Eigenvalue Location Condition for a Polytope of Matrices in a Sector

Member Takehiro Mori (Kyoto Institute of Technology)
Member Hideki Kokame (Osaka Institute of Technology)

Key words: Eigenvalue location, Polytope, Sector, Stability Robustness, Uncertainty

1. Introduction

Stability of interval matrices or a polytope of matrices has been paid considerable attention in these several years associated with stability robustness problems (see, for example, Ref. (1), (8)). To the authors' knowledge so far, however, only sufficient conditions for stability are in general available for both interval matrices and a polytope of matrices (e.g., Ref. (4), (6), (9)).

In this paper, a sufficient condition is provided to ensure the eigenvalue location of a polytope of matrices in a sector of the complex plane. When the sector coincides with the left half complex plane, the result obtained arrives at an existing stability condition. When it reduces to the left half real axis, the result gives an aperiodicity condition. Checking the condition is carried out by computing a matrix measure of a certain matrix. In what follows, for a matrix $X \in \mathbb{R}^{n \times n}$, $X'$ denotes the transpose of $X$ and $\lambda(X)$ stands for the eigenvalue of $X$. The matrix measure of $X$ used here is defined as $\mu(X) = \max \{\lambda_i(X + X')\}/2$ (for details of matrix measures, see Ref. (2)). $I_n$ is used to denote the unit matrix of order $n$. The symbol, $\mathcal{H}$, means the whole family of Hurwitz matrices.

2. Main result

Consider a polytope $\mathcal{A}$ of matrices, $A_i \in \mathbb{R}^{n \times n}$, $i = 1, \cdots, m$ defined below.

$$\mathcal{A} := \left\{ A \in \mathbb{R}^{n \times n} : A = \sum_{i=1}^{m} a_i A_i, \quad \sum_{i=1}^{m} a_i = 1, a_i \geq 0 \right\} \quad \cdots (1)$$

We would like to know whether all the eigenvalues of any member of $\mathcal{A}$ lie in a sector on the complex plane shown in Fig. 1. The sector has the vertex $(-p, 0)$ on the real axis and makes an angle $\delta$ ($0 \leq \delta < \pi/2$) with the imaginary axis. A sufficient condition is given by the following theorem.

(Theorem 1)

Any matrix $A$ in $\mathcal{A}$ has all its eigenvalues in the sector, if

$$\mu(D(A_i, \delta)) + p < 0, \quad i = 1, \cdots, m \quad \cdots (2)$$

where,

$$D(A_i, \delta) := \begin{bmatrix} A_i & -A_i \tan \delta \\ A_i \tan \delta & A_i \end{bmatrix} \quad \cdots (3)$$

This theorem comes from combining the two lemmas shown below.

(Lemma 1)

Let $B$ be an $n \times n$ real matrix. Then all the $\lambda(B)$ exist in the sector of Fig. 1, if and only if

$$W(B, \delta) + p \cos \delta a_i \in \mathcal{H}, \quad \cdots \cdots \cdots \cdots (4)$$

where

$$W(B, \delta) := \begin{bmatrix} B \cos \delta & -B \sin \delta \\ B \sin \delta & B \cos \delta \end{bmatrix} \quad \cdots (5)$$

(Lemma 2)

For a polytope $\mathcal{A}$ given by Eq. (1) to satisfy, $\mathcal{A} \subset \mathcal{H}$, \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (6)

it is sufficient that

$$W(B, \delta) + p \cos \delta a_i \in \mathcal{H}, \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (7)$$

Fig. 1. A sector on the complex plane.
The result of (Lemma 2) is originally developed for interval matrices, but it is actually more general to cover a polytope of matrices. Now, we are in position to prove (Theorem 1).

[Proof of Theorem 1]: In light of (Lemma 1), it is sufficient to show,
\[ W(A, \delta) + p \cos \delta I_n \in \mathcal{X}, \quad \forall A \in \mathcal{A}, \quad \cdots \quad (8) \]
or,
\[ D(A, \delta) + p I_n \in \mathcal{X}, \quad \forall A \in \mathcal{A}. \quad \cdots \quad (9) \]
In view of Eq. (3), we see that the left hand side of the above relation is indeed a matrix polytope constructed from
\[ D(A_i, \delta) + p I_n, \quad i=1, \ldots, m. \quad \cdots \quad (10) \]
(Lemma 2) now can be used to confirm that
\[ \mu(D(A_i, \delta) + p I_n) < 0, \quad i=1, \ldots, m \quad \cdots \quad (11) \]
is sufficient for the relation, Eq. (9). Noting the fact, \[ \mu(X + aI_n) = \mu(X) + a, \quad \forall X \in \mathbb{R}^{n \times n}, \quad \forall a \in \mathbb{R} \]
which is due to the definition of \( \mu(X) \), Eq. (11) immediately gives Eq. (2).

Several comments on the theorem are in order.

(1) As stated in section 1, any interval matrix can be represented as a polytope of matrices and therefore (Theorem 1) is in force for interval matrices as well.

(2) As seen from the proof, if an exact stability condition for a polytope of matrices is on our hand, our result would become exact as well. Unfortunately, however, this is not the case as of now except \( m=2 \).

(3) Setting \( \delta=0 \) and \( p=0 \) in (Theorem 1), the condition reduces to that of (Lemma 2). In this sense, (Theorem 1) is an extension of (Lemma 2).

(4) Letting \( p=0 \) and \( \delta=\pi/2 \) in (Theorem 1) yields an aperiodicity condition. Here, aperiodicity of a matrix implies that all of its eigenvalues lie in the open left half real axis. The reasoning in the proof gives the condition as follows:
\[ \mu(W(A_i, \pi/2)) < 0, \quad i=1, \ldots, m. \quad \cdots \quad (12) \]
More commonly accepted notion of aperiodicity includes, however, one more condition, no repetition of the eigenvalues (see Ref. (5)).

(5) It should be noted that no cross-terms between \( p \) and \( \delta \) exist in Eqs (2) and (3) and that \( \delta \) appears in the off-diagonal block matrices in \( D \) and \( p \) does in the diagonal matrix independent from \( D \). This enables us to use (Theorem 1) for obtaining an appropriate \( p \) or \( \delta \) consecutively.

3. Concluding remarks

A sufficient condition is derived to test the location of all the eigenvalues of a polytope of matrices in a specified sector on the complex plane. The condition can be checked simply by computing a matrix measure for certain vertex matrices of doubled size. (Manuscript received Jan. 21, '91)

References


Takehiro Mori (Member)

He received B. Eng., M. Eng. and Dr. Eng. all from Kyoto Univ. in 1968, 1970 and 1977, respectively. He is now a Professor of Kyoto Institute of Technology.

Hideki Kokame (Member)

He received B. Eng., M. Eng. and Dr. Eng. all from Kyoto Univ. in 1968, 1970 and 1978, respectively. He is now a Professor of Osaka Institute of Technology.