Edge-Coloring Problems for Graphs

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The edge-coloring problem is one of the fundamental problems on graphs, which often appears in various scheduling problems like the file transfer problem on computer networks. In this paper, we survey old and new results on the classical edge-coloring problem as well as the generalized edge-coloring problems, called the f -coloring and fg -coloring problems. In particular we especially review various upper bounds on the minimum number of colors required to edge-color graphs, and present efficient algorithms to edge-color graphs with a number of colors not exceeding the upper bounds.

KEYWORDS: graph, algorithm, edge-coloring

1. A survey of the edge-coloring problem

1.1 A history of the edge-coloring problem

The edge-coloring problem is one of the fundamental problems on graphs. A graph $G = (V, E)$ is an ordered pair of vertex set V and edge set E . An edge in E joins two vertices in V . Fig. 1 depicts a graph of seven vertices and twelve edges, where vertices are drawn by circles and edges by lines. The edge-coloring problem is to color all edges of a given graph with the minimum number of colors so that no two adjacent edges are assigned the same color. Fig. 1 illustrates an edge-coloring of a graph with four colors. A set of edges which are not adjacent each other is called a *matching*. Since each set of edges colored with the same color is a matching, an edge-coloring of a graph is indeed a partition of E to matchings.

The concept of a matching often appears in various situations, for example the date-match problem (See Fig. 2). Suppose that there are several boys and girls, and each girl favors some of the boys. What condition is necessary to make a date-matching in which every girl is assigned to one of her favorite boys. Consider a graph in which boys and girls correspond to vertices and a vertex corresponding to a girl is joined to vertices corresponding to boys whom the girl favors. Then there is a date-matching above if and only if the graph has a matching containing all vertices corresponding to girls. Since such a matching problem appears in various situations, the edge-coloring problem which is a partition of a graph to matchings is also a basic and natural problem.

We then historically review the edge-coloring problem. The edge-coloring problem was posed in 1880 in relation with the well-known four-color conjecture. The four-color conjecture is that every map could be colored with four colors so that any neighboring countries have different colors. It took more than 100 years to prove the conjecture affirmatively in 1976 with the help of computers since it was posed in 1852. The first paper that dealt with the edge-coloring problem was written by Tait in 1889 [Fiorini and Wilson (1977)]. In this paper Tait proved that the four-color problem is equivalent with the problem of edge-coloring every planar 3-connected cubic graph with three colors. The minimum number of colors needed to color edges of G is called the *chromatic index* $\chi'(G)$ of G . The maximum degree of graph G is denoted by $\Delta(G)$. Obviously $\chi'(G) \geq \Delta(G)$ since all edges incident to the same vertex must be assigned different colors. König (1916) proved that every bipartite graph can be edge-colored with exactly $\Delta(G)$ colors, that is $\chi'(G) = \Delta(G)$. Shannon (1949) proved that every graph can be edge-colored with at most $3\Delta(G)/2$ colors, that is $\chi'(G) \leq 3\Delta(G)/2$. Vizing (1964) proved that $\chi'(G) \leq \Delta(G) + 1$ for every simple graph. A few other upper bounds on $\chi'(G)$ have been known [Andersen (1977), Goldberg (1984), Nishizeki and Kashiwagi (1990)].

We next review the results on the edge-coloring problem. According to the rapid progress of computers, the research on computer algorithms has become active with emphasis on the efficiency and complexity, and efficient algorithms have been developed for various graph problems. However Holyer (1980) proved that the edge-coloring problem is *NP*-complete, and hence it is very unlikely that there is a polynomial time algorithm for solving the problem [Aho *et al.* (1974)]. Hence a good approximation algorithm would be useful. Approximation algorithms are evaluated by the approximation ratio and the complexity. The polynomial time algorithm having the best approximate ratio so far was given by Nishizeki and Kashiwagi (1990), whose approximation ratio is asymptotically 1.1. Gabow *et al.* (1985) gave the most efficient algorithm which edge-colors simple graph G with at most $\Delta(G) + 1$ colors in $O(|E| \sqrt{|V|} \log |V|)$ time where V is the set of vertices and E is the set of edges in G . Furthermore sequential and parallel algorithms have been obtained for various classes of graphs, such as bipartite graphs [Cole and Hopcroft (1982), Gabow and Kariv (1982)], planar graphs [Chrobak and Nishizeki (1990), Gabow *et al.* (1985)], series-parallel graphs [Terada and Nishizeki (1982), Seymour (1990),

Fig. 1. An edge-coloring of a graph with four colors.

