Time-Variant Haptic Simulator Based on Fine and Fast Real-Time Discretization Using FDTD Method

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This paper proposes fine and fast discretization, which is based on the Finite Difference Time Domain (FDTD) method without a matrix exponential function. The proposed method is more accurate than the conventional discretization based on the first-order Pade approximation. Moreover, compared with that of the conventional method, the computational cost of the proposed method is very small. Although the Pade approximation is impossible to realize broadband real-time control, the FDTD method achieves the broadband real-time control. The proposed method and the conventional Pade approximation are analyzed by the Taylor expansion in a symbolic expression. According to the theoretical analysis, the error of the FDTD discretization method is smaller than that of the conventional method. Numerical simulations also compare the errors and computational cost associated with the proposed and conventional methods. The FDTD discretization method is applied in admittance control for rendering the time-variant haptic sensation. In the case of the conventional method, owing to the limitation due to computation cost, the parameter update is not completed within the sampling period. On the other hand, the FDTD method completes the discretization within the sampling period. The gain and phase characteristics of the proposed method are close to the ideal responses. The validity of the method is verified by an experiment involving linear motor systems.

Keywords: real-time discretization, time-variant system, FDTD method, multi-inertia resonance system, admittance control, hapticsimulator

1. Introduction

Many systems in the real world are modeled as time-variant systems, and are described by state differential equations. Robots, traveling cranes, tether satellites, engines, elevators, and airplanes have dynamic systems in which the state transition matrix, matrix A , is composed of time-variant parameters⁽¹⁾⁽²⁾. The parameters of the robot change according to its own posture⁽³⁾. In Ref. (3), although the robot was modeled as a time-invariant system with fixed constant parameters, the time varying factors depending on the posture were robustly controlled by a sliding mode controller. On the other hand, a method of actively controlling against time variations has also been studied⁽⁴⁾⁽⁵⁾. In position control of the three orthogonal axes, the resonance frequency depends on the position of each axis. Vibration suppression has been realized by the variable gain according to the time-varying resonance frequency. However, it is necessary to calculate the time-varying gain in advance and store the gain table in the computer's memory. When large parameter fluctuation occurs, the preservation amount of the time-varying gain becomes remarkably large. Furthermore, the vibration suppression controller for elevators cranes and tether satellites, which have variable rope length, are expressed as time-variant parameters⁽⁶⁾⁽⁷⁾. As with the traveling crane, a method such as gain scheduling has been adopted according to the time-variant

parameter⁽⁸⁾.

In these conventional studies, to implement the controller for a time-variant system, the calculation cost is reduced by using tables calculated in advance. However, the discretization of high-order systems in every sampling period has not been conducted owing to the calculation cost.

In a training simulator operated by a human, it is necessary to accurately construct the physical model and set the sampling time to be short. For example, a real-time flight simulator and driving simulator have been developed to reproduce the reaction force of control surfaces or steering wheels⁽⁹⁾⁽¹⁰⁾. In a rehabilitation robot that uses force feedback, because the virtual physical model in accordance with operation by the human is updated in real time, it is necessary to perform discretization in every sampling period⁽¹¹⁾. In this case, it is necessary to adjust the size of the system so that the calculation time does not exceed the control sampling time.

The calculation cost increases in the implementation owing to the matrix exponential function. When the continuous state equation, including the physical parameters, is discretized, it is equivalently transformed into a matrix exponential function. The matrix exponential function is generally implemented on a computer by using the Pade approximation⁽¹²⁾⁽¹³⁾. However, the relation between calculation precision and calculation cost is a tradeoff in the Pade approximation. Therefore, when the size of matrix A increases, the computation cost increases exponentially. For that reason, this paper applies the FDTD method⁽¹⁴⁾ to discretize a spring mass system with higher order, time varying parameters, and

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proposes the fast real-time discretization with a low calculation cost. Unlike the conventional method, the proposed method does not use a matrix exponential function and a table calculated in advance. In addition, one advantage of the proposed method is that the discrete-time state equation derived by the FDTD method is simply expressed by continuous physical parameters.

2. Discretization of a Continuous System

The continuous time state differential equation of the target system is shown as follows:

$$\dot{x}(t) = A_c x_c(t) + B_c u(t) \quad (1)$$

$$y(t) = C_c x_c(t) \quad (2)$$

where A , B and C denote the system matrices. The subscript c and d represent continuous and discrete. x , u and y denote the state vector, input, and output signals, respectively.

