Quantification and Visualization of Diamond Brilliance

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1. Introduction

"Beauty" is one of the conditions necessary for diamond as a gem, and the brilliance is the most important factor of it. Diamond has high refractive index among the minerals. This property results in the peculiar brightness that consists of unique fire, brilliancy and scintillation. To make a diamond brighter, many types of diamond cutting had been attempted, and now, the cut appraised highest is the Round Brilliant Cut 58 hedron. This attractive brightness of diamonds is caused by reflection of light penetrated in the stones, and so the degree of brightness depends on the amount of light reflected. Thus you can quantify the difference of brightness between diamonds by calculating the amount of light reflected.
I have succeeded quantifying the "degree of brilliancy" by estimating intensity of light reflected in stones. Computer algorithm was applied to calculate ray trajectories of Round Brilliant Cut in three dimensional space.
In this paper, I introduce (1) some principles to calculate the intensity of light reflected in three dimensional models of diamond of Round Brilliant Cut and (2) results of computer simulation on some stones of different proportions by this method, advocating indispensability of quantification of brilliancy for grading diamonds.

2. Properties of Light and Diamond

Reflection and Refraction of Light on Diamond
Light passes straight through material, but reflects and/or refracts when it crosses over the boundary of different materials (e.g. air and diamond). It is known as the Fresnel's rule.
As shown in Figure 1, incident ray at point A on a plane of crown refracts, that is, alters its course, then passes straight through the inside of the diamond and arrives at point B on plane of pavilion. When reaching at this point, the ray reflects at the same angle with that of incidence, and go straight to the next plane. At point C, the ray reflects the same manner at point B and finally leaves the diamond with refracting at point D. Any ray whose angle of incident is less than 90° on diamond plane refracts. Although the ray at a right angle with normal can not strike diamond plane, the ray that has angle of incident slightly less than 90° maximizes the angle of refraction. This angle of refraction is called maximum angle of refraction.

Let the angle of incidence be \( \alpha \), and the angle of refraction be \( \beta \). Refraction index of diamond is 2.4173 for Sodium D-Line (wavelength 5893 Å), so that sin \( \alpha \) divided by sin \( \beta \) always equals 2.4173 about any \( \alpha \) and \( \beta \) on the conditions that \( 0 < \beta < 90 \) (refractive law of light). When \( \alpha \) is approximately 90°, \( \beta \) reaches maximum angle of refraction and we can obtain this value as follows.

\[
\frac{\sin \alpha}{\sin \beta} = \frac{\sin (90°)}{\sin \beta} = \frac{1.0}{\sin \beta} = 2.4173
\]

\[
\sin \beta = \frac{\frac{1}{2.4173}} \therefore \beta = 24.43°
\]

This value of \( \beta = 24.43° \) is called critical angle of diamond.
Conversely, when light strikes the interface from inside of diamond, the ray having an angle of incidence less than critical angle exits the stone with refracting between 0° ~ 90°. However, since angle of refraction cannot exceed 90°, the
ray striking outside of the critical angle is totally reflected and back into the stone (see Figure 2).

3. The Definition of Brilliance of Diamonds

Everyone must agree with the opinion that the beauty of diamonds is resulted from their brilliance. This brilliance emerges from the effect of light entered and repeatedly reflected within a diamond. This effect of light consists of fire, brilliance and scintillation. The degree of this unique brilliance of diamonds depends on the amount of light reflected, and this affects their beauty.

Thus, we define the brilliance of diamonds as follows:

\[
\text{brilliance of the diamond} = \frac{\text{amount of light left}}{\text{amount of light striking}}
\]

4. Methods of Numeration

4-1 The Way of thinking

[Numeration of amount of light that exits from the upper surface of a diamond tracing the light passage through it]

The light entered to a diamond from its source finally leaves from the surface after repeating reflection and refraction in the gem. If we make the intensity of the light that struck a point of diamond surface be constant, e.g. 100 units, the amount of light which leaves from the upper surface can be obtained by calculating decrements inside of the stone. All courses and incident points of light can be determined by ray tracing algorithm that follows the path of leaving light inversely.

1) [Tracing paths of light in a diamond]
   1) Direction of the ray entered a diamond is calculated by taking the angle of incidence with plane of the stone where the ray from the light source strikes.
   2) Once light enters in diamond, it goes straight inside and strikes the opposite planes can be struck by the ray depending on its direction and location. We have to discriminate the plane hit by confirming all the planes successively whether each plane satisfies conditions necessary to be struck or not. Then, we calculate the angle of incidence on the plane hit by the light and compare it with the critical angle.
   3) When the angle of incidence of the ray is greater than the critical angle, the ray is reflected totally. But if not so, it leaves the diamond with refracting.
      (i). When the ray refracts, we calculate the direction of it from refractive index.
      (ii). When the ray reflects, we calculate the direction of it and search the plane struck next.
   4) Unlike other planes, girdle is a curved surface, but calculating procedure is the same. If the ray strikes girdle, we draw the normal of that point (any of it is a horizontal line pointed to center of the stone) and calculate the angle of incidence, and the angle of reflection.
   5) We can trace all courses of light in diamond from penetration to escape by repeating the above mentioned procedure until the light leaves the stone.

(2) [Intensity distribution of light in a diamond]

In this section, we are quantifying the brilliance caused by the light exited from a diamond using the ray tracing algorithm. All the parallel light ray which leaves at a right angle with the table from the upper surface of a diamond, contributes to the brilliance. We calculate the intensity of a unit ray as follows:

1) When a ray refracts, the amount of its energy changes according to the Fresnel's rule. Since the area of cross section of refracted light beam varies depending on the angle of incidence, the
intensity of the refracted light (amount of energy per unit area of cross section) varies in proportion to the ratio of cross section area of incident light to that of refracted beam. For example, let us suppose that the amount of energy of incident light is divided to 20% and 80% for light reflected and refracted respectively (Figure 3). Therefore, the intensity of refracted light is 80\( \times (S_1/S_2) \) percent of that of incident light.

2) We suppose that intensity of light does not change when it reflects.

3) When, in diamond, angle of incidence is within the critical angle, we assume that the light beam leaves the stone by the same mode as described in (1).

Example (Figure 4)

To calculate the intensity of light (E), to be exact, we must consider the effect of all beams, say, light (A), (B), (C), (D) and so on. Though, we trace the light of principal axis (A) only, neglecting absorption and diffused reflection, and ignoring B, C, and D due to their little effect on the main beam. This treatment shortens the time required for calculation.

We suppose that the light reflects totally when its angle of incidence exceeds the critical angle, and if not so, the light leaves the diamond. But, in reality, small amount of light begins to leak out from the diamond when its angle of incidence is equal to the critical angle. Intensity of light reflected and refracted becomes nearly even when the angle of refraction of leaving light is about 80° (see reference data).

Procedures of calculation described above is precise only if there is no decrease of light energy (energy of reflected light + energy of refracted light = energy of incident light). But actually, from 5 to more than 10% of the incident light energy may be lost by absorption and/or diffused reflection according to the condition of diamond surface.

4-2 The Conditions of Numeration

Practical conditions for numeration of brilliancy are as follows:

1. Model diamond is expressed in normalized orthogonal co-ordinate system so that the cross section at the lower margin of the girdle plane lies on the \( X-Y \) plane and the center axis of the stone is on the \( Z \) axis.

2. Ratio of lights reflecting and refracting on the plane of diamond is determined according to Fresnel’s rule.

3. Incident point of light is determined by ray tracing algorithm.

4. Refractive index adopted in calculation is 2.4173.

5. Neglecting absorption and diffused reflection, we trace the main axis of light only.

6. We assume that the light beams contributing to the diamond brilliancy are the parallel beams released from surface of a diamond at a right angle with the table plane.