Fig. 2. A date-match problem.

Zhou *et al.* (1982) (1993a)], partial k -trees [Bodlaender (1990), Zhou *et al.* (1993b)], degenerated graphs and bounded-genus graphs [Zhou and Nishizeki (1994)].

On the other hand, various generalizations of edge-coloring have been introduced and investigated. In 1970's Hilton and de Werra obtained many notable results on "equitable and edge-balanced colorings" in which each color appears at each vertex uniformly. In 1980's Hakimi *et al.* studied the following f -coloring problem. An f -coloring of a graph G is a coloring of edges of G such that each color appears at each vertex $v \in V$ at most $f(v)$ times [Hakimi and Kariv (1986)]. Figure 3 depicts an f -coloring with four colors where numbers in circles mean $f(v)$ for vertices v . An ordinary edge-coloring is a special case of an f -coloring in which $f(v) = 1$ for every vertex $v \in V$. The minimum number of colors needed to f -color G is called the f -chromatic index $\chi_f'(G)$ of G . Since deciding the chromatic index of G is NP -complete [Holyer (1980)], deciding the f -chromatic index of G is also NP -complete in general. Hakimi and Kariv (1986) obtained an upper bound on $\chi_f'(G)$. Another upper bound on χ_f' has been known [Nakano *et al.* (1988)].

The file transfer problem on computer networks introduced by Coffman *et al.* (1985) is related to the edge-coloring problem. The file transfer problem is modeled as follows. Each computer v has a limited number $f(v)$ of communication ports. For each pair of computers there are a number of files which are transferred between the pair of computers. In such a situation the problem is how to schedule the file transfers so as to minimize the total time for the overall transfer process. The general problem is NP -complete since the simple version of the problem can be reduced to the edge-coloring problem. Coffman *et al.* obtained simple approximate algorithms

Fig. 3. An f -coloring of a graph with four colors.

which solve the file transfer problem. The file transfer problem in which each file has the same length is formulated as an f -coloring problem for a graph as follows. Vertices of the graph correspond to nodes of the network, and edges correspond to files to be transferred between the endpoints. Such a graph G describes the file transfer demands. Assume that each computer v has $f(v)$ communication ports, and transferring any file takes an equal amount of time. Under these assumptions, the schedule to minimize the total time for the overall transfer process corresponds to an f -coloring of G with the minimum number of colors. Note that the edges colored with the same color correspond to files that can be transferred simultaneously.

However, in computer networks, we must also consider the capacity of channels. Thus an f -coloring is generalized to an fg -coloring so as to treat the file transfer problem with channel constraints [Nakano *et al.* (1990)]. The capacity of a channel between vertices v and w is denoted by $g(vw)$. An fg -coloring of G is a coloring of edges such that each vertex v has at most $f(v)$ edges colored with the same color and each set $E(vw)$ of multiple edges contains at most $g(vw)$ edges colored with the same color. Figure 4 depicts an fg -coloring with four colors. The minimum number of colors needed to fg -color G is called the fg -chromatic index $\chi'_{fg}(G)$ of G . Several upper bounds on χ'_{fg} have been known [Nakano *et al.* (1990), (1991)].

