On Formal Expression of Spatial Distribution of Strata Using Boundary Surfaces — C_1 and C_2 Type of Contact —*

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Abstract : In order to formulate the distribution of strata in a survey area Ω in terms of subspaces $b_1, b_2, \dots, b_n$ bounded by boundary surfaces $S_1, S_2, \dots, S_{n-1}$, we introduce a new concept called C_1 and C_2 type of boundary surface as a mathematical model of conformity and unconformity, respectively. Let S_i be a boundary surface which divides a given successive sequence of subspaces $(b_r, b_{r+1}, \dots, b_k)$ into two successive sequences $(b_r, \dots, b_i)$ and $(b_{i+1}, \dots, b_k)$. Then a surface S_k is called C_1 type of boundary surface if and only if S_i also divides a successive sequence $(b_r, \dots, b_k, b_{k+1})$ into two successive sequences $(b_r, \dots, b_i)$ and $(b_{i+1}, \dots, b_k, b_{k+1})$. On the other hand, the boundary surface S_k is called C_2 type of boundary surface if and only if S_k divides a successive sequence $(b_1, b_2, \dots, b_k, b_{k+1})$ into a successive sequences $(b_1, \dots, b_k)$ and a single subspace b_{k+1} . It is proved that all subspaces $b_1, b_2, \dots, b_n$ are uniquely defined by boundary surfaces $S_1, S_2, \dots, S_{n-1}$ if subspaces are bounded by either C_1 or C_2 type of boundary surfaces. According to the formulation of strata in terms of subspaces bounded by boundary surfaces, we can define a function g which assigns a label corresponding to a stratum to every point in Ω . The formulation of subspaces and the labeling function provide theoretical bases of the computerized geologic mapping system "CIGMA".

Key Words : Computer Geologic Mapping, Distribution of Strata, Boundary Surface, Conformity, Unconformity, Mathematical Model, CIGMA

1. Introduction

As a theoretical basis of a computerized system for geologic mapping, we have developed an algorithm to construct a labeling function which assigns a unique geologic unit to every point within a survey area Ω .

If we can determine a geologic unit to occur in every grid cell, it is easy to draw the distribution of geologic units in a form of a geologic map through a simple repetition of painting a grid cell with a color appropriate to the corresponding geologic unit. Therefore, it is important in the automatic mapping system to incorporate the function which determines the geologic unit at any given point. It is difficult to store the correspondence between points in a three dimensional space and geologic units

within physical memories of finite size. A realistic way is to "compute" which geologic unit occurs at the given point, referring to a logical model defining the area of geologic unit by two dimensional surfaces which can be easily stored within memories of small size. Generally speaking, geologic units are considered as subspaces which are bounded by some sheets of boundary surfaces. Therefore it is expected that we can define the distribution of geologic units by known boundary surfaces under some geologically reasonable assumptions.

In this paper, we present a preliminary approach to a principle to define the distribution of geologic units by boundary surfaces, based on fundamental properties of sedimentary strata. It will be proved that the distribution of strata is defined uniquely by given boundary surfaces when boundary surfaces have some special attributes. Although the present model is only preliminary, it provides a theoretical basis of our computer system "CIGMA" for automatic mapping of geologic structures described in Sakamoto *et. al.* (1993).

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2. Basic terminology

a) Surface dividing a space Ω

Let Ω be a subspace of three dimensional Euclidian space R^3 . For simplicity of discussion, it is assumed here that the space Ω is a topologically “connected” and “closed” space in which a Cartesian coordinate system O -xyz with a vertical z -axis is defined.

A surface that divides a space Ω into two subspaces is referred to as a boundary surface dividing Ω (See Fig.1). The subspaces that lie in the lower and upper sides of a boundary surface S dividing Ω are denoted by S^- and S^+ , respectively. It is assumed that both S^- and S^+ include the surface S . Then it is clear that

$$S^- \cap S^+ = S,$$

$$S^- \cup S^+ = \Omega.$$

<Example 1>

If a surface S is expressed by a single-valued function $z = s(x, y)$ in a Cartesian coordinates with a vertical z -axis, then S^- and S^+ are

$$S^- = \{(x, y, z) \mid z \leq s(x, y)\},$$

$$S^+ = \{(x, y, z) \mid z \geq s(x, y)\}.$$

b) Relation between subspaces of Ω

In the present paper, we discuss spatial relations between subspaces of Ω defined by boundary surfaces dividing Ω . It is assumed that subspaces are also “connected” and “closed” spaces which include boundary points.

If two subspaces a and b satisfy

$$a \cap b \neq \phi,$$

then we say that a is in contact with b . Further, if a is in contact with b and there exists a boundary surface S dividing Ω such that

$$a \cap b \subset S,$$

then we say that a is in contact with b on S or that a boundary surface S passes through the contact surface between a and b .

Fig.1 A boundary surface dividing a space Ω into the upper subspace S^+ and the lower one S^- .

If there exists a boundary surface S such that

$$a = (a \cup b) \cap S^-,$$

$$b = (a \cup b) \cap S^+,$$

then we say that a is lower than b with respect to S , or simply a is lower than b . We say that a is directly under b , denoted by $a \rightarrow b$, if a is in contact with b and is lower than b (See Fig.2). A fact that $a_i \rightarrow a_{i+1}$ for all $i = 1, \dots, n-1$ is denoted by $a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_n$.

c) Successive sequence of subspaces

If $a \rightarrow b$, then (a, b) is called a successive pair of subspaces. A set of subspaces $\{b_1, b_2, \dots, b_n\}$ which satisfies

$$b_1 \rightarrow b_2 \rightarrow \dots \rightarrow b_n$$

is called a successive sequence of subspaces or simply a successive sequence and is denoted by $(b_1, b_2, \dots, b_n)$.

We say that subspaces $b_1, b_2, \dots, b_n$ cover Ω if

$$b_1 \cup b_2 \cup \dots \cup b_n = \Omega,$$

and we say that $(b_1, b_2, \dots, b_n)$ is a successive sequence covering Ω if $b_1, b_2, \dots, b_n$ cover Ω . Fig.3 shows four examples of successive sequences covering Ω .