On these conditions, we imagine a straight line vertical to the table plane connects any point in the diamond upper surface with a point of sight. We trace this line inversely and inside the stone...
 reckon the angle of refraction from the angle of incidence and the refractive index. Then, we repeat calculations for all the paths of the light until obtaining the light penetrating point on diamond and angle of incidence at this point. When the penetrating point of the ray traced is at pavilion (i.e. Z co-ordinates of incident point is zero or negative), we treat intensity of the light as 0% at the start point, since no light can penetrate into a diamond from pavilion. Alternatively, if the penetrating point of a light is above lower edge of girdle plane, we treat the intensity of the light at the starting point as the value obtained by this calculation.

4-3 Formulation
-Expression of refraction and reflection as equations-
(1) Reflection of light on diamond surface
1. Reflection of light
If an angle of incidence is \( \theta \), the angle of reflection is also \( \theta \).

Two Dimensional Reflection (Figure 5)
Suppose the X axis is the reflecting surface and normal of it is Z axis, then \( X_1 \), the X component of a incident ray vector equals to \( X_2 \), that of reflecting ray vector. Z components of these ray vectors have also the same scalar, but their direction is opposite. Namely, there is relationships among these components that \( X_1 = X_2 \), and \( Z_1 = Z_2 \).

Three Dimensional Reflection (Figure 6)
Suppose the X-Y plane is the reflecting surface and normal of it is Z axis. And assume the components \( X, Y \) and \( Z \) of the incident ray vector be \( X_1, Y_1 \) and \( Z_1 \), respectively, and that of reflected ray vector be \( X_2, Y_2 \) and \( Z_2 \), respectively. The relationships among these components become \( X_1 = X_2, Y_1 = Y_2 \) and \( Z_1 = Z_2 \). We express reflection on diamond surface by those equations.

(2) Refraction of light at diamond surface
Two Dimensional Refraction (Figure 7)
Let the scalar of the vector expressing rays be \( R \) and refractive index be \( ND \), then

\[
R^2 = X_1^2 + Z_1^2 = X_2^2 + Z_2^2
\]

\[
\sin(\alpha) = \frac{X_1}{R}
\]

\[
\sin(\beta) = \frac{X_2}{R} \quad \therefore ND = \frac{X_1}{X_2}
\]

Three Dimensional Refraction

\[
R^2 = X_1^2 + Y_1^2 + Z_1^2 = X_2^2 + Y_2^2 + Z_2^2
\]

\[
\sin(\alpha) = \frac{\sqrt{X_1^2 + Y_1^2}}{R}
\]

\[
\sin(\beta) = \frac{\sqrt{X_1^2 + Y_1^2}}{R} \quad \therefore ND = \frac{\sqrt{X_1^2 + Y_1^2}}{\sqrt{X_2^2 + Y_2^2}}
\]

As it is evident from the diagram depicted by three dimensional way (Figure 6), reflection and
refraction can be expressed by equations which express the relationships between rays and the refracting surface and its normal. Since a diamond is a three dimensional object, we calculate the direction of light reflected and refracted by three dimensional equations as follows:

A) Assume the normalized orthogonal co-ordinate system which expresses a diamond surface plane (X-Y plane) and normal of it (Z axis).

Let the fundamental vectors of X-Y-Z co-ordinate system be \((f, t, l), (g, j, m), (h, k, n)\) in \(X'-Y'-Z'\) co-ordinate system, then

\[
\begin{pmatrix}
c \\
d \\
e'
\end{pmatrix} = \begin{pmatrix}
fg \\
ij \\
im
\end{pmatrix} \begin{pmatrix}
c' \\
d' \\
e'
\end{pmatrix}
\]

where \((c, d, e)^T\) is a vector expressed in the X-Y-Z co-ordinate system and \((c', d', e')^T\) is the same vector expressed in the \(X'-Y'-Z'\) co-ordinate system.

B) Refracted ray vector at entering a diamond

On the \(X'-Y'-Z'\) co-ordinates system, let the vector of refracted ray be \((c'', d'', e'')\), then

\[
c'' = c' \times ND, \\
d'' = d' \times ND, \\
e'' = \frac{e'}{|e'|} \sqrt{c'^2 + d'^2 + e'^2 - c''^2 - d''^2}
\]

C) Conditions of reflection and refraction in a diamond

To reflect a ray in a diamond, it must be that

\[
\frac{\sqrt{c'^2 + d'^2}}{\sqrt{c'^2 + d'^2 + e'^2}} < \frac{1}{ND}
\]

To refract in a diamond, it must be that

\[
\frac{\sqrt{c'^2 + d'^2}}{\sqrt{c'^2 + d'^2 + e'^2}} > \frac{1}{ND}
\]

D) When the ray leaves a diamond with refracting,

\[
c'' = c' \times ND, \\
d'' = d' \times ND, \\
e'' = \frac{e'}{|e'|} \sqrt{c'^2 + d'^2 + e'^2 - c''^2 - d''^2}
\]

E) When the ray reflects inside of the diamond,

\[
c'' = c', \\
d'' = d', \\
e'' = -e'.
\]

F) The ray vector \(O = (c'', d'', e'')^T\), in \(X'-Y'-Z'\) co-ordinates system, is transformed to vector \((c'', d'', e'')^T\) in X-Y-Z co-ordinates system. The relation between the components of this vector can be written as follows:

\[
\begin{pmatrix}
c'' \\
d'' \\
e''
\end{pmatrix} = \begin{pmatrix}
f \\
g \\
i
\end{pmatrix} \begin{pmatrix}
h \\
 j \\
l
\end{pmatrix} \begin{pmatrix}
c'' \\
d'' \\
e''
\end{pmatrix}
\]

Generally, light entering in diamond reflects repeatedly due to its high refractive index. We calculate again and again until the entered light leaves the stone. Using above equations, we can calculate the intensity of light on each point that consists the diamond surface. Thus, we can obtain the intensity distribution of light across the upper surface of a diamond, and the distribution pattern obtained directly shows the brilliance of the gem.

5. Principles of Numeration of the Brilliance by Computer

5-1 Principles

[1] Expression of diamond surface

Diamond surface consists of 57 flat planes and one curved girdle plane (cylindrical surface). Each corner of these planes is numbered and co-ordinates of these points are expressed as COD (I, J).

\[I = 1: X\text{ co-ordinates, } I = 2: Y\text{ co-ordinates, } I = 3: Z\text{ co-ordinates, } J:\text{ number of each corner}\]

The flat planes are triangle, tetragon or octagon. To simplify calculation, we express all flat planes as combination of triangles. The tetragon and octagon are divided into two and six triangles, respectively. As a result, 57 planes are divided into 78 triangles, and the curved girdle plane is expressed by cylindrical co-ordinates and is numbered 79.

[2] The co-ordinate system

To express the model diamond, we use a right handed \(X-Y-Z\) normalized orthogonal co-ordinate system so that the cross section at the lower margin of the girdle plane lie on \(X-Y\) plane and the center axis of the stone is on \(Z\) axis.


All measurements of diamonds used in equations are expressed as the percentage of the girdle diameter as follows:

\[
\text{Journ.Gemmol.Soc.Japan, Vol.20} \quad 157
\]
Table diameter(%) 
Star/Upper girdle Ratio 
[Length of the projection of an upper girdle facet / (Length of the projection of star facet + an upper girdle facet)]
Lower girdle diameter(%) 
Crown height(%) 
Pavilion depth(%) 
Pavilion height(%) 
Girdle thickness(%) 

When cut symmetrical, angle of crown, angle of pavilion and angle of star facet are calculated from these measures.

[4] Co-ordinates of triangles on diamond surface 
We express a triangle as matrix \( CD(i, j, k) \) that consists of co-ordinates of each apex as components.