1.2 Definitions

In this paper we deal with multigraphs which may have multiple edges but have no selfloops. We simply call such a multigraph a *graph*. $G = (V, E)$ denotes a graph with vertex set V and edge set E . We denote the degree of vertex v by $d(v)$ and the *maximum degree* of G by $\Delta(G) = \max_{v \in V} d(v)$. An edge joining vertices v and w is denoted by vw . $E(vw)$ is the set of multiple edges joining vertices v and w , and $p(vw)$ is the cardinality of the set $E(vw)$, that is $p(vw) = |E(vw)|$. We write $H \subseteq G$ if H is a subgraph of G . $V(H)$ denotes the set of vertices of H , and $E(H)$ denotes the set of edges of H . A graph in which at most one edge joins any pair of vertices is called a *simple graph*. A *bipartite graph* $G = (V_1, V_2, E)$ is a graph whose vertex set can be partitioned into two sets V_1 and V_2 so that $e \in V_1 \times V_2$ for each edge $e \in E$. The graph in Fig. 2 is bipartite. $G - e$ denotes the graph obtained from G by deleting an edge e .

A *path* $P = v_0 e_1 v_1 e_2 v_2 \cdots v_{k-1} e_k v_k$ is a sequence of distinct vertices and edges, where $e_i \in E(v_{i-1} v_i)$ for $1 \leq i \leq k$. Vertex v_0 is the *start vertex* of P and v_k the *end vertex*. The *length* of P is the number of edges in P , and denoted by $|P|$.

A *walk* $W = v_0 e_1 v_1 e_2 v_2 \cdots v_{k-1} e_k v_k$ is a sequence of vertices and distinct edges, where $e_i \in E(v_{i-1} v_i)$ for $1 \leq i \leq k$, and vertices $v_0, v_1, \dots, v_k$ are not necessarily distinct. For simplicity we often denote W by $v_0 v_1, v_1 v_2, \dots, v_{k-1} v_k$ if no confusion may occur. A walk W is called a *cycle* if $v_0 = v_k$.

We shall use $\lfloor x \rfloor$ for the largest integer not greater than x , and $\lceil x \rceil$ for the smallest integer not smaller than x .

Let C be the set of colors available for an edge-coloring. An edge colored with a color $\alpha \in C$ is called an α -edge. If there is no α -edge which is incident to vertex v , then we say “vertex v misses the color α ” or “ α is a *missing color* at vertex v .”

Fig. 4. An fg -coloring of a graph with four colors.

2. Edge-coloring

2.1 Edge-colorings of bipartite graphs

The edge-coloring problem for bipartite graphs can be used to model a time table problem: assume that, given a set of teachers and a set of classes, it is known which classes and how many hours each teacher must teach, then construct a time table to minimize the schooltime. Represent teachers and classes as vertices of a graph. When a teacher must teach a class h hours, join the vertices corresponding to the teacher and the class by h multiple edges. The edges colored with a same color correspond to classes that can be held simultaneously. Thus the time table to minimize the schooltime corresponds to an edge-coloring of the graph with the minimum number of colors.

Every bipartite graph G can be edge-colored with $\Delta(G)$ colors, that is $\chi'(G) \leq \Delta(G)$ [König (1916)]. On the other hand clearly $\chi'(G) \geq \Delta(G)$. Therefore $\chi'(G) = \Delta(G)$. One can easily prove the inequality $\chi'(G) \leq \Delta(G)$ above using a technique of “switching an alternating path,” as follows.

An $\alpha\beta$ -alternating path is a maximal path whose edges are colored with distinct colors α and β alternately. Switching an $\alpha\beta$ -alternating path P is to interchange the colors α and β in P . Switching P may change the missing color only at the start and end vertices. Any vertex at which either α or β is missing is a start or end vertex of an $\alpha\beta$ -alternating path.

We now prove that every bipartite graph G can be edge-colored with $\Delta(G)$ colors. Let $e \in E$ be an arbitrary edge in the bipartite graph $G = (V, E)$, and $G - e$ is a graph obtained from G by deleting an edge e . It is sufficient to show if the graph $G - e$ is edge-colored with $\Delta(G)$ colors, G also can be edge-colored with $\Delta(G)$ colors. Assume that $G - e$ is edge-colored with $\Delta(G)$ colors and $e \in E(vw)$. Let C be the set of $\Delta(G)$ colors. Since vertex v has at most $\Delta(G) - 1$ colored edges, v has at least one missing color $\alpha \in C$. Similarly vertex w has a missing color $\beta \in C$. If $\alpha = \beta$, then coloring the uncolored edge e with α would complete an edge-coloring of G . Thus one may assume that a β -edge is incident to v and α -edge is incident to w . Vertex v is a start vertex of an $\alpha\beta$ -alternating path P . Exactly one of colors α and β is missing at the end vertex of P . None of colors α and β is missing at intermediate vertex in P . Assume without loss of generality that V is bipartitioned to V_1 and V_2 , that is, $G = (V_1, V_2, E)$ and $v \in V_1$ and $w \in V_2$. Since β is a missing color at w , w is not on P . After switching P , β becomes a missing color at v and β still remains a missing color at w . Coloring the uncolored edge e with β would complete an edge-coloring of G .