<Example 2>

Suppose that there exist $n-1$ layers of boundary surfaces dividing Ω and no pairs of the surfaces intersect each other. Then the surfaces are numbered such a manner that they satisfy

$$S_1^- \subset S_2^- \subset \dots \subset S_{n-1}^-$$

or

$$S_{n-1}^+ \subset \dots \subset S_2^+ \subset S_1^+.$$

Subspaces $b_1, \dots, b_n$ defined by

$$b_1 = S_1^-$$

$$b_i = S_{i-1}^+ \cap S_i^-, \quad i = 2, \dots, n-1$$

$$b_n = S_{n-1}^+$$

satisfy the following :

(1) b_i is in contact with b_{i+1} on S_i .

(2) $b_i \cap b_j = \begin{cases} S_i, & j = i+1 \\ \phi, & j > i+1 \end{cases}$

(3) $(b_1, \dots, b_n)$ is a successive sequence covering Ω .

Fig.2 Spatial relation between subspaces a and b . a is in contact with b on S . a is lower than b with respect to S . Therefore a is directly under b ; $a \rightarrow b$.

Fig.3 Examples of successive sequences covering Ω .

(a) $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4$ (b) $b_1 \rightarrow b_2 \rightarrow b_3 \rightarrow b_4$ (c) $c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow c_4$ (d) $d_1 \rightarrow d_2 \rightarrow d_3 \rightarrow d_4$

3. Logical relation between three subspaces and two boundary surfaces

As a preliminary inspection on a logical relation between boundary surfaces and successive sequence of subspaces covering Ω , let us consider relation between three subspaces and two boundary surfaces.

Let (a, b, c) be a successive sequence covering Ω such that

$$a \cup b \cup c = \Omega, \quad (3.1)$$

and suppose that a is in contact with b on X , and b is in contact with c on Y (See Fig.4(a)). Then from definitions, we have both

$$a = (a \cup b) \cap X^-, \quad (3.2a)$$

$$b = (a \cup b) \cap X^+, \quad (3.2b)$$

and

$$b = (b \cup c) \cap Y^-, \quad (3.3a)$$

$$c = (b \cup c) \cap Y^+. \quad (3.3b)$$

However, we cannot define subspaces a , b and c from the above equations, but we need some additional information on the relation between subspaces a and c .

The following (1) and (2) are examples of the relation between a and c .

(1) Suppose that a is in contact with $b \cup c$ on X (See Fig.4(b)). Then we have

$$a = (a \cup (b \cup c)) \cap X^-, \quad (3.4a)$$

$$b \cup c = (a \cup (b \cup c)) \cap X^+. \quad (3.4b)$$

It is noted that equations (3.4a,b) are derived from a substitution of $b \cup c$ for b in the equations (3.2). Substitution of the equation (3.1) into the equations (3.4) gives

$$a = X^-, \quad (3.5a)$$

$$b \cup c = X^+. \quad (3.5b)$$

Further substituting the equation (3.5b) into equations (3.3a,b), we have finally

$$c = Y^+ \cap X^+,$$

$$b = Y^- \cap X^+,$$

$$a = X^-.$$

(2) On the other hand, suppose that $a \cup b$ is in contact with c on Y (Fig.4(c)). Then we have

$$a \cup b = ((a \cup b) \cup c) \cap Y^-, \quad (3.6a)$$

$$c = ((a \cup b) \cup c) \cap Y^+. \quad (3.6b)$$

Substitution of the equation (3.1) into equations (3.6a,b), we have

$$a \cup b = Y^-,$$

$$c = Y^+,$$

and finally we have

$$c = Y^+,$$

$$b = Y^- \cap X^+,$$

$$a = Y^- \cap X^-.$$

Above examples (1) and (2) suggest a possibility that we can define subspaces by boundary surfaces when we introduce some assumptions on the relation between subspaces and boundary surfaces.

Fig.4 Two possibility of successive sequences (a, b, c) . (a) a is in contact with b on X and b is in contact with c on Y . But it is not clear whether a is in contact with c on X or Y . (b) a is in contact with $b \cup c$ on X or a is in contact with c on X . (c) $a \cup b$ is in contact with c on Y or a is in contact with c on Y .

4. C_1 type of boundary surface

Let $(b_1, b_2, \dots, b_n)$ be a successive sequence covering Ω and let $S_1, S_2, \dots, S_{n-1}$ be boundary surfaces dividing Ω such that S_i passes through the contact surface between b_i and b_{i+1} ($i=1, \dots, n-1$).

From the definitions, we have

$$b_i = (b_i \cup b_{i+1}) \cap S_i^-,$$

$$b_{i+1} = (b_i \cup b_{i+1}) \cap S_i^+,$$

indicating that a subspace $(b_i \cup b_{i+1})$ is divided by a boundary surface S_i into two subspaces; the lower b_i and the upper b_{i+1} . Similarly, the fact that a subspace $(b_r \cup b_{r+1} \cup \dots \cup b_k)$ is divided by a boundary surfaces S_t ($r \leq t \leq k-1$) into two subspaces; the lower $(b_r \cup \dots \cup b_t)$ and the upper $(b_{t+1} \cup \dots \cup b_k)$ can be formulated as follows :

$$b_r \cup \dots \cup b_t = (b_r \cup \dots \cup b_t \cup b_{t+1} \cup \dots \cup b_k) \cap S_t^-,$$

$$b_{t+1} \cup \dots \cup b_k = (b_r \cup \dots \cup b_t \cup b_{t+1} \cup \dots \cup b_k) \cap S_t^+.$$

If the above equations hold true for a successive sequence $(b_r, b_{r+1}, \dots, b_k)$, we say that a boundary surface S_t divides a successive sequence $(b_r, b_{r+1}, \dots, b_k)$ into two sets of successive sequences $(b_r, \dots, b_t)$ and $(b_{t+1}, \dots, b_k)$.

