Optimal Control Problems for a Class of Nonlinear Descriptor Systems

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Summary

In this paper, we consider optimal control problems for a class of nonlinear descriptor systems. A necessary condition which may be regarded as the maximum principle for descriptor systems is derived, and some problems related to this necessary condition, such as the continuity and differentiability of the hamiltonian in this necessary condition, the sufficient and necessary condition on linear descriptor systems, and so on, are discussed. Our results may include some ones in optimal control problems for conventional systems. Finally, an illustrative example is also given.

Key words: Descriptor Systems, Maximum Principle, Sufficient and Necessary Condition, Function Norm, Bellman-Gronwall Inequality

1. Introduction

In recent years, the control problems for descriptor systems (or singular systems, or generalized state-space systems) have drawn the considerable attention of many researchers due to extensive applications of descriptor systems in large-scale systems, singular perturbation theory, electrical networks, economic systems, macroeconomic systems, and other areas (e.g. see Refs. (1)~(7)). Many papers and works dealing with descriptor systems have appeared. For a fairly comprehensive introduction to descriptor systems, and for a motivation for studying them, one may see Refs. (1), (2) and (8). Some theoretic aspects of the topic have been given by Campbell(9)(10) which discussed descriptor systems with aid of the matrix generalized inverse called the "Drazin inverse".

In this paper, we will consider optimal control problems for a class of nonlinear descriptor systems. That is, descriptor systems of the following form will be considered:

\[ E \frac{dx(t)}{dt} = f(x(t), u(t), t) \]  \hspace{1cm} (1)

where \( E \) is a constant square matrix and \( E \) may be singular.

Few efforts have been made to discuss optimal control problems for nonlinear descriptor systems of the form (1). In Ref. (19), for example, Lovass-Nagy, et al. investigated the problem of optimal control of linear descriptor systems while minimizing a quadratic cost functional, and derived the necessary conditions for optimality of control. But in Ref. (19), control vector \( u(t) \) is unconstrained. Particularly, in a recent paper(11), Lin and Yang studied the problem of optimal control of the systems (1). In their paper(11), by making use of the transform of the forms \( \ddot{x}(t) = Q\dot{x}(t) = [\dot{x}_1(t), \dot{x}_2(t)] \) and \( \ddot{f} = P^{-1}f \) where \( Q \) and \( P \) are \( n \times n \) non-singular matrices, a necessary condition for optimality of control was derived, but in their necessary condition, a restrictive assumption that \( (\partial \ddot{f}/\partial \dot{x}_2)' \) is independent of \( \dot{x}_1(t), \dot{x}_2(t) \) and \( u(t) \) is required.

In this paper, we discuss the problem of optimal control of a class of nonlinear descriptor systems with the form Eq. (1). The well-known Pontryagin's maximum principle in optimal control problem for conventional systems is extended to the
Descriptor systems (1), and some problems related to this principle, such as the continuity and differentiability of the Hamiltonian in this maximum principle, the sufficient and necessary condition on linear descriptor systems, and so on, are discussed. The concept of the function norm and the Bellman-Gronwall inequality will be employed to derive this maximum principle for descriptor systems. Our results may include some ones in the optimal control problems for conventional systems. Finally, an illustrative example is also given.

This paper is organized as follows. In Section 2, the optimal control problems to be tackled in this paper are precisely stated, and the concept of the function norm and the Bellman-Gronwall Lemma are given. In Section 3, we derive a necessary condition, i.e. the maximum principle for descriptor systems with the form Eq. (1), and discuss some problems related to this necessary condition. In Section 4, an illustrative example is given. The paper is concluded in Section 5 with a very brief discussion on the above results.

