}^{pl} - 0.5d\varepsilon_{11}^{pl} - 0.5d\varepsilon_{11}^{pl} = 0$), while dilatancy is generated when the dilatancy angle is larger than $0^\circ$. 

The strain for plain concrete obtained through the compression tests was decomposed into elastic strain and plastic strain according to the method depicted in Figure 3.26. The relation between the axial plastic strain and the lateral plastic strain is shown in Figures 3.28 to 3.32.

It was found that when the axial plastic strain increases, in other words, when the maximal loading is approached, the tangent stiffness \( \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} \) increases, and the dilatancy angle increases in accordance with the stress ratio (stress/compressive strength for each temperature).

\( \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} \) was obtained by differentiating the curves in Figures 3.28 to 3.32. The relations between \( \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} \) and the stress ratios are shown in Figures 3.33 to 3.35. As already mentioned, the lower limit of \( \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} \) is 0.5, and therefore the tangent stiffness for the curves in Figures 3.28 to 3.32 has been plotted for values above 0.5.

Following the increase of the stress, there are many cases where \( \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} \) increases as a quadratic function. The mean value of \( \frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}} \) increased with the decrease of the water-binder ratio.

The dilatancy angle was calculated using the relation in (3.9), and the results are shown in Figures 3.36 to 3.38. The dilatancy angle increased following the increase in the stress, and it was found that it converges to a range between around 50° and 65° at maximal loading.
Figure 3.28  Axial and lateral plastic strain (ambient temperature)

Figure 3.29  Axial and lateral plastic strain (200°C)

Figure 3.30  Axial and lateral plastic strain (400°C)
Figure 3.31 Axial and lateral plastic strain (600°C)

Figure 3.32 Axial and lateral plastic strain (800°C)
Figure 3.33 Dependency of $\frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}}$ on the stress (W/B=22.5%)

Figure 3.34 Dependency of $\frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}}$ on the stress (W/B=30%)

Figure 3.35 Dependency of $\frac{d\varepsilon_{22}^{pl}}{d\varepsilon_{11}^{pl}}$ on the stress (W/B=50%)
Figure 3.36 Dependency of the dilatancy angle on the stress (W/B=22.5%)

Figure 3.37 Dependency of the dilatancy angle on the stress (W/B=30%)

Figure 3.38 Dependency of the dilatancy angle on the stress (W/B=50%)
3.4.2 Deriving the plastic flow rule for confined concrete

As in the case of plain concrete, we have derived the plastic strain increment ratio and the dilatancy angle for confined concrete. Since in the case of confined concrete the compressive stress and the elastic component of the strain emerge in both radial (lateral) and circumferential direction, it is necessary to consider the influence of the elastic strain for the portion of the Poisson ratio from other directions with respect to both the longitudinal and the radial direction. As a result, the longitudinal elastic strain $\varepsilon_{11}^{el}$ and the lateral elastic strain $\varepsilon_{22}^{el}$ are obtained from the following equations, and the longitudinal and the lateral plastic strain was obtained from the measured strain by eliminating the elastic strain.

$$\varepsilon_{11}^{el} = \frac{1}{E} \left[ \sigma_{11} - 2\nu \sigma_{22} \right]$$

$$\varepsilon_{22}^{el} = \frac{1}{E} \left[ \sigma_{22} - \nu (\sigma_{11} + \sigma_{22}) \right]$$

In addition, the experimental results for plain concrete were used for the Young modulus. The plastic strain increment ratio and the dilatancy angle for confined concrete with respect to each experimental temperature are shown in Figures 3.39 to 3.53. For the purpose of comparison, the experimental results for plain concrete have been plotted with a black line on those figures.

It becomes clear from Figures 3.39 to 3.53 that for temperatures above 400°C, the plastic strain increment ratio and the dilatancy angle do not depend on the presence or absence of hoops for any of the water-binder ratios, except in the case of W/B=22.5%. Furthermore, the curves almost converge in the upward right direction, and it was found that the confinement by hoops do not exert any influence on the plastic strain increment ratio and the dilatancy angle.

The values for the plastic strain increment ratio and the dilatancy angle for temperatures below 200°C depend on the presence or absence of hoops. However, since there was no consistent tendency such as, e.g., the plastic strain increment ratio and the dilatancy angle increase or decrease together with the number of hoops, it was considered that the dependency ranges fluctuated, it has been conjectured that the presence or absence of lateral confinement does not exert any influence on the plastic strain increment ratio and the dilatancy angle.