Definition 1 : C_1 type of boundary surface

Let S_t be a boundary surface which divides a successive sequence $(b_r, b_{r+1}, \dots, b_k)$ into two successive sequences $(b_r, \dots, b_t)$ and $(b_{t+1}, \dots, b_k)$. Then a surface S_k is called C_1 type of boundary surface or simply " C_1 " if and only if S_t also divides a successive sequence $(b_r, \dots, b_k, b_{k+1})$ into two successive sequences $(b_r, \dots, b_t)$ and $(b_{t+1}, \dots, b_k, b_{k+1})$.

According to the definition, if S_k is the C_1 type of boundary surface and we have relations

$$b_r \cup \dots \cup b_t = (b_r \cup \dots \cup b_t \cup b_{t+1} \cup \dots \cup b_k) \cap S_t^-, \quad (4.1a)$$

$$b_{t+1} \cup \dots \cup b_k = (b_r \cup \dots \cup b_t \cup b_{t+1} \cup \dots \cup b_k) \cap S_t^+. \quad (4.1b)$$

then we can expect additional relations as follows :

$$b_r \cup \dots \cup b_t =$$

$$(b_r \cup \dots \cup b_t \cup b_{t+1} \cup \dots \cup b_k \cup b_{k+1}) \cap S_t^-, \quad (4.2a)$$

$$b_{t+1} \cup \dots \cup b_k \cup b_{k+1} =$$

$$(b_r \cup \dots \cup b_t \cup b_{t+1} \cup \dots \cup b_k \cup b_{k+1}) \cap S_t^+. \quad (4.2b)$$

It is worth while to note that we get the equations (4.2) by substitution of $b_k \cup b_{k+1}$ for b_k in the equations (4.1).

As Fig.5 shows, when subspaces represent sedimentary strata, the C_1 type of boundary surface is interpreted to represent a sedimentary surface of a stratum.

<Example 3>

Let (b_1, b_2, b_3, b_4) be a successive sequence covering Ω and S_i be the boundary surface through the contact surface between b_i and b_{i+1} ($i=1,2,3$). If boundary surfaces S_1, S_2 and S_3 are the C_1 type of boundary surface, we can define subspaces b_1, b_2, b_3 and b_4 as follows :

Regardless of the type of boundary surfaces, we have

$$b_1 \cup b_2 \cup b_3 \cup b_4 = \Omega, \quad (4.3)$$

$$b_4 = (b_3 \cup b_4) \cap S_3^+, \quad (4.4a)$$

$$b_3 = (b_3 \cup b_4) \cap S_3^-, \quad (4.4b)$$

$$b_3 = (b_2 \cup b_3) \cap S_2^+, \quad (4.5a)$$

$$b_2 = (b_2 \cup b_3) \cap S_2^-, \quad (4.5b)$$

$$b_2 = (b_1 \cup b_2) \cap S_1^+, \quad (4.6a)$$

$$b_1 = (b_1 \cup b_2) \cap S_1^-. \quad (4.6b)$$

Since S_2 is " C_1 ", substituting $b_2 \cup b_3$ for b_2 in equations (4.6a,b), we have

$$b_2 \cup b_3 = (b_1 \cup b_2 \cup b_3) \cap S_1^+, \quad (4.7a)$$

$$b_1 = (b_1 \cup b_2 \cup b_3) \cap S_1^-. \quad (4.7b)$$

In the same way, substituting $b_3 \cup b_4$ for b_3 in equations (4.7a,b), we have

$$b_2 \cup b_3 \cup b_4 = (b_1 \cup b_2 \cup b_3 \cup b_4) \cap S_1^+, \quad (4.8a)$$

$$b_1 = (b_1 \cup b_2 \cup b_3 \cup b_4) \cap S_1^-. \quad (4.8b)$$

Substituting the equation (4.3) into equations (4.8a,b), we obtain

$$b_2 \cup b_3 \cup b_4 = S_1^+, \quad (4.9a)$$

$$b_1 = S_1^-. \quad (4.9b)$$

Since S_3 is " C_1 ", substitution of $b_3 \cup b_4$ for b_3 in equations (4.5a,b) gives

$$b_3 \cup b_4 = (b_2 \cup b_3 \cup b_4) \cap S_2^+, \quad (4.10a)$$

$$b_2 = (b_2 \cup b_3 \cup b_4) \cap S_2^-. \quad (4.10b)$$

From equations (4.9a) and (4.10a,b), we obtain

$$b_3 \cup b_4 = S_2^+ \cap S_1^+, \quad (4.11a)$$

$$b_2 = S_2^- \cap S_1^+. \quad (4.11b)$$

From equations (4.4a,b) and (4.11a), we obtain

$$b_4 = S_3^+ \cap S_2^+ \cap S_1^+,$$

$$b_3 = S_3^- \cap S_2^+ \cap S_1^+.$$

Fig.5 A successive sequence (b_1, b_2, b_3, b_4) bounded by the C_1 type of boundary surfaces S_1, S_2 and S_3 .

Thus, subspaces are defined by S_1, S_2, S_3 as follows :

$$\begin{aligned} b_4 &= S_3^+ \cap S_2^+ \cap S_1^+, \\ b_3 &= S_3^- \cap S_2^+ \cap S_1^+, \\ b_2 &= S_2^- \cap S_1^+, \\ b_1 &= S_1^-. \end{aligned}$$

This result can be expressed in a tabular form as shown in Table 1. Generalizing this example, we have the following theorem.

