Modeling and Analysis of a Half-Bridge IGBT Inverter for High-Frequency Induction Heating Applications

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This paper presents the description and analysis of a half-bridge IGBT inverter suitable for heating ferromagnetic materials at high-frequency. The series-parallel scheme is adopted and, an optimum mode of operation is selected which results in a maximum current gain and practically no voltage spikes in the devices at turn-off. Using the state variable approach, the inverter system circuit model was analyzed including the effect of branch stray inductances and snubber components.

The actual performance was tested on a 50-150 kHz prototype rated at 6 kW. The developed hybrid inverter is characterized by its simplicity of design and operation, yet is versatile in performance. Experimental evidence is presented to confirm the validity of the inverter system model.

Also, an analytical model for the long billet-short coil induction heating geometry is briefly described. The resulting expressions for the induction heating load parameters are obtained in the form of rapidly converging series. The theoretical results obtained have been experimentally verified.

Key words: Half-bridge inverter, insulated gate bipolar transistors (IGBT’s), induction heating load, compensating capacitors, phase-locked loop (PLL)

1. Introduction

Recently, there has been growing interest in the development of inverters capable of supplying high-power at frequencies ranging from 10 to 200 kHz for the surface hardening of tubes, bars and many other industrial applications of induction heating. A variety of different operating principles and inverter circuit configurations exist, each of which have their own particular merits.

This paper describes a 50-150 kHz half-bridge inverter for induction heating applications employing insulated gate bipolar transistors (IGBT’s). The actual performance of the system was tested on a prototype whose power rating (6 kW) is within the range of the actual requirements of industrial applications and allows significant scaling for larger implementations.

Power electronic circuits can be simulated by adapting general purpose circuit analysis software packages such as PSPICE, ECAP and so on. However, the major disadvantage of such a procedure is that many of the device models for power electronic circuit elements may not be readily available. The paper presents a rather general and practical method to predict inverter systems performances. Theoretical results are compared with the measured ones in order to illustrate the validity of the method.

2. Circuit description and the operating principle

A circuit diagram of the basic system, as shown in Fig.1, comprises essentially a three-phase full-bridge diode rectifier, a single-phase half-bridge inverter, an induction heating load and a phase-locked loop (PLL) control circuit.

![Fig.1. System configuration.](image)

The induction heating load constitutes a 0.42% carbon...
steel billet placed inside a 7-turn water-cooled copper coil at a specific air gap. The magnetic field, induced in the coil when energized by the high-frequency inverter, causes eddy currents to occur in the billet, and these give rise to the heating effect. A high-power density is then applied at the surface of the billet, and the heat is concentrated quickly in the localized region.

The insulated gate bipolar transistor (IGBT) is nowadays gaining popularity for its relatively high speed and low gate power requirements. The used IGBT's are TOSHIBA type MG25N2YS1 devices. Two IGBT's with internal antiparallel diodes are mounted in the same module. The switches are actually made up of eight devices in parallel to satisfy power requirements and to increase the switching speed, since the transistors are typically two or three times faster when operated below 60% of their rated currents.

Snubber R-C circuits are connected in parallel with the IGBT's to avoid any excessive voltage spike during device turn-off.

The work-piece, along with the heating coil, presents a highly inductive load to the power source. In order to minimize the reactive loading, series and parallel compensating capacitors \((C_s\) and \(C_p\)) are used in the output circuit. The series-parallel scheme has the desirable characteristics of the series and parallel ones. The load short circuit and the no-load regulation are possible.

In operation, each IGBT conducts for the period corresponding to half the total cycle time. The phase angle between the output current and voltage of the inverter depends on the operating frequency which is the switching rate of the IGBT's. The frequency is controlled in a phase-locked loop (PLL) circuit in sympathy with changing load characteristics.