- \( k \) is the number of triangle (1~78) 
- column \( j \) is the number of each apex of a triangle (1~3) 
- row \( i \) = 1 is X co-ordinates of each apex of triangle 
- row \( i \) = 2 is Y co-ordinates of each apex of triangle 
- row \( i \) = 3 is Z co-ordinates of each apex of triangle

[5] Co-ordinate transformation matrices for triangles 
All planes but table consisting diamond surface are arranged with some inclination to X-Y plane. Since the degree of inclination of each triangle is unique, the co-ordinate system of the vector of incident light has to be transformed to apply the relationship between light of incident and reflected light described in section 4-3. 
Let the transformation matrix of each triangle be \( VP(i, j, k) \), where

- \( k \) is the number of triangle (1~78) 
- column \( j \) = 1 is the vector pointed from the apex 1 to apex 2 of a triangle, 
- column \( j \) = 2 is the vector on the triangle plane and intersects at right angles to \( j = 1 \), 
- column \( j \) = 3 is the normal vector of the triangle and points in the direction of outside of the stone, 
- row \( i \) = 1 is X components of each vector, 
- row \( i \) = 2 is Y components of each vector, 
- row \( i \) = 3 is Z components of each vector.

The norms of vector in transformation matrices must be normalized in order not to change during co-ordinate transformation.

[6] Ray vectors 
Co-ordinates of the source of light and reflecting point for incident light are expressed as \( CORIN(j) \) and \( COROUT(j) \), respectively. Using \( VERIN(j) \) and \( VEROUT(j) \), we denote each ray vector pointed from the source of light to reflecting point and that of reflecting ray respectively, where 

- \( j = 1 \) is X component, 
- \( j = 2 \) is Y component, 
- \( j = 3 \) is Z component.

[7] Calculation of the incident points on the planes
CorIN (i) : co-ordinates of light source
(i=1, 2, 3 are X, Y, Z co-ordinates respectively)
VerIN (i) : vector of incident ray
CorOUT (i) : co-ordinates of incident point
( reflecting point)
CD (i, j, k) : matrix expressing k th triangle
(this consists of co-ordinates of each apex,
  j=1, 2, 3)
VP (i, j, k) : matrix consists of fundamental
  vectors of co-ordinate system on k th triangle

Let arbitrary point P (i) on the vector of incident
ray express as vector, then

P (i) = CorIN (i) + C1 × VerIN (i)

where C1 is constant, and arbitrary point P (j)
on a triangle plane can be written

P (j) = CD (j, l, k) + C2 × VP (i, l, k) + C3 × VP (j, l, k)

where C2, C3 are constants. Then we eliminate
constants C1, C2, C3, and find a intersection of a
triangle plane and ray vector released from

CorIN (i).

Next, we confirm whether this intersection falls
in the triangle or not.

Co-ordinate : CorIN (i)
Vector : VerIN (i)

(i) When the point of P (i) falls in the triangle,
the area of the triangle S equals to entire area
of three triangles, S1, S2, S3, which are divided by
lines that connects the point of P (i) and each
 apex of the original triangle. Namely,

\[ S = S_1 + S_2 + S_3. \]

(ii) Alternatively, if the point of P (i) falls out
of the triangle, the area is less than entire area
of three triangles divided by lines that connects
the point of P (i) and each apex of the original
triangle. Namely,

\[ S < S_1 + S_2 + S_3. \]

(iii). When point of P (i) falls in the triangle,
CorOUT (i) = P (i). If there is not terminal
point of P (i) in the triangle, light of incidence
do not meet this triangle. Thus we seek the triangle
in which terminal point of P (i) falls, until we
find it.

[8] Co-ordinate transformation of incident ray
vector to triangle plane co-ordinates system
Since any triangle plane is not parallel with X-Y
plane, it is inconvenient to get the angles of incidence,
angle of reflection, and angle of refraction. Hence, we transform
the X-Y-Z co-ordinate system of incident ray vector to co-ordinate
system on triangle plane so that we can use the relationship
described in section 4-3.

VerIN (i) : incident ray vector
VerA (i) : incident ray vector expressed in
X-Y-Z co-ordinate system
VerB (i) : incident ray vector expressed in
co-ordinate system on triangle plane
CorOUT (i) : co-ordinates of the point struck
by a ray (reflecting point) expressed in X-Y-Z co-ordinate system

\[ VP(i, j, k) : \text{a transformation matrix that has fundamental vectors of the co-ordinate system on } k \text{th triangle (these vectors are expressed in X-Y-Z co-ordinate system)} \]

At first, \( VERA(i) \) equals \( VERIN(i) \). Then, we find \( BERB=(b_1, b_2, b_3)^T \) from \( BERA=(a_1, a_2, a_3)^T \) and \( VP=\{v_{ij} | i=1, 2, 3; j=1, 2, 3 \} \), that is

\[ VERB(i) = \sum_{j=1}^{3} VERA(i) \times VP(i, j, k) \]

[9] Reflected and refracted ray vectors at incidence

We get the vectors of ray reflected and refracted from the calculation of the incident ray vector expressed in co-ordinate system on the triangle that obtained in [8].

(i) Variables to calculate a reflection vector

\( VERB(i) \) : transformed incident ray vector to triangle plane co-ordinate systems

\( VERC(i) \) : reflection vector expressed in triangle plane co-ordinate systems

\( VERD(i) \) : reflection vector expressed in triangle plane co-ordinate systems

(ii) Calculation of the angle of incidence

Assume that the angle of incident is \( a \), we can express that

\[ \sin a = \frac{\sqrt{VERB(1)^2 + VERB(2)^2}}{\sqrt{VERB(1)^2 + VERB(2)^2 + VERB(3)^2}} \]

(iii) Total reflection

The refractive index of diamond is 2.413. The incident light on diamond surface is not totally reflected. Some parts of it penetrate into the stone with refraction. However, inside the stone, light reached to the diamond surface reflects totally if the angle of incident is greater than critical angle, but refract if its angle is not so.

On the total reflection, \( X \) components and \( Y \) components of reflected ray vector are the same with those of the incident ray vector, though, their \( Z \) components have the same scalar but the opposite sign. Hence, relations between components of each vector is that \( VERC(1)=VERB(1) \), \( VERC(2)=VERB(2) \), and \( VERC(3)=-VERB(3) \). In addition, \( VERD(i)=0 \), because the light do not refract at this time.

(iv) Refraction (when light enters to a gem from air)

At refraction, \( X \) components and \( Y \) components of refracted ray vector reduce to \( 1/ND \) of those of incident ray vector, but scalar, \( SCL \), of these vectors are identical. Hence, relation between components of each vector can be written as follows:

\[ SCL=\sqrt{VERB(1)^2 + VERB(2)^2 + VERB(3)^2} \]

\[ VERD(1) = \frac{VERD(1)}{ND}, VERD(2) = \frac{VERD(2)}{ND} \]

\[ VERD(3) = SCL^2 - VERD(1)^2 - VERD(2)^2 \]

We suppose that the light do not refract in this situation. Thus \( VERC(i)=0 \).

(v) Refraction (when light leaves to air)

At refraction, \( X \) components and \( Y \) components of refracted ray vector becomes \( ND \) times as much as those of incident ray vector, but scalar, \( SCL \), of these vectors are identical. Hence, relation between components of each vector can be written as follows:

\[ SCL=\sqrt{VERB(1)^2 + VERB(2)^2 + VERB(3)^2} \]

\[ VERD(1) = VERD(1) \times ND, VERD(2) \times ND \]

\[ VERD(3) = SCL^2 - VERD(1)^2 - VERD(2)^2 \]

We suppose that the light do not refract in this situation. Thus \( VERC(i)=0 \).

[10] Transformation from the co-ordinate system on triangle plane to the X-Y-Z co-ordinate system of reflected ray vector and refracted ray vector

Reflected ray vector and refracted ray vector expressed in the co-ordinates system on triangle plane are transformed to the X-Y-Z co-ordinate system using inverse transformation matrix.

\[ \cdot VERC(i) : \text{reflected ray vector expressed in co-ordinate system on triangle plane} \]

\[ \cdot VERD(i) : \text{reflected ray vector expressed in co-ordinate system on triangle plane} \]

\[ \cdot VEROUT(i) : \text{reflected ray vector expressed in the X-Y-Z co-ordinate system} \]
\[ \text{VERREF}(i) : \text{reflected ray vector expressed in the X-Y-Z co-ordinate system} \]
\[ \text{VP}(i, j, k)^{\perp} : \text{the transformation matrix from the triangle plane co-ordinate system to the X-Y-Z co-ordinate system} \]

Reflected ray vector and refracted ray vector expressed in the X-Y-Z co-ordinates system can be written as

\[ \text{VERREF}(j) = \sum_{i=1}^{3} \text{VERC}(i) \times \text{VP}(i, j, k)^{\perp} \]

Where \( \text{VERC}(1), \text{VERC}(2) \) and \( \text{VERC}(3) \) are \( X \) component, \( Y \) component and \( Z \) component of \( \text{VERC}(i) \) respectively, and where \( \text{VERD}(1), \text{VERD}(2), \) and \( \text{VERD}(3) \) are \( X \) component, \( Y \) component and \( Z \) component of \( \text{VERD}(i) \), respectively.