Similarly to the case of plain concrete, the plastic strain increment ratio and the dilatancy angle at 400°C for the confined specimen with W/B= 22.5% are different than those at other temperatures, whereby the plastic strain increment ratio exceeded 4 for relatively low stress ratios. The reason for this is as yet unknown.

For all water-binder ratios, the dilatancy angle did not change between ambient temperature and 200°C, while it increased between 200°C and 600°C and decreased between 600°C and 800°C.

It was found that the lower the water-binder ratio (i.e., the higher the strength at ambient temperature), the higher the plastic strain increment ratio and the dilatancy angle.
Figure 3.39 Plastic strain increment ratio and dilatancy angle (W/B=22.5%, ambient temperature)

Figure 3.40 Plastic strain increment ratio and dilatancy angle (W/B=22.5%, 200°C)

Figure 3.41 Plastic strain increment ratio and dilatancy angle (W/B=22.5%, 400°C)
Figure 3.42 Plastic strain increment ratio and dilatancy angle (W/B=22.5%, 600°C)

Figure 3.43 Plastic strain increment ratio and dilatancy angle (W/B=22.5%, 800°C)
Figure 3.44 Plastic strain increment ratio and dilatancy angle (W/B=30%, ambient temperature)

Figure 3.45 Plastic strain increment ratio and dilatancy angle (W/B=30%, 200°C)

Figure 3.46 Plastic strain increment ratio and dilatancy angle (W/B=30%, 400°C)
Figure 3.47 Plastic strain increment ratio and dilatancy angle (W/B=30%, 600°C)

Figure 3.48 Plastic strain increment ratio and dilatancy angle (W/B=30%, 800°C)
Figure 3.49 Plastic strain increment ratio and dilatancy angle (W/B=50%, ambient temperature)

Figure 3.50 Plastic strain increment ratio and dilatancy angle (W/B=50%, 200°C)

Figure 3.51 Plastic strain increment ratio and dilatancy angle (W/B=50%, 400°C)
3.4.3 **Verifying the derived plastic flow rule**

The stress-strain relation was calculated by performing an analysis, in order to verify the accuracy of the plastic flow rules derived in the preceding section. ABAQUS ver. 6.5-4 was used for the analysis. The model considered the symmetry of the cylinder, which was divided by using axisymmetric solid elements into a mesh containing 1 element in the longitudinal direction and 10 elements in the radial direction (Figure 3.54). In order to account for the Bernoulli-Euler theory, a multi-point constraint was imposed in order for the longitudinal displacement of the 11 nodal points marked with ● in Figure 3.54 to match, and the lateral displacement was left free.

The following three points were given as properties of the concrete:

1. The relation between the stress and the longitudinal strain is the multilinear model shown in Figures 3.55 to 3.57.
2. The Poisson ratio (cf. Table 3.2)
3. The plastic flow rules for plain concrete (cf. Figures 3.36 to 3.38)

In addition, the stress in the multilinear model shown in Figures 3.55 to 3.57 represents the true stress, where the changes in the cross-sectional area during the loading are taken into account. Therefore, in comparison with the relation between the engineering
stress and the strain, the strain takes slightly lower values for high temperatures where there is notable volumetric expansion.

The default settings in ABAQUS do not reflect the plastic flow rule in $\varnothing$, which depends on the temperature and the stress, and a user subroutine was prepared in order to account for it.

Figures 3.58 to 3.72 show comparisons between the analytical and the experimental results. Since the analytical values of the longitudinal and the axial strain matched the experimental ones, it was confirmed that sufficient accuracy was achieved by using the material models presented in $\varnothing$ to $\varnothing$ above.

![Analytical model (units: mm)](image)

*Figure 3.54 Analytical model (units: mm)*
Figure 3.55 Stress-strain relation used in the analysis (W/B=22.5%)

Figure 3.56 Stress-strain relation used in the analysis (W/B=30%)

Figure 3.57 Stress-strain relation used in the analysis (W/B=50%)
Figure 3.58 Experimental and analytical stress-strain relation (W/B=22.5%, ambient temperature)

Figure 3.59 Experimental and analytical stress-strain relation (W/B=22.5%, 200°C)

Figure 3.60 Experimental and analytical stress-strain relation (W/B=22.5%, 400°C)
Figure 3.61 Experimental and analytical stress-strain relation (W/B=22.5%, 600°C)