3. **Optimum mode of inverter operation**

Fig.2 shows typical experimental frequency characteristics of the inverter system illustrating the optimum operating conditions. From top to bottom, the former shows the variations of the inverter output current \(I_1\), heating coil current \(I_2\), current gain \(I_2/I_1\), phase angle \(\theta\) of the inverter output current with respect to the gate signal and IGBT collector to emitter voltage \(V_{CE}\) as a function of the inverter operating frequency. As seen from the figure, the angle \(\theta\) is zero at points A, B and C. Operation at B is desirable as the current gain is at its peak value and the voltage spike, in the switching device at turn-off, practically suppressed. The phase-locked loop (PLL) control circuit was designed to track only point B, in the 50 to 150 kHz range, irrespective of load variations.

The maximum current gain is attained at

\[
f_m = \frac{1}{2\pi} \sqrt{\frac{1}{L_2 C_p} \cdot \frac{R_i^2}{2 L_4^2}} \quad (1)
\]

and its corresponding expression is

\[
\left[ \frac{I_2}{I_1} \right]_{\text{max}} = \left[ \frac{C_p R_i^2}{L_4} \left( 1 - \frac{C_p R_i^2}{4 L_4} \right) \right]^{1/2} \quad (2)
\]

where \(I_1\) and \(I_2\) are respectively the RMS values of \(i_1\) and \(i_2\). \(R_i\) and \(L_i\) are the equivalent resistance and inductance of the induction heating load, respectively. Eq. (2) shows that the maximum current gain depends upon the induction heating load parameters and the parallel compensating capacitor \(C_p\). For our practical cases,

\[
\frac{C_p R_i^2}{L_4} \ll 1 \quad (3)
\]

\[
\begin{align*}
\text{Inverter operating frequency [kHz]} & \quad \text{Fig.2. Typical experimental inverter frequency characteristics} \\
& \quad (E=40 \text{ V, } C_p=1.8 \text{ } \mu\text{F and } C_s=0.6 \text{ } \mu\text{F}).
\end{align*}
\]
Then, Eq. (2) reduces to
\[
\left[ \frac{I_2}{I_1} \right]_{\text{max}} = \frac{1}{R_L} \sqrt{ \frac{L_2}{C_p} } \tag{4}
\]

As stated by Eq. (1), the maximum current gain frequency \(f_m\) is independent of the choice of the series compensating capacitor \(C_p\). However, the frequency at point \(B, f_B\) changes as \(C_s\) changes \(5\). Therefore, by a proper adjustment of \(C_s\), \(f_B\) can be set to \(f_m\).

4. Inverter circuit analysis

In this section, a generalized analysis using the state variable approach is briefly presented. Fig.3 shows the equivalent circuit model of the inverter system where the induction heating load is modeled by \(G_2 (= 1/R_L)\) and \(L_2 (= L_2)\). These parameters depend on several variables including the shape of the heating coil, the spacing between the workpiece and coil, their electrical conductivities and magnetic permeabilities.

For the sake of analysis, the eight identical paralleled IGBT modules have been treated as one. A practical equivalent circuit, for modeling the operations of IGBT1 and IGBT2, is shown in the figure where \(C_7\) and \(C_8\) are respectively their collector to emitter capacitances. Conductances \((G_5, G_6)\) and inductances \((L_4, L_5)\) belong to their bulks and cannot be neglected at switching. The inductances \(L_4\) and \(L_5\) can influence the magnitude of the collector to emitter voltage during rapid switching of large currents.

\[
i_R = f(v_R, v_{GE}) = \begin{cases} 
T(v_R, v_{GE}) & v_R > 1.2 \text{ and } v_{GE} > v_T \\
0 & -0.7 < v_R \leq 1.2 \\
D(v_R) & v_R > 1.2 \text{ and } v_{GE} \leq v_T \\
\end{cases}
\tag{5}
\]

where \(T(v_R, v_{GE})\) and \(D(v_R)\) are respectively the functions describing the characteristics of the IGBT and the antiparallel diode,

\[
T(v_R, v_{GE}) = \beta \left( \frac{v_{GE} - v_T}{v_{GE} - v_T} \right)^2 \tag{6}
\]

\[
D(v_R) = a v_R^2 + b v_R + c
\]

\(i_R\) and \(v_R\) are the nonlinear resistor current and voltage, respectively; \(v_{GE}\) is the gate-emitter voltage; \(v_T\) is the threshold voltage; \(\alpha, \beta, \gamma, \text{ and } K\) are other device constants to be determined for any given transistor; \(a, b, \text{ and } c\) are the coefficients of the quadratic equation approximating the diode characteristic.