By this transformation, we obtain the vector of ray refracted which expressed in original X-Y-Z co-ordinate system.

[11] Repetition of calculation on reflected ray vector in the stone

Following the steps (7) to (10) discussed above, we can obtain the reflecting point \( \text{COROUT}(i) \) and reflected ray vector \( \text{VERREF}(i) \). On the calculation next, in turn, these reflecting point and reflected ray vector become the light source and incident ray vector, respectively. Thus we may get all the paths of any ray from invasion into to escape from the diamond by repeating these sets.

5.2 Conditions for Simulation

[1] Proportion of a diamond to simulate is expressed by ratios (%) of the following items to the girdle diameter.
1. Diameter of table, 2. Ratio of facets of an upper girdle and a star (ratio of length on the projection), 3. Distance between an apex of lower girdle facet and the point of contact of pavilion facets, 4. Height of crown, 5. Depth of pavilion, 6. Thickness of girdle.

[2] A diamond is expressed in the normalized orthogonal co-ordinate system (X-Y-Z) so that the cross section at the lower margin of the girdle plane is on the X-Y plane and the center axis of the stone is on the Z axis.

[3] We assume that all the surface of a gem is struck by lights from all directions uniformly with intensity of 10 unit.

[4] We postulate that human eyes see only the parallel light rays that released vertically upward from the "top" (crown) of the diamond. We follow trajectories of these rays using ray tracing algorithm, and finally get the entering point of the ray on the diamond surface. Assuming the initial intensity of light to be 100 unit, we calculate the reduction of light intensity when they leave the stone. On any cases of reflection or refraction, only main axis of light is traced with neglecting absorption and diffused reflection.

[5] Calculations described in section 4-3 are performed for all the crown surface of a diamond. We postulate that the summation of these results is the total energy of light (intensity) reaching to human eyes.


[7] Refractive index adopted in calculation is 2.4173.

6. Computer Simulation

[Setting the proportion of diamond]
1. This program proposed here calculates co-ordinates of each facets of Round Brilliant Cut
diamond (58 hedron) and shows its shape on CRT (Photo. 3) from inputted data on the dimensions (Photo. 1) and coordinates of 57 point of the stone (Photo. 2)

[Ray trajectories within the diamond]

1. On CRT, projection of the stone and the lateral view of its three dimensional figure are displayed. You can set any angle of lateral view within 0° ~ 180° (Photo. 4).

2. By inputting the location of light source and incident point in the image of lateral view, you can recognize three dimensional trajectories of light in the diamond.

- You can look the trajectories by three kinds of light that have different wavelength (Photo. 5)
- To input plural light source and incident points, you can see their trajectories in three dimensional way as a result of the computer calculation of reflections and refractions (Photo. 6).
- By dividing the diamond surface into small portions, this program calculates reflections and refractions on each point, and shows these trajectories in three dimensional way (Photo. 7).
face area from which the light released and we get the value. Since this value was closely related to the proportions of stones, we concluded that the average value is an effective index to evaluate the brilliance of diamonds.

1) Five typical diamonds used
   I. a lumpy stone
   II. an ideal cut
   III. a cut of common proportion
       (in the market) A
   IV. a cut of common proportion B
   V. a cut of common proportion C
2) Measurements
   Proportions of diamonds were measured by the aid of a digital measurement microscope, Olympus model STM5.
3) Effective light.
   Among the rays penetrated into the gem at crown (portion of positive Z co-ordinates), the effective light we called is the light beam which escaped from the crown being vertically to the table plane.(we supposed that this light reached to human eyes)
4) Analysis
   (a) Energy analysis for effective light
       This is a method to analyze the distribution of energy of effective light, by calculating the intensity of light energy on each co-ordinates point of the stone surface.
   (b) Analysis of effective light by angle of incidence
       This method aims to analyze the distribution of the angle of incident with Z axis for effective light.
   (c) Analysis of effective light by angle of incidence on each facets
       This is an analysis of distribution of incident angled on each facet plane for effective light.

7. Computer Analysis on Five Typical Diamonds

We compared five diamonds¹) on the market using computer simulation of light behavior in these stones. After inputting measured²) proportions of these gems, distributions of effective light³) were analyzed⁴) in three ways. Based on these results, we obtained the average brightness value of these diamonds. Divide that the total energy of light released from crown by the sur-

¹) Five typical diamonds used
   I. a lumpy stone
   II. an ideal cut
   III. a cut of common proportion
       (in the market) A
   IV. a cut of common proportion B
   V. a cut of common proportion C

²) Measurements
   Proportions of diamonds were measured by the aid of a digital measurement microscope, Olympus model STM5.

³) Effective light.
   Among the rays penetrated into the gem at crown (portion of positive Z co-ordinates), the effective light we called is the light beam which escaped from the crown being vertically to the table plane.(we supposed that this light reached to human eyes)

⁴) Analysis
   (a) Energy analysis for effective light
       This is a method to analyze the distribution of energy of effective light, by calculating the intensity of light energy on each co-ordinates point of the stone surface.
   (b) Analysis of effective light by angle of incidence
       This method aims to analyze the distribution of the angle of incident with Z axis for effective light.
   (c) Analysis of effective light by angle of incidence on each facets
       This is an analysis of distribution of incident angled on each facet plane for effective light.
Sample I. Lumpy Stone

<table>
<thead>
<tr>
<th>Proportions of Sample Diamond</th>
<th>Frequency Distribution of Brightness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured Portion</td>
<td>Proportion (%)</td>
</tr>
<tr>
<td>TB (Table diameter)</td>
<td>64.48</td>
</tr>
<tr>
<td>SU (Star/Upper girdle ratio)</td>
<td>56.0</td>
</tr>
<tr>
<td>LG (Lower girdle diameter)</td>
<td>10.4</td>
</tr>
<tr>
<td>CH (Crown height)</td>
<td>12.89</td>
</tr>
<tr>
<td>PD (Pavilion depth)</td>
<td>46.86</td>
</tr>
<tr>
<td>GH (Girdle thickness)</td>
<td>4.55</td>
</tr>
<tr>
<td>Crown angle</td>
<td>rear</td>
</tr>
</tbody>
</table>

Remarks

The brightest portion (≥80% of the light intensity) was limited to 0.37% of the total area. The area without effective light ("rear" in table) reached about a quarter of the upper surface of the stone. The low average brightness of this diamond (45.10%, less than half) spoiled the intrinsic brilliance by excess reflection of light at the surface.

This diamond appeared to be smaller than its real size since the portions of "rear" surrounded the crown.

Sample II. Ideal Cut

<table>
<thead>
<tr>
<th>Proportions of Sample Diamond</th>
<th>Frequency Distribution of Brightness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured Portion</td>
<td>Proportion (%)</td>
</tr>
<tr>
<td>TB (Table diameter)</td>
<td>54.48</td>
</tr>
<tr>
<td>SU (Star/Upper girdle ratio)</td>
<td>56.0</td>
</tr>
<tr>
<td>LG (Lower girdle diameter)</td>
<td>15.00</td>
</tr>
<tr>
<td>CH (Crown height)</td>
<td>15.59</td>
</tr>
<tr>
<td>PD (Pavilion depth)</td>
<td>43.79</td>
</tr>
<tr>
<td>GH (Girdle thickness)</td>
<td>2.48</td>
</tr>
<tr>
<td>Crown angle</td>
<td>rear</td>
</tr>
</tbody>
</table>

Remarks

Portions of high brightness over 70% were very frequent as about 38%. The area of brightness exceeding 60% reached to 61.2%. The class of rear (light energy zero) occupied only 7.55%. Thus the average brightness is high as 55.82%.

