Thermal Conductivity of Several Liquid Foods

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The thermal conductivity of eight kinds of liquid foods of total solid content between 10 and 60 wt% was measured by a transient heat flow method using twin probes at temperatures ranging from 10 to 50°C. The effects of temperature and total solid content on the thermal conductivity of sample solutions were investigated. The thermal conductivity of the samples could be represented as a function of both temperature and total solid content. The observed values were compared with values calculated from the three typical structural models for thermal conductivity, i.e., series, parallel, and random heat transfer models. A new model to predict the thermal conductivity of liquid foods, defined as a combination of parallel and random models, was developed, and the observed data agreed well with the values calculated from the new model.

Keywords: Liquid Food, Thermal conductivity, Twin probe method, Heat transfer model

Introduction

Thermal treatments such as pasteurization, concentration, drying, cooling and others are frequently used in food processing, transportation, storage and cooking. Knowledge of the thermal conductivity of food is thus important not only for the process design but also for the prediction and control of various changes occurring in food during thermal processing, and precise data of thermal conductivity are required both for existing foods and for new products and processes. To date, the thermal conductivity of many foods has been measured; some data were summarized by Sweat (1994). However, because of the wide variation across foods, it is practically impossible to obtain reliable thermal conductivity data for all kinds of food from the measured data. A predictive method for the thermal conductivity of foods is therefore needed.

The thermal conductivities of milk (More and Prasad, 1988; Reddy and Datta, 1994; Muramatsu et al., 2003) and fruit juice (Bhumbla et al., 1989; Constenla et al., 1989; Telis-Romero et al., 1998; Muramatsu et al., 2000) have been measured, and have been represented as a function of temperature, total solid content or moisture content. To date, the thermal conductivities of most liquid foods have been analyzed as a two-component system, a solid and a liquid. However, the thermal conductivity of the liquid food is influenced by not only the moisture content but also the composition of the solid part, i.e., the protein, fat, carbohydrate, dietary fiber, and mineral content. Moreover, the thermal conductivity of food depends on the structure (spatial distribution) of the components, so a structural model of thermal conductivity would be valuable for practical use. However, the thermal conductivity of most liquid foods is reported only as a function of temperature and total solid content, and there have been few reports on structural models that predict the thermal conductivity of liquid food. Recently, Bhumbla et al. (1989), Constenla et al. (1989) and Choi and Okos (1986) modeled the thermal conductivity of fruit juice and concentrated milk using the parallel heat transfer model. Muramatsu et al. (2003) reported that a multiphase mixture model (Cheng and Vachon, 1969) well represented the thermal conductivity of whole milk, skim milk, and whey.

In this study, the thermal conductivities of eight kinds of liquid food were measured for various total solid contents and temperatures. The thermal conductivity was determined based on the transient heat flow method using twin probes (Kasubuchi, 1977; Muramatsu et al., 2000, 2002, 2003). The objectives of this study were to (1) determine for several sample solutions the relationship of the thermal conductivity to the total solid content and temperature, and (2) to develop a structural model to predict the thermal conductivity of the sample solutions.

Materials and Methods

Sample In this study, eight kinds of liquid food adjusted to various total solid contents were used, i.e., a fructose solution (total solid content 10, 20, 30, 40, 50, 60 wt %), sucrose solution (10, 20, 30, 40, 50, 60 wt%), grape juice (10, 20, 30, 40, 50 wt%), orange juice (10, 20, 30, 40 wt%), pineapple juice (10, 20, 30, 45 wt%), peach juice (10, 15, 20, 25 wt%), cream (10, 20, 30, 40, 45 wt%), and condensed cream (10, 20, 30, 40, 45, 50 wt%). The fructose solution and the sucrose solution were made by dissolving guaranteed reagents manufactured by Kanto Kagaku Co., Ltd. in pure water. For the fruit juices, in order to obtain
different solid contents, commercially obtained concentrated juice was diluted with distilled water. Cream was made by mixing commercially obtained cream (Snow Brand Milk Products Co., Ltd; Composition: non-fat solids, 5 wt%; fat, 45 wt%; water, 50 wt%), skim milk powder (Snow Brand Milk Products Co., Ltd), and distilled water. Compound cream at various solid contents was made from corn oil (Ajinomoto Co., Ltd), distilled water, and emulsifier (Mitsubishi-Kagaku Foods Co., Ltd., Tokyo, Japan). The compositions of these sample solutions were calculated from measured data and data from the standard table of food compositions in Japan (5th edition, 2002). As an example, the composition of each sample solution at 10 wt % is shown in Table 1.