The constants values are given in Table 1. These were obtained using the data sheet provided by the manufacturer.

| \(v_T = 4.0\), \(\alpha = 0.15\), \(\beta = 28.0\), \(\gamma = 1.39\), \(K = 2.52\) |
| \(-a = -181.4\), \(-b = -199.2\), \(-c = -49.7\) |

Stray or leakage inductances, related to the circuit layout, inevitably appear in the various parts of practical inverter circuits. Although minimized, their effect at high-frequency is significant on the selection and rating of the inverter components. The presence of a certain amount of these inductances is actually required in order to limit the rate of rise of current in the IGBT’s to an acceptable level and reduce the \(dv/dt\) stresses. In our case, the branch inductances, i.e., Lumped stray and leads inductances, are profitably used as part of the resonating inductances.
Stray inductance in a circuit that experiences a large $\frac{di}{dt}$ should be kept as low as possible to minimize transient voltages. An effective measure to reduce the stray inductance is to connect the semi-conductor modules using multilayer sheets or copper strips with a thin insulator sandwiched between them. The modules must be mounted as close to each other as allowed by the need to ensure adequate heat sink.

In terms of the desired state variables $v_C$ and $i_L$, the state equations written in matrix form are given by

$$\begin{bmatrix}
C & 0 \\
0 & L
\end{bmatrix}
\begin{bmatrix}
\frac{d}{dt}v_C \\
\frac{d}{dt}i_L
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
v_C \\
i_L
\end{bmatrix}
+ \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}f(v_C, i_L, v_{GE}) \tag{7}
$$

where

$$v_C = \begin{bmatrix} v_{C1}, v_{C2}, ..., v_{C8} \end{bmatrix}^T \tag{8}$$

$$i_L = \begin{bmatrix} i_{L1}, i_{L2}, ..., i_{L8} \end{bmatrix}^T \tag{9}$$

$C$ is the tree-branch capacitance matrix and $L$ the loop inductance matrix for the normal tree.

$$C = \text{diag}[C_1, C_2, ..., C_8] \tag{10}$$

$$L = \begin{bmatrix}
L_1 + \Gamma_1 + \Gamma_2 & -\Gamma_1 & -\Gamma_1 & 0 & 0 \\
-\Gamma_1 & L_2 + \Gamma_1 & \Gamma_1 & 0 & 0 \\
-\Gamma_1 & \Gamma_1 & L_3 + \Gamma_1 & \Gamma_2 & 0 & 0 \\
0 & 0 & 0 & L_4 & 0 \\
0 & 0 & 0 & 0 & L_5
\end{bmatrix} \tag{11}$$

$$A_{11} = 0, \quad A_{12} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{12}$$

$$A_{21} = A_{12}^T \tag{13}$$

$$A_{22} = \begin{bmatrix}
-G_1^{-1} \cdot G_3^{-1} & 0 & G_1^{-1} \\
0 & -G_2^{-1} & 0 \\
G_1^{-1} & 0 & -G_1^{-1} \cdot G_4^{-1} \otimes G_3^{-1} & 0 \\
0 & 0 & G_4^{-1}
\end{bmatrix} \tag{14}$$

$$B_1 = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix} \tag{15}$$

$$B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & G_5^{-1} & 0 & 0 \\
0 & 0 & 0 & G_6^{-1} & 0
\end{bmatrix} \tag{16}$$

$f(v_C, i_L, v_{GE})$ is the expression governing the link resistive branches,

$$f(v_C, i_L, v_{GE}) = [i_{R1}, i_{R2}, ..., i_{R5}]^T \tag{17}$$

and the vector $v_{GE}$ controls the states of the devices,

$$v_{GE} = [v_{GE1}, v_{GE2}]^T \tag{18}$$

where $v_{GE1}$ and $v_{GE2}$ are the gate signals applied to IGBT$_1$ and IGBT$_2$, respectively.