The whole crown surface of this stone looked shining evenly.

Sample II. Cut of Common Proportion A

<table>
<thead>
<tr>
<th>Proportions of Sample Diamond</th>
<th>Frequency Distribution of Brightness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured Portion</td>
<td>Proportion (%)</td>
</tr>
<tr>
<td>TB (Table diameter)</td>
<td>64.40</td>
</tr>
<tr>
<td>SU (Star/Upper girdle ratio)</td>
<td>55.0</td>
</tr>
<tr>
<td>LG (Lower girdle diameter)</td>
<td>15.00</td>
</tr>
<tr>
<td>CH (Crown height)</td>
<td>12.75</td>
</tr>
<tr>
<td>PD (Pavilion depth)</td>
<td>44.30</td>
</tr>
<tr>
<td>GH (Girdle thickness)</td>
<td>4.03</td>
</tr>
<tr>
<td>Crown angle</td>
<td>rear</td>
</tr>
</tbody>
</table>

Remarks

This cut is most common in market. The brightest area (over 80%) did not exceed 1.2% of whole crown surface. The area of "rear" reaching to 15.19% distributed mainly on the peripheral region of the table plane. The average brightness
of this stone was low (50.67%). This stone looked to be notched partly around the margin. The inside of the table plane looked dark.

Sample V. Common Proportion B

<table>
<thead>
<tr>
<th>Proportions of Sample Diamond</th>
<th>Frequency Distribution of Brightness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured Portion</td>
<td>proportion (%)</td>
</tr>
<tr>
<td>Talb (Table diameter)</td>
<td>62.72</td>
</tr>
<tr>
<td>$SU$/Upper girdle Ratio</td>
<td>56.0:44.0</td>
</tr>
<tr>
<td>LG (Lower girdle diameter)</td>
<td>19.67</td>
</tr>
<tr>
<td>CH (Crown height)</td>
<td>14.09</td>
</tr>
<tr>
<td>PD (Pavilion depth)</td>
<td>44.89</td>
</tr>
<tr>
<td>GI (Girdle thickness)</td>
<td>5.56</td>
</tr>
<tr>
<td>Pavilion angle</td>
<td>12.71</td>
</tr>
</tbody>
</table>

Remarks
This type of proportion is very common in market. The portions showing relatively high brightness (60~80%) exceed a half of the total crown area (51.99%). However, area of "rear" class was so much as 18.2% and these portion were found on marginal parts of the table and crown. The average brightness was less than half as 49.12%. This sample looked like less roundish due to many notched shadows of the stone margin. Because of the weak brightness inside, the table plane was seen as reflecting plane.

Sample V. Common Proportion B

<table>
<thead>
<tr>
<th>Proportions of Sample Diamond</th>
<th>Frequency Distribution of Brightness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured Portion</td>
<td>proportion (%)</td>
</tr>
<tr>
<td>Talb (Table diameter)</td>
<td>62.72</td>
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<tr>
<td>$SU$/Upper girdle Ratio</td>
<td>56.0:44.0</td>
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<tr>
<td>LG (Lower girdle diameter)</td>
<td>19.67</td>
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<tr>
<td>CH (Crown height)</td>
<td>14.09</td>
</tr>
<tr>
<td>PD (Pavilion depth)</td>
<td>44.89</td>
</tr>
<tr>
<td>GI (Girdle thickness)</td>
<td>5.56</td>
</tr>
<tr>
<td>Pavilion angle</td>
<td>12.71</td>
</tr>
</tbody>
</table>

Remarks
The high brightness area (60~) exceeded a half of total crown area. Although the area of relatively dark portions (<40%) plus area of sections belonging "rear" class occupied about a quarter of the entire surface (26.4%), the average brightness of this stone was 52.81% which was the highest value among the common cuts. Consequently, this cut is comparatively excellent among the stones of common proportions in market.

This sample looked like somewhat angular, since there are some notched parts over the stone margin.
Sample VI. Stone cut based on the ideal proportion following the present system

<table>
<thead>
<tr>
<th>Proportions of Sample Diamond</th>
<th>Frequency Distribution of Brightness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured Portion</td>
<td>proportion (%)</td>
</tr>
<tr>
<td>TB (Table diameter)</td>
<td>55.00</td>
</tr>
<tr>
<td>SD (Star/Upper girdle Ratio)</td>
<td>57.043.0</td>
</tr>
<tr>
<td>LG (Lower girdle diameter)</td>
<td>20.00</td>
</tr>
<tr>
<td>CH (Crown height)</td>
<td>15.40</td>
</tr>
<tr>
<td>PD (Pavilion depth)</td>
<td>43.00</td>
</tr>
<tr>
<td>GT (Girdle thickness)</td>
<td>2.10</td>
</tr>
<tr>
<td>Crown angle</td>
<td>34.40°</td>
</tr>
<tr>
<td>Pavilion angle</td>
<td>40.70°</td>
</tr>
</tbody>
</table>

Remarks
The proportion is obtained based on the results of proportion analyses of a few hundreds combinations. Incident ray angle and the width of reflected ray are stably balanced, giving the maximum value to the average brightness, 60.11%. Rear stone appears bright.
The sample cut exactly following these proportions shows an image of table facet at the center of the stone, shadows of star facets appearing in a form of bow-tie surrounding the octagonal table facet, and furthermore, pavilion facets appearing in eight directions giving image of eight arrows. Beautiful scintillation effect appears when the stone is moved. It was confirmed that the stone appears even in different hue, depending on light source, due to its strong brilliance and fire.

7. Conclusion
As we stated in the introduction, “beauty” is one of the necessary conditions for a diamond as a gem, in which brilliance is the most important factor. Since a diamond is a hexagon, it is inappropriate to evaluate it by such values as measurements of the cross section or a portion of the stone, or by the finish of the surface, all of which have been so far used for a grading of a diamond. I thought that another method is needed to evaluate the brilliancy of diamond properly. In this paper, I described how to calculate the behavior of light within diamonds, and how to express the brilliancy of diamonds using those values obtained. As a result of the analysis, I concluded that the index obtained from this method well expressed the difference of brilliancy between diamonds with different proportions.

The goal of our work is to know what conditions needed for beautiful brilliancy and to devise the technique that enable to satisfy these conditions. When we reach this goal, diamonds will show us more fascinating and the most shining appearance.

Symbols

\[ \alpha \] angle of incident
\[ \beta \] angle of refraction
\[ \{c', d', e'\} \] incident ray vector expressed in X-Y-Z co-ordinate system
\[ \{c'', d'', e''\} \] reflected or refracted ray vector expressed in X'- Y'- Z' co-ordinate system
\[ \{c'''', d''', e''''\} \] reflected or refracted ray vector expressed in X-Y-Z co-ordinate system

\[ \begin{pmatrix} f \& g \& h \\
                    i \& j \& k \\
                    l \& m \& n \end{pmatrix} \]

A matrix having fundamental vectors of x-y-z co-ordinate system as culm vectors x-y plane of this co-ordinate system lies on the diamond surface

ND refractive index of diamond

ND = 2.4173

References
2) Three dimensional treatment of CT image of head: Development of display the transparent object).
5) Yamada, G., Knowledges of Optics.
6) Dodson, J.S. The statistical brilliancy, spakliness and fire of round brilliant-cut diamond. Blakett Laboratory Imperial College, London.
Reference Data

<table>
<thead>
<tr>
<th>Color</th>
<th>Wavelength(Å)</th>
<th>Color</th>
<th>Wavelength(Å)</th>
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<td>deep</td>
<td>7594</td>
<td>green</td>
<td>5270</td>
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<td>crimson</td>
<td>6667</td>
<td>deep blue</td>
<td>4851</td>
</tr>
<tr>
<td>orange</td>
<td>6363</td>
<td>indigo blue</td>
<td>4308</td>
</tr>
<tr>
<td>yellow</td>
<td>5893</td>
<td>violet</td>
<td>3968</td>
</tr>
</tbody>
</table>