Figure 3.62 Experimental and analytical stress-strain relation (W/B=22.5%, 800°C)
Figure 3.63 Experimental and analytical stress-strain relation (W/B=30%, ambient temperature)

Figure 3.64 Experimental and analytical stress-strain relation (W/B=30%, 200°C)

Figure 3.65 Experimental and analytical stress-strain relation (W/B=30%, 400°C)
Figure 3.66 Experimental and analytical stress-strain relation (W/B=30%, 600°C)

Figure 3.67 Experimental and analytical stress-strain relation (W/B=30%, 800°C)
Figure 3.68 Experimental and analytical stress-strain relation (W/B=50%, ambient temperature)

Figure 3.69 Experimental and analytical stress-strain relation (W/B=50%, 200°C)

Figure 3.70 Experimental and analytical stress-strain relation (W/B=50%, 400°C)
3.5 Failure criterion

3.5.1 Drucker-Prager failure criterion modified with high-temperature strength

Based on the relation between the compressive strength and the lateral stress obtained from the experiments, we will discuss the high-temperature failure criterion for laterally confined concrete. Here, the examination will be performed with the assumption that the failure is defined as the maximal loading.

If the failure criterion for concrete is defined as a function of the stress on the columns, it is possible to generalize it regardless of the direction of the stress and to apply it to the analysis. Therefore, by focusing on the first stress variant $I_1$ and the second invariant of the deviatoric stress $J_2$, the relation between the lateral stress and the compressive strength can be adjusted.

\[ I_1 = \sigma_1 + \sigma_2 + \sigma_3 \]  
(3.10)

\[ J_2 = \frac{1}{6} \left( (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right) \]  
(3.11)

Here, $\sigma_1$ : compressive strength (compression is considered positive)

$\sigma_2, \sigma_3$ : Lateral stress with respect to compression (compression is considered positive)
The first stress invariant $I_1$ and the square root of the second stress invariant $\sqrt{J_2}$ have been made dimensionless by dividing them by the plain compressive strength $\sigma_b(T)$ at each experimental temperature $T$. $I_1/\sigma_b(T)$ and $\sqrt{J_2}/\sigma_b(T)$ are shown in Table 3.5, and the relations between $I_1/\sigma_b(T)$ and $\sqrt{J_2}/\sigma_b(T)$ are shown in Figure 3.73 to 3.75. The linear regression formula is also shown in the figures.

From those figures, it has been confirmed that the relation between $I_1/\sigma_b(T)$ and $\sqrt{J_2}/\sigma_b(T)$ is linear for all water-binder ratios. Therefore, this shows that, for a practical amount of lateral hoop reinforcement, the Drucker-Prager failure criterion[3] is applicable, as well as that the slope of the line, i.e., the constant $\alpha$ in Equation 3.12 is not influenced by the temperature rise, remaining constant from ambient temperature to high temperature. The relations between $\alpha$ on one hand and the water-binder ratio and the compressive strength at ambient temperature on the other are shown in Figure 3.76. From the same figure, the lower the water-binder ratio (i.e., the higher the strength), the higher the value of $\alpha$.

The physical meaning of $\alpha$ is the dependency of the compressive strength on the hydrostatic pressure, and a large value of $\alpha$ indicates a large increase in strength due to lateral stress.

$$\sqrt{J_2} = \alpha I_1 + k$$

(3.12)

Here, $\alpha, k = \text{const.}$
Figure 3.73 Relation between $I_1/\sigma_b(T)$ and $\sqrt{J_2}/\sigma_b(T)$ ($W/B=22.5\%$)

Figure 3.74 Relation between $I_1/\sigma_b(T)$ and $\sqrt{J_2}/\sigma_b(T)$ ($W/B=30\%$)

Figure 3.75 Relation between $I_1/\sigma_b(T)$ and $\sqrt{J_2}/\sigma_b(T)$ ($W/B=50\%$)
Table 3.5 $I_1 / \sigma_s(T) \text{ and } \sqrt{J_2 / \sigma_s(T)}$