It is a simple matter to show that

$$\begin{bmatrix}
[i_{R1}] \\
i_{R2} \\
i_{R3}
\end{bmatrix} = \begin{bmatrix}
-R_1^{-1} & -R_1^{-1} & -R_1^{-1} & 0 \\
0 & -R_2^{-1} & 0 \\
0 & 0 & -R_3^{-1}
\end{bmatrix} \begin{bmatrix}
v_{C1} \\
v_{C2} \\
v_{C3}
\end{bmatrix} + \begin{bmatrix}
E \\
0 \\
0
\end{bmatrix} \tag{19}$$

and

$$i_{R4} = -G_5 v_{R4} + G_5 v_{C7} + i_{L4} \tag{20}$$

$$i_{R5} = -G_6 v_{R5} + G_6 v_{C8} + i_{L5} \tag{21}$$

The two nonlinear resistors $R_4$ and $R_5$ are both voltage-controlled and specified by Eqs (5) and (6). Hence,

$$i_{R4} = f(v_{R4}, v_{GE1}) \tag{22}$$

$$i_{R5} = f(v_{R5}, v_{GE2}) \tag{23}$$

Substituting Eq. (20) in Eq. (19) yields
Eqs (21) and (22) are nonlinear equations of the form
\[ g(x) = 0 \]  
(23)
To solve for \( v_R \) and \( v_{RS} \), the following Newton-Raphson iteration formula was used
\[ x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \]  
(24)
where \( g' \) is the derivative of \( g \). As \( f(v_R, v_{GE}) \) is a single-valued and well-behaved function, rapid convergence is achieved.

5. Derivation of the induction heating load parameters

The \( N \)-turn induction heating coil surrounding the cylindrical work-piece, as depicted in Fig.1, can be approximated by the model shown in Fig.4(a). The work-piece, of radius \( r_w \) and arbitrary length, is replaced by an infinitely long cylinder in the model. It is assumed that the magnetic permeability \( \mu \) and electrical conductivity \( \sigma \) are uniform throughout the cylinder.

The actual coil geometry will not be considered. Rather, the coil will be approximated by a short current sheet of radius \( r_c \) and axial length \( L_c \) with its axis in the z-direction and center at the origin \( o \).

The exact coil current distribution is not generally known prior to numerical calculation. Consequently, an assumed distribution must be used. One approximation(3) would be that of a uniform distribution having a magnitude of \( (Nl/l_c) \). The function \( J(z) \) then has the form shown in Fig.4(b).

\[ J(z) = \sum_{n=1}^{\infty} J_n \cos \lambda_n z \quad n = 1, 3, 5, \ldots \]  
(25)
where
\[ J_n = \frac{4NI}{n\pi l_c} \sin \frac{\lambda_n l_c}{2} \]  
(28)
It follows from the Appendix that the \( \phi \) component of the magnetic vector potential at \( r = r_c \) is given by the following infinite series solution:
\[ A_\phi(r_c, z) = \sum_{n=1}^{\infty} \left[ B_{1n} l_1(\lambda_n r_c) + B_{2n} K_1(\lambda_n r_c) \right] \cos \lambda_n z \]  
(29)
wherein \( n \) is odd. \( l_1 \) and \( K_1 \) are the modified Bessel functions of order one. \( B_{1n} \) and \( B_{2n} \) are constants obtained from the magnetic boundary conditions.

Knowing the vector potential distribution at \( r = r_c \) it becomes a relatively simple matter to determine the impedance of the loaded coil. The coil voltage distribution can be expressed as:
\[ v(z) = j\omega \frac{2\pi r_c N}{l_c} A_\phi(r_c, z) \]  
(30)
The total coil voltage is obtained by integrating \( v(z) \) between the limits \( \pm L_c/2 \):
\[ V = \int_{-L_c/2}^{L_c/2} v(z) dz \]  
(31)
The loaded coil complex impedance is obtained by dividing \( V \) by the current \( I \). Thus,
\[ R_L + j\omega L_L = j\omega I^2 \left[ \frac{16 \mu_0 \pi r_c^2 N^2}{T l_c^2} \right. \]
\[ \times \sum_{n=1}^{\infty} R_n K_1^2(\lambda_n r_c) \left( \frac{\sin \left( \frac{\lambda_n L_c}{2} \right)}{\lambda_n} \right)^2 \]  
(32)
\( L_c \), the inductance of the empty coil, is given by