**Thermal conductivity measurement** Fig. 1 shows a schematic diagram of the heat probe (Tokyo Riko Co., Ltd., Tokyo, Japan) used in this study. This probe consists of a constantan heater wire (0.1 mm diameter, resistance 14.6 Ω) and a T type thermocouple (0.1 mm diameter) in a stainless steel tube (1 mm in outer diameter, 100 mm in length). The internal space of the stainless steel tube is filled with silicon oil. Because the ratio of the length of the probe to the diameter is negligible (100:1), the length of the heat source can be considered to be infinity (Hooper and Lepper, 1950).

A schematic diagram of the thermal conductivity measurement apparatus is shown in Fig. 2. The apparatus consists of three units: heating, temperature control, and recording units. The heating unit consists of two probes, a DC power source (Regulated DC power supply, Model 526, Metronix Co., Ltd., Tokyo, Japan), a DC ammeter (Portable direct current meter / voltmeter, Model 2012, Yokogawa Electric Co., Ltd., Tokyo, Japan), and a switch. To ensure consistent heating, the heater wires for the probes were connected in series. To maintain a constant temperature throughout the test, a circulating water bath was used (Thermocirculator, Model TC-100, Tokyo Riko Co., Ltd., Tokyo, Japan), which had a temperature control accuracy of 0.004°C. The temperature change in the heat source, namely the change in thermo-electromotive force, was amplified with an OP-amplifier (DC amplifier, Model CA-25F, Tokyo Riko Co., Ltd., Tokyo, Japan) and was recorded with a data logger (Data logger, Model R7326B, Advantest Co., Ltd., Tokyo, Japan).

Measurement based on the transient heat flow method using twin probes, called the twin probe method, is a relative measurement, and can measure thermal conductivity with high accuracy (Kasubuchi, 1977; Muramatsu et al., 2000, 2002, 2003). In this method, the temperature variation is simultaneously measured for probes separately inserted in a reference material and the sample, and the effective thermal conductivity of the sample is calculated by multiplying the thermal conductivity of the reference material by the ratio of the temperature increase of the reference probe to that of the sample probe. In this study, measurements of thermal conductivity were conducted for several total solid contents ranging from 10 to 60 wt% and five temperatures (10, 20, 30, 40, 50°C) for each total solid content. To avoid natural convection, 1 wt% agar was added to make a gel for sample solutions other than the compound cream, for which 1 wt% agar and waste oil disposal agent were added. The 1 wt% agar gel was also used as the reference material and was considered equivalent to water as it contained 99 wt% water. After the water temperature in the water bath reached a preset temperature, one probe was inserted into the center of the reference material (1 wt% agar gel), and the other into the sample. The sample container was submerged in the water bath and thus the temperatures of the sample, reference material, and reference point thermocouple (cold junction) were precisely controlled. Because of the high thermal conductivity requirement, the cylindrical sample container and reference point container were made of 0.3 mm-thick copper plate. The contain-