The proof above immediately yields an $O(|E||V|)$ time algorithm to edge-color a bipartite graph G with $\Delta(G)$ colors. There exists a more efficient algorithm which, based on the divide and conquer, edge-colors a bipartite graph in $O(|E|\log|E|)$ time [Cole and Hopcroft (1982)].

2.2 Edge-colorings of simple graphs

Vizing (1964) proved that every simple graph G can be edge-colored with $\Delta(G) + 1$ colors, that is,

$$\chi'(G) \leq \Delta(G) + 1.$$

Since $\chi'(G) \geq \Delta(G)$, every simple graph G can be edge-colored with either $\Delta(G)$ or $\Delta(G) + 1$ colors. His proof depends on a technique of “shifting a fan.” Let $e_0 = wv_0$ be an uncolored edge. Then a fan F is a sequence of distinct edges $e_0, e_1, \dots, e_k$ incident to a vertex w such that there is a sequence of distinct colors $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ satisfying the following conditions (a)–(c), where $v_{i,0} \leq i \leq k$, is the end of e_i other than w . (See Fig. 5).

- (a) α_i is a missing color at v_i , $0 \leq i \leq k - 1$;
- (b) e_i , $1 \leq i \leq k$, is colored with α_{i-1} ; and
- (c) $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ are distinct.

Since G is a simple graph, vertices $v_0, v_1, \dots, v_k$ are distinct. Shifting a fan F means to recolor e_i with α_i for each i , $0 \leq i \leq k - 1$, and erase the color α_{k-1} of e_k . Shifting F yields another edge-coloring of G in which e_k instead of e_0 is uncolored.

We are now ready to prove Vizing’s upper bound. It is sufficient to show that if $G - e$ is edge-colored with $\Delta(G) + 1$ colors then G also can be edge-colored with $\Delta(G) + 1$ colors. Let $e_0 = wv_0$, and let C be a set of $\Delta(G) + 1$ colors. Since vertex w has at most $\Delta(G) - 1$ colored edges, w has at least two missing colors. Let $\beta \in C$ be one of such colors. Similarly, v_0 has at least two missing colors. Let $\alpha_0 \in C$ be one of them. If $\alpha_0 = \beta$, then coloring the uncolored edge e_0 would complete an edge-coloring of G . Furthermore one may assume that α_0 is not a missing color at w and β is not a missing color at v_0 ; otherwise, one can complete an edge-coloring of G similarly as above.

An α_0 -edge $e_1 = wv_1$ is incident to w . Since vertex v_1 has at most $\Delta(G)$ colored edges, v_1 has at least one missing color, say, $\alpha_1 \in C$. Now $F = e_0, e_1$ is a fan which has a color sequence α_0 . If α_1 is a missing color at w , then shifting the fan $F = e_0, e_1$ would make α_1 a missing color at both ends of the uncolored edge $e_1 = wv_1$, and consequently coloring e_1 with α_1 would complete an edge-coloring of G . Otherwise, an α_1 -edge $e_2 = wv_2$ is incident to w . Let α_2 be a missing color at v_2 . Now $F = e_0, e_1, e_2$ is a fan which has a color sequence α_0, α_1 . Thus assume in general that a fan $F = e_0, e_1, \dots, e_k$ has been constructed so far and it has a color sequence $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$. If a missing color α_k at v_k is also a missing color at w , then shifting the fan F would make α_k a missing color at both ends of the uncolored edge $e_k = wv_k$ and consequently coloring e_k with α_k would complete an edge-coloring of G . Thus one can assume that α_k is not a missing color at w . Furthermore there must exist j , $0 \leq j < k$, such that $\alpha_k = \alpha_j$; otherwise, F could be enlarged although F is assumed to be a maximal fan. Let P be an $\alpha_k\beta$ -alternating path starting at v_k . F does not have a β -edge and has exactly one $\alpha_k (= \alpha_j)$ -edge e_{j+1} . Furthermore $\alpha_0, \dots, \alpha_{k-1}$ are all distinct. Thus P passes through none of edges of F except for an α_k -edge e_{j+1} .