\[
L_c = \frac{16 \mu_0 \pi^2 N^2}{T^2} \sum_{n=1}^{\infty} i_1(\lambda_n r_e) \times K_1(\lambda_n r_c) \left( \frac{\sin \left( \frac{L_c}{\lambda_n} \right)}{\lambda_n} \right)^2
\]

and

\[
R_n = \frac{B_{2n}}{B_{2n}} = \frac{I_1(\gamma_n r_w)}{r_w \Phi_n K_1(\lambda_n r_w)} \frac{I_1(\lambda_n r_w)}{K_1(\lambda_n r_w)} \tag{34}
\]

where

\[
\Phi_n = \left( \frac{\gamma_n I_0(\gamma_n r_w) - K_0(\lambda_n r_w)}{\lambda_n I_1(\gamma_n r_w) K_1(\lambda_n r_w)} \right) \tag{35}
\]

and \( \gamma_n \) is a parameter defined as:

\[
\gamma_n = \sqrt{\lambda_n^2 + j \mu \omega \sigma} \tag{36}
\]

The second term of Eq. (32) represents the reflected impedance of the work-piece. Although the loaded coil impedance is derived as a function of \( T \), this parameter does not affect the value to which the series converges. The accuracy offered by the present method of analysis is best illustrated by comparing predicted and measured values of \( R_L \) and \( L_L \). Several billet configurations will be considered. Although the derivations are limited to cylindrical loads, the method can also handle complicated geometries. For example, it is a simple matter to include multi-section coils. In contrast to the classical method\(^{(7)}\), no empirical correction factors were used.

### 6. Results and discussion

A series of practical tests were conducted on a 6 kW, 50-150 kHz prototype inverter to confirm the validity of the proposed inverter system model. The heating coil is a multi-turn water-cooled one made from round copper tubing, and the work-pieces are cylindrical billets. The heating coil and work-pieces data are given in Table 2.

The billets were purposely chosen longer than the heating coil to simulate billet heaters employing true continuous feed and many other applications of industrial interest. The sample billets were cooled, for most of the experimental tests, by passing water through them to keep the magnetic permeability and electrical resistivity of the ferromagnetic material constant.

<table>
<thead>
<tr>
<th>Turn number: 7</th>
<th>Axial length: 200 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial length: 100 mm</td>
<td>Diameters: 20, 25, 30, 35 and 40 mm</td>
</tr>
<tr>
<td>Inner diameter: 51 mm</td>
<td>Material: 0.42% carbon steel</td>
</tr>
<tr>
<td>Tube external diameter: 10 mm</td>
<td></td>
</tr>
<tr>
<td>Tube internal diameter: 9.1 mm</td>
<td></td>
</tr>
<tr>
<td>Spacing between turn: 5 mm</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5 shows the induction heating load parameters \( R_L \) and \( L_L \) as a function of frequency with a work-piece diameter equal to 25 mm. Fig. 6 illustrates the effect of changing the billet size on the \( R_L \) and \( L_L \) parameters at 100 kHz.

A series RLC type circuit was used to experimentally measure \( R_L \) and \( L_L \). A dc voltage was applied for a period of time and then removed, for the circuit to perform free damped oscillations. From the transient response of the circuit current (underdamped case), the damped natural frequency and the amplitudes of two successive cycles were measured from which \( R_L \) and \( L_L \) were evaluated.

Figs 5 and 6 also compare the simulated values of \( R_L \) and \( L_L \) with the experimental ones. The results were found to be satisfactory and gave good support to the presented theory. The main discrepancies in results were attributed to the
assumption of neglecting the work-piece end effects in the analysis, i.e., replacing the finite length billet by an infinitely long cylinder in the model.

The components values for the circuit of Fig.3 are summarized in Table 3. Some of these values were obtained experimentally. The induction heating load parameters $G_2$ and $L_2$ were obtained directly from Eq. (32) as they varied with frequency and billet size.