Theorem 1 Let $(b_1, b_2, \dots, b_n)$ be a successive sequence of subspaces covering Ω . If all boundary surfaces S_i through the contact surface between b_i and b_{i+1} ($i=1, \dots, n-1$) are the C_1 type of boundary surface, then we have

$$\begin{aligned} b_n &= S_1^+ \cap \dots \cap S_{n-2}^+ \cap S_{n-1}^+, \\ b_i &= S_1^+ \cap \dots \cap S_{i-1}^+ \cap S_i^-, \quad i=2, \dots, n-1, \\ b_1 &= S_1^-. \end{aligned}$$

5. C_2 type of boundary surface

Definition 2 : C_2 type of boundary surface

The boundary surface S_k is called C_2 type of boundary surface or simply “ C_2 ”, if and only if S_k divides a successive sequence $(b_1, b_2, \dots, b_k, b_{k+1})$ into a successive sequences $(b_1, \dots, b_k)$ and a single subspace b_{k+1} .

According to the definition, if S_k is the C_2 type of boundary surface, then we have equations as follows :

$$b_1 \cup \dots \cup b_k = (b_1 \cup \dots \cup b_k \cup b_{k+1}) \cap S_k^-, \quad (5.1a)$$

$$b_{k+1} = (b_1 \cup \dots \cup b_k \cup b_{k+1}) \cap S_k^+. \quad (5.1b)$$

As Fig.6 shows, when the successive sequence $(b_1, \dots, b_n)$ represents the distribution of strata, the C_2 type of boundary surface is considered to represent an unconformable contact generated when the sedimentary surface of the youngest layer has been completely eroded.

Table 1 Logical relation between subspaces and boundary surfaces for successive sequence (b_1, b_2, b_3, b_4) bounded by the C_1 type of boundary surfaces S_1, S_2 and S_3 .

	S_3	S_2	S_1
b_4	+	+	+
b_3	—	+	+
b_2		—	+
b_1			—

Fig.6 A successive sequence (b_1, b_2, b_3, b_4) bounded by the C_2 type of boundary surfaces S_1, S_2 and S_3 .

<Example 4>

Let (b_1, b_2, b_3, b_4) be a successive sequence of subspaces covering Ω and S_i be the boundary surface through the contact surface between b_i and b_{i+1} ($i=1, 2, 3$). If boundary surfaces S_1, S_2 and S_3 are the C_2 type of boundary surface, we can define subspaces b_1, b_2, b_3 and b_4 as follows :

Regardless of the type of boundary surface, we have equations (4.3) ~ (4.6). Since S_3 and S_2 are “ C_2 ”, we have both

$$b_4 = (b_1 \cup b_2 \cup b_3 \cup b_4) \cap S_3^+, \quad (5.2a)$$

$$b_1 \cup b_2 \cup b_3 = (b_1 \cup b_2 \cup b_3 \cup b_4) \cap S_3^-, \quad (5.2b)$$

and

$$b_3 = (b_1 \cup b_2 \cup b_3) \cap S_2^+, \quad (5.3a)$$

$$b_1 \cup b_2 = (b_1 \cup b_2 \cup b_3) \cap S_2^-. \quad (5.3b)$$

Substitution of the equation (4.3) into equations (5.2a,b) gives

$$b_4 = S_3^+, \quad (5.4a)$$

$$b_1 \cup b_2 \cup b_3 = S_3^-. \quad (5.4b)$$

Substituting the equation (5.4b) into equations (5.3a,b), we have

$$b_3 = S_3^- \cap S_2^+,$$

$$b_1 \cup b_2 = S_3^- \cap S_2^-.$$

Further, the substitution of the equation (5.5b) into equations (4.6a,b) gives

$$b_2 = S_3^- \cap S_2^- \cap S_1^+,$$

$$b_1 = S_3^- \cap S_2^- \cap S_1^-.$$

Thus, we obtain

$$b_4 = S_3^+,$$

$$b_3 = S_3^- \cap S_2^+,$$

$$b_2 = S_3^- \cap S_2^- \cap S_1^+,$$

$$b_1 = S_3^- \cap S_2^- \cap S_1^-.$$

This result can be expressed in a tabular form as shown in Table 2. Generalizing this example, we have the following theorem.

Table 2 Logical relation between subspaces and boundary surfaces for successive sequence (b_1, b_2, b_3, b_4) bounded by the C_2 type of boundary surfaces S_1, S_2 and S_3 .

	S_3	S_2	S_1
b_4	+		
b_3	-	+	
b_2	-	-	+
b_1	-	-	-

Theorem 2 Let $(b_1, b_2, \dots, b_n)$ be a successive sequence of subspaces covering Ω . If all boundary surfaces S_i through the contact surface between b_i and b_{i+1} ($i=1, \dots, n-1$) are the C_2 type of boundary surface, we have

$$\begin{aligned} b_n &= S_{n-1}^+, \\ b_i &= S_{n-1}^- \cap \dots \cap S_i^- \cap S_{i-1}^+, \quad i = n-1, \dots, 2, \\ b_1 &= S_{n-1}^- \cap \dots \cap S_2^- \cap S_1^-. \end{aligned}$$

6. General case including both C_1 and C_2 types of boundary surfaces

Subspaces constituting a successive sequence $(b_1, b_2, \dots, b_n)$ can be defined by boundary surfaces $S_1, S_2, \dots, S_{n-1}$ as far as boundary surfaces are either C_1 or C_2 type of boundary surface.

<Example 5>

Consider a successive sequence $(b_1, b_2, b_3, b_4, b_5)$ as follows :

$$\begin{array}{cccccc} C_2 & C_1 & C_1 & C_2 & C_1 & \\ b_1 \rightarrow b_2 \rightarrow b_3 \rightarrow b_4 \rightarrow b_5 \rightarrow b_6 & & & & & \\ S_1 & S_2 & S_3 & S_4 & S_5 & \end{array}$$

where $S_1, S_2, \dots, S_5$ are boundary surfaces, and C_1 and C_2 show the type of each boundary surface (See Fig.7).