2.1 Time-Invariant System The continuous time system is strictly discretized by the sampler and zero-order hold. The discrete time system is equivalently derived by the matrix exponential function as follows:

$$x[i+1] = A_d x_d[i] + B_d u[i] \quad (3)$$

$$y[i] = C_d x_d[i] \quad (4)$$

$$A_d = e^{A_c T_s} \quad (5)$$

$$B_d = \int_0^{T_s} e^{A_c \tau} d\tau B_c \quad (6)$$

$$C_d = C_c \quad (7)$$

where T_s denote the sampling period. The state, input, and output are constant values in the sampling period, and are represented as $x_c(t) = x[iT_s]$, $u(t) = u[iT_s]$, and $y(t) = y[iT_s]$, where i denotes the discrete time. In a time-invariant system, the discretization is performed only once before the control loop. Therefore, parameter changes of the system after discretization are not considered.

2.2 Time-Variant System The system matrices matrix $A_c(t)$ and matrix $B_c(t)$ of the continuous time differential equation including the time-variant parameter are expressed by the time functions as follows:

$$x[i+1] = A_d[i] x_d[i] + B_d[i] u[i] \quad (8)$$

$$y[i] = C_d x_d[i] + D_d u[i] \quad (9)$$

$$A_d[i] = e^{A_c(i)T_s} \quad (10)$$

$$B_d[i] = \int_0^{T_s} e^{A_c(i)\tau} d\tau B_c \quad (11)$$

When the internal parameters are changed depending on the time, it is necessary to discretize the system according to the parameter change. Thus, Eq. (8) to Eq. (11) should be calculated in every sampling period.

2.3 Example of an n th-Inertia Resonance System

As an example, the state equation of the n th-inertia resonance system with time-variant parameters is shown as follows:

$$\begin{bmatrix} \dot{f}_n \\ \ddot{x}_n \\ \vdots \\ \dot{f}_1 \\ \ddot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & k_n(t) & & 0 \\ -\frac{1}{m_n(t)} & 0 & & 0 \\ & & \ddots & \\ 0 & & 0 & k_1(t) \\ & & -\frac{1}{m_1(t)} & 0 \end{bmatrix} \begin{bmatrix} f_n \\ \dot{x}_n \\ \vdots \\ f_1 \\ \dot{x}_1 \end{bmatrix}$$

Fig. 1. n th-inertia environmental model with time-variant parameters

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{m_1(t)} \end{bmatrix}^T u(t) \quad (12)$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} f_n \\ \dot{x}_n \\ \vdots \\ f_1 \\ \dot{x}_1 \end{bmatrix} \quad (13)$$

where k and m stand for time-variant stiffness and time-variant mass, respectively. The input, output, and each parameter are assumed as constant values within the sampling interval. Figure 1 shows the mechanical configuration and the block diagram of the n th-inertia resonance system. Ideal discretization based on analytic solutions is difficult to implement as the solutions are long and complicated. In the next section, Pade approximations are shown, and the problems are summarized.

3. Conventional Discretization Method

As a conventional method, the l th order Pade approximation is shown.

3.1 Pade Approximation The Pade approximation of the matrix exponential function e^H is expressed as follows:

$$e^H \cong (I + \alpha_1 H + \cdots + \alpha_l H^l)^{-1} (I - \alpha_1 H + \cdots - \alpha_l H^l) \quad (14)$$

where α and l denote the coefficient of the Pade approximation and order l , which are given by

$$\alpha_l = \alpha_{l-1} \frac{l-j+1}{j(2l-j+1)}, \quad (\alpha_0 = 1, j = 1, 2, 3, \dots, l) \quad (15)$$

In this paper, the system is discretized by the first-order Pade approximation. The first-order Pade approximation is derived by A_c as follows:

$$A_{Pade} = \left(I - \frac{T_s}{2} A_c(t) \right)^{-1} \left(I + \frac{T_s}{2} A_c(t) \right), \quad (H = A_c T_s) \quad (16)$$

Although the higher order approximation improves accuracy, matrix A_{pade} becomes very long and complicated.