Table 3. Components values of the inverter system circuit model.

| $E = 40$ V | $R_1 = 0.58$ Ω, $R_2 = R_3 = 500$ Ω |
| $L_1 = 1.33$ μH, $L_2 = 0.94$ μH, $L_3 = L_4 = 0.54$ μH |
| $C_1 = C_2 = 120$ μF, $C_3 = 1.8$ μF, $C_4 = 0.6$ μF |
| $C_5 = C_6 = C_7 = G_1 = 3$ nF |
| $G_1 = 0.18$ μH, $G_2 = 1.53$ μH |
| $G_3 = 4.0$ Ω$^{-1}$, $G_4 = G_5 = 0.04$ Ω$^{-1}$ |
| $G_6 = 0.17$ Ω$^{-1}$ |

From top to bottom, Figs 7(a) and 7(b) show respectively at 50 and 102 kHz, using the 30 mm diameter sample billet, the experimental and simulated waveforms of the instantaneous inverter output current $i_1$, heating coil current $i_2$, IGBT1 collector to emitter voltage $v_{CE1}$ and inverter output voltage $v_o$. The waveforms predict the transient behavior of the system and show the transition to steady state. At 102 kHz (around B), the voltage spike occurring in the transistors at turn-off is practically suppressed.

Fig.7. Actual and predicted waveforms of the instantaneous inverter output current $i_1$, heating coil current $i_2$, IGBT1 collector to emitter voltage $v_{CE1}$ and inverter output voltage $v_o$ at a) 50 kHz and b) 102 kHz (Optimum operating frequency).

(b) At 102 kHz (Optimum operating frequency).

Fig.8. Predicted and measured RMS values of the inverter output and heating coil currents.

output voltage $v_o$. The waveforms predict the transient behavior of the system and show the transition to steady state. At 102 kHz (around B), the voltage spike occurring in the transistors at turn-off is practically suppressed.
Fig. 8 compares the analytical RMS values of the inverter output current and those of the heating coil current with the respective observed ones as the inverter operating frequency changes from 60 to about 135 kHz.

The results obtained from the theory developed are reasonably in good agreement with the practical ones, confirming the validity of the inverter system model.

7. Conclusions

A 50-150 kHz laboratory scale (6 kW) half-bridge inverter for induction heating applications, employing insulated gate bipolar transistors as the switching devices, was built. The behavior of the prototype was observed under load conditions in the mentioned frequency range. Optimum inverter operations could be achieved by proper adjustments of the compensating capacitors $C_s$ and $C_p$. The PLL control circuit was designed and constructed to track only the frequency at the desirable operating point; namely $B$, irrespective of load variations.

The experimental results gave good support to the theory developed for modeling the inverter system. The approach, presented in this paper, will be of importance since the major problem in the analysis of power electronic circuits is the topological changes due to the switch operation.

Closed-form expressions for the induction heating load parameters have been derived. The expected results are in good agreement with the experimental ones. Besides avoiding empirical correction factors, the developed method can be easily extended to other geometries. The major limitations of the method is that work-piece end effects are neglected.

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Appendix

Azimuthal components of the magnetic vector potential in the induction heating load

The azimuthal components of the magnetic vector potential, in the induction heating coil and billet, have the following forms in the three regions indicated in Fig.4(a).

$$
A_p(r, z) = \begin{cases} 
\sum_{n=1}^{\infty} A_n l_1(\gamma_n r) \cos \lambda_n z, & 0 \leq r \leq r_w \\
\sum_{n=1}^{\infty} [B_n l_1(\gamma_n r) + B_{2n} K_1(\gamma_n r)] \cos \lambda_n z, & r_w \leq r \leq r_c \\
\sum_{n=1}^{\infty} [C_n l_1(\gamma_n r) + C_{2n} K_1(\gamma_n r)] \cos \lambda_n z, & r_c \leq r
\end{cases}
$$

wherein $n$ is odd. $A_n$, $B_{1n}$, $B_{2n}$, $C_{1n}$ and $C_{2n}$ are constants obtained by applying the magnetic boundary conditions.

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