As S_4 is " C_2 ", we obtain

$$\begin{aligned} b_5 &= (b_1 \cup b_2 \cup b_3 \cup b_4 \cup b_5) \cap S_4^+, \\ b_1 \cup b_2 \cup b_3 \cup b_4 &= (b_1 \cup b_2 \cup b_3 \cup b_4 \cup b_5) \cap S_4^-. \end{aligned}$$

As S_5 is " C_1 ", substituting of $(b_5 \cup b_6)$ for b_5 in the equations, we obtain

$$\begin{aligned} b_5 \cup b_6 &= (b_1 \cup b_2 \cup b_3 \cup b_4 \cup b_5 \cup b_6) \cap S_4^+, \\ b_1 \cup b_2 \cup b_3 \cup b_4 &= (b_1 \cup b_2 \cup b_3 \cup b_4 \cup b_5 \cup b_6) \cap S_4^-. \end{aligned} \quad (6.1a)$$

$$\begin{aligned} b_1 \cup b_2 \cup b_3 \cup b_4 &= (b_1 \cup b_2 \cup b_3 \cup b_4 \cup b_5 \cup b_6) \cap S_4^-. \end{aligned} \quad (6.1b)$$

Substituting

$$b_1 \cup b_2 \cup b_3 \cup b_4 \cup b_5 \cup b_6 = \Omega$$

into equations (6.1a,b), we have

$$b_5 \cup b_6 = S_4^+, \quad (6.2a)$$

Fig.7 A successive sequence $(b_1, b_2, b_3, b_4, b_5, b_6)$. S_2, S_3, S_5 : " C_1 ". S_1, S_4 : " C_2 ".

$$b_1 \cup b_2 \cup b_3 \cup b_4 = S_4^-. \quad (6.2b)$$

As S_2 and S_3 are " C_1 ", we obtain

$$b_2 \cup b_3 \cup b_4 = (b_1 \cup b_2 \cup b_3 \cup b_4) \cap S_1^+, \quad (6.3a)$$

$$b_1 = (b_1 \cup b_2 \cup b_3 \cup b_4) \cap S_1^-. \quad (6.3b)$$

Equations (6.2a,b) and (6.3a,b) give

$$\begin{aligned} b_5 \cup b_6 &= S_4^+, \\ b_2 \cup b_3 \cup b_4 &= S_4^- \cap S_1^+, \\ b_1 &= S_4^- \cap S_1^-. \end{aligned}$$

Since S_2, S_3 and S_5 are C_1 , finally we have

$$\begin{aligned} b_6 &= S_5^+ \cap S_4^+, \\ b_5 &= S_5^- \cap S_4^+, \\ b_4 &= S_4^- \cap S_3^+ \cap S_2^+ \cap S_1^+, \\ b_3 &= S_4^- \cap S_3^- \cap S_2^+ \cap S_1^+, \\ b_2 &= S_4^- \cap S_2^- \cap S_1^+, \\ b_1 &= S_4^- \cap S_1^-. \end{aligned}$$

This result is summarized in Table 3. Generalizing this example, we obtain the following theorem.

Theorem 3 Let $(b_1, b_2, \dots, b_n)$ be a successive sequence of subspaces covering Ω . If all boundary surfaces S_i through the contact surface between b_i and b_{i+1} ($i=1, \dots, n-1$) are either C_1 or C_2 type of boundary surface, then we can define all subspaces $b_1, b_2, \dots, b_n$ uniquely by boundary surfaces $S_1, S_2, \dots, S_{n-1}$.

(Proof)

As shown in Theorems 1 and 2, the theorem holds true in the case that all boundary surfaces are the same type.

Let us consider here the general case that there exist both C_1 and C_2 types of boundary surfaces.

Suppose that a successive series of boundary surfaces

Table 3 Logical relation between subspaces and boundary surfaces for Example 5. S_2, S_3, S_5 : “C₁”. S_1, S_4 : “C₂”.

	S_5	S_4	S_3	S_2	S_1
b_6	+	+			
b_5	−	+			
b_4		−	+	+	+
b_3		−	−	+	+
b_2		−		−	+
b_1		−			−

$S_i, S_{i+1}, \dots, S_{i+k-1}$ ($k > 0$) are the C₁ type of boundary surface and both S_{i-1} and S_{i+k} are “C₂”. Then as S_{i-1} is “C₂”, we have directly

$$b_i = (b_1 \cup \dots \cup b_{i-1} \cup b_i) \cap S_{i-1}^+,$$

$$b_1 \cup \dots \cup b_{i-1} = (b_1 \cup \dots \cup b_{i-1} \cup b_i) \cap S_{i-1}^-.$$

As S_i is “C₁”, substituting $(b_i \cup b_{i+1})$ for b_i in the above equations, we have

$$b_i \cup b_{i+1} = (b_1 \cup \dots \cup b_{i-1} \cup b_i \cup b_{i+1}) \cap S_{i-1}^+,$$

$$b_1 \cup \dots \cup b_{i-1} = (b_1 \cup \dots \cup b_{i-1} \cup b_i \cup b_{i+1}) \cap S_{i-1}^-.$$

Similarly, since surfaces $S_{i+1}, \dots, S_{i+k-1}$ are “C₁”, we have finally

$$b_i \cup \dots \cup b_{i+k} =$$

$$(b_1 \cup \dots \cup b_{i-1} \cup b_i \cup \dots \cup b_{i+k}) \cap S_{i-1}^+, \quad (6.5a)$$

$$b_1 \cup \dots \cup b_{i-1} =$$

$$(b_1 \cup \dots \cup b_{i-1} \cup b_i \cup \dots \cup b_{i+k}) \cap S_{i-1}^-. \quad (6.5b)$$

On the other hand, since S_{i+k} is “C₂”, we have

$$b_{i+k+1} = (b_1 \cup \dots \cup b_{i+k+1}) \cap S_{i+k}^+, \quad (6.6a)$$

$$b_1 \cup \dots \cup b_i \cup \dots \cup b_{i+k} = (b_1 \cup \dots \cup b_{i+k+1}) \cap S_{i+k}^-. \quad (6.6b)$$