3.2 Problem of Conventional Method The first-order Pade approximation is widely used for low-order matrix exponential functions. However, in higher order models, the computational cost increases exponentially. Furthermore, the accuracy also deteriorates. When using the Pade

approximation, the implementation is limited owing to the trade-off relation between the discretization time and accuracy. As the order l increases, the accuracy of the matrix exponential function increases. Therefore, it is possible to accurately discretize a control system designed in the continuous time domain. However, the absolute value of the computational cost is drastically increased. When performing a high-precision Pade approximation, it is necessary to set a long control period; therefore, it is not possible to implement a high-bandwidth control system.

4. Proposed Real-time Discretization

4.1 FDTD Method The FDTD method has been used for electromagnetic analysis of metamaterials and simulations of the sound field ⁽¹⁵⁾⁽¹⁶⁾.

The algorithm of the FDTD method is applied to the spring-mass system shown in Fig. 1. The FDTD method is based on the wave equation. Therefore, it is effective for discretization of a multi-inertia spring mass system. The modeling in this paper is based on the following standard “wave equations”:

$$\frac{\partial \dot{x}(t, x)}{\partial t} = -\frac{1}{m_n(t)} \frac{\partial f(t, x)}{\partial x} \dots\dots\dots (17)$$

$$\frac{\partial f(t, x)}{\partial t} = -k_n(t) \frac{\partial \dot{x}(t, x)}{\partial x} \dots\dots\dots (18)$$

which are expressed as “partial derivatives” of the time and space(position) instead of simple Hooke’s law ($f(t) = k_n(t)x(t)$).

In this paper, each subscript and element index of the force are respectively defined as follows:

$$f_{\text{spatial index}[\text{time index}]} \dots\dots\dots (19)$$

As with the force, the subscript and element index of the velocity is also defined. Each differential equation is discretized by the algorithm of the FDTD method as follows:

$$\frac{\dot{x}_n[i+1] - \dot{x}_n[i]}{\Delta t} = -\frac{1}{m'_n[i]} \frac{f_n[i] - f_{n-1}[i]}{\Delta x} \dots\dots\dots (20)$$

$$\frac{f_n[i+1] - f_n[i]}{\Delta t} = -k'_n[i] \frac{\dot{x}_{n+1}[i] - \dot{x}_n[i]}{\Delta x} \dots\dots\dots (21)$$

The relationship between each element parameters and Δx is as follows:

$$m_n[i] = m'_n[i] \Delta x \dots\dots\dots (22)$$

$$k_n[i] = \frac{k'_n[i]}{\Delta x} \dots\dots\dots (23)$$

Eq. (20) and Eq. (21) are rewritten as follows:

$$\dot{x}_n[i+1] = \dot{x}_n[i] + \frac{T_s}{m_n[i]} \{f_{n-1}[i] - f_n[i]\} \dots\dots\dots (24)$$

$$f_n[i+1] = f_n[i] + T_s k_n[i] \{\dot{x}_n[i+1] - \dot{x}_{n+1}[i+1]\} \dots\dots (25)$$

where f and $\dot{x}$ denote the force and velocity, respectively. Δt is equivalent to T_s . The subscript n stands for the numbering of the multi-inertia. The calculated value of the velocity is used for the force expression within the same control period. In the FDTD method, the velocity and force are alternately

calculated.

The discrete equations of the FDTD method, discrete equations are implemented by applying the time-variant analog parameters $m_n[i]$, and $k_n[i]$ to Eq. (24) and Eq. (25). It is not necessary to consider the matrix exponential function as in the conventional method. Nevertheless, the calculation cost is small with a simple discretization.

4.2 State Equation of the FDTD Method Equations (24) and (25) are generally transformed into a state equation. Substitute Eq. (24) for $\dot{x}_n[i+1]$ of Eq. (25) as follows:

$$\dot{x}_n[i+1] = \frac{T_s}{m_n[i]} f_{n-1}[i] + \dot{x}_n[i] - \frac{T_s}{m_n[i]} f_n[i] \dots\dots (26)$$

$$f_n[i+1] = \frac{k_n[i] T_s^2}{m_{n+1}[i]} f_{n+1}[i] - k_n[i] T_s x_{n+1}[i] + \left(1 - \frac{k_n[i] T_s^2}{m_n[i]} - \frac{k_n[i] T_s^2}{m_{n+1}[i]}\right) f_n[i] + k_n[i] T_s x_n[i] + \frac{k_n[i] T_s^2}{m_n[i]} f_{n-1}[i] \dots\dots\dots (27)$$