Refractive index of diamond for some light colors (from J.S. Ames)

<table>
<thead>
<tr>
<th>Color</th>
<th>Refractive Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>deep red</td>
<td>2.4024</td>
</tr>
<tr>
<td>crimson</td>
<td>2.4073</td>
</tr>
<tr>
<td>orange</td>
<td>2.4100</td>
</tr>
<tr>
<td>yellow</td>
<td>2.4173</td>
</tr>
<tr>
<td>green</td>
<td>2.4269</td>
</tr>
<tr>
<td>deep blue</td>
<td>2.4354</td>
</tr>
<tr>
<td>indigo blue</td>
<td>2.4514</td>
</tr>
<tr>
<td>violet</td>
<td>2.4645</td>
</tr>
</tbody>
</table>

Percentage of light reflected and refracted on the surface of diamond (from E. Gibeau)

<table>
<thead>
<tr>
<th>Angle of incident from air (°)</th>
<th>Light reflected(%)</th>
<th>Light reflected(%)</th>
<th>Angle of incident from air (°)</th>
<th>Light reflected(%)</th>
<th>Light reflected(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1°</td>
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<td>18.73</td>
<td>81.27</td>
<td></td>
</tr>
<tr>
<td>10°</td>
<td>17.23</td>
<td>82.77</td>
<td>21.12</td>
<td>78.88</td>
<td></td>
</tr>
<tr>
<td>20°</td>
<td>17.36</td>
<td>82.64</td>
<td>24.85</td>
<td>75.15</td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td>17.77</td>
<td>82.23</td>
<td>27.21</td>
<td>72.29</td>
<td></td>
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<tr>
<td>40°</td>
<td>17.77</td>
<td>82.23</td>
<td>43.33</td>
<td>56.67</td>
<td></td>
</tr>
</tbody>
</table>

[Fresnel's rule]
This is the rule of about refraction established by Fresnel in 1626.

[Optical character]
Refractive index of diamond is 2.4173 for Sodium D-Line (wavelength 5893 Å)
ダイヤモンドの輝きの数値化

井上一夫
株式会社 ジェイ・ディ・カンパニー、東京都台東区台東2-14-5-501

1. 序文

ダイヤモンドが宝石として欠かせない条件に「美しさ」があるが、その中でも最も重要なのが輝きである。
鉱物の中で非常に高い屈折率を持つダイヤモンドから生まれた輝きは、他の宝石には見られないファイヤーやブリリアンシー、シンチレーションとしてみる人を楽しませてくれる。

その光を効率的に発揮させるためにいろいろなカット形状が試みられているが、現状では58面体ラウンド・ブリリアント・カットが最も高い評価を得ている。

このダイヤモンドの美しい輝きとは、ダイヤモンド内を通過した光の反射によるものであり、輝きの強さは光の反射量によって決定される。したがって光の反射量を求めることが出来れば、輝きの強さを数値化できるはずである。

我々は、ラウンド・ブリリアントにカットされたダイヤモンド内部での光の三次元的な挙動をコンピュータで計算し、ダイヤモンド内部からの反射光強度を求めることによって、輝きの強さを数値化することができた

三次元の計算方法の設定原理とそれによる数値の近似のコンピュータ・シミュレーションを紹介し、輝きの数値化を提案するものである。

2. 光とダイヤモンドの性質

ダイヤモンドの反射と屈折
光には物質の中を直進する性質があるが、物質と物質の境界（例えば、空気とダイヤモンドの境界）では、反射や屈折という現象を生じることはずれの法則で周知の通りである。

Figure 1に於いて、クラウン面のA点に入射した光線は屈折を起こし、光の方向を変えてダイヤモンドの内部を直進し、パビジオン面のB点に達する。ここで反射を起こしたとき、この面で光は入射角と反射角が同じとなる方向に向かえ次に当たる面まで直進する。C点でもB点と同様、反射をし、入射角と反射角が互いに光りを向かい、最後にD点で屈折を起こし、ダイヤモンドの外部へ出てくる。

外部からダイヤモンドの表面に90度より小さな角度で入射する光は全て屈折を起こす。

丁度90度の光線、ダイヤモンド表面と平行なため面に当たることはないが90度よりわずかに小さい角度を持つ光線は最も屈折が大であり、最大屈折角と呼ばれる。

ダイヤモンドの屈折率はナトリウムのD線（波長5893Å）に対して2.4173なので、

\[
\frac{\sin \alpha}{\sin \beta} = 2.4173
\]

と表され、αがほぼ90度のときβは最大屈折角となり次のように求められる。

[Figure 2]

\[
\frac{\sin \alpha}{\sin \beta} = \frac{\sin(90^\circ)}{\sin \beta} = \frac{1}{\sin \beta} = 2.4173
\]

\[
\sin \beta = \frac{1}{2.4173} \therefore \beta = 24.43^\circ
\]

このβは24.43度、ダイヤモンドの臨界角と呼ばれる。逆にダイヤモンド内部から臨界角より小さい角度で表面に当たった光は屈折して0度～90度の範囲で外へ出て行くが、臨界角より大きな角度でダイヤモンドの面に当たる光は屈折角が90度を越える事が無いため、全反射して再びダイヤモンドの内部を反対の方向へと進む。

3. ダイヤモンドの輝きの定義

ダイヤモンドの美しさは「輝き」によって生まれ
表1
光源数値
と臨界角100とし、数値があらわれ、逆することができる。
ダイヤモンドの輝きを定義すれば

ダイヤモンドの輝き = 出射光量 / 入射光量

となる。

4. 数値化の方法

4-1 数値化の考え方
ダイヤモンド内部を通過し視点に届く光の量を求め数値で表す。
光源から出た光がダイヤモンドに入射後、反射と屈折を繰り返しながら、ダイヤモンドの内部へと出てくる。この入射点に当たる光の強度を100とした場合、表面から出る光の強度を計算すれば、ダイヤモンドから出てくる光の量を数値で表すことができる。
この出る光の経路を逆にたどっていく光線追跡法（ray-tracing algorithm）によって、全ての光源からの経路と入射点を求めることができる。
（1）ダイヤモンド内部の面積を求めること

1）光源から出た光がダイヤモンドに当たった表面での光強度、と面の角度から屈折の計算をして、ダイヤモンド内部での光の進行方向を求めること。

2）ダイヤモンド内部では光の方向性が直進し反射側の面に当たるが、光の方向性が変化して、どの方向になるか確認をし入射点の角度から屈折の計算をして、面の角度と光の方向から面への入射角を求め、臨界角と比較する。

3）臨界角より大きければ、その面では反射し再びダイヤモンド内部を光が進むが、臨界角より小さければ屈折して外へ出てくる。

・屈折する場合、屈折率から外部へ出る光の方向を計算する。

・反射する場合、入射角から反射光の方向を計算し次に光が当たる面を求める。

4）ガードル面は面であり他の平面とは異なるが計算手順は同じであり、ガードル面に光が当たったら、その点の法線（常にダイヤモンドの中心に向かう水平線）を計算し、入射角、屈折角あるいは反射角を計算する。

5）この反射計算を光が外部へ出るまで繰り返すことで光源からのダイヤモンド内部の軌跡経路を求めることができる。

（2）内部光の強度分布
ダイヤモンドを上から見た投影面積全体について光線追跡法により計算し屈折光による輝きの度合いを定量化する。単位光線ごとの光の強度は次の法則に従って計算する。

1）光線が屈折する場合の屈折光のエネルギーはフレネルの法則に従う。更に屈折角に応じて、入射光と屈折光の面積変化に比例し単位面積当たりの光の強度が変化する。

Figure 3 によって説明すると、100％のエネルギーを有する入射光の内20％が反射光となり80％が屈折光となった場合、屈折光のエネルギーは80％であるが、単位面積当たりのエネルギーは、80×S1/S2となる。