2. Problem formulation

2.1 Optimal control problem in descriptor systems

Let us consider a nonlinear optimal control problem in descriptor systems which is described by the following differential equation with descriptor form.

\[
E \frac{dx(t)}{dt} = f(x(t), u(t), t) \quad (2a)
\]

\[
x(t_0) = x_0, \quad t \in [t_0, t_f] \quad (2b)
\]

where

\[
u(t) \in U(t), \quad t \in [t_0, t_f] \quad (3)
\]

Here, initial time \(t_0\), initial state \(x_0\), and final time \(t_f\) are fixed; \(u(\cdot)\) is a \(m\)-vector control function; the components of \(u(t)\) may be piecewise continuous functions; \(x(\cdot)\) is the state vector of dimension \(n\), \(U(t), t \in [t_0, t_f]\) is the subsets of \(R^m\). It is worth noting that there is no rank assumption on constant matrix \(E\) which may be singular.

The cost functional is given by \(J\), where

\[
J(u, x) = F(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t) dt \quad (4)
\]

and at \(t_f\), the constraint condition is given by

\[
\phi(x(t_f), t_f) = 0 \quad (5)
\]

If for any initial state \(x(t_0) = x_0\) and any control vector \(u(t) \in U(t)\), the differential Eq. (2) has one and only one continuous state vector \(x(t)\), and there is one and only one continuous adjoint state vector \(\lambda(t)\), then this initial state \(x(t_0) = x_0\) is called an admissible initial state, this control vector is called an admissible control, and \(U(t)\) the admissible control set.

Remark 1

This definition is concerned with the existence of the solution of the Eq. (2). In a linear descriptor system, for example, this existence is to be understood in the distributional sense, and this is equivalent to demanding that the pencil \((sE-A)\) be regular, i.e., \(\det(sE-A) \neq 0\). Further, \(u \in C^0([t_0, t_f])\) guarantees a solution devoid of impulses, except possibly at \(t = t_0\). In the nonlinear case, we require existence of a \(C^1([t_0, t_f])\) solution for all \(x(t_0) = x_0\) and \(u \in U\), since distributional solutions to nonlinear Eq. (2) cannot be defined (e.g. see Refs. (12) and (13)).

Now, the problem can be stated as follows: We want to find an admissible control \(u(t) \in U(t)\) to minimize the cost functional (4) subject to system Eq. (2) with an admissible initial state \(x(t_0) = x_0\), and subject to terminal constraint condition (5).

For this optimal control problem, if the admissible control \(u^*(t) \in U(t)\) and corresponding state trajectory \(x^*(t)\) have the property that for all admissible control \(u(t) \in U(t)\),

\[
J(u^*(t), x^*(t)) \leq J(u(t), x(t)) \quad (6)
\]

then, \(u^*(t)\) is called an optimal control, and \(x^*(t)\) an optimal state trajectory. In the next section, we will derive a necessary condition for optimality of this control \(u^*(t)\), which is similar to the maximum principle in optimal control problems for conventional systems.

2.2 Mathematical preliminaries

Here, we introduce the concepts of the function norm and corresponding matrix norm, and list the lemma for the Bellman-Gronwall inequality.

Definition 2

For a real-valued vector \(x(t) \in R^n\), i.e. \(x(t) = (x_1, x_2, \ldots, x_n)\), the function norm is defined by

\[
\|x(t)\| = \sum_{i=1}^{n} |x_i(t)| \quad (7)
\]

and the matrix norm is defined by

\[
\|A\| = \sup_{x \neq 0} \frac{|A x|}{\|x\|} \quad (8)
\]

where \(A\) is a \(n \times n\) matrix.

Lemma (Bellman-Gronwall inequality)

If \(\phi(t)\) is a non-negative function on \([0, T]\), and if \(\phi(t)\) satisfies the differential inequality

\[
\frac{d\phi(t)}{dt} \leq -\gamma(t)\phi(t) + \eta(t) \quad (9)
\]

for \(0 \leq t \leq T\), where \(\gamma(t)\) and \(\eta(t)\) are non-negative functions on \([0, T]\), then

\[
\phi(t) \leq e^{-\int_{0}^{t} \gamma(s) ds} \left[ e^{\int_{0}^{T} \gamma(s) ds} \phi(0) - \int_{0}^{T} e^{\int_{0}^{s} \gamma(s) ds} \eta(s) ds \right] \quad (10)
\]

for \(0 \leq t \leq T\).
we define the following vector norm
\[ \|x(t)\| = \sum_{j=1}^{n} |x_{j}(t)| \] ........................................ (7)

The induced matrix norm corresponding to the norm \( \|\cdot\| \) is defined as follows.