As an example, the continuous expression Eq. (12) is discretized by the FDTD method as follows:

$$x[i+1] = A_{FDTD}[i] x[i] + B_{FDTD} u[i] \dots\dots\dots (28)$$

$$y[i] = C x[i] \dots\dots\dots (29)$$

$$A_{FDTD}[i] =$$

$$\begin{bmatrix} a_{1,1}[i] & a_{1,2}[i] & & & \\ a_{2,1}[i] & a_{2,2}[i] & & & 0 \\ & & \ddots & & \\ 0 & 0 & & a_{2n-1,2n-1}[i] & a_{2n-1,2n}[i] \\ & & & a_{2n,2n-1}[i] & a_{2n,2n}[i] \end{bmatrix}$$

$$a_{1,1}[i] = 1 - \frac{k_n[i] T_s^2}{m_n[i]} \dots\dots\dots (30)$$

$$a_{1,2}[i] = k_n[i] T_s \dots\dots\dots (31)$$

$$a_{2,1}[i] = -\frac{T_s}{m_n[i]} \dots\dots\dots (32)$$

$$a_{2,2}[i] = 1 \dots\dots\dots (33)$$

$$\vdots$$

$$a_{2n-1,2n-1}[i] = 1 - \frac{k_1[i] T_s^2}{m_1[i]} - \frac{k_1[i] T_s^2}{m_2[i]} \dots\dots\dots (34)$$

$$a_{2n-1,2n}[i] = k_1[i] T_s \dots\dots\dots (35)$$

$$a_{2n,2n-1}[i] = -\frac{T_s}{m_1[i]} \dots\dots\dots (36)$$

$$a_{2n,2n}[i] = 1 \dots\dots\dots (37)$$

$$B_{FDTD}[i] = \begin{bmatrix} 0 & 0 & \dots & \frac{k_1[i] T_s^2}{m_1[i]} & \frac{T_s}{m_1[i]} \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$x[i] = \begin{bmatrix} f_n[i] & \dot{x}_n[i] & \dots & f_1[i] & \dot{x}_1[i] \end{bmatrix} \dots\dots\dots (38)$$

The FDTD method of Eq. (26) and Eq. (27) and the state equation are completely equivalent.

The FDTD discretization method is simply expressed by the continuous physical parameter. The FDTD method does not use the matrix exponential function, so calculation time is small due to direct discretization of the wave equation. The FDTD method completes the discretization within a sampling period.

Fig. 2. Simulation of 1st Pade approximation(conv.): (i) Force input obtained by experiment, (ii) Time-variant parameter m_1 and m_2 , (iii) Time-variant parameter k_1 , (iv) Velocity responses $\dot{x}_1$, (v) Error between 1st Pade approximation and the ideal responses

5. Numerical Validation

The responses of the Pade approximation and proposed method are compared by simulations.

5.1 Simulations Setup The state equation of Eq. (12) is discretized at each sampling time. The force obtained from the experimental results is used as the input for the simulations. The parameters m_1 , m_2 , and k_1 are set to be time-variant from 1 s to 3 s. The parameters of each time domain are as follows:

Time-invariant phase ($iT_s \leq 1s$) ($3s < iT_s$);

$$\begin{cases} k_{1..6}[iT_s] = 1200 \text{ N/m} \\ m_{1..6}[iT_s] = 0.3 \text{ kg} \end{cases}$$

Time-variant phase ($1s < iT_s \leq 3s$);

$$\begin{cases} k_1[iT_s] = 1000 \cos(0.001iT_s) + 1200 \text{ N/m} \\ m_{1..2}[iT_s] = 0.2 \sin(0.001iT_s) + 0.3 \text{ kg} \\ k_{2..6}[iT_s] = 1200 \text{ N/m} \\ m_{3..6}[iT_s] = 0.3 \text{ kg.} \end{cases}$$

5.2 Comparison of Simulation Results The velocity responses $\dot{x}_1$ of each discretization method are shown in Figs. 2 and 3. The ideal velocity responses are the thirteenth-order Pade approximation. The errors are the differences between the results of each discretization method and the ideal response. Figure 4 shows the errors between the discretization methods. The vertical axis is the square of the error, which should be a small value for the accuracy of the discretization to be high. Because the first-order Pade approximation has a large error, the accuracy of the discretization becomes low. On the other hand, because the FDTD method has a small error, the accuracy of discretization becomes high. The FDTD method does not degrade the accuracy of higher order systems.