From equations (6.5a,b) and (6.6a,b), we have

$$b_{i+k+1} = (b_1 \cup \dots \cup b_{i+k+1}) \cap S_{i+k}^+, \quad (6.7a)$$

$$b_i \cup \dots \cup b_{i+k} =$$

$$(b_1 \cup \dots \cup b_{i+k+1}) \cap S_{i+k}^- \cap S_{i-1}^+, \quad (6.7b)$$

$$b_1 \cup \dots \cup b_{i-1} =$$

$$(b_1 \cup \dots \cup b_{i+k+1}) \cap S_{i+k}^- \cap S_{i-1}^-. \quad (6.7c)$$

Thus a successive sequence $(b_1, \dots, b_{i+k+1})$ is divided into two sets of successive sequences $(b_1, \dots, b_{i-1})$, $(b_i, \dots, b_{i+k})$ and a single subspace b_{i+k+1} by the C₂ type of boundary surfaces S_{i-1} and S_{i+k} . It is noted that $(b_i, \dots, b_{i+k})$ is a successive sequences of subspaces bounded by “C₁”. This indicates that a successive sequence bounded by “C₁” can be grouped consistently into a subspace which is bounded by “C₂”.

Let $B_1, B_2, \dots, B_N$ be subspaces grouped in such a man-

ner as mentioned above, and let S'_J be a boundary surface through the contact surface between B_J and B_{J+1} ($J = 1, \dots, N-1$). It is noted that S'_J is one of boundary surfaces $S_1, S_2, \dots, S_{n-1}$. Since $S'_1, S'_2, \dots, S'_{N-1}$ are the C₂ type of boundary surface, Theorem 2 shows that the subspaces $B_1, \dots, B_N$ are defined by boundary surfaces $S'_1, S'_2, \dots, S'_{N-1}$ as follows :

$$B_N = S'_{N-1}^+,$$

$$B_J = S'_{N-1}^- \cap \dots \cap S'_{J-1}^-, \quad J=2, \dots, N-1,$$

$$B_1 = S'_{N-1}^- \cap \dots \cap S'_2^- \cap S'_1^-.$$

Suppose that a subspace B_J ($J=1, \dots, N$) consists of subspaces $b_{i(J)}, b_{i(J)+1}, \dots, b_{i(J)+k(J)}$. Then Theorem 2 shows that each subspace is defined by

$$b_{i(J)+k(J)} =$$

$$B_J \cap S_{i(J)}^+ \cap \dots \cap S_{i(J)+k(J)-2}^+ \cap S_{i(J)+k(J)-1}^+,$$

$$b_j = B_J \cap S_{i(J)}^+ \cap \dots \cap S_{j-1}^+ \cap S_j^-,$$

$$j=i(J)+1, \dots, i(J)+k(J)-1$$

$$b_{i(J)} = B_J \cap S_{i(J)}^-.$$

Thus, all subspace $b_1, b_2, \dots, b_n$ are defined uniquely by $S_1, S_2, \dots, S_{n-1}$. $\square$

<Example 6>

The proof of Theorem 3 provides a mechanical procedure to determine each subspaces. Let us consider again a successive sequence $(b_1, b_2, b_3, b_4, b_5)$ shown in Example 5.

Let B_1, B_2 and B_3 be three sets of successive sequence of subspaces

$$B_1 = b_1,$$

$$B_2 = b_2 \cup b_3 \cup b_4,$$

$$B_3 = b_5 \cup b_6.$$

Since B_1, B_2 and B_3 are bounded by the C₂ type of boundary surfaces S_1 and S_4 , we have

$$B_1 = S_4^- \cap S_1^-,$$

$$B_2 = S_4^- \cap S_1^+,$$

$$B_3 = S_4^+.$$

Using the results, we have

$$b_6 = B_3 \cap S_5^+ = S_5^+ \cap S_4^+,$$

$$b_5 = B_3 \cap S_5^- = S_5^- \cap S_4^+,$$

$$b_4 = B_2 \cap S_3^+ \cap S_2^+ = S_4^- \cap S_3^+ \cap S_2^+ \cap S_1^+,$$

$$b_3 = B_2 \cap S_3^- \cap S_2^+ = S_4^- \cap S_3^- \cap S_2^+ \cap S_1^+,$$

$$b_2 = B_2 \cap S_2^- = S_4^- \cap S_2^- \cap S_1^+,$$

$$b_1 = B_1 = S_4^- \cap S_1^-.$$

7. Discussions

a) Contact surface between any pair of subspaces

As mentioned above, each boundary surface S_i ($i=1, \dots, n-1$) is defined as a surface through the contact surface between b_i and b_{i+1} , dividing the three dimensional space Ω . This definition does not directly mention about the contact surface between every pair of subspaces. However, we can determine the contact surface between every pair of subspaces because we can define all subspaces covering Ω by boundary surfaces as far as the surfaces are either “C₁” or “C₂”. As shown in Example 5, two subspaces b_3 and b_6 are expressed as follows :

$$\begin{aligned} b_6 &= S_5^+ \cap S_4^+, \\ b_3 &= S_4^- \cap S_3^- \cap S_2^+ \cap S_1^+. \end{aligned}$$

Therefore we have

$$\begin{aligned} b_6 \cap b_3 &= (S_5^+ \cap S_4^+) \cap (S_4^- \cap S_3^- \cap S_2^+ \cap S_1^+) \\ &= S_4 \cap (S_5^+ \cap S_3^- \cap S_2^+ \cap S_1^+) \\ &\subset S_4. \end{aligned}$$

This indicates that b_3 is in contact with b_6 on S_4 if $b_3 \cap b_6 \neq \phi$.

<Example 7>

Table 4 gives contact surfaces between all pairs of subspaces constituting of a successive sequence (b_1, b_2, b_3, b_4, b_5) in Example 5.

Generalizing this discussion, we get the following theorem.

Theorem 4 Let ($b_1, b_2, \dots, b_n$) be a successive sequence of subspaces covering Ω . If every boundary surface S_i through the contact surface between b_i and b_{i+1} ($i=1, \dots, n-1$) is either “C₁” or “C₂”, then we can determine uniquely the boundary surface through the contact surface between an arbitrary pair of subspaces.