If P ends at neither w nor v_j , then switch P and subsequently shift $F = e_0, e_1, \dots, e_k$. Since β becomes a missing color at both ends of the uncolored edge $e_k = wv_k$, coloring e_k with β would complete an edge-coloring of G .

If P ends at v_j , then switch P and subsequently shift the subfan $F' = e_0, e_1, \dots, e_j$. Since β becomes a missing color at both ends of the uncolored edge $e_j = wv_j$, coloring e_j with β would complete an edge-coloring of G .

If P ends at w , then switch P and subsequently shift the subfan $F' = e_0, e_1, \dots, e_j$. Since $\alpha_k = \alpha_j$ becomes a missing color at both ends of the uncolored edge $e_j = wv_j$, coloring e_j with α_k would complete an edge-coloring of G .

The proof above immediately yields an $O(|E||V|)$ time algorithm to edge-color a simple graph G with $\Delta(G) + 1$ colors. There exist more efficient edge-coloring algorithms for simple graphs. The most efficient one takes time $O(|E|\sqrt{|V|\log|V|})$ [Gabow *et al.* (1985)].

The following results are known for planar simple graphs G . If $\Delta(G) \leq 5$ then $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$. If $\Delta(G) \geq 8$ then $\chi'(G) = \Delta(G)$. It is open whether there exists a planar simple graph G with $6 \leq \Delta(G) \leq 7$ such that $\chi'(G) = \Delta(G) + 1$. For the case $\Delta(G) \geq 8$ there is an $O(|V|^2)$ time algorithm for edge-coloring a pla-

Fig. 5. A fan.

nar simple graph G with $\Delta(G)$ colors [Terada and Nishizeki (1982)]. Furthermore for the case $\Delta(G) \geq 9$ there is a more efficient algorithm of time-complexity $O(|V|\log|V|)$ [Chrobak and Nishizeki (1990)].

On the other hand, for a partial k -tree G , a linear algorithm for edge-coloring G with $\chi'(G)$ colors is given for any fixed k [Zhou *et al.* (1993b)].

2.3 Edge-colorings of multigraphs

Vizing (1964) obtained the following upper bound on the chromatic index of multigraphs:

$$\chi'(G) \leq \max_{v, w \in V} \{d(w) + p(vw)\}.$$

On the other hand Shannon (1949) obtained the following upper bound:

$$\chi'(G) \leq \left\lfloor \frac{3}{2} \Delta(G) \right\rfloor.$$

We next present a proof for the Shannon's bound. It is sufficient to show that if $G - e_0$ is edge-colored with $\lfloor \frac{3}{2} \Delta(G) \rfloor$ colors then G also can be edge-colored with $\lfloor \frac{3}{2} \Delta(G) \rfloor$ colors. Let $e_0 \in E(wv_0)$. Since vertex w has at most $\Delta(G) - 1$ colored edges, w has at least $\lfloor \frac{1}{2} \Delta(G) \rfloor + 1$ missing colors. Similarly, v_0 has at least $\lfloor \frac{1}{2} \Delta(G) \rfloor + 1$ missing colors. If there exists a common missing color at v_0 and w , then coloring e_0 with such a color would complete an edge-coloring of G . Otherwise, choose a missing color α at v_0 . Since α is not a missing color at w , there exists an α -edge wv_1 . If there is a common missing color β at w and v_1 , then recoloring e_1 with β and coloring e_0 with α would complete an edge-coloring of G . Thus one may assume that there is no common missing color at w and v_1 . Vertex v_1 has at least $\lfloor \frac{1}{2} \Delta(G) \rfloor$ missing colors. Thus v_0 and v_1 have at least one common missing color γ ; otherwise, the number of used colors would exceed $\lfloor \frac{3}{2} \Delta(G) \rfloor$. Since v_1 has an α -edge wv_1 , $\gamma \neq \alpha$. Color γ is not missing at w . Choose a missing color δ at w . Color δ is missing at neither v_0 nor v_1 . Then w is a start vertex of a $\delta\gamma$ -alternating path P . Switch P . If P does not end at v_0 , then, γ becomes a common missing color at w and v_0 and consequently coloring e_0 with γ would complete an edge-coloring of G . If P ends at v_0 , then γ becomes a common missing color at w and v_1 and consequently recoloring e_1 with γ and coloring e_0 with α would complete an edge-coloring of G .

The proof above immediately yields an $O(|E|(\Delta(G) + |V|))$ time algorithm to edge-color a multigraph G with $\lfloor \frac{3}{2} \Delta(G) \rfloor$ colors.

There is another upper bound on the chromatic index of multigraphs [Andersen (1977)]:

$$\chi'(G) \leq \max \left\{ l(G), \left\lfloor \frac{5\Delta(G) + 2}{4} \right\rfloor \right\},$$

where

$$l(G) = \max_{H \subseteq G, |V(H)| \geq 3} \left\lfloor \frac{|E(H)|}{\left\lfloor \frac{|V(H)|}{2} \right\rfloor} \right\rfloor.$$

Clearly at most $\lfloor |V(H)|/2 \rfloor$ edges can be colored with the same color. Therefore $l(G)$ is a lower bound on $\chi'(G)$: $\chi'(G) \geq l(G)$.

Furthermore the following better upper bounds are known [Hochbaum *et al.* (1986), Goldberg (1984), Nishizeki and Kashiwagi (1990)]:

$$\begin{aligned} \chi'(G) &\leq \max \left\{ l(G), \left\lfloor \frac{7\Delta(G) + 4}{6} \right\rfloor \right\}, \\ \chi'(G) &\leq \max \left\{ l(G), \left\lfloor \frac{9\Delta(G) + 6}{8} \right\rfloor \right\}, \quad \text{and} \\ \chi'(G) &\leq \max \left\{ l(G), \left\lfloor \frac{11\Delta(G) + 8}{10} \right\rfloor \right\}. \end{aligned}$$

The proofs for these upper bounds are similar. In this paper we sketch the proof in [Goldberg (1984)]. Define a margin $m(v, \alpha) = 1$ if α is a missing color at v , and $m(v, \alpha) = 0$ otherwise.

If $\Delta(G) \leq 1$ then $\chi'(G) = \Delta(G)$. Therefore we can assume $\Delta(G) \geq 2$. Let $e = vw \in E$ be an arbitrary edge in $G = (V, E)$. It suffices to prove that if $G - e$ is edge-colored with

$$u(G) = \max \left\{ l(G), \left\lfloor \frac{9\Delta(G) + 6}{8} \right\rfloor \right\}$$

colors then G is also edge-colored with $u(G)$ colors. Construction of such an edge-coloring depends on various cases. We present only some of the cases in detail. Let C be the set of $u(G)$ colors.

Since $\Delta(G) \geq 2$,

$$u(G) \geq \left\lfloor \frac{9\Delta(G) + 6}{8} \right\rfloor \geq \Delta(G) + 1.$$

For any vertex x

$$\sum_{\alpha \in C} m(x, \alpha) = u(G) - d(x) \geq u(G) - \Delta(G) \geq 1.$$

This way every vertex has at least one missing color.

Case 1: vertices v and w have a common missing color α .

In this case coloring e with α would complete an edge-coloring of G .

In the following cases we assume that v has a missing color α , w has a missing color β , and $\alpha \neq \beta$. There exists an $\alpha\beta$ -alternating path P starting at v .

Case 2: P does not