(Definition 3) \[ \|A\| = \max \sum_{j=1}^{n} |a_{ij}| \quad (column \ sum) \] ........................................ (8)
where \( a_{ij} \) for \( i, j = 1, 2, \ldots, n \) denotes the entry of the matrix \( A \).

Finally, we directly state a lemma for the Bellman-Gronwall inequality as follows. For the proof, see Ref. (15).

(Lemma 1 (The Bellman-Gronwall lemma))

Let \( v(t), f(t) \) and \( c(t) \) be real continuous functions of \( t \), and let \( f(t) > 0 \).
\[ v(t) \leq c(t) + \int_{0}^{t} f(\tau)v(\tau)d\tau \] ........................................ (9)
then
\[ v(t) \leq c(t) + \int_{0}^{t} f(s)\exp\left[ \int_{s}^{t} f(\tau)d\tau \right] c(s)ds \] ........................................ (10)

In the light of Lemma 1, we can then have the following two results given by (Corollary 1) and (Corollary 2).

(Corollary 1)
If \( f(t) = f = constant \), then
\[ v(t) \leq c(t) + f \int_{0}^{t} v(\tau)d\tau \] ........................................ (11)
implies
\[ v(t) \leq c(t) + f \int_{0}^{t} \exp(f(t-s))c(s)ds \] ........................................ (12)

(Corollary 2)
If both \( f(t) \) and \( c(t) \) are constants, i.e. \( f(t) = f \) and \( c(t) = c \), then
\[ v(t) \leq c + f \int_{0}^{t} v(\tau)d\tau \] ........................................ (13)
implies
\[ v(t) \leq c \exp(f t) \] ........................................ (14)

2.3 Some assumptions

(Assumption 1)
The functions \( f, L, F \) and \( \phi \) are continuous and twice differentiable with respect to \( t \) and \( x \); \( f \) and \( \partial f/\partial x \) satisfy Lipschitz conditions with respect to \( u \) and \( v \). Furthermore, for the function \( f \), the Lipschitz condition is denoted by
\[ \|f(x + \delta x, u + \delta u, t) - f(x, u, t)\| \leq \|K_{1}\|\|\delta x\| + \|K_{2}\|\|\delta u\| \] ........................................ (15)
where \( K_{i} \) for \( i = 1, 2 \) are Lipschitz constant matrices.

(Assumption 2)
\( U(t) \) is the subsets of \( R^{m} \), and is closed and bounded.

3. Main results

In this section, we derive the necessary condition for the optimal control \( u^{*}(t) \), and discuss some problems related to this necessary condition.

(Theorem 1)
Consider the optimal control problem, described in Section 2.1, and let (Assumptions 1 and 2) be satisfied. Suppose that the admissible control \( u^{*}(t) \) is an optimal control, with \( x^{*}(t) \) denoting the corresponding optimal state trajectory. Then there exists an adjoint state vector function \( \lambda(t) : [t_{0}, t_{f}] \rightarrow R^{n} \) such that the following relations are satisfied.
\[ \frac{d}{dt}\lambda(t) = -L(x, u) + \lambda f(x, u, t) \] ........................................ (16 a)
\[ x(t_{0}) = x_{0}, \quad t \in [t_{0}, t_{f}] \] ........................................ (16 b)
\[ \max_{u \in U(t)} H(t, \lambda(t), x^{*}(t), u(t)) = H(t, \lambda(t), x^{*}(t), u^{*}(t)) \] ........................................ (17)
\[ \frac{d}{dt}H(t, \lambda(t), x^{*}(t), u^{*}(t)) \] ........................................ (18 a)
\[ \frac{\partial}{\partial x}H(t, \lambda(t), x^{*}(t), u^{*}(t)) = -\frac{\partial}{\partial x}\phi(x^{*}(t), t_{f}) \] ........................................ (18 b)
where
\[ H(t, \lambda, x, u) = -L(x, u, t) + \lambda f(x, u, t) \] ........................................ (19)
\[ \phi(x, t) = F(x, t) + \nu \phi(x, t) \] ........................................ (20)
where \( \nu \) is any positive constant.