<table>
<thead>
<tr>
<th>Water-binder ratio</th>
<th>Temperature °C</th>
<th>$p_w = 0%$</th>
<th>$p_w = 0.3%$</th>
<th>$p_w = 0.6%$</th>
<th>$p_w = 1.5%$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$I_1 / \sigma_s(T)$</td>
<td>$\sqrt{J_2 / \sigma_s(T)}$</td>
<td>$I_1 / \sigma_s(T)$</td>
<td>$\sqrt{J_2 / \sigma_s(T)}$</td>
<td>$I_1 / \sigma_s(T)$</td>
</tr>
<tr>
<td>W/B=22.5%</td>
<td>Ambient temperature</td>
<td>1.00 0.58</td>
<td>1.02 0.58</td>
<td>1.12 0.63</td>
<td>1.25 0.66</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>1.00 0.58</td>
<td>- -</td>
<td>1.26 0.69</td>
<td>1.54 0.81</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>1.00 0.58</td>
<td>- -</td>
<td>1.33 0.73</td>
<td>1.64 0.83</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>1.00 0.58</td>
<td>1.03 0.58</td>
<td>1.15 0.64</td>
<td>- -</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>1.00 0.58</td>
<td>0.89 0.51</td>
<td>0.93 0.53</td>
<td>- -</td>
</tr>
<tr>
<td>W/B=30%</td>
<td>Ambient temperature</td>
<td>1.00 0.58</td>
<td>1.06 0.60</td>
<td>1.13 0.62</td>
<td>1.29 0.66</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>1.00 0.58</td>
<td>- -</td>
<td>1.17 0.64</td>
<td>1.45 0.74</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>1.00 0.58</td>
<td>- -</td>
<td>1.25 0.67</td>
<td>1.60 0.80</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>1.00 0.58</td>
<td>1.06 0.60</td>
<td>1.19 0.66</td>
<td>- -</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>1.00 0.58</td>
<td>0.96 0.55</td>
<td>1.02 0.58</td>
<td>- -</td>
</tr>
<tr>
<td>W/C=50%</td>
<td>Ambient temperature</td>
<td>1.00 0.58</td>
<td>1.00 0.55</td>
<td>1.15 0.61</td>
<td>1.51 0.72</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>1.00 0.58</td>
<td>1.11 0.61</td>
<td>1.27 0.67</td>
<td>1.61 0.77</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>1.00 0.58</td>
<td>- -</td>
<td>1.22 0.64</td>
<td>1.62 0.77</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>1.00 0.58</td>
<td>1.08 0.61</td>
<td>1.16 0.63</td>
<td>- -</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>1.00 0.58</td>
<td>1.06 0.61</td>
<td>1.15 0.65</td>
<td>- -</td>
</tr>
</tbody>
</table>

Figure 3.76 Relation between the gradient of the line on one hand and the water-binder ratio and the compressive strength at ambient temperature on the other

The results from the high-temperature compressive strength tests for compressive strength at ambient temperature of about 45N/mm$^2$ and 65N/mm$^2$ performed by Kubota[4] are also given in Figure 3.75. According to this figure, the results from the
experiments performed by Kubota also indicated that the relation between $I_1 / \sigma_b(T)$ and $\sqrt{J_2} / \sigma_b(T)$ is linear, however, the slope of the line is larger than that obtained in the present experiment. Regarding the water-binder ratio and other factors as exerting influence on the slope of the line, it is necessary to collect and examine more data in the future.

As mentioned previously, since the effects of the friction of the bearing surface on the lateral stress the have not been considered in the present research, there is a possibility that the lateral stress has been overestimated. Therefore, one needs to be cautious of the possibility of overestimating the strength when using the values obtained here for calculating the increase in strength induced by the confined effect.

In the present experiments, we have calculated $\alpha$ in Equation 3.12 at the so-called compressive meridians in the case where the longitudinal stress is higher than the lateral stress. If the failure criterion from Equation 3.12 is rendered in the principal stress space, the planes which are perpendicular to the hydrostatic axis (the deviatoric planes) in the cones for which the hydrostatic axis passes through the center are circular in shape. However, it is known that $\alpha$ in Equation 3.12 becomes small in cases where the lateral stress is larger than the longitudinal compressive stress at ambient temperature, i.e., on the tensile meridians. In other words, if the failure plane is drawn on the deviatoric plane, the failure plane has the shape of a rounded triangle rather than a round shape, and therefore it is considered that if the lateral stress is larger than the longitudinal stress, the constant $\alpha$ as obtained here can not be applied. There are very few experimental data relating to the compressive meridians even for ambient temperature, and none for high temperatures. This will be left as a task for the future.

3.5.2 Outcome of the associated flow rule

When elastoplastic analysis is performed for concrete, it is customary to use the associated flow rule which introduces the yield surface to the plastic potential of the plastic flow rule, based on the principle of maximal plastic work.