5.3 Evaluation of Discretization Time To ensure real-time control, this paper uses the Linux software named

Fig. 3. Simulation of FDTD method(prop.): (i) Force input obtained by experiment, (ii) Time-variant parameter m_1 and m_2 , (iii) Time-variant parameter k_1 , (iv) Velocity responses $\dot{x}_1$, (v) Error between FDTD method and the ideal responses

Fig. 4. Squaring error between 1st Pade approximation and FDTD method

as Advanced Robot Control System (ARCS). The specifications of the computer on which ARCS is implemented are Intel(R) Core(TM) i7-4770S at 3.10 GHz, memory 3.7 GB. In this paper, the control period is set to $100\mu s$. The calculation cost on ARCS is $6952\mu s$ in the conventional method and $78\mu s$ in the proposed method. Therefore, the calculation is completed within only $100\mu s$ in the proposed method.

This paper changes the number of the mass to and compare the discretization time. The discretization time of the 1- to 10-inertia spring mass systems are shown in Fig. 5. Matrix A_c of the n -th inertia spring-mass system is a square matrix with $2n$ rows and $2n$ columns. The Pade approximation increases the discretization time exponentially when the matrix A_c increases. In the case of the six inertia spring mass system, matrix A_c is a twelfth order state transition matrix. The discretization time of the six-inertia spring mass system is shown in the Table 1. When the sampling time is $100\mu s$, calculation of the first-order Pade approximation is

Fig. 5. Discretization time of 1st pade approximation and FDTD method. Calculations are performed using ARCS that runs in a single thread

Table 1. Computation time for discretizing six inertia spring mass system by using the ARCS

Method	1st Pade approx.	FDTD method
Discretization time [s]	6952 μ s	68 μ s

Fig. 6. Admittance control for haptic rendering

not completed within the control period. On the other hand, the proposed method completes the discretization within 100 μ s.

6. Application for Haptic Simulator

This paper applies the FDTD real-time discretization method to the haptic simulator, which reproduces the haptic sensation of the target object in the virtual space. As the shape of the object changes, the internal mass and stiffness parameters also change. Therefore, the time-variant environment is discretized by the FDTD method to realize a high-speed simulator directly by using the analog parameters.

6.1 Admittance Control for Rendering The virtual mechanical impedance is reproduced by implementing admittance control on the liner motor⁽¹⁷⁾. The time-variant environmental admittance $\mathcal{L}\{Y_e(t)\}$ expressed by the discrete time state equation is virtually reproduced via the linear motor. The structure of admittance control for rendering the environment is shown in Fig. 6. $G_A(s)$ is the transfer function from the external force f_0 to the position response x_1^{res} :

$$\begin{aligned}
 G_A(s) &= \frac{x_1^{res}(s)}{f_0(s)} \\
 &= -\frac{\mathcal{L}\{Y_e(t)\}}{s} \frac{1}{s^2 + C_p(s)} \left\{ C_p(s) G_{obs}(s) \right. \\
 &\quad \left. + \frac{s}{\mathcal{L}\{Y_e(t)\} M_n} (1 - G_{obs}(s)) \right\} \dots \dots \dots (39)
 \end{aligned}$$

Fig. 7. Actual configuration for haptic simulator

Table 2. Setup parameter for conventional and proposed method

Param.	Descriptions	Conv.	Prop.
M_n	Mass of linear motor[kg]	0.35	0.35
K_{fn}	Force constant[N/A]	4.17	4.17
g_{pd}	Bandwidth of the pseudo differentiator[rad/s]	300	3000
g_{obs}	Bandwidth of the DOB and RFOB[rad/s]	150	1300
T_s	Sampling time [μ s]	7000	100
	Resolution of the linear encoder [nm]	100	100
g_p	PD controller poles [rad/s]	30	200
K_p	Proportional gain of the PD controller[1/s ²]	900	40000
K_v	Differential gain of the PD controller[1/s]	60	400
$Y_e(s)$	Six inertial environments for rendering		
m_e	Mass of environments [kg]	0.3	0.3
k_e	Stiffness of environments [N/m]	700	700

$G_{obs}(s)$ and $C_p(s)$ represent the disturbance observer (DOB) and position controllers, which are given by

$$G_{obs}(s) = \frac{\hat{f}_0(s)}{f_0(s)} = \frac{g_{obs}}{s + g_{obs}}, \dots \dots \dots (40)$$

$$C_p(s) = K_p + K_v s = K_p + K_v \frac{g_{pd} s}{s + g_{pd}} \dots \dots \dots (41)$$

where g_{pd} denotes the Bandwidth of the differentiator. K_p and K_v denote the position proportional gain and the position integral gain, respectively.