Table 4 Boundary surface between arbitrary pair of subspaces for Example 5. S_2, S_3, S_5 : “C₁”. S_1, S_4 : “C₂”.

	b_6	b_5	b_4	b_3	b_2	b_1
b_6	*	S_5	S_4	S_4	S_4	S_4
b_5	S_5	*	S_4	S_4	S_4	S_4
b_4	S_4	S_4	*	S_3	S_2	S_1
b_3	S_4	S_4	S_3	*	S_2	S_1
b_2	S_4	S_4	S_2	S_2	*	S_1
b_1	S_4	S_4	S_1	S_1	S_1	*

(Proof)

Case (1) Suppose that all boundary surfaces are “C₁”. Then from Theorem 1 we have

$$\begin{aligned} b_j &= S_j^- \cap S_{j-1}^+ \cap \dots \cap S_i^+ \cap \dots \cap S_1^+, \quad 2 \leq i < j \leq n, \\ b_i &= S_i^- \cap S_{i-1}^+ \cap \dots \cap S_1^+. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} b_j \cap b_i &= (S_j^- \cap S_{j-1}^+ \cap \dots \cap S_i^+ \cap \dots \cap S_1^+) \cap \\ &\quad (S_i^- \cap S_{i-1}^+ \cap \dots \cap S_1^+) \\ &= S_j^- \cap S_{j-1}^+ \cap \dots \cap S_{i+1}^+ \cap S_i \\ &\quad \cap S_{i-1}^+ \cap \dots \cap S_1^+ \\ &\subset S_i. \end{aligned}$$

This indicates that b_i is in contact with b_j on S_i if $b_j \cap b_i \neq \phi$. As $b_1 = S_1^-$, it is clear that b_1 is in contact with b_j on S_1 if $b_1 \cap b_i \neq \phi$ ($j=2, \dots, n$).

Case (2) Suppose that all boundary surfaces are “C₂”. Then it is found from Theorem 2 that b_i is in contact with b_j on S_{j-1} ($1 \leq i < j \leq n$) in the similar manner as in the case (1).

Case (3) Suppose that there exists some “C₂” among a successive series of surfaces $S_1, S_2, \dots, S_{n-1}$. As shown in the proof of Theorem 3, successive sequences of subspaces bounded by “C₁” can be grouped into sets of subspaces bounded by “C₂”, say $B_1, \dots, B_N$. Let S'_j be a boundary surface between B_j and B_{j+1} , then from Theorem 2, we have

$$\begin{aligned} B_j &= S'_{N-1} \cap \dots \cap S'_j \cap S'_{j-1}, \\ B_I &= S'_{N-1} \cap \dots \cap S'_I \cap S'_{I-1} \cap \dots \cap S'_I \cap S'_{I-1}, \\ &\quad 2 \leq I < J \leq N-1. \end{aligned}$$

① Suppose that two subspaces b_i and b_j ($i < j$) are included in the same group, say B_I . Then it is clear that b_i is in contact with b_j on S_i from the result of the case (1).

② Suppose that two subspaces b_i and b_j ($i < j$) are included in different groups, say b_i is in B_I and b_j is in B_J .

When $I > 1$, we have

$$\begin{aligned} B_J \cap B_I &= (S'_{N-1} \cap \dots \cap S'_J \cap S'_{J-1}) \cap \\ &\quad (S'_{N-1} \cap \dots \cap S'_I \cap S'_{I-1} \cap \dots \cap S'_I \cap S'_{I-1}) \\ &= S'_{N-1} \cap \dots \cap S'_J \cap S'_{J-1} \cap \\ &\quad S'_J \cap \dots \cap S'_I \cap S'_{I-1} \\ &\subset S'_{J-1}. \end{aligned}$$

When $I = 1$, as

$$B_I = S'_{N-1} \cap \dots \cap S'_I \cap S'_{I-1} \cap \dots \cap S'_2 \cap S'_1,$$

we have $B_J \cap B_I \subset S'_{J-1}$.

Therefore B_I is in contact with B_J on S'_{J-1} ($1 \leq I < J \leq N-1$). This indicates that b_i is in contact with b_j on S_{j-1} . $\square$

The logical relation between subspaces and boundary surfaces is used to construct data sets for determination of boundary surfaces based on locations of observation points in the computerized geologic mapping system "CIGMA" (Sakamoto *et. al.*, 1993; Sakamoto, 1994).

b) Construction of a function $g: \Omega \rightarrow B$

According to the above discussions, if all boundary surfaces are either "C₁" or "C₂" then all subspaces $b_1, \dots, b_n$ are uniquely defined by boundary surfaces $S_1, \dots, S_{n-1}$. The relations between subspaces and boundary surfaces are expressed by $(n-1)$ digit character strings

$$c_i = d_{in-1} \dots d_{i2} d_{i1},$$

where d_{ij} is a character "0", "1" or "*" given by

$$d_{ij} = \begin{cases} 0 & ; b_i \subset S_j^- \\ 1 & ; b_i \subset S_j^+ \\ * & ; \text{others} \end{cases}$$

Substitution of 1 or 0 for * in the string c_i provides a $(n-1)$ digit binary number which corresponds a minset bounded by $n-1$ sheets of boundary surfaces as follows :

$$M_{d_{n-1} \dots d_{2d_1}} = h_{n-1}(d_{n-1}) \cap \dots \cap h_2(d_2) \cap h_1(d_1),$$

where

$$h_i(d_i) = \begin{cases} S_i^- & ; d_i = 0 \\ S_i^+ & ; d_i = 1. \end{cases}$$

It is clear that if we have a $(n-1)$ digit binary number $d_{n-1} \dots d_2 d_1$ by substituting 0 or 1 for every * in the string c_i then a minset $M_{d_{n-1} \dots d_{2d_1}}$ is a subset of a subspaces b_i .