2）光線が全反射する場合には強度の変化はないものとする。

3）ダイヤモンド内部の反射光がダイヤモンド表面に入射する場合の入射角が臨界角以下になった場合は（1）と同様の計算によって求める強度の屈折光で外部に出るものをとする。

例）（Figure 4）外部から見える（E）の光の強度を求める場合は（A）（B）（C）（D）等、全ての光について、その影響を計算しなければならないが、乱反射及び吸収によるロスは無視し（D）は除外し（B）（C）は、その影響が少ないものので計算時間を関係から無視し主軸光（A）のみ限定して追求する。

光がダイヤモンドの内部から外部に出る場合、入射光が臨界角以上になると内部で全反射し臨界角以下では、外部へ出るものとして計算するが、実際には臨界角で外部に出る光が漏れ始めるものの大き
4-3 数値化の数式
[反射と屈折を数式によりあらわす]
(1) ダイヤモンドにおける光の反射
光の反射
反射面上に垂直な法線と入射光の角度をθとするとき、反射光と法線との角度も同じθとなる。
二次元反射 [Figure 5]
いま反射面をx軸に法線をZ軸に取ると、入射光のX軸成分X1は、反射光のX軸成分X2で同じだが、Z成分の大きさは同じでも方向は逆になっている。
即ちX1=X2；Z1=-Z2関係が成り立つ。
三次元反射 [Figure 6]
反射面をX軸とY軸に法線をZ軸にとり、入射光のX·Y·Z軸の成分X1, Y1, Z1とし反射光のX·Y·Z成分をX2, Y2, Z2とすると、それぞれの関係は
X1=X2, Y1=Y2, Z1=-Z2となる。
ダイヤモンドの表面での反射は全て上式で表現する。
(2) ダイヤモンドにおける光の屈折
二次元屈折
\[ R^2 = X_1^2 + Y_1^2 = X_2^2 + Z_2^2 \]
\[ \sin(\alpha) = \frac{X_1}{R} \]
\[ \sin(\beta) = \frac{X_2}{R} \because ND = \frac{X_1}{X_2} \]
三次元屈折
\[ R^2 = X_1^2 + Y_1^2 + Z_1^2 = X_2^2 + Y_2^2 + Z_2^2 \]
\[ \sin(\alpha) = \frac{\sqrt{X_1^2 + Y_1^2}}{R} \]
\[ \sin(\beta) = \frac{\sqrt{X_2^2 + Y_2^2}}{R} \because ND = \frac{\sqrt{X_1^2}}{Y_2^2} \]
三次元図で明らかのように反射や屈折は反射面 (xy平面) と法線 (Z軸) に対する光線の関係式を用いて表すことが出来る。ダイヤモンドは立体であり反射や屈折は二次元ではないので、下記の三次元式で計算する。
A) ダイヤモンドのある平面上にX軸、Y軸を取りその法線をZ軸とする正規直交座標を考える。
\[\begin{pmatrix} fgh \\ ijk \\ lmn \end{pmatrix}\] と表すと
光線の${X'}\cdot{Y'}\cdot{Z'}$座標系でのベクトル

$$\begin{pmatrix} c \\ d \\ e \end{pmatrix}$$

と${X'}\cdot{Y'}\cdot{Z'}$座標系のベクトル

$$\begin{pmatrix} c' \\ d' \\ e' \end{pmatrix}$$

には、式（1）の関係が成り立つ。

B) 光が外部からダイヤモンドに入れる場合の屈折計算は式（2）で与えられる

（注）$ND$は屈折率

C) ダイヤモンド内部での反射または屈折は式（3），（3'）式で与えられる。

D) ダイヤモンド内部から光が屈折して出る場合は式（4）で与えられる。

E) ダイヤモンド内部で光が反射する場合には式（5）の関係がある。

F) $X'\cdot{Y'}\cdot{Z'}$座標系で求めた光のベクトル

$$O = (c', d', e')$$

を式（6）により$X\cdot{Y}\cdot{Z}$座標系に戻す。

$$\begin{pmatrix} c' \\ d' \\ e' \end{pmatrix} = \begin{pmatrix} fg \\ ij \\ lmn \end{pmatrix}^{-1} \begin{pmatrix} c \\ d \\ e \end{pmatrix}$$

ダイヤモンドは屈折率が高いため、ダイヤモンド内部で反射を何度も繰り返すことが多く、ダイヤモンドに入ったり出るまでの反射の計算を繰り返す必要がある。

ダイヤモンドの表面を微細化し各々の点については、上記の計算式を基に光の強度を求めれば、ダイヤモンドの輝きを示す光の強度分布のパターン分析することが出来る。[Figure 7]

5. コンピュータでの輝きの数値化設定原理

[1] ダイヤモンド表面の表現

ダイヤモンド表面は、27の平面とガードル面（円柱面）から構成されている。ダイヤモンド表面の角にそれぞれ番号を付け、各点の座標をCOD（I, J）で表す。

（J=1X座標，I=2Y座標，J=32座標，J各点の番号）[Figure 8]

ダイヤモンドの平面は三角形と四角形と八角形を含んでいる。計算上の理由から三角形と八角形はそれぞれ二個の三角形と六個の三角形に分割し、平面をすべて三角形の組み合わせで表現する。したがって57の平面は78の三角形に分割されガードル面は曲面なので、この面だけは円柱座標系で表し79番目の面とする。（Figure 9）

[2] 座標系

$X\cdot{Y}\cdot{Z}$右手系正直交座標系を用い、中心軸をZ軸に合わせ、$Z=0$をガードル面の下線にとる。（Figure 10）

[3] ダイヤモンド寸法

ガードル直径を100％とし、他の寸法はそれに相当する％で表す。（Figure 11）

で表され、シュメトリナーカットの場合は、クラウン高さ、パピリオン角度、スターフェッジ角度、などがこれらの寸法から算出される。

[4] ダイヤモンド表面を表す三角形の座標

三角形の各頂点の座標をCD（i，j，k）とする。

$k$は三角形の番号（1〜78）

j是三角形の各頂点の番号（1〜3）

$i=1$は三角形の各頂点の$X$座標の値

$i=2$は三角形の各頂点の$Y$座標の値

$i=3$は三角形の各頂点の$Z$座標の値

[5] 三角形の座標変換行列

ダイヤモンドの表面は、テーブル面を除き全ての表面がX軸とY軸からなるXY平面に対して傾斜して配置されている。各々の三角形は平面の角度がそれぞれ異なっているので、（1）の出した入射光と反射光の関係を利用することにより、入射光ベクトルの座標系の変換を行う。それぞれの三角形は変換行列を持ており、その変換行列を$VP(i,j,k)$


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とする。

☆ \( k \) は三角形の番号（1〜78）
☆ \( j=1 \) は三角形の頂点1から頂点2へのベクトル
☆ \( j=2 \) は三角形の平面上にあり、\( j=1 \) と直交するベクトル
☆ \( j=3 \) は三角形の法線ベクトルで方向をダイヤモンドの内面から外面へとする。
☆ \( i=1 \) は各ベクトルの \( X \) 座標の値
☆ \( i=2 \) は各ベクトルの \( Y \) 座標の値
☆ \( i=3 \) は各ベクトルの \( Z \) 座標の値

変換するベクトルのノルムが変換前後で変わらないよう、ベクトルの変換行列のノルムは正規化しておく。

[6] 光線ベクトル

入射光は光源座標を CORIN (j) ダイヤモンド表面での反射座標を COROUT (j) で表す。入射光の光源から反射点へのベクトルを VERIN (j) 反射後の光源ベクトルを VEROUT (j) で表す。ここで、それぞれの変数の添字 \( j \) は、

\[
\begin{align*}
    j = 1 & \text{は} X \text{座標または} X \text{成分} \\
    j = 2 & \text{は} Y \text{座標または} Y \text{成分} \\
    j = 3 & \text{は} Z \text{座標または} Z \text{成分}
\end{align*}
\]