6.2 Experimental Setup This paper implements the time-variant environment discretized by the FDTD method to admittance control. The force estimated by the DOB is input into the environment. The environmental velocity $\dot{x}_1$ of the motor with admittance control is then measured by the optical encoder. An overview of the experimental setup is shown in Fig. 7, and the experimental parameters are shown in Table 2. The time-variant parameters are the same as those of the simulations. The sampling period of the Pade approximation and the FDTD method are 7000 μ s and 100 μ s, respectively.

6.3 Experimental Results Figures 8 and 9 show the results of actual experiments with the Pade approximation and the FDTD method. In the conventional method, it is impossible to realize broadband admittance control with a sampling period of 100 μ s. The estimation of the force applied by the operator is shown at the top of the figure. The operator feels the reaction force from the time-variant environment. The first-order Pade approximation has a large error, the accuracy of the discretization becomes low. On the other hand, because the FDTD method has a small error, the accuracy of discretization becomes high. In the proposed method, the time-variant system is discretized in real-time, and admittance control is realized.

6.4 Evaluation of High-definition Frequency Responses The frequency response of each method is measured by the Frequency Response Analysis (FRA)⁽¹⁸⁾. The sampling time of each method depends on the calculation

Fig. 8. Experimental results of the Pade approximation ($T_s = 7000 \mu s$): (i) Force estimated by disturbance observer, (ii) Time-variant parameter m_1 and m_2 , (iii) Time-variant parameter k_1 , (iv) Velocity responses $\dot{x}_1$, (v) Error between the FDTD method and the ideal response

Fig. 9. Experimental results of FDTD method ($T_s = 100 \mu s$): (i) Force estimated by disturbance observer, (ii) Time-variant parameter m_1 and m_2 , (iii) Time-variant parameter k_1 , (iv) Velocity responses $\dot{x}_1$, (v) Error between the FDTD method and the ideal response

cost. Therefore, the bandwidth of each controller is limited by the sampling time. Note that the environments are set to be time invariant for the linear frequency response measurement. The set up parameters are shown in Table 2. The counter motor for the external force input is connected with the motor with admittance control via a coupling in the Fig. 10. The frequency responses from the external force to the position response in Eq. (39) are shown in the Fig. 11. In the conventional method, the anti-resonant frequency appears at different points. In addition, the phase is also shifted to a low frequency. The gain and phase of the proposed method are closed to the ideal responses. According to the conventional method, owing to the limitation of calculation cost, the parameter update does not end within a $100 \mu s$ sampling

Fig. 10. Actual configuration for FRA

Fig. 11. Frequency responses haptic reproduction

period. Only the proposed method realizes broadband admittance control. The broadband admittance control accurately reproduces the gain and phase characteristics. Thus, the reproducibility is improved by the FDTD method.

7. Conclusions

In a conventional system, the matrix exponential function is implemented by the Pade approximation, which has large errors and large discretization time for higher-order systems.

In particular, the Pade approximation exponentially increases discretization time when the system matrix, matrix A , becomes large. The proposed method does not degrade the accuracy of higher order systems. Furthermore, because the discretization time is small, high speed discretization is possible. In the experiments, the FDTD method is applied to the haptic simulator, and the time-variant system is accurately reproduced in real-time. The effectiveness of the proposed method is demonstrated by simulations and experiments.

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Appendix

In this section, the discretization accuracy of the FDTD method is theoretically evaluated in the continuous time domain. The discrete time state differential equations are inversely converted to continuous time.

1. Discretization Error of FDTD Method

For instance, the continuous target system is defined as follows:

$$A_c = \begin{bmatrix} 0 & k_1 \\ -\frac{1}{m_1} & 0 \end{bmatrix} \dots\dots\dots (A1)$$

Eq. (A1) is discretized by the FDTD method as follows:

$$A_d^{FDTD} = \begin{bmatrix} 1 - \frac{k_1 T_s^2}{m_1} & k_1 T_s \\ -\frac{T_s}{m_1} & 1 \end{bmatrix} \dots\dots\dots (A2)$$

where the eigenvalues of λ_1 and λ_2 are as follows:

$$\lambda_1 = \frac{-k_1 T_s^2 + 2m_1 + \sqrt{T_s^4 k_1^2 - 4k_1 T_s^2 m_1}}{2m_1} \dots\dots (A3)$$

$$\lambda_2 = \frac{-k_1 T_s^2 + 2m_1 - \sqrt{T_s^4 k_1^2 - 4k_1 T_s^2 m_1}}{2m_1} \dots\dots (A4)$$