<p>| Table 1. Composition of each sample at 10 wt% total solid content. |</p>
<table>
<thead>
<tr>
<th>Sample</th>
<th>Moisture, wt%</th>
<th>Protein, wt%</th>
<th>Fat, wt%</th>
<th>Carbohydrate, wt%</th>
<th>Fiber, wt%</th>
<th>Ash, wt%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fructose solution</td>
<td>90.0</td>
<td>0.0</td>
<td>0.0</td>
<td>10.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Sucrose solution</td>
<td>90.0</td>
<td>0.0</td>
<td>0.0</td>
<td>10.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Grape juice</td>
<td>90.0</td>
<td>0.2</td>
<td>0.1</td>
<td>9.6</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>Orange juice</td>
<td>90.0</td>
<td>0.4</td>
<td>0.1</td>
<td>9.1</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>Pineapple juice</td>
<td>90.0</td>
<td>0.2</td>
<td>0.0</td>
<td>9.4</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>Peach juice</td>
<td>90.0</td>
<td>0.4</td>
<td>0.0</td>
<td>8.2</td>
<td>0.4</td>
<td>1.0</td>
</tr>
<tr>
<td>Cream</td>
<td>90.0</td>
<td>1.8</td>
<td>5.0</td>
<td>2.8</td>
<td>0.0</td>
<td>0.4</td>
</tr>
<tr>
<td>Compound cream</td>
<td>90.0</td>
<td>0.0</td>
<td>10.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Apple juice a</td>
<td>90.0</td>
<td>0.1</td>
<td>0.0</td>
<td>9.7</td>
<td>0.0</td>
<td>0.2</td>
</tr>
<tr>
<td>Milk b</td>
<td>90.0</td>
<td>2.6</td>
<td>2.7</td>
<td>4.1</td>
<td>0.0</td>
<td>0.6</td>
</tr>
<tr>
<td>Skim milk b</td>
<td>90.0</td>
<td>3.5</td>
<td>0.1</td>
<td>5.6</td>
<td>0.0</td>
<td>0.8</td>
</tr>
<tr>
<td>Whey b</td>
<td>90.0</td>
<td>1.2</td>
<td>0.5</td>
<td>7.4</td>
<td>0.0</td>
<td>0.9</td>
</tr>
</tbody>
</table>

a Muramatsu et al., 2000
b Muramatsu et al., 2003
er for the reference material was 30 mm in diameter and 200 mm in height. When the variation in the thermoelectromotive force of both sample and reference material was less than 0.4 μV, the materials were assumed to have reached the preset temperature, that is, they were considered to be in thermal equilibrium. Approximately 90 minutes was necessary for this state of equilibrium to be achieved. At this point, the heater wires of the probes were energized, increasing the probe temperatures. The temperature variation of the probes was amplified by the amplifier (Specifications: Input range (sensibility), ±25, 50, 100, 250, 500 μV, ±1, 2.5, 10, 25, 50, 100 mV; Input impedance, 100 kΩ; Noise, ±0.4 μV; Speed of response, 1 sec or less in the 25 μV range; Linearity, 0.5% of full scale; Output, ±10 mV or ±10 V for the μV or mV range, respectively and recorded with the data logger. In this study, when the input range and the output of the amplifier were adjusted to 100 μV and 10 mV, respectively, and the current applied to the heater wire of the probe was adjusted to 130 mA, the temperature of the probe was raised by about 2°C. Current was supplied to the heater wires of the probes for 180 seconds.

Blackwell (1954) gave the following equation to express the temperature rise of a probe.

\[
\theta_t - \theta_k = \frac{q}{4πλ_s} \left[ \ln \left( \frac{4kt}{b^2} \right)^{-\gamma} + \frac{2λ_s}{bH} + \frac{b^2}{2κ_l} \left\{ \ln \left( \frac{4kt}{b^2} \right)^{-\gamma} + 1 \right\} \right] - \frac{M c_p}{πb^2} \left[ \ln \left( \frac{4kt}{b^2} \right)^{-\gamma} + \frac{2λ_s}{bH} \right] + \left( \frac{b_1}{k^2T^2} \right)
\]  

In Eq. (1), the influences of the contact resistance between the sample and the probe, and the heat capacity of the probe are considered. When two probes with the same characteristics are inserted separately into the sample and the reference material, and the current is supplied to the heater wires of the probes simultaneously, the ratio of the temperature increases of the two probes is represented by the following equation based on Eq. (1).

\[
\frac{θ_s - θ_k}{θ_i - θ_k} = \frac{q/4πλ_s}{q/4πλ_w} \left[ \frac{\ln (4kt/b^2)^{-\gamma} + 2λ_s/bH}{2κ_l} \right] + \left( \frac{b_1}{k^2T^2} \right) \frac{1 + \frac{M c_p}{πb^2} (\ln (4kt/b^2)^{-\gamma} + 2λ_s/bH) + a (b_1/κ_l)}{1 + \frac{M c_p}{πb^2} (\ln (4kt/b^2)^{-\gamma} + 2λ_s/bH) + a (b_1/κ_l)}
\]

\[
θ_s - θ_k \quad θ_i - θ_k
\]