<Example 8>

We can define strings $c_1, c_2, \dots, c_6$ for subspaces $b_1, b_2, \dots, b_6$ given in Example 5 as follows :

$$c_6 = 11*** \quad \text{for } b_1$$

$$c_5 = 01*** \quad \text{for } b_5$$

$$c_4 = *0111 \quad \text{for } b_4$$

$$c_3 = *0011 \quad \text{for } b_3$$

$$c_2 = *0*01 \quad \text{for } b_2$$

$$c_1 = *0**0 \quad \text{for } b_1.$$

It is clear that minsets $M_{00001}, M_{00101}, M_{10001}, M_{10101}$ are in b_2 .

Inspecting all strings $c_1, c_2, \dots, c_n$, we can assign a unique subspace to every $(n-1)$ digit binary number. The result can be expressed by a function $g_1: I \rightarrow B$, where $I = \{0, 1, \dots, 2^{n-1}-1\}$ and $B = \{b_1, b_2, \dots, b_n\}$.

Suppose that boundary surfaces $S_{n-1}, \dots, S_2, S_1$ are expressed by single-valued functions $s_{n-1}(x, y), \dots, s_2(x, y), s_1(x, y)$, respectively. Then we can assign a $(n-1)$ digit binary number $d_{n-1} \dots d_2 d_1$ to every point $P(x_P, y_P, z_P)$ in a

space Ω as follows :

$$d_j = \begin{cases} 0 & ; z_P < s_j(x_P, y_P) \\ 1 & ; z_P \geq s_j(x_P, y_P). \end{cases}$$

This correspondence between points in Ω and binary numbers is expressed by a function $g_2: \Omega \rightarrow I$.

Therefore, a convolution of functions $g_1: I \rightarrow B$ and $g_2: \Omega \rightarrow I$

$$g(x_P, y_P, z_P) = g_1(g_2(x_P, y_P, z_P))$$

provides a rule to define the subspace that includes a given point P . This shows that we can define a rule to assign a label corresponding to a stratum to every point in a survey area Ω if subspaces are interpreted to represent strata. The labeling function g is used to draw a geologic map in the computerized geologic mapping system "CIGMA" (Sakamoto *et. al.*, 1993; Sakamoto, 1994).

8. Summary

In order to formulate the distribution of strata in terms of subspaces bounded by boundary surfaces, we introduce new concepts called C₁ and C₂ types of boundary surfaces as mathematical models of conformity and unconformity, respectively.

Let S_i be a boundary surface which divides a successive sequence $(b_r, b_{r+1}, \dots, b_k)$ into two successive sequences $(b_r, \dots, b_i)$ and $(b_{i+1}, \dots, b_k)$. Then a surface S_k is called C₁ type of boundary surface if and only if S_i divides a successive sequence $(b_r, \dots, b_k, b_{k+1})$ into two successive sequences $(b_r, \dots, b_i)$ and $(b_{i+1}, \dots, b_k, b_{k+1})$.

The boundary surface S_k is called C₂ type of boundary surface if and only if S_k divides a successive sequence $(b_1, b_2, \dots, b_k, b_{k+1})$ into a successive sequences $(b_1, \dots, b_k)$ and a single subspace b_{k+1} . It is proved that all subspaces are uniquely defined by boundary surfaces if they are bounded by either C₁ or C₂ type of boundary surfaces. According to the logical relation between subspaces and boundary surfaces, we can define a function g which assigns a subspace to every point in Ω when all boundary surfaces are given by single valued functions. The logical relation between subspaces and boundary surfaces and the labeling function g provide the theoretical bases of the computerized geologic mapping system "CIGMA".

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要旨：境界面を使った地層の分布域の定式表現について — C_1 タイプおよび C_2 タイプの境界面 —

： 塩野清治・升本眞二・坂本正徳

調査地域を表す3次元空間 Ω 内における地層の分布を境界面で境された部分空間という形式で定式表現する。整合および不整合という地質概念の数学モデルとして、 Ω を覆う順序づけられた部分空間 $b_1, b_2, \dots, b_n$ を境する境界面 $S_1, S_2, \dots, S_{n-1}$ に関して、 C_1 タイプの境界面および C_2 タイプの境界面という概念を導入した。すなわち、境界面 S_i が連続する部分空間の列 $(b_r, b_{r+1}, \dots, b_k)$ を2つの連続する部分空間の列 $(b_r, \dots, b_i)$ と $(b_{i+1}, \dots, b_k)$ に分割するとき、 b_{k+1} を追加した部分空間の列 $(b_r, \dots, b_k, b_{k+1})$ もまた S_i によって2つの連続する部分空間の列 $(b_r, \dots, b_i)$ と $(b_{i+1}, \dots, b_k, b_{k+1})$ に分割されるならば、 b_k と b_{k+1} の境界面 S_k を C_1 タイプの境界面と呼ぶ。一方、 b_k と b_{k+1} の境界面 S_k が連続する部分空間の列 $(b_1, b_2, \dots, b_k, b_{k+1})$ を $(b_1, \dots, b_k)$ と単一の部分空間 b_{k+1} に分割するとき、 S_k を C_2 タイプの境界面と呼ぶ。もし、3次元空間 Ω を覆うすべての部分空間 $b_1, b_2, \dots, b_n$ が C_1 タイプあるいは C_2 タイプのいずれかであるような境界面 $S_1, S_2, \dots, S_{n-1}$ で境されているとき、すべての部分空間 $b_1, b_2, \dots, b_n$ は境界面 $S_1, S_2, \dots, S_{n-1}$ を使って一意に定式表現される。また、その定式表現にしたがえば、調査地域を表す3次元空間 Ω の任意の点に地層を表すラベルをつける関数 g を定めることができる。部分空間の定式表現と関数 g は地質図自動作成システム“CIGMA”の基本構造に対する理論的根拠を与える。

キーワード：コンピュータ地質図，地層の分布，境界面，整合，不整合，数学モデル，CIGMA

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