The eigenvalues are converted from discrete time to continuous time as follows:

$$\eta_1 = \ln(\lambda_1)/T_s \dots\dots\dots (A5)$$

$$\eta_2 = \ln(\lambda_2)/T_s \dots\dots\dots (A6)$$

The eigenvalues of the FDTD method, which are inversely

transformed by the ideal eigenvector T , are converted into a continuous matrix. The matrix of the FDTD method A_c^{FDTD} converted to continuous time is as follows:

$$A_c^{FDTD} = T \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix} T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{2T_s}(-2\ln(2) + \ln(2\lambda_1) + \ln(2\lambda_2)) & \\ -\frac{\sqrt{-m_1 k_1}}{2k_1 T_s m_1}(-\ln(2\lambda_1) + \ln(2\lambda_2)) & \\ -\frac{m_1 k_1}{2\sqrt{-m_1 k_1} T_s}(-\ln(2\lambda_1) + \ln(2\lambda_2)) & \\ \frac{1}{2T_s}(-2\ln(2) + \ln(2\lambda_1) + \ln(2\lambda_2)) & \end{bmatrix} \dots\dots\dots (A7)$$

To compare the difference between Eq. (A7) and Eq. (A1), Eq. (A7) is expanded by the Taylor expansion with T_s , as follows:

$$A_{11} = 0 \dots\dots\dots (A8)$$

$$A_{12} = k_1 + \underbrace{\frac{k_1^2 T_s^2}{24 m_1} + \frac{3k_1^3 T_s^4}{640 m_1^2} + \frac{5k_1^4 T_s^6}{7168 m_1^3}}_{\text{Error terms}} \dots\dots (A9)$$

$$A_{21} = -\frac{1}{m_1} - \underbrace{\frac{k_1 T_s^2}{24 m_1^2} - \frac{3k_1^2 T_s^4}{640 m_1^3} - \frac{5k_1^3 T_s^6}{7168 m_1^4}}_{\text{Error terms}} \dots\dots\dots (A10)$$

$$A_{22} = 0 \dots\dots\dots (A11)$$

Here, the first term of each element represents Eq. (A1). In addition, T_s is approximated to zero, and Eq. (A7) represents the original system of Eq. (A1) as follows:

$$\lim_{T_s \rightarrow 0} A_c^{FDTD} = \begin{bmatrix} 0 & k_1 \\ -\frac{1}{m_1} & 0 \end{bmatrix} = A_c \dots\dots\dots (A12)$$

2. Discretization Error of the Pade Approximation

As with the FDTD method, the discretization accuracy of the first-order Pade approximation is evaluated in the continuous time domain. The matrix A_c discretized by the first-order Pade approximation is as follows:

$$A_d^{Pade} = \begin{bmatrix} \frac{-k_1 T_s^2 + 4m_1}{k_1 T_s^2 + 4m_1} & \frac{4k_1 T_s m_1}{k_1 T_s^2 + 4m_1} \\ -\frac{4T_s}{k_1 T_s^2 + 4m_1} & \frac{-k_1 T_s^2 + 4m_1}{k_1 T_s^2 + 4m_1} \end{bmatrix} \dots\dots\dots (A13)$$

where the eigenvalues of λ'_1 and λ'_2 are as follows:

$$\lambda'_1 = \frac{-k_1 T_s^2 + 4\sqrt{-m_1 k_1} T_s + 4m_1}{k_1 T_s^2 + 4m_1} \dots\dots\dots (A14)$$

$$\lambda'_2 = \frac{-k_1 T_s^2 - 4\sqrt{-m_1 k_1} T_s - 4m_1}{k_1 T_s^2 + 4m_1} \dots\dots\dots (A15)$$