(Proof)
Here, we only state the main steps.

(i) For this optimal control problem with the terminal constraint, we first make a generalized cost functional of the form
\[ J(u, x) = \phi(x(t_{f}), t_{f}) + \int_{t_{0}}^{t_{f}} L(x, u(\tau), \tau)d\tau \] ........................................ (21)
(ii) Let $u^*(t)$ and $x^*(t)$ be the optimal control and the optimal state trajectory, respectively; and let $u(t) = u^*(t) + \delta u(t)$ and $x(t) = x^*(t) + \delta x(t)$ be an admissible control and the admissible state trajectory, respectively. Then, from Eqs. (6) and (21), we have

$$\Delta J \doteq \int_{t_1}^{t_2} \left[ f(x(t), u(t)) - f(x^*(t), u^*(t)) \right] \, dt \geq 0 \quad \cdots (22)$$

(iii) Now, we introduce a time-varying multiplier (or adjoint state vector) $\lambda(t)$ of dimension $n$, and consider the following cost functional.

$$\begin{align*}
\check{f}(u, x) &= \psi(x(t_1), t_1) + \int_{t_0}^{t_1} \left[ L(x(t), u(t)) - \lambda^T(t) \left( E \frac{dx(t)}{dt} \right) \right] \, dt \cdots (23)
\end{align*}$$

Then, the variation of the cost functional Eq. (23) is as follows.

$$\Delta \check{f} = \phi(x^*(t), t) - \phi(x(t), t) + \int_{t_0}^{t_1} \lambda^T(t) E \delta x(t) \, dt \quad \cdots (24)$$

where

$$\delta x = \frac{d \delta x}{dt}$$

By making use of integration by parts, noting $Sx(t_0) = 0$, and adding $\pm H(t, \lambda, x^*, u^* + \delta u)$ and $\pm H_x(t, \lambda, x^*, u^*)$ to Eq. (24), then we can have

$$\begin{align*}
\Delta \check{f} &= \phi(x^*(t), t) - \phi(x(t), t) + \int_{t_0}^{t_1} \lambda^T(t) E \delta x(t) \, dt \\
&\quad + \int_{t_0}^{t_1} \left[ E \frac{dx(t)}{dt} \right] \delta x(t) \, dt \\
&\quad - \int_{t_0}^{t_1} \left[ (H(t, \lambda, x^*, u^* + \delta u)) \right] \delta x(t) \, dt \\
&\quad - \int_{t_0}^{t_1} \left[ (H_x(t, \lambda, x^*, u^*)) \right] \delta x(t) \, dt \\
&\quad - H_x(t, \lambda, x^*, u^*) \delta x(t) \, dt \quad \cdots (25)
\end{align*}$$

where

$$H_x(\cdot) = \frac{\partial}{\partial x} H(\cdot)$$

Furthermore, by using a Taylor series expansion, and by making use of Eq. (18), from Eq. (25), we can have

$$\begin{align*}
\Delta \check{f} &= (1/2) \Delta x^T \phi_{xx}(x^*, \theta, \delta x, t) \delta x, t \cdots (26)
\end{align*}$$

where

$$\begin{align*}
0 < \theta_1 \leq \theta_2 \leq 1
\end{align*}$$

and

$$H_{xx}(\cdot) = \frac{\partial^2}{\partial x^2} H(\cdot), \quad \phi_{xx}(\cdot) = \frac{\partial^2}{\partial x^2} \phi(\cdot)$$

(iv) On the other hand, for $t_0 \leq t_1 \leq t_2 \leq t_f$, let $\delta u(t) = 0$ for all $t \in (t_0, t_f)$ and $\delta u(t) \neq 0$ for all $t \in (t_1, t_2)$. Then, $\delta x(t) = 0$ for all $t \in (t_0, t_f)$ - $(t_1, t_2)$.

From Eq. (2), we have

$$\begin{align*}
E \frac{d \delta x(t)}{dt} &= f(x^* + \delta x, u^* + \delta u, t) \\
&\quad - f(x^*, u^*, t) \quad \cdots (27)
\end{align*}$$

Furthermore, noting $\delta x(t_f) = 0$, we can have

$$\begin{align*}
E \delta x(t) = \int_{t_1}^{t_2} \left[ (f(x^* + \delta x, u^* + \delta u, t) - f(x^*, u^*, t) \right] \, dt \quad \cdots (28)
\end{align*}$$

In the light of [Assumption 1], for any $t \in (t_1, t_2)$, from Eq. (28) we can obtain the following inequality.

$$\begin{align*}
\| E \delta x(t) \| &\leq \int_{t_1}^{t_2} \| K \| \| \delta x(r) \| \, dr \\
&\quad + \int_{t_1}^{t_2} \| K_t \| \| \delta u(r) \| \, dt \quad \cdots (29)
\end{align*}$$

From Eq. (29), we can also have

$$\begin{align*}
\| \delta x(t) \| &\leq M_0 + M_1 \int_{t_1}^{t_2} \| \delta x(r) \| \, dt \\
&\quad + M_2 \int_{t_1}^{t_2} \| \delta u(r) \| \, dt \quad \cdots (30)
\end{align*}$$

where $M_i (i = 0, 1, 2)$ are the positive constants defined by

$$\begin{align*}
\sup_{t_0 \leq t \leq t_f} \| \delta x(t) \| &\leq M_0 \\
M_i = \| K \| / \| E \| \quad (i = 1, 2)
\end{align*}$$

By making use of Bellman-Gronwall Lemma, furthermore, we obtain

$$\begin{align*}
\| \delta x(t) \| &\leq (M_0 + M_1 \int_{t_1}^{t_2} \| \delta u(r) \| \, dr) \\
&\quad \times \exp (M_i (t - t_1)) \quad \cdots (31)
\end{align*}$$
Denote $|\delta u(t)|=|\delta u_1(t)|+M_3/m$ for $t \in (t_1, t_2)$, and denote $|\delta u_1(t)|=0$ for all $t \in (t_0, t_1)-(t_1, t_2)$. Then for any $t \in (t_1, t_2)$, we have

$$
\|\delta x(t)\| \leq M \int_{t_1}^{t} \|\delta u(t)\| dt
$$

i.e.

$$
|\delta x(t)| \leq M \int_{t_1}^{t} \sum_{i=1}^{n} |\delta u_i(t)| dt
$$

$(i=1, \ldots, n, t_1 \leq t \leq t_2)$ \quad (32)

where $M_3$ and $M$ are positive constants, and defined by

$$
M_3 = \frac{M_1}{M} (h_2-h_0), \quad M = M_2 \exp(M_1 (h_2-h_0))
$$

(34)

Thus, taking account of the fact that $\Delta f$ is positive, and that the matrix $E$ is not concerned in the right-hand side of Eq. (26), then by making use of Eq. (32), we may treat Eq. (26) in a way similar to Ref. (16) to obtain the necessary condition for optimal control $u^*(t)$, i.e. for any $t \in [t_1, t_2]$, $u^*(t)$ must satisfy

$$
\max_{u \in U} H(t, \lambda, x^*, u) = H(t, \lambda, x^*, u^*) \quad (33)
$$

Thus, we complete the proof of this theorem.

(Remark 2)

If the matrix $E$ is not singular, then the system equation described by Eq. (2) can become

$$
\frac{dx(t)}{dt} = f(x(t), u(t), t) \quad (34)
$$

where

$$
f(\cdot) = E^{-1} f(\cdot) \quad (35)
$$

That is, descriptor systems are reduced to conventional systems for this optimal control problem, and the corresponding necessary condition given by (Theorem 1) is also reduced to the well-known Pontryagin's maximum principle. Therefore, we may say that the above theorem is the maximum principle for descriptor systems, and furthermore, the Pontryagin's maximum principle for conventional systems is a particular case of (Theorem 1).

(Remark 3)

It is worth noting that for the proof of (Theorem 1), we do not use the transform of any form for descriptor systems (2), and there is no restrictive assumption. In Ref. (11), for example, the restrictive assumption that $(\delta f/\delta x)(2)'$ is independent of $\dot{x}(t)$, $x(2)(t)$ and $u(t)$ is required (see Ref. (11), Theorem 3.1). Therefore, our results are more general.

(Remark 4)

When there are the constraints on the control vectors $u(t)$, in the light of Assumption 2, the necessary condition given by (Theorem 1) is still valid. This case is different from that of the papers which consider optimal control problem for descriptor systems (e.g. see Refs. (17) ~ (19)).

(Remark 5)

In the light of the procedures of the proof of (Theorem 1), when the cost functional $J(u, x)$ becomes

$$
J(u, x) = \int_{0}^{\infty} L(x(t), u(t), t) dt \quad (36)
$$

where

$$
\|L(x(t), u(t), t)\| < \infty \quad (37)
$$

and $x(t) \to x'$, a given terminal state, as $t \to \infty$, the necessary condition given by (Theorem 1) is still valid.

(Remark 6)

Here, we consider the following linear descriptor system

$$
E \frac{dx(t)}{dt} = A(t) x(t) + C(t) \quad (38)
$$

Assume that the functions $L(\cdot), F(\cdot)$ and $\phi(\cdot)$ are linear in $x$. For such linear descriptor systems, we can obtain the following sufficient and necessary condition for control $u^*$, given by the following theorem.

(Theorem 2)

Consider the optimal control problem of Eqs. (38), (4) and (5) with the above linear assumption. Then, a sufficient and necessary condition for the control $u^*(t)$ which provides an optimal solution, with $x^*(t)$ denoting the corresponding optimal state trajectory, is that there exists an adjoint state vector function $\lambda(t)$ : $[a_0, t_f] \to \mathbb{R}^n$ such that the following relations are satisfied.

$$
E \frac{dx^*(t)}{dt} = A(t) x^*(t) + C(u^*(t), t) \quad (39 a)
$$

$$
x^*(t_0) = x^0 \quad t \in [t_0, t_f] \quad (39 b)
$$

$$
\max_{u \in U} H(t, \lambda, x^*, u) = H(t, \lambda, x^*, u^*) \quad (40)
$$

$$
\lambda^T(t) A(t) = 0 \quad (41 a)
$$
\[
E^\ast \lambda(t_f) = -\frac{\partial}{\partial x} \phi(x^\ast(t_f), t_f) \quad \cdots \quad (41b)
\]

(Proof)

From Eq. (26) and the assumption that \( L(\cdot), F(\cdot) \) and \( \phi(\cdot) \) are linear in \( x \), we can obtain
\[
\Delta f = -\int_{t_0}^{t_f} \left[ H(t, \lambda, x^\ast, u^\ast + \Delta u) - H(t, \lambda, x^\ast, u^\ast) \right] dt \quad \cdots \quad (42)
\]
It can easily be proved that the sufficient and necessary condition for \( u^\ast(t) \) to be optimal is that for any \( t \in [t_0, t_f] \), the following condition holds.
\[
\max_{u \in U} H(t, \lambda, x^\ast, u) = H(t, \lambda, x^\ast, u^\ast) \quad \cdots \quad (43)
\]
In addition, in the rest parts of this section, we will discuss the properties of the hamiltonian \( H(\cdot) \). For this, we will have the following theorem.

(Theorem 3)

If admissible control \( u^\ast(t) \) with \( x^\ast(t) \) denoting the corresponding state trajectory satisfies the condition given in (Theorem 1), then the function \( M(t) \), where
\[
M(t) = H(t, \lambda(t), x^\ast(t), u^\ast(t)) \quad \cdots \quad (44)
\]
is a continuous function of time \( t \), \( t \in [t_0, t_f] \); and furthermore for \( t \in [t_0, t_f] \), at which \( u^\ast(t) \) is continuous, there will be
\[
\frac{dM(t)}{dt} = \frac{\partial}{\partial t} H(t, \lambda(t), x^\ast(t), u^\ast(t)) \quad \cdots \quad (45)
\]
(Proof)

Consider the following difference
\[
\Delta M(t) = H(t + \Delta t, \lambda(t + \Delta t), x^\ast(t + \Delta t), u^\ast(t + \Delta t)) - H(t, \lambda(t), x^\ast(t), u^\ast(t)) \quad \cdots \quad (46)
\]
From Eq. (17), we can obtain
\[
H(t + \Delta t, \lambda(t + \Delta t), x^\ast(t + \Delta t), u^\ast(t + \Delta t)) - H(t, \lambda(t), x^\ast(t), u^\ast(t)) \quad \leq \Delta M(t)
\]
\[
\leq \Delta H(t + \Delta t, \lambda(t + \Delta t), x^\ast(t + \Delta t), u^\ast(t + \Delta t)) - H(t, \lambda(t), x^\ast(t), u^\ast(t + \Delta t)) \quad \cdots \quad (47)
\]
In the light of Eq. (47), we can make the following analysis.

(i) Taking account of the continuity of \( \lambda(t) \) and \( x(t) \), and noting the fact that in the left- and right-hand sides of Eq. (47), \( u^\ast(\cdot) \) takes its value at the same time, it follows from Eq. (47) that when \( \Delta t \to 0 \), \( \Delta M(t) \to 0 \), i.e., \( M(t) \) is a continuous function of time \( t \).

(ii) Dividing Eq. (47) by \( \Delta t \), when \( \Delta t \to 0 \), the left- and right-hand sides of Eq. (47) have a limit of the same form
\[
H_x^\ast(\cdot) \frac{dx^\ast(t)}{dt} + H_{x^\ast}(\cdot) \frac{dx^\ast(t)}{dt} + H(\cdot)
\]
\[
\quad \cdots \quad (48)
\]
where
\[
H(\cdot) = H(t, \lambda(t), x^\ast(t), u^\ast(t))
\]
Taking into consideration that
\[
H_x^\ast(\cdot) \frac{dx^\ast(t)}{dt} + H_{x^\ast}(\cdot) \frac{dx^\ast(t)}{dt} = -E^\ast \frac{d\lambda(t)}{dt}
\]
thereby, we can obtain Eq. (45).

(Comment)

When the hamiltonian \( H(\cdot) \) does not depend explicitly on time variable \( t \), we then have
\[
\frac{\partial H}{\partial t} = 0
\]
for \( t \in [t_0, t_f] \), at which \( u^\ast(t) \) is continuous.

In the light of the continuity of the function \( M(t) \) given in (Theorem 3), we can obtain that for any \( t \in [t_0, t_f] \),
\[
H(\cdot) \equiv \text{const} \quad \cdots \quad (49)
\]

4. An illustrative example

In this section, we give an illustrative example as follows.

The systems considered in this example will be
\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \frac{dx_i}{dt} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} x_i + \begin{bmatrix}
0 \\
1
\end{bmatrix} u \quad \cdots \quad (50)
\]
and the given admissible initial condition will be
\[
x(0) = (x_{01}, x_{02})^T \quad \cdots \quad (51)
\]
This system was considered by several researchers (see, for. Refs. (1), (11), (17), (19) and (20)). With the aid of state feedback and a Riccati-type matrix differential equation, Ref. (20) has derived an optimal control \( u(t) \) and an optimal state trajectory \( x(t) \) that minimize the cost functional \( J \), where
\[
J = \int_0^T \left[ x_1^2(t) + x_2^2(t) + u^2(t) \right] dt \quad \cdots \quad (52)
\]
Furthermore, we require that the terminal state \( x^T \) reach zero point, i.e.
\[
x^T = (0, 0)^T \quad \cdots \quad (53)
\]
Here, we derive generally the same optimal con-
control \( u(t) \) and optimal state trajectory \( x(t) \) as those derived in Ref. (20) for this optimal control problem, by making use of the necessary condition developed in this paper.

Applying the necessary condition given in (Theorem 1), we have a hamiltonian of the form

\[
H(t, \lambda, x, u) = -\{x_1^2 + x_2^2 + u^2\} + \lambda_1 x_1 + \lambda_2 (x_2 + u) \tag{54}
\]

Because there are no constraints on \( u(t) \) in this example, according to (Theorem 1), from \( \partial H/\partial u = 0, \ \partial^2 H/\partial u^2 = -1 < 0 \) we can obtain

\[
u^*(t) = (1/2) \lambda(t) \tag{55}
\]

In addition, the adjoint equation will be

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix}
= 
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix}
+ 
\begin{bmatrix}
2x_1^* \\
2x_2^*
\end{bmatrix}
\]

\[
\tag{56}
\]

where \( x^*(t) = (x_1^*(t), x_2^*(t))^T \) is the optimal state trajectory, and satisfies the following system equations.

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{dx_1^*}{dt} \\
\frac{dx_2^*}{dt}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1^* \\
x_2^*
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
u^*
\end{bmatrix}
\]

\[
\tag{57}
\]

Substitution of Eqs. (55) and (56) into Eq. (57) yields

\[
\frac{dx_1^*}{dt} = 2x_2^*(t) \tag{58a}
\]

\[
\frac{dx_2^*}{dt} = x_1^*(t) \tag{58b}
\]

Then from Eq. (58), we can obtain

\[
x_2^*(t) = C_1 \exp(-\sqrt{2}t) + C_2 \exp(\sqrt{2}t) \tag{59}
\]

In order to guarantee that \( x^*(t) \rightarrow 0 \) as \( t \rightarrow \infty \), we can obtain

\[
x_1^*(t) = C_1 \exp(-\sqrt{2}t) \tag{60}
\]

By making use of the given admissible initial condition Eq. (51), we finally obtain the optimal state trajectory as follows.

\[
x^*(t) = \begin{bmatrix}
-x_2 \exp(-\sqrt{2}t) \\
x_2 \exp(-\sqrt{2}t)
\end{bmatrix} \tag{61}
\]

In addition, we can obtain the solution of the adjoint Eq. (56) as follows.

\[
\lambda(t) = \begin{bmatrix}
-2\sqrt{2}x_2 \\
-2x_2
\end{bmatrix} \exp(-\sqrt{2}t) \tag{62}
\]

Thus, we can write the optimal control of the form

\[
u^*(t) = -x_2 \exp(-\sqrt{2}t) \tag{63}
\]

These results of Eqs. (61) and (63) are identical with those derived in Ref. (20)

5. Conclusion

In this paper, we consider optimal control problems for descriptor systems with a general form. Under the assumptions of the admissible initial state and the admissible control (see Definition 1), a necessary condition which may be regarded as the maximum principle for descriptor systems is derived. It is worth noting that there is no assumption on admissible initial state in optimal control problems for conventional systems, and the assumption on admissible control in descriptor systems is also different from the one in conventional systems. These facts result in important differences between the optimal control problems of descriptor systems and those of conventional systems, and are restrictions for descriptor systems.

In addition, we also discuss some problems related to this necessary condition, such as the continuity and differentiability of the hamiltonian in this necessary condition. Furthermore, the sufficient and necessary condition on optimal control of linear descriptor systems has been developed (Theorem 2). Our results may include some ones (e.g. see Remark 2) in the optimal control problems for conventional systems. The example of a linear time-invariant descriptor systems is also given, which illustrates the application of our method. The results obtained in this paper may be expected to have some applications for optimal control problems of descriptor systems.

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