The effective thermal conductivity of the sample can be determined using Eq. (3). When \(θ_s - θ_k\) is plotted against \(θ_i - θ_k\) the general features of the temperature variation of the probe can be seen, as shown in Fig. 3. After the first few seconds, a linear relationship is obtained, with the slope equal to \(θ_s - θ_i/θ_i - θ_k\) (the ratio of the variation in temperature of the reference probe to that of the sample probe). In this study, the thermal conductivity of water in Eq. (3) was taken from the literature (Choi and Okos, 1986). The measurements of the effective thermal conductivity were repeated five times for each experimental
Results and Discussion

Relationship between thermal conductivity and total solid content and temperature: The thermal conductivity of eight kinds of liquid food was measured by the transient heat flow method using twin probes. Measurement of the four kinds of fruit juice, the fructose solution, and the sucrose solution gave almost the same value under the same conditions (temperature and total solid content) because the compositions of these samples are similar. For the same total solid content, the thermal conductivity of sample solutions decreased with increasing fat content.

The thermal conductivities of liquid foods such as milk, fruit juice and others can be expressed as functions of temperature and total solid content. The relationship between thermal conductivity and both temperature and total solid content for four sample solutions are shown in Fig. 4. As shown in Fig. 4, the thermal conductivities of all sample solutions decreased as total solid content increased, and a linear or quadratic relationship existed between the thermal conductivity and total solid content. The relationship between thermal conductivity and temperature of the fructose solution is shown in Fig. 5. It can be seen that the thermal conductivity of the fructose solution increased linearly with increasing temperature. Similar results to Figs. 4 and 5 were obtained for all sample solutions. Therefore, we derived the following empirical equation to represent the relationship between thermal conductivity, total solid content, and temperature (Muramatsu et al., 2000).

\[
\lambda_s = aC^sT + bC^s + dCT + eC + fT + g
\]  
(5)

The measured data for each sample were fitted by a linear least squares method to Eq. (5). The solid lines in Fig. 5 show the calculated results from Eq. (5). For all sample solutions, the calculations from Eq. (5) agreed well with the measured data. The parameters and standard error (S.E.) of Eq. (5) for each sample are summarized in Table 2. The standard error is calculated from Eq. (6).

\[
S.E. = \sqrt{\frac{\sum (Y - E)^2}{n}}
\]  
(6)

The relationship between thermal conductivity and both total solid content and temperature for sample could be represented by Eq. (5).

Modeling of thermal conductivity: Thermal conductivity of food is affected by the composition, temperature, and structure (spatial distribution) of the components. Therefore, a structural model of thermal conductivity would be valuable for practical use. Although the thermal conductivity of each sample can be expressed by Eq. (5) as a function of temperature and total solid content, this equation is empirical and is not a structural model. A parallel structural model has been previously applied to estimate the thermal conductivity of fruit juice or concent-
trated milk (Bhumbla et al., 1989; Constenla et al., 1989; Choi and Okos, 1986). In this study, theoretical values of thermal conductivity of each sample solution were calculated from series, parallel, and random heat transfer models, using the volume fraction and thermal conductivity of each component.

The series model (Sweat, 1994):

\[ \frac{1}{\lambda_s} = \sum_i \frac{\varphi_i}{\lambda_i} \]  

(7)

The parallel model (Sweat, 1994):

\[ \lambda_p = \sum_i \varphi_i \lambda_i \]  

(8)

The random model (Woodside and Messmer, 1961):

\[ \lambda_r = \lambda_{r1} \varphi_1 \lambda_{r2} \varphi_2 \cdots \lambda_{rn} \varphi_n \]  

(9)

In the series model (Eq. (7)), heat conduction is assumed to occur perpendicular to alternating layers of each component. In the parallel model (Eq. (8)), the layers of each component are arranged parallel to the direction of heat flow. The parallel model yields the highest values of thermal conductivity, while the lowest values are obtained by the series model. In the random model (Eq. (9)), the phases of each component are assumed to be randomly mixed and the thermal conductivity is the geometric mean of the values for the individual phases. In this study, the volume fractions of each component and temperature (10, 20, 30, 40, 50°C) in Eq. (7)-(9) were obtained by substituting the values of mass fraction (standard table of food compositions in Japan (5th edition, 2002)) and density (Choi and Okos, 1986) for each component into Eq. (10).

\[ \varphi_i = \frac{X_i/\rho_i}{\sum_i X_i/\rho_i} \]  

(10)

The values of intrinsic thermal conductivity of each component and temperature in Eq. (7)-(9) were taken from a reference (Choi and Okos, 1986). In this study, the thermal conductivity of corn oil, expressed by Eq. (11), was measured at 10, 20, 30, 40, 50°C.

\[ \lambda_{Co} = 6.393 \times 10^{-6}T + 1.493 \times 10^{-1} \]  

(11)

In Eq. (7)-(9), the thermal conductivity of corn oil used the calculated value from Eq. (11). Fig. 6 shows a comparison of the measured results (○ symbol) and those calculated by the three different models (Eq. (7)-(9); solid line) at 10 wt% for cream. In this case, the error between the measured and predicted values was about 10% for the series model (Eq. (7)), and 2% for the parallel and random models (Eq. (8)-(9)). The errors increased with increasing total solid content. For example, the errors for the series, parallel, and random models were about 10%, 2%, 2% at 10 wt%; 17%, 5%, 4% at 20 wt%; 20%, 8%, 5% at 30 wt%; 22%, 11%, 6% at 40 wt%; and 24%, 12%, 8% at 45 wt%, respectively. In all measured conditions, Eq. (7)-(9) did not predict the thermal conductivities of the sample solutions with good accuracy.

To predict the effective thermal conductivity of granular starches (Maroulis et al., 1990), fruits, vegetables and meat (Rahman and Chen, 1995; Rahman, 1992; Rahman et al., 1997), granular materials (Krokida et al., 2001), and powdered foods (Murakami and Okos, 1989; Rahman et al., 1991), models combining Eq. (7)-(9) were proposed. We propose a new model to easily predict the thermal conductivity. Because all data points fall within the range defined by the predicted values of the random and parallel models, a new model (Eq. (12)) can be defined by combining the random and parallel models.

\[ \lambda_i = F \times \lambda_{Ra} + (1 - F) \times \lambda_{Pa} \]  

(12)

---

**Table 2.** The parameters and standard error (S.E.) of Equation (5) for each sample.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$a \times 10^3$</th>
<th>$b \times 10^3$</th>
<th>$d \times 10^3$</th>
<th>$e \times 10^3$</th>
<th>$f \times 10^3$</th>
<th>$g \times 10^3$</th>
<th>S.E. $\times 10^3$ W/m °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fructose solution</td>
<td>-1.371</td>
<td>-0.1257</td>
<td>3.815</td>
<td>-3.374</td>
<td>1.349</td>
<td>5.758</td>
<td>1.689</td>
</tr>
<tr>
<td>Sucrose solution</td>
<td>-1.182</td>
<td>-3.341</td>
<td>-2.511</td>
<td>-3.006</td>
<td>1.360</td>
<td>5.758</td>
<td>2.014</td>
</tr>
<tr>
<td>Grape juice</td>
<td>-1.698</td>
<td>1.255</td>
<td>3.457</td>
<td>-3.287</td>
<td>1.344</td>
<td>5.757</td>
<td>1.408</td>
</tr>
<tr>
<td>Orange juice</td>
<td>-1.052</td>
<td>-2.073</td>
<td>-0.9723</td>
<td>-3.172</td>
<td>1.348</td>
<td>5.765</td>
<td>1.288</td>
</tr>
<tr>
<td>Pineapple juice</td>
<td>-0.3894</td>
<td>-11.02</td>
<td>-4.722</td>
<td>-2.729</td>
<td>1.372</td>
<td>5.759</td>
<td>1.296</td>
</tr>
<tr>
<td>Peach juice</td>
<td>-0.8696</td>
<td>-9.194</td>
<td>1.334</td>
<td>-2.953</td>
<td>1.364</td>
<td>5.759</td>
<td>0.9043</td>
</tr>
<tr>
<td>Cream</td>
<td>3.108</td>
<td>3.187</td>
<td>-46.40</td>
<td>-4.746</td>
<td>1.404</td>
<td>5.783</td>
<td>5.452</td>
</tr>
<tr>
<td>Compound cream</td>
<td>2.126</td>
<td>28.74</td>
<td>-30.90</td>
<td>-6.779</td>
<td>1.406</td>
<td>5.760</td>
<td>2.598</td>
</tr>
</tbody>
</table>
In this model, heat conduction is assumed to take place by a combination of random and parallel heat flow. The parameter $F$ is a distribution coefficient and represents the volumetric fraction of the material random to the direction of heat flow; thus $(1-F)$ is the fraction of the material parallel to the heat flow. The mixed model (Eq. (12)) reduces to the parallel model for $F > 0$ or to the random model for $F = 1$. In Eq. (12), Parameter $F$ is generally unknown. Therefore, by substituting the values $\lambda_{pa}$ and $\lambda_{pr}$ calculated from Eq. (8) and (9), respectively, and the measured values of thermal conductivity into Eq. (12), the values of parameter $F$ can be determined for each measured condition of each sample, and the relationship between $F$ and the total solid content and temperature can be examined. The results of this analysis, summarized in Table 3, showed that the values of parameter $F$ were nearly constant for each sample solution regardless of the total solid content and temperature. A comparison between the measured results and the results calculated from Eq. (12) using $F$ values from Table 3 for cream is shown in Fig. 7. The calculated results agreed well with the measured results. The standard errors between the measured values and the values calculated from Eq. (12) using $F$ values for each sample solution are shown in Table 4. From the standard error values shown in Table 4, it can be seen that the thermal conductivity calculated from Eq. (12) corresponded well with the measured data.

In addition, the thermal conductivity values of sample solutions were calculated using the following equation, in which the value of $F$ in Eq. (12) was replaced with the constant 0.5 to give the arithmetic mean of the parallel and random models.

$$\lambda_s = 0.5(\lambda_{pa} + \lambda_{pr})$$

This model (Eq. (13)) recognizes that parallel model (Eq. (8)) and random model (Eq. (9)) are distributed in a 1:1 relationship parallel to the direction of heat flow. The dashed line in Fig. 7 is the value calculated from Eq. (13). The measured data agreed well with the values calculated from Eq. (13). When the results calculated from Eq. (13) were compared with the results calculated from Eq. (12),
no significant difference was observed. The standard errors of Eq. (13) for each sample solution are shown in Table 3. From Fig. 7 and the values of standard error of Eq. (13), it was confirmed that the thermal conductivities of the 12 kinds of sample solutions could be predicted by Eq. (13).

Thus, Eq. (13) is simple in form, and is both convenient and suitable for predicting the thermal conductivity of liquid food with sufficient accuracy for practical use. Eq. (13) will be useful in the design of equipment and operations for processing liquid food.

**Nomenclature**

- $C$: Total solid content (wt%)
- $E_i$: Calculated result
- $n$: Number of data points
- $T$: Temperature ($^\circ$C)
- $X$: Mass fraction (——)
- $Y_i$: Observed result
- $\phi$: Volume fraction (——)
- $\theta$: Probe temperature ($^\circ$C)
- $\lambda$: Thermal conductivity (W/m$^\circ$C)
- $\rho$: Density (kg/m$^3$)
- $a, b, c, d, e, f, g$: Constants

**Subscript**

- $a$: Ash
- $c$: Carbohydrate
- $co$: Corn oil
- $d$: Disperse phase
- $f$: Fat
- $i, j$: Components
- $p$: Protein
- $P_\parallel$: Parallel model
- $P_\perp$: Random model
- $s$: Sample
- $w$: Water
- 0: Initial value

**References**