Eq. (A14) and Eq. (A14) are converted to continuous time by Eq. (A5). The eigenvalues of the first-order Pade approximation, which are inversely transformed by the ideal eigenvector T , are converted into a continuous matrix. The matrix of the first-order Pade approximation A_c^{Pade} converted to continuous time is as follows:

$$A_c^{Pade} = T \begin{bmatrix} \eta'_1 & 0 \\ 0 & \eta'_2 \end{bmatrix} T^{-1} = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2T_s} \ln(\lambda'_1) + \frac{1}{2T_s} \ln(\lambda'_2) \\ \frac{\sqrt{-m_1 k_1}}{2k_1 T_s m_1} \ln(\lambda'_1) - \frac{\sqrt{-m_1 k_1}}{2k_1 T_s m_1} \ln(\lambda'_2) \\ \frac{m_1 k_1}{2\sqrt{-m_1 k_1} T_s} \ln(\lambda'_1) - \frac{m_1 k_1}{2\sqrt{-m_1 k_1} T_s} \ln(\lambda'_2) \\ \frac{1}{2T_s} \ln(\lambda'_1) + \frac{1}{2T_s} \ln(\lambda'_2) \end{bmatrix} \dots \dots \dots (A16)$$

Similarly, the Taylor expansion of A_c^{Pade} , which is inversely transformed, becomes as follows:

$$A'_{11} = 0 \dots \dots \dots (A17)$$

$$A'_{12} = k_1 \underbrace{-\frac{k_1^2 T_s^2}{12 m_1} + \frac{k_1^3 T_s^4}{80 m_1^2} - \frac{k_1^4 T_s^6}{448 m_1^3}}_{\text{Error terms}} \dots \dots \dots (A18)$$

$$A'_{21} = -\frac{1}{m_1} + \underbrace{\frac{k_1^2 T_s^2}{12 m_1^2} - \frac{k_1^3 T_s^4}{80 m_1^3} + \frac{k_1^4 T_s^6}{448 m_1^4}}_{\text{Error terms}} \dots \dots \dots (A19)$$

$$A'_{22} = 0 \dots \dots \dots (A20)$$

Here, the first term of each element represents Eq. (A1). In addition, T_s is approximated to zero. Eq. (A16) represents the original system of Eq. (A1) as follows:

$$\lim_{T_s \rightarrow 0} A_c^{Pade} = \begin{bmatrix} 0 & k_1 \\ -\frac{1}{m_1} & 0 \end{bmatrix} = A_c \dots \dots \dots (A21)$$

According to Eq. (A1), Eq. (A12) and Eq. (A21), both the FDTD method and the Pade approximation are correct.

3. Comparison

This paper compares the coefficients of Taylor expansion. A comparison of A_{12} and A'_{12} is shown as follows:

(Left side : (Right side :

1st Pade approximation) FDTD method)

$$k_1 = k_1 \dots \dots \dots (A22)$$

$$\left| -\frac{k_1^2 T_s^2}{12 m_1} \right| > \left| +\frac{k_1^2 T_s^2}{24 m_1} \right| \dots \dots \dots (A23)$$

$$\left| +\frac{k_1^3 T_s^4}{80 m_1^2} \right| > \left| +\frac{3k_1^3 T_s^4}{640 m_1^2} \right| \dots \dots \dots (A24)$$

$$\left| -\frac{k_1^4 T_s^6}{448 m_1^3} \right| > \left| +\frac{5k_1^4 T_s^6}{7168 m_1^3} \right| \dots \dots \dots (A25)$$

A comparison of A_{21} and A'_{21} is shown as follows:

(Left side : (Right side :

1st Pade approximation) FDTD method)

$$-\frac{1}{m_1} = -\frac{1}{m_1} \dots \dots \dots (A26)$$

$$\left| +\frac{k_1 T_s^2}{12 m_1^2} \right| > \left| -\frac{k_1 T_s^2}{24 m_1^2} \right| \dots \dots \dots (A27)$$

$$\left| -\frac{k_1^2 T_s^4}{80 m_1^3} \right| > \left| -\frac{3k_1^2 T_s^4}{640 m_1^3} \right| \dots \dots \dots (A28)$$

$$\left| +\frac{k_1^3 T_s^6}{448 m_1^4} \right| > \left| -\frac{5k_1^3 T_s^6}{7168 m_1^4} \right| \dots \dots \dots (A29)$$

The Pade approximation is represented on the left side and the FDTD method is represented on the right side. In A_{12} , A'_{12} and A_{21} , A'_{21} terms, the FDTD method has all its coefficients, which are smaller than those of the first-order Pade approximation. In other words, the discretization accuracy of the FDTD method is higher than that of the Pade approximation.

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