表す。

[7] 光線の平面上への到着点の計算

| CORIN (i) | 入射光源の座標  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1, 2, 3 ) は ( X, Y, Z ) 軸</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>VERIN (i)</th>
<th>入射光のベクトル</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1, 2, 3 ) は三角形の各頂点</td>
<td></td>
</tr>
</tbody>
</table>

| CD (i, j, k) | 三角形の座標  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1, 2, 3 ) は三角形の各頂点</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>VP (i, j, k)</th>
<th>三角形の平面ベクトルと法線ベクトル</th>
</tr>
</thead>
</table>

ベクトルで表現すると、
入射光の任意の点 \( P (i) \) は

\[
P (i) = \text{CORIN} (i) + (C_1) \times \text{VERIN} (i)
\]

定数 \( C_1 \)、\( C_2 \)、\( C_3 \) を消去し、CORIN (i) を通る光線と三角形面との交点を求める。
次にこの交点が三角形の中にあるか否かをチェックする。

① \( P (i) \) が三角形の中に在ると、三角形の各頂点と \( P (i) \) を結ぶ直線によって分けられた三つの小さな三角形の総面積は元の三角形の面積に等しい。

\[
\text{面積} S = S_1 + S_2 + S_3
\]

② \( P (i) \) が三角形の外に在ると、三角形の各頂点と \( P (i) \) を結ぶ直線によって作られた三つの小さな三角形の総面積は元の三角形の面積より大きい。

\[
\text{面積} S < S_1 + S_2 + S_3
\]

③ 点 \( P (i) \) が三角形の中に在れば、COROUT (i) = P とする。
点 \( P (i) \) が三角形の外に在れば、その三角形と入射光は交わらないので、\( P (i) \) が三角形の中に在るように三角形が見つかるまで調査する。

[8] 入射光ベクトルの三角形平面座標系への変換

三角形平面は、テーブル面を除き \( XY \) 平面と平行にしていない。このため入射光と平面の成す角度、反射角、屈折角を計算するのに不便であり、3）項のダイヤモンド表面の表現の関係を使うため入射光ベクトルをXYZ座標系から三角形平面座標系に変換する。

<table>
<thead>
<tr>
<th>( \star ) VERIN (i)</th>
<th>入射光のベクトル</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( \star ) VERA (i)</th>
<th>変換前入射光ベクトル</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( \star ) VERB (i)</th>
<th>変換後入射光ベクトル</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>( \star ) COROUT (i)</th>
<th>到着点（反射点）の座標</th>
</tr>
</thead>
</table>

| \( \star \) VP (i, j, k) | 三角形（番号k）の平面ベクトルと法線ベクトル  
|----------------------|-----------------------------------|

（変換ベクトル）
はじめにVERA \((i)\) = VERIN \((i)\) とする。

行列VERA
\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
\end{pmatrix}
\]
変換行列
\[
VP
\begin{pmatrix}
V_{11}, V_{12}, V_{13} & V_{21}, V_{22}, V_{23} & V_{31}, V_{32}, V_{33}
\end{pmatrix}
\]
から

行列VERB
\[
\begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
\end{pmatrix}
\]
を求め式(7)が考えられる。

[9] 入射光の反射ベクトルと屈折ベクトル

(8)で求めた三角形平面座標系に変換した入射光ベクトルを用いて反射光と屈折光を求める。

反射ベクトル計算変数
VERB \((i)\)：三角形平面座標系に変換した入射光ベクトル
VERC \((i)\)：三角形平面での反射光ベクトル
VERD \((i)\)：三角形平面での屈折光ベクトル

入射角の計算
入射角を \(a\) とすると、\(\sin(a)\) は式(8)と表せる。

3全反射
ダイヤモンドの屈折率は、2.4173であり、ダイヤモンドの外から表面で反射した光は全反射せず屈折してダイヤモンドの中へ入る。一方、ダイヤモンドの中から表面に反射した光は、入射角が臨界角 \(\alpha\) より大きければ全反射し、小さければ屈折する。
全反射の場合反射光のX成分とY成分は変化せず、Z成分は同じスカラーを持ち符号は逆である。
従ってVERC \((1)\) = VERB \((1)\)、VERC \((2)\) = VERB \((2)\)、VERC \((3)\) = VERB \((3)\) となる。
この時屈折はないので、VERD \((i)\) = 0とする。

4屈折(外部から内へ)
屈折の場合、スカラー \((SCL)\) はわずかX成分とY成分はそれぞれND分の一になるので、各ベクトル成分間の関係は式(9)のように表される。
この時反射は無いものとしてVERC \((i)\) = 0とする。

5屈折(中から外部へ)
屈折の場合、スカラー \((SCL)\) はわずかX成分とY成分はそれぞれND倍になるので各ベクトル成分間の関係は式(10)で表される。
この時反射は無いものとしてVERC \((i)\) = 0とする。

[10] 反射ベクトルと屈折ベクトルの三角形平面座標からX・Y・Z座標への変換

三角形平面座標系で計算した反射ベクトルと屈折ベクトルを変換行列を用い、X・Y・Z座標系に戻す。式(11)および式(12)。

\[
\begin{align*}
\text{VER}(i) & = X\cdot Y\cdot Z \Rightarrow \text{三角形平面への変換} \\
\text{VERA}(i) & = X\cdot Y\cdot Z \Rightarrow \text{三角形平面での反射光ベクトル} \\
\text{VERB}(i) & = X\cdot Y\cdot Z \Rightarrow \text{三角形平面での屈折光ベクトル} \\
\text{VERC}(i) & = X\cdot Y\cdot Z \Rightarrow \text{三角形平面での反射光ベクトル} \\
\text{VERD}(i) & = X\cdot Y\cdot Z \Rightarrow \text{三角形平面での屈折光ベクトル}
\end{align*}
\]

この変換により、もとのX・Y・Z座標系での反射光ベクトル及び屈折光ベクトルが得られる。

【11】内部繰り返し反射の計算

式(7)～式(10)までの計算で反射点COROUT \((i)\) 及び反射ベクトルVERREF \((i)\) が得られる。この反射点及び反射ベクトルは、次の計算の光源及び光線ベクトルであり、再び式(7)～式(10)を繰り返すことより、光線がダイヤモンドの外に出るまで、その光線の軌道を求めることができること。

5-2 〔設定条件〕

[1] 表示するダイヤモンドのプロパーションの設定は、ガードル直径を100%としたとき各々の比率
1. テーブル径
2. スターとアッパーサセット比
(投影長さの比率)
3. ローガードルサセットとバピリオンサセットとの接点比率
4. クラウン高さ
5. バピリオン深さ
6. ガードル厚さ

[2] 表示するダイヤモンドを正規直交座標系(X・Y・Z)の中心におき、ダイヤモンドの中心軸をZ軸に、またガードル面の下側を原点とする。

[3] ダイヤモンドには、その周囲からすべての部分に均一に強度100の光があたっているものとする。
[4] ダイヤモンドをTOP（上面）から見た時、
ダイヤモンドから出た目に届く光を平行光線と仮定し、目に届いた光線の軌道を光線
追跡法で逆にたどり、光源からダイヤモンドへの入射点を求め、強度100の光が入射した
場合の上面から出てくるまでの光の減衰を計算する。なお、乱反射及び吸収による
ロスは無視し、反射あるいは屈折光についても主軸光のみとする。
[光の減衰率の計算前提]
・ダイヤモンド表面での反射光及び屈折光
の比率計算はフレーヌの法則によるがその
計算は、E. Gübelinの資料【資料参照】
を使用した。
・ダイヤモンド内部では減衰は無いと仮定
する。
・光量の変化は屈折による光線断面積の変
化に逆比例する。
[5][3]での計算はダイヤモンド上面全面総て
について行い、その総和がダイヤモンドか
ら目に届いた光のエネルギー総量（強度）で
あるとする。
[6] 計算に使用した光の波長は2893Ａ黄色の単
色光（D光線）
[7] ダイヤモンドの屈折率は、2.4173として計
算している。