The failure criterion shown in Figures 3.73 to 3.75 can be considered as yield surfaces when the loading is maximal. We will examine the phase between the failure criterion from Figures 3.73 to 3.75 and the plastic potential (Equation 3.2).

Since the relation between the Mises stress $t$ from Equation 3.2 and $\sqrt{J_2}$ from Equation 3.12 is $\sqrt{J_2} = t / \sqrt{3}$ and the relation between the hydrostatic stress from Equation 3.2 and $I_1$ from Equation 3.12 is $I_1 = 3p$, if we express Equation 3.12 through the Mises stress $t$ and the hydrostatic stress $p$, we obtain $t = 3\sqrt{3\alpha p} + \sqrt{3k}$.

If the failure criterion is adopted into the plastic flow rule, plastic strain is generated perpendicularly to the failure criterion, and $\tan \Psi$ from Equation 3.2 equals $3\sqrt{3\alpha}$. The dilatancy angle and the plastic strain increment ratio deduced from the failure criterion are shown in Table 3.6. A comparison with the dilatancy angle derived in Section 3.4 is shown in Figure 3.77. From the same figure, it is clear that dilatancy angle deduced from the failure criterion matches the dilatancy angle for maximal loading obtained
in the experimental results, the generated plastic flow rule is perpendicular to the failure criterion, and the associated flow rule is valid in the case of maximal loading. However, the dilatancy angle and the plastic potential derived from the failure criterion for low stress levels do not match, and phenomena such as stress-dependent increase of the dilatancy angle are not observed. Therefore, the dilatancy is overestimated if elastoplastic analysis is performed by using the yield surface for maximal loading in the plastic flow rule of concrete.

Table 3.6 Dilatancy angle and plastic strain increment ratio deduced from the failure criterion

<table>
<thead>
<tr>
<th>Water-binder ratio</th>
<th>α</th>
<th>tan φ</th>
<th>φ (deg)</th>
<th>$\frac{ds_{p}^{0}}{ds_{p}^{0}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.5%</td>
<td>0.42</td>
<td>2.18</td>
<td>65</td>
<td>4.50</td>
</tr>
<tr>
<td>30%</td>
<td>0.37</td>
<td>1.92</td>
<td>63</td>
<td>3.18</td>
</tr>
<tr>
<td>50%</td>
<td>0.32</td>
<td>1.66</td>
<td>59</td>
<td>2.37</td>
</tr>
</tbody>
</table>

Figure 3.77 Dilatancy angle in the case where the associated flow rule is adopted for the plastic flow rule

3.5.3 Relations with other failure criteria

(1) The Richart formula[5]

The following equation is sometimes used with regards to the increase of the compressive strength due the confined effect, where the confined compressive strength $\sigma_c$, the plain compressive strength $\sigma_p$, the constant $c$ and the lateral stress $p$ are explanatory variables.
\[ u \sigma_b = \sigma_b + c \cdot p \] (3.13)

The value of the constant \( c \) above was obtained as 4.1\[5\] by Richart et al. Furthermore, with regards to the influence of the concrete strength, the value of the constraint factor \( c \) is 2~3 for low strength and 6 for high strength, which indicates that the higher the strength of the concrete, the higher the value of \( c \)[6]. In addition, according to the research of Yasuda et al. [7], the value of \( c \) for high-strength concrete which has silica fume mixed into it is 8.3.

Although both Equation 3.12 and Equation 3.13 represent the increase in the compressive strength due to the confined effect, the shapes of the functions are different. We will examine the relation between Equation 3.12, Equation 3.13 and the phase.

When lateral stress \( p \) is applied to concrete with plain strength \( \sigma_b \), the compressive strength according to Equation 3.13 becomes \( \sigma_b + c \cdot p \). In other words, the stress on the columns follows the equation below.

\[ \sigma_1 = \sigma_b + c \cdot p \] (3.14)

\[ \sigma_2 = \sigma_3 = p \] (3.15)

If Equation 3.14 and Equation 3.15 are substituted into Equation 3.10 and Equation 3.11 and are converted into dimensionless variables by dividing them by the plain strength \( \sigma_b \), then \( I_1 / \sigma_b(T) \) and \( \sqrt{J_2} / \sigma_b(T) \) can be expressed through the following equations.

\[ I_1 / \sigma_b = 1 + \frac{(c+2) \cdot p}{\sigma_b} \] (3.16)

\[ \sqrt{J_2} / \sigma_b = \frac{1}{\sqrt{3}} \left( 1 + \frac{(c-1) \cdot p}{\sigma_b} \right) \] (3.17)

If Equation 3.12 is converted into a dimensionless variable through the division by the plain strength \( \sigma_b \), the graph corresponding to the result of the uniaxial compressive strength test necessarily passes through the point with coordinates \( (I_1, \frac{1}{\sqrt{3}}) \), as shown in Figure 3.78. Therefore, \( k \) from Equation 3.12 becomes \( \left( \frac{1}{\sqrt{3}} - \alpha \right) \sigma_b \), and Equation 3.12 can be written as

\[ \sqrt{J_2} = \alpha I_1 + \left( \frac{1}{\sqrt{3}} - \alpha \right) \sigma_b \] (3.18)

With regards to the relation between \( I_1 / \sigma_b(T) \) and \( \sqrt{J_2} / \sigma_b(T) \), the coordinates as determined from Equation 3.16 and Equation 3.17 are \( \left( 1 + \frac{(c+2) \cdot p}{\sigma_b}, \frac{1}{\sqrt{3}} \left( 1 + \frac{(c-1) \cdot p}{\sigma_b} \right) \right) \).

As mentioned above, since the graph corresponding to the relation between \( I_1 / \sigma_b(T) \) and \( \sqrt{J_2} / \sigma_b(T) \) necessarily passes through the \( (I_1, \frac{1}{\sqrt{3}}) \) point, Equation 3.13 can be expressed as a straight line connecting the points \( (I_1, \frac{1}{\sqrt{3}}) \) and \( \left( \frac{1}{\sqrt{3}} \left( 1 + \frac{(c-1) \cdot p}{\sigma_b} \right), 1 + \frac{(c+2) \cdot p}{\sigma_b} \right) \).
The slope of this line corresponds to $\alpha$ in Equation 3.12, and can be written as the following equation. The relation between $\alpha$ and the constraint factor $c$ is shown in Figure 3.79.

$$\alpha = \frac{(c-1) \cdot p}{\sqrt{3} (c+2) \cdot p} = \frac{1}{\sqrt{3}} \cdot \frac{c-1}{c+2}$$

Equation 3.13 is equivalent to the following equation if we acknowledge the fact that there is a linear relation between $I_1 / \sigma_b(T)$ and $\sqrt{J_2 / \sigma_b(T)}$

$$\sqrt{J_2 / \sigma_b} = \frac{1}{\sqrt{3}} \left( \frac{c-1}{c+2} \right) I_1 / \sigma_b + \frac{1}{\sqrt{3}} \left( \frac{c-1}{c+2} \right)$$

(2) The Mohr-Coulomb failure criterion

The Mohr-Coulomb internal friction angle $\phi$ can be used as a measure of the confined effect. For example, in order to determine the internal friction angle corresponding to the stress-strain relation of confined concrete with compressive strength at normal temperature of 30N/mm$^2$, Yoshida et al.[8] have performed a finite element analysis using the internal friction angle as a parameter, and have determined that a value of 53° to be the most adequate.

The following relation exists between the internal friction angle $\phi$ and $\alpha$ from Equation 3.12[9], and this relation is shown in Figure 3.80. The constraint factor $c$ and the internal friction angle obtained from this experiment are shown in Table 3.7.

$$\alpha = \frac{2 \sin \phi}{\sqrt{3} (3 - \sin \phi)}$$

<table>
<thead>
<tr>
<th>Water-binder ratio</th>
<th>$\alpha$</th>
<th>$c$</th>
<th>($^\circ$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>W/B=22.5%</td>
<td>0.42</td>
<td>9.0</td>
<td>53</td>
</tr>
<tr>
<td>W/B=30%</td>
<td>0.37</td>
<td>6.4</td>
<td>47</td>
</tr>
<tr>
<td>W/B=50%</td>
<td>0.32</td>
<td>4.7</td>
<td>41</td>
</tr>
</tbody>
</table>

Figure 3.78 Equation (3.12) converted into a dimensionless variable through division by the plain strength $\sigma_b$. 
3.6 Verifying the derived plastic flow rule and failure criterion

When longitudinal stress is applied to concrete reinforced with hoops, the concrete shrinks in longitudinal direction while expanding in lateral direction at the same time. Since the hoops confine the expansion of concrete in lateral direction, lateral stress is generated in the concrete, which results in increased strength of the concrete. This behavior is a combination of lateral expansion and passive confinement, and represents a coupled effect of the plastic flow and the increase in strength due to multiaxial stress.

The plastic flow rule derived in Section 3.4 determines the amount of expansion of the concrete in lateral direction, and the failure criterion derived in Section 3.5 determines the increase of the compressive strength in accordance with the lateral confinement stress. Those have been obtained independently.

Consequently, in order to explore whether the increase of the compressive strength and the value of the strain at peak stress for concrete confined with hoops can be predicted by using the plastic flow rule and the failure criterion, a finite element analysis was performed, and the experimental results for the relation between the stress on one hand and the longitudinal and the lateral strain on the other for the confined specimens were compared with the analytical results.

The finite element mesh is identical with Figure 3.54, whereby hoops were distributed in the elements in the right part of the same figure. On the basis of Figures 2.2 to 2.6, the stress-strain relation for the hoops has been set as the trilinear stress-strain relation shown in Figure 3.81. In addition, although the stress in Figures 2.2 to 2.6 is the
engineering stress, which is the load divided by the cross-section of the fundament, the stress shown in Figure 3.81 is the true stress which takes into account the changes in the cross-section.

The results from the analysis, which include the compressive strength, the longitudinal and the lateral strain at peak stress and the lateral stress induced by the confinement with hoops, are shown in Table 3.8. The comparisons between the analytical and the experimental values of the compressive strength, the longitudinal and the lateral strain at peak stress and the lateral stress induced by the confinement with hoops are presented in Figures 3.82 to 3.85 (excepting the analytical results for plain concrete). The experimental and the analytical results for the stress-strain relation are shown in Figures 3.86 to 3.100. The stress-strain relation for the hoops shown in Figure 3.81 is the relation obtained by the analysis of the \(\sigma\) plot at peak stress.

Figures 3.82 to 3.85 indicate that there is an accurate match with the experimental results for the compressive strength and the lateral stress with respect to all water-binder ratios, which allows for the lateral stress induced by the lateral expansion of the concrete and the hoop confinement, as well as the increase of the strength induced by the lateral stress, to be predicted with a relatively high accuracy. Also, the values for the longitudinal and the lateral strain in the cases of W/B = 22.5% and W/B = 30% mostly matched the experimental results, which confirmed the validity of the derived constitutive laws.

The analytical values for both the longitudinal and the lateral stress in the case of W/B = 50% were lower than the experimentally obtained ones. This is attributable to the fact that although the stress-strain relation of plain concrete adopted for the analysis was based on the experimental results, during the experiments the strain in the vicinity of the peak stress accelerated, and the measurements could not be performed for peak stress. An improvement is desirable whereby the strain flow at peak stress can be measured by lowering the loading speed or by shortening the time gap between measurements further.

Looking at the stress-strain relations shown in Figures 3.86 to 3.100, the analysis can mostly reproduce the shape of the stress-strain relations, and it can be said that the plastic flow rule and the failure criterion derived in sections 3.4 and 3.5 are valid.
### Table 3.8 List of the analytical results

<table>
<thead>
<tr>
<th>Water-binder ratio</th>
<th>Temperature °C</th>
<th>$p_w=0.3%$</th>
<th>$p_w=0.6%$</th>
<th>$p_w=1.5%$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Strength (N/mm²)</td>
<td>Axial strain ($\times 10^6$)</td>
<td>Lateral strain ($\times 10^6$)</td>
</tr>
<tr>
<td>$W/B=22.5%$</td>
<td>20</td>
<td>125</td>
<td>-3400</td>
<td>951</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>92.5</td>
<td>-4174</td>
<td>1812</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>65.7</td>
<td>-10162</td>
<td>11239</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>42.5</td>
<td>-6732</td>
<td>13765</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>41.9</td>
<td>-22763</td>
<td>25419</td>
</tr>
<tr>
<td>$W/B=30%$</td>
<td>20</td>
<td>95.5</td>
<td>-2891</td>
<td>1063</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>70.3</td>
<td>-3770</td>
<td>1578</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>54.6</td>
<td>-5318</td>
<td>5196</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>34.7</td>
<td>-6156</td>
<td>7826</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>33.3</td>
<td>-19569</td>
<td>27216</td>
</tr>
<tr>
<td>$W/C=50%$</td>
<td>20</td>
<td>55.7</td>
<td>-2624</td>
<td>2178</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>44.0</td>
<td>-2577</td>
<td>754</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>41.8</td>
<td>-4473</td>
<td>3019</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>21.2</td>
<td>-5148.8</td>
<td>6172</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>16.0</td>
<td>-5175</td>
<td>6669</td>
</tr>
</tbody>
</table>

**Figure 3.82** Comparison between the experimental and the analytical values of the compressive strength, the lateral stress and the strain (all cases)
Figure 3.83 Comparison between the experimental and the analytical values of the compressive strength, the lateral stress and the strain (W/B=22.5%)

Figure 3.84 Comparison between the experimental and the analytical values of the compressive strength, the lateral stress and the strain (W/B=30%)
Figure 3.85 Comparison between the experimental and the analytical values of the compressive strength, the lateral stress and the strain (W/B=50%)

Figure 3.86 Experimental and analytical values of the stress-strain relation (W/B=22.5%, ambient temperature)

Figure 3.87 Experimental and analytical values of the stress-strain relation (W/B=22.5%, 200°C)

Figure 3.88 Experimental and analytical values of the stress-strain relation (W/B=22.5%, 400°C)
Figure 3.89 Experimental and analytical values of the stress-strain relation (W/B=22.5%, 600°C)

Figure 3.90 Experimental and analytical values of the stress-strain relation (W/B=22.5%, 800°C)

Figure 3.91 Experimental and analytical values of the stress-strain relation (W/B=30%, ambient temperature)

Figure 3.92 Experimental and analytical values of the stress-strain relation (W/B=30%, 200°C)

Figure 3.93 Experimental and analytical values of the stress-strain relation (W/B=30%, 400°C)
Figure 3.94 Experimental and analytical values of the stress-strain relation (W/B=30%, 600°C)

Figure 3.95 Experimental and analytical values of the stress-strain relation (W/B=30%, 800°C)

Figure 3.96 Experimental and analytical values of the stress-strain relation (W/B=50%, ambient temperature)

Figure 3.97 Experimental and analytical values of the stress-strain relation (W/B=50%, 200°C)

Figure 3.98 Experimental and analytical values of the stress-strain relation (W/B=50%, 400°C)
3.7 Summary of the chapter

Aiming at understanding the plastic strain behavior and the failure criterion for concrete at high temperatures, we have performed high-temperature compressive strength tests at ambient to high (800°C) temperatures with respect to plain and confined concrete with compressive strength at ambient temperature between 50 and 115 N/mm², which is made using andesite as grave aggregate, and have subsequently derived the plastic flow rule and the failure criterion. Also, we have implemented finite element analysis in order to verify the validity of the derived plastic flow rule and the failure criterion, and the following conclusions were drawn upon discussing the results.

- The longitudinal and the lateral stress of the concrete increase together with the temperature, and the higher the strength, the higher the rate of the increase.
- The proportion of the lateral strain to the longitudinal strain becomes larger for higher temperatures, and at 800°C the lateral strain exceeds the longitudinal strain.
- From the behavior of the volumetric strain, it can be considered that the higher the strength of the concrete, the greater the decrease of the creep failure stress.
- The dilatancy angle for the plastic flow rule derived from the stress-strain relation at high temperatures increases in accordance with the stress relation (stress/compressive strength at each temperature), and the higher the temperature, the higher the dilatancy angle.
- The dilatancy angle decreases between ambient temperature and 200°C, increases between 200°C and 600°C, and again decreases between 600°C and 800°C.
- For a practical amount of hoop reinforcement, the dimensionless Drucker-Prager failure criterion obtained through the plain compressive strength can be applied to concrete for ambient to high temperatures.
The loser the water-binder ratio, the greater the increase of the strength induced by the lateral stress, and the increase of the compressive strength induced by the confined effect is more prominent.

By reproducing the confined experiment through a finite element analysis using the derived plastic flow rule and the failure criterion, a relatively accurate match was achieved with regards to the results for the increase of the strength due to the hoop reinforcement. Furthermore, the stress-strain relation was also reproduced with good accuracy, and it can be said that the derived plastic flow rule and the failure criterion are valid.

REFERENCES

5. Richart, F.E., Brandtzaeg, A. and Brown, R.L., A Study of the Failure of Concrete under Combined Compressive Stress, Bulletin No.185, Univ. of Illinois Engineering Experiment Station, 1928