Adaptive Robust State Feedback Controllers for Linear Dynamical Systems with Bounded Uncertainties

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The problem of robust stabilization of a class of time-varying dynamical systems with disturbance and uncertain parameters is considered. The bounds of the disturbance and uncertain parameters are assumed to be unknown. For such uncertain dynamical systems, a class of continuous adaptive robust state feedback controllers are proposed. It is shown that the resulting closed-loop dynamical systems are stable in the sense of uniform ultimate boundedness. A numerical example is given to demonstrate the synthesis procedure for adaptive stabilizing controller proposed in this paper. In particular, by making use of the conventional robust state feedback controllers reported in the control literature, one cannot stabilize the system given in this example. However, by employing the adaptive robust control scheme proposed in this paper, it is possible to stabilize this system.

Keywords: Robust stabilization, adaptive control, robust control, linear systems, uncertainty, state feedback, uniform ultimate boundedness.

1. Introduction

Robust stabilization of dynamical systems with significant uncertainties has been widely investigated over the last decades. For a class of uncertain dynamical systems, i.e. the system state vector is available, the so-called matching conditions are satisfied, and the upper bound of the uncertainty vector norm is supposed to be known, several types of state feedback controllers have been proposed by making use of the techniques based on Lyapunov's direct method (see, e.g. Refs. (1)~(3), (7), (8), (18), (19) and the references therein). In general, the upper bound of the uncertainty vector norm is employed in such state feedback controllers to lead to some desired performance, e.g. asymptotic stability, ultimate boundedness, and so on, of the resulting closed-loop systems with uncertainties.

In the practical control problems, the bounds of the uncertainties might not be exactly known. Therefore, for such a class of uncertain dynamical systems whose uncertainty bounds are partially known, adaptive control schemes have been introduced to update these unknown bounds. In Ref. (9), for example, three types of dynamical systems containing uncertain elements are considered due to imperfect knowledge about the model and the input, and the corresponding adaptive robust controllers are proposed. In Ref. (15), a new class of state observers for nonlinear uncertain dynamical systems are proposed by using an adaptive scheme. In Ref.

(16), a class of adaptive robust controllers are designed for uncertain dynamical systems with fast time-varying uncertainties and measurement noise. In Ref. (17), a saturation-type adaptive robust controller is proposed for a class of uncertain dynamical systems, where the uncertainty bounds are partially known. In Ref. (20), for a class of nonlinear systems with partially known uncertainties, a adaptive robust controller is proposed by using a dead-zone and a hysteresis function.

In this paper, we consider the problem of robust stabilization of a class of time-varying dynamical systems with disturbance and uncertain parameters. Here, the bounds of the disturbance and the uncertain parameters are assumed to be unknown. For such uncertain dynamical systems, we propose a class of continuous adaptive robust state feedback controllers which can stabilize the resulting closed-loop dynamical systems in the sense of uniform ultimate boundedness. In particular, by employing the adaptive robust state feedback controller proposed in this paper, we do not require that the bound $\mu(t)$ of the uncertain $\Delta B(\cdot)$ is known. Furthermore, we do not assume that $\mu(t) > -1$ for any $t \in R^+$, which is a necessary condition for robust stability of uncertain systems in the control literature on the the problem of robust stabilization.

The organization of this paper is as follows. In Section 2, the problem to be tackled is stated and some standard assumptions are introduced. In Section 3, a class of adaptive robust control schemes are proposed

and the corresponding stability analysis is made. In Section 4, a numerical example is given to illustrate the use of the results. Finally, the paper will be concluded in Section 5 with a brief discussion of the results.

2. Problem Formulation and Assumptions

Consider a class of uncertain dynamical systems described by the following differential equations:

$$\begin{aligned} \frac{dx(t)}{dt} = & [A(t) + \Delta A(v, t)]x(t) \\ & + [B(t) + \Delta B(v, t)]u(t) \\ & + C(t)w(t) \dots\dots\dots (1) \end{aligned}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, $w(t) \in R^s$ is the external disturbance vector bounded by $\|w(t)\| \leq \theta_w(t)$ where $\|\cdot\|$ is the Euclidean norm of a vector “.”, $(v, \nu) \in \Psi$ is the uncertain vector, $\Psi \subset R^L$ is a compact set, $A(t)$, $B(t)$, $C(t)$ are continuous matrices of appropriate dimensions, and the matrices $\Delta A(\cdot)$, $\Delta B(\cdot)$ represent the system uncertainties and are continuous in all their arguments.

Provided that all states are available, the state feedback controller can be represented by a function:

$$u(t) = p(x(t), t) \dots\dots\dots (2)$$

Now, the question is to how to synthesize a state feedback controller $u(t)$ that can guarantee the stability of dynamical system (1) in the presence of the uncertain $\Delta A(\cdot)$, $\Delta B(\cdot)$, $w(\cdot)$.

Before proposing our controllers, we introduce for system (1) the following standard assumptions.

Assumption 2.1. The pair $\{A(\cdot), B(\cdot)\}$ given in system (1) is uniformly completely controllable.

Assumption 2.2. For all $(x, t) \in R^n \times R$ there exist continuous matrix functions $H(\cdot)$, $E(\cdot)$, $D(\cdot)$ of appropriate dimensions such that

$$\begin{aligned} \Delta A(v, t) &= B(t)H(v, t) \\ \Delta B(v, t) &= B(t)E(v, t) \\ C(t) &= B(t)D(t) \end{aligned}$$

For convenience, we now introduce the following notations which represent the bounds of the uncertainties.

$$\begin{aligned} \rho(t) &:= \max_v \|H(v, t)\| \\ \mu(t) &:= \min_{\nu} \left[\frac{1}{2} \lambda_{\min} (E(\nu, t) + E^T(\nu, t)) \right] \end{aligned}$$

where $\|\cdot\|$ is the spectral norm of a matrix “.”, and $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of the matrix “.”, respectively. Moreover, $\rho(t)$, $\mu(t)$, $\theta_w(t)$ are assumed to be continuous and bounded for any $t \in R^+$.

Remark 2.1. It is well known that *Assumption 2.1* is standard and denotes the internally stabilizability of the nominal system, i.e., the system in the absence of the uncertain $\Delta A(\cdot)$, $\Delta B(\cdot)$, $w(\cdot)$. *Assumption 2.2* defines the matching condition about the uncertainties, and is a rather standard assumption for robust control problem (see, e.g. Refs. (1)~(3), (10), (11), (18), (19)). For dynamical system with matched uncertainties, one can always design some types of stabilizing state feedback controllers. In Ref. (19), for example, a class of continuous state feedback controllers are proposed such that ultimate uniform boundedness for the system can be guaranteed. In Ref. (10), another class of continuous state feedback controllers with simpler structure are introduced to guarantee the practical stability (or ultimate uniform boundedness) of the system. The controllers guaranteeing asymptotic and exponential stability of uncertain systems are presented in Ref. (11) and Ref. (18), respectively. However, this assertion is not valid for systems with unmatched uncertainties. For such uncertain systems, one must find some conditions such that some types of stability can be guaranteed (see, e.g. Refs. (12)~(14)).

Remark 2.2. The stabilizing state feedback controllers proposed in the control literature for system (1) are based on the fact that the bounds of the uncertainties are known. That is, $\rho(t)$, $\mu(t)$, $\theta_w(t)$ are assumed to be the known continuous and bounded functions, and the proposed control laws include such bounds $\rho(t)$, $\mu(t)$, $\theta_w(t)$. However, in a number of practical control problems, such bounds may be unknown. Therefore, some updating laws to such unknown bounds must be introduced to construct adaptive robust controllers (see, e.g. Refs. (9), (15)~(17)). In this paper, we propose another class of adaptive robust state feedback controllers for uncertain system (1).

3. Adaptive Robust Controller

Since the bounds $\rho(t)$, $\mu(t)$, $\theta_w(t)$ have been assumed to be continuous and bounded, we can suppose that there exist some constants ρ^* , μ^* , θ^* such that for any $t \in R^+$,

$$\rho(t) \leq \rho^*, \quad \theta_w(t) \leq \theta^*, \quad \mu(t) \geq \mu^* \dots\dots\dots (3)$$

Here, it is worth pointing out that the constants ρ^* , μ^* , θ^* are unknown.

It follows from *Assumption 2.1* that for any symmetric positive definite matrix $Q \in R^{n \times n}$, the matrix Riccati equation of the form

$$\frac{dP(t)}{dt} + A^\top(t)P(t) + P(t)A(t) - P(t)B(t)B^\top(t)P(t) = -Q \dots\dots (4)$$

has a solution which satisfies

$$\alpha_1 I \leq P(t) \leq \alpha_2 I \dots\dots\dots (5)$$

for all t , where α_1 and α_2 are positive numbers (see, e.g. Refs. (5), (6)).

Thus, for system (1) we propose the following adaptive robust state feedback controller:

$$\begin{aligned} u(t) &= p(x(t), t) \\ &= -\frac{1}{2} k(t)B^\top(t)P(t)x(t) \dots\dots\dots (6a) \end{aligned}$$

where the control gain function $k(t)$ is given by

$$k(t) = \frac{1 + \delta_1^2 \hat{\rho}^2(t) + \delta_2^2 \|D(t)\|^2 \hat{\theta}^2(t)}{1 + \hat{\mu}(t)} \dots\dots (6b)$$

$$\frac{1}{\delta_1^2} < \lambda_{\min}(Q) \dots\dots\dots (6c)$$

and where $P(t) \in R^{n \times n}$ is the solution of the Riccati equation described by (4), δ_1 , δ_2 are two positive constants, and $\hat{\rho}(\cdot)$, $\hat{\theta}(\cdot)$, $\hat{\mu}(\cdot)$ are respectively the estimates of the unknown parameters ρ^* , θ^* , μ^* , which are updated respectively by the following adaptive laws:

$$\begin{aligned} \frac{d\hat{\rho}(t)}{dt} &= -\sigma_1 \gamma_1 \hat{\rho}(t) \\ &+ \gamma_1 \|B^\top(t)P(t)x(t)\| \|x(t)\| \dots\dots (7a) \end{aligned}$$

$$\begin{aligned} \frac{d\hat{\theta}(t)}{dt} &= -\sigma_2 \gamma_2 \hat{\theta}(t) \\ &+ \gamma_2 \|B^\top(t)P(t)x(t)\| \|D(t)\| \dots\dots (7b) \end{aligned}$$

$$\begin{aligned} \frac{d\hat{\mu}(t)}{dt} &= -\sigma_3 \gamma_3 \hat{\mu}(t) \\ &- \frac{1}{2} \gamma_3 k(t) \|B^\top(t)P(t)x(t)\|^2 \dots\dots (7c) \end{aligned}$$

where σ_i , γ_i , $i = 1, 2, 3$, are any positive constants, $\hat{\rho}(t_0)$, $\hat{\theta}(t_0)$, $\hat{\mu}(t_0)$ are finite, and specially, $\hat{\mu}(t_0) \geq 0$ and σ_3 , γ_3 are selected so that $\hat{\mu}(t) > -1$ for any $t \in R^+$.

Remark 3.1. It is worth pointing out that for uncertain system (1) we do not assume that $\mu(t) > -1$ for any $t \in R^+$. It is well known that $\mu(t) > -1$ for any $t \in R^+$ is a necessary condition for robust stability of

uncertain system (1) (see, e.g. Refs. (1)~(3), (10) and the references relative to robust stabilization of uncertain systems).

Remark 3.2. It is obvious that the solution to (7c) can be represented as

$$\begin{aligned} \hat{\mu}(t) &= \exp\{-\sigma_3 \gamma_3 (t - t_0)\} \hat{\mu}(t_0) \\ &- \frac{1}{2} \gamma_3 \int_{t_0}^t \exp\{-\sigma_3 \gamma_3 (t - \tau)\} k(\tau) \\ &\times \|B^\top(\tau)P(\tau)x(\tau)\|^2 d\tau \dots\dots (8) \end{aligned}$$

It follows from (8) that if letting $\hat{\mu}(t_0) > 0$ and if system (1) can be stabilized, we can select always some parameters σ_3 and γ_3 such that the solutions to (7c) are larger than minus one, i.e. $\hat{\mu}(t) > -1$ for any $t \in R^+$. For instance, for a sufficient small γ_3 , an appropriate σ_3 , and a sufficient large $\hat{\mu}(t_0)$, from (8) we can guarantee that $\hat{\mu}(t) > -1$ for any $t \in R^+$. However, though the existence of such $\hat{\mu}(t_0)$, γ_3 , σ_3 can be guaranteed theoretically, how to select them, i.e. a systematic selection method, may be necessary for further investigation. In the illustrative example given in Section 4 we have employed a trial and error method to select such $\hat{\mu}(t_0)$, γ_3 , σ_3 that have guaranteed $\hat{\mu}(t) > -1$ for any $t \in R^+$.

Applying (6) to (1) yields a closed-loop system of the form:

$$\begin{aligned} \frac{dx(t)}{dt} &= \left[A(t) - \frac{1}{2} k(t)B(t)B^\top(t)P(t) \right] x(t) \\ &+ \left[\Delta A(v, t) - \frac{1}{2} k(t)\Delta B(v, t)B^\top(t)P(t) \right] x(t) \\ &+ C(t)w(t) \dots\dots\dots (9) \end{aligned}$$

On the other hand, letting

$$\begin{aligned} \psi(t) &= \hat{\rho}(t) - \rho^* \\ \varphi(t) &= \hat{\theta}(t) - \theta^* \\ \phi(t) &= \hat{\mu}(t) - \mu^* \end{aligned}$$

we can rewrite (7) as the following error system

$$\begin{aligned} \frac{d\psi(t)}{dt} &= -\sigma_1 \gamma_1 \psi(t) - \sigma_1 \gamma_1 \rho^* \\ &+ \gamma_1 \|B^\top(t)P(t)x(t)\| \|x(t)\| \dots\dots (10a) \end{aligned}$$

$$\begin{aligned} \frac{d\varphi(t)}{dt} &= -\sigma_2 \gamma_2 \varphi(t) - \sigma_2 \gamma_2 \theta^* \\ &+ \gamma_2 \|B^\top(t)P(t)x(t)\| \|D(t)\| \dots\dots (10b) \end{aligned}$$

$$\begin{aligned} \frac{d\phi(t)}{dt} &= -\sigma_3 \gamma_3 \phi(t) - \sigma_3 \gamma_3 \mu^* \\ &- \frac{1}{2} \gamma_3 k(t) \|B^\top(t)P(t)x(t)\|^2 \dots\dots (10c) \end{aligned}$$

Then, the following theorem can be obtained which shows the uniform ultimate boundedness of uncertain closed-loop system (9) and error system (10).

Theorem 3.1. Consider the uncertain closed-loop system (9) and the error system (10) satisfying *Assumption 2.1* and *Assumption 2.2*. Then, the solutions $(x, \psi, \varphi, \phi)(t; t_0, x(t_0), \psi(t_0), \varphi(t_0), \phi(t_0))$ of the closed-loop system (9) and error system (10) are uniformly ultimately bounded in the presence of the disturbance and uncertain parameters.

Proof: We first define for systems (9) and (10) a Lyapunov function candidate as follows.

$$V(x, \Psi) = x^\top(t)P(t)x(t) + \Psi^\top(t)\Gamma^{-1}\Psi(t) \quad (11)$$

where $P(t)$ is the solution of Riccati differential equation (4), Γ^{-1} is a positive definite matrix, and $\Gamma^{-1}, \Psi(\cdot)$ are defined as

$$\Psi(\cdot) = \begin{bmatrix} \psi(\cdot) \\ \varphi(\cdot) \\ \phi(\cdot) \end{bmatrix}, \quad \Gamma^{-1} = \begin{bmatrix} \gamma_1^{-1} & 0 & 0 \\ 0 & \gamma_2^{-1} & 0 \\ 0 & 0 & \gamma_3^{-1} \end{bmatrix}$$

Let $(x(t), \Psi(t))$ be the solution of closed-loop system (9) and error system (10) for $t \geq t_0$. Then by taking the derivative of $V(\cdot)$ along the trajectories of (9) and (10) it is obtained that

$$\begin{aligned} \frac{dV(x, \Psi)}{dt} &= x^\top(t) \left[\frac{dP(t)}{dt} + A^\top(t)P(t) + P(t)A(t) \right. \\ &\quad \left. - k(t)P(t)B(t)B^\top(t)P(t) \right] x(t) \\ &\quad + 2x^\top(t)P(t)\Delta A(v, t)x(t) \\ &\quad - k(t)x^\top(t)P(t)\Delta B(v, t)B^\top(t)P(t)x(t) \\ &\quad + 2x^\top(t)P(t)C(t)w(t) \\ &\quad + 2\Psi^\top(t)\Gamma^{-1}\frac{d\Psi(t)}{dt} \\ &= x^\top(t) \left[\frac{dP(t)}{dt} + A^\top(t)P(t) + P(t)A(t) \right. \\ &\quad \left. - k(t)P(t)B(t)B^\top(t)P(t) \right] x(t) \\ &\quad + 2x^\top(t)P(t)B(t)H(v, t)x(t) \\ &\quad - k(t)x^\top(t)P(t)B(t)E(v, t)B^\top(t)P(t)x(t) \\ &\quad + 2x^\top(t)P(t)B(t)D(t)w(t) \\ &\quad + 2\Psi^\top(t)\Gamma^{-1}\frac{d\Psi(t)}{dt} \\ &\leq x^\top(t) \left[\frac{dP(t)}{dt} + A^\top(t)P(t) + P(t)A(t) \right. \end{aligned}$$

$$\begin{aligned} &\quad \left. - k(t)P(t)B(t)B^\top(t)P(t) \right] x(t) \\ &\quad + 2\|B^\top(t)P(t)x(t)\| \|H(v, t)\| \|x(t)\| \\ &\quad - \lambda_{\min} \left[\frac{1}{2} (E(v, t) + E^\top(v, t)) \right] k(t) \\ &\quad \times \|B^\top(t)P(t)x(t)\|^2 \\ &\quad + 2\|B^\top(t)P(t)x(t)\| \|D(t)\| \|w(t)\| \\ &\quad + 2\Psi^\top(t)\Gamma^{-1}\frac{d\Psi(t)}{dt} \\ &\leq x^\top(t) \left[\frac{dP(t)}{dt} + A^\top(t)P(t) + P(t)A(t) \right. \\ &\quad \left. - k(t)P(t)B(t)B^\top(t)P(t) \right] x(t) \\ &\quad + 2\rho(t) \|B^\top(t)P(t)x(t)\| \|x(t)\| \\ &\quad - \mu(t)k(t) \|B^\top(t)P(t)x(t)\|^2 \\ &\quad + 2\theta_w(t) \|B^\top(t)P(t)x(t)\| \|D(t)\| \\ &\quad + 2\Psi^\top(t)\Gamma^{-1}\frac{d\Psi(t)}{dt} \\ &\leq x^\top(t) \left[\frac{dP(t)}{dt} + A^\top(t)P(t) + P(t)A(t) \right. \\ &\quad \left. - k(t)P(t)B(t)B^\top(t)P(t) \right] x(t) \\ &\quad + 2\rho^* \|B^\top(t)P(t)x(t)\| \|x(t)\| \\ &\quad - \mu^*k(t) \|B^\top(t)P(t)x(t)\|^2 \\ &\quad + 2\gamma_1^{-1}\psi(t)\frac{d\psi(t)}{dt} + 2\gamma_2^{-1}\varphi(t)\frac{d\varphi(t)}{dt} \\ &\quad + 2\gamma_3^{-1}\phi(t)\frac{d\phi(t)}{dt} \dots\dots\dots (12) \end{aligned}$$

From (10c), we can obtain

$$\begin{aligned} &-\mu^*k(t) \|B^\top(t)P(t)x(t)\|^2 + 2\gamma_3^{-1}\phi(t)\frac{d\phi(t)}{dt} \\ &= -\hat{\mu}(t)k(t) \|B^\top(t)P(t)x(t)\|^2 \\ &\quad - 2\sigma_3\phi^2(t) - 2\sigma_3\phi(t)\mu^* \\ &= -\hat{\mu}(t)k(t)x^\top(t)P(t)B(t)B^\top(t)P(t)x(t) \\ &\quad - 2\sigma_3\phi^2(t) - 2\sigma_3\phi(t)\mu^* \dots\dots\dots (13) \end{aligned}$$

For convenience, we define

$$\tilde{k}(t) := \delta_1^2\hat{\rho}^2(t) + \delta_2^2\|D(t)\|^2\hat{\theta}^2(t)$$

Then, substitution of (13) into (12) and using (6b) yields

$$\begin{aligned}
\frac{dV(x, \Psi)}{dt} &\leq x^\top(t) \left[\frac{dP(t)}{dt} + A^\top(t)P(t) + P(t)A(t) \right. \\
&\quad \left. - P(t)B(t)B^\top(t)P(t) \right] x(t) \\
&\quad - \bar{k}(t) \|B^\top(t)P(t)x(t)\|^2 \\
&\quad + 2\rho^* \|B^\top(t)P(t)x(t)\| \|x(t)\| \\
&\quad + 2\theta^* \|B^\top(t)P(t)x(t)\| \|D(t)\| \\
&\quad + 2\gamma_1^{-1}\psi(t) \frac{d\psi(t)}{dt} + 2\gamma_2^{-1}\varphi(t) \frac{d\varphi(t)}{dt} \\
&\quad - 2\sigma_3\phi^2(t) - 2\sigma_3\phi(t)\mu^* \\
&= -x^\top(t)Qx(t) - \sigma_3\phi^2(t) \\
&\quad - \left[\delta_1^2\hat{\rho}^2(t) + \delta_2^2\|D(t)\|^2\hat{\theta}^2(t) \right] \|B^\top(t)P(t)x(t)\|^2 \\
&\quad + 2\rho^* \|B^\top(t)P(t)x(t)\| \|x(t)\| \\
&\quad + 2\theta^* \|B^\top(t)P(t)x(t)\| \|D(t)\| \\
&\quad + 2\gamma_1^{-1}\psi(t) \frac{d\psi(t)}{dt} + 2\gamma_2^{-1}\varphi(t) \frac{d\varphi(t)}{dt} \\
&\quad - \sigma_3\phi^2(t) - 2\sigma_3\phi(t)\mu^* \dots\dots\dots (14)
\end{aligned}$$

Moreover, from (10a) and (10b) we can obtain

$$\begin{aligned}
&-\delta_1^2\hat{\rho}^2(t) \|B^\top(t)P(t)x(t)\|^2 + 2\gamma_1^{-1}\psi(t) \frac{d\psi(t)}{dt} \\
&\quad + 2\rho^* \|B^\top(t)P(t)x(t)\| \|x(t)\| \\
&= -\delta_1^2\hat{\rho}^2(t) \|B^\top(t)P(t)x(t)\|^2 \\
&\quad + 2\hat{\rho}(t) \|B^\top(t)P(t)x(t)\| \|x(t)\| \\
&\quad - 2\sigma_1\psi^2(t) - 2\sigma_1\psi(t)\rho^* \\
&= - \left[\delta_1\hat{\rho}(t) \|B^\top(t)P(t)x(t)\| - \frac{1}{\delta_1}\|x(t)\| \right]^2 \\
&\quad + \frac{1}{\delta_1^2}\|x(t)\|^2 - 2\sigma_1\psi^2(t) - 2\sigma_1\psi(t)\rho^* \\
&\leq \frac{1}{\delta_1^2}\|x(t)\|^2 - 2\sigma_1\psi^2(t) - 2\sigma_1\psi(t)\rho^* \dots\dots (15)
\end{aligned}$$

and

$$\begin{aligned}
&-\delta_2^2\|D(t)\|^2\hat{\theta}^2(t) \|B^\top(t)P(t)x(t)\|^2 \\
&\quad + 2\theta^* \|B^\top(t)P(t)x(t)\| \|D(t)\| \\
&\quad + 2\gamma_2^{-1}\varphi(t) \frac{d\varphi(t)}{dt} \\
&= -\delta_2^2\|D(t)\|^2\hat{\theta}^2(t) \|B^\top(t)P(t)x(t)\|^2 \\
&\quad + 2\hat{\theta}(t) \|B^\top(t)P(t)x(t)\| \|D(t)\| \\
&\quad - 2\sigma_2\varphi^2(t) - 2\sigma_2\varphi(t)\theta^*
\end{aligned}$$

$$\begin{aligned}
&= - \left[\delta_2\|D(t)\|\hat{\theta}(t) \|B^\top(t)P(t)x(t)\| - \frac{1}{\delta_2} \right]^2 \\
&\quad + \frac{1}{\delta_2^2} - 2\sigma_2\varphi^2(t) - 2\sigma_2\varphi(t)\theta^* \\
&\leq \frac{1}{\delta_2^2} - 2\sigma_2\varphi^2(t) - 2\sigma_2\varphi(t)\theta^* \dots\dots\dots (16)
\end{aligned}$$

Therefore, applying (15) and (16) to (14) yields

$$\begin{aligned}
\frac{dV(x, \Psi)}{dt} &\leq -\eta\|x(t)\|^2 - \Psi^\top(t)\Sigma\Psi(t) \\
&\quad - \sigma_1\psi^2(t) + 2\sigma_1|\psi(t)||\rho^*| \\
&\quad - \sigma_2\varphi^2(t) + 2\sigma_2|\varphi(t)||\theta^*| \\
&\quad - \sigma_3\phi^2(t) + 2\sigma_3|\phi(t)||\mu^*| + 1/\delta_2^2 \\
&\leq -\eta\|x(t)\|^2 - \Psi^\top(t)\Sigma\Psi(t) + \varepsilon \dots\dots (17)
\end{aligned}$$

where

$$\eta := \lambda_{\min}(Q) - \frac{1}{\delta_2^2} > 0 \dots\dots\dots (18a)$$

$$\varepsilon := \sigma_1|\rho^*|^2 + \sigma_2|\theta^*|^2 + \sigma_3|\mu^*|^2 + 1/\delta_2^2 \dots\dots (18b)$$

$$\Sigma := \text{diag}\{\sigma_1, \sigma_2, \sigma_3\} \dots\dots\dots (18c)$$

On the other hand, since $P(t)$ is symmetric positive definite, it is obtained from the Rayleigh principle⁽⁴⁾ that for any $t \in R^+$,

$$\begin{aligned}
\lambda_{\min}(P(t))\|x(t)\|^2 &\leq x^\top(t)P(t)x(t) \\
&\leq \lambda_{\max}(P(t))\|x(t)\|^2 \dots\dots\dots (19)
\end{aligned}$$

Then, in terms of (5) we can know that there exists a constant $\alpha_{\max}$ such that for any $t \in R^+$,

$$\lambda_{\max}(P(t)) \leq \alpha_{\max} \dots\dots\dots (20)$$

Thus, from (20) and (17) we obtain

$$\frac{dV(x, \Psi)}{dt} \leq -\tilde{\mu} V(x, \Psi) + \varepsilon \dots\dots\dots (21)$$

where

$$\tilde{\mu} := \min \left\{ \alpha_{\max}^{-1}\eta, \lambda_{\min}^{-1}(\Gamma)\lambda_{\min}(\Sigma) \right\}$$

From (21), it is obvious that $V(x, \Psi)$ decreases monotonically along any solution of closed-loop system (9) and error system (10) until the solution reaches the compact set

$$\Omega_f := \left\{ (x, \Psi) : V(x, \Psi) \leq V_f \right\} \dots\dots\dots (22)$$

where

$$V_f := \tilde{\mu}^{-1}\varepsilon \dots\dots\dots (23)$$

Therefore, it can be concluded that the solution $(x, \psi, \varphi, \phi)(t; t_0, x(t_0), \psi(t_0), \varphi(t_0), \phi(t_0))$ of closed-loop system (9) and error system (10) are uniformly ultimately bounded with respect to the bound V_f given by (23). ■

Remark 3.3. It is worth pointing out that the parameters σ_i , $i = 1, 2, 3$, and δ_i , $i = 1, 2$, will be selected by the system designer. Therefore, by properly choosing these parameters, we can guarantee to obtain a better stability results for the adaptive systems. In fact, it can be known from (18b) that a smaller ε can be guaranteed by choosing the parameters σ_i , $i = 1, 2, 3$, which are small enough, and by making δ_2 large enough. However, we should note that making σ_i small will allow adaptation gain to become high and making δ_2 large will lead to a high gain controller (see (6)), resulting in a tradeoff between the better stability results and large gains.

4. An Illustrative Example

Consider an uncertain time-varying linear dynamical system described by

$$A(t) = \begin{bmatrix} -1 + \frac{0.5}{1+t} & 1 \\ 0 & 0.5 + \frac{0.5}{1+t} \end{bmatrix} \quad \dots (24a)$$

$$B(t) = [0 \quad 1]^\top, \quad C(t) = [0 \quad 1]^\top \quad \dots (24b)$$

$$w(t) = 0.1 \sin(t) \quad \dots (24c)$$

$$\Delta A(\cdot) = \begin{bmatrix} 0 & 0 \\ 0.5 \cos(3t) & 0.5 \cos(3t) \end{bmatrix} \quad \dots (24d)$$

$$\Delta B(\cdot) = [0 \quad 1 - 2.5 \sin(t)]^\top \quad \dots (24e)$$

It can be seen that for this system the matching conditions are satisfied with

$$H(\cdot) = [0.5 \cos(3t) \quad 0.5 \cos(3t)]^\top \quad \dots (25a)$$

$$E(\cdot) = 1 - 2.5 \sin(t) \quad \dots (25b)$$

$$D(\cdot) = 1 \quad \dots (25c)$$

Letting $Q = I$. Since the pair $\{A(\cdot), B(\cdot)\}$ given in (24a) and (24b) is uniformly completely controllable, from (4) we can obtain a solution of $P(t)$ which is symmetric positive definite.

Remark 4.1. It is obvious from (25b) that for any $t \in R^+$,

$$\mu(t) = 1 - 2.5 \sin(t)$$

That is, we cannot guarantee that for any $t \in R^+$,

$$\mu(t) > -1$$

For instance, at $t = \frac{\pi}{2}$, $\mu(t) = -1.5 < -1$. Therefore, it is concluded that by making use of conventional (adaptive) robust state feedback controllers, one cannot stabilize this system (see Remark 3.1). However, it can be seen below that by employing the adaptive robust control scheme proposed in this paper, we can stabilize this system.

Moreover, for the controller given in (6) we select the parameters δ_1 and δ_2 as

$$\delta_1 = \delta_2 = 3$$

and for the adaptive laws given in (7) we choose the following parameters:

$$\Sigma = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 2.0 & 0 & 0 \\ 0 & 2.0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

For simulation, the initial conditions of the system and the adaptive laws are given by

$$x(t_0) = [5.5 \quad 5.5]^\top$$

and

$$\hat{\rho}(t_0) = 8.0, \quad \hat{\theta}(t_0) = 8.0, \quad \hat{\mu}(t_0) = 50.0$$

respectively.

With the chosen parameter settings, the results of a simulation are shown in Figs.1-4.

Fig.1. Response of state variable $x(t)$ for the closed-loop system.

Fig.2. History of the updating parameter $\hat{\rho}(t)$.

Fig.3. History of the updating parameter $\hat{\theta}(t)$.

Fig.4. History of the updating parameter $\hat{\mu}(t)$.

It can be observed from Fig.1 that the closed-loop dynamical system resulting from the design in this paper is indeed stable in the sense of uniform ultimate boundedness. It can be seen from Fig.4 that under the chosen parameters σ_3 , γ_3 , and initial condition $\hat{\mu}(t_0)$, we have guaranteed that for any $t \in R^+$, $\hat{\mu}(t) > -1$.

5. Conclusion

The problem of robust stabilization of a class of time-varying dynamical systems with disturbance and uncertain parameters has been considered. The bounds of the disturbance and uncertain parameters are assumed to be unknown. For such uncertain dynamical systems, a class of continuous adaptive robust state feedback controllers have been proposed which can guarantee the uniform ultimate boundedness of the resulting closed-loop dynamical systems.

Finally, a numerical example is given to demonstrate the synthesis procedure for adaptive stabilizing linear controller proposed in this paper. In particular, by making use of the conventional robust state feedback controllers reported in the control literature, one cannot stabilize the system given in this example. However, we have shown that by employing the adaptive robust control scheme proposed in this paper, it is possible to stabilize this system. It is shown from the example and the results of its simulation that the results obtained in this paper are effective and feasible. Therefore, our results may be expected to have some applications to some practical control problems of uncertain systems with disturbance and uncertain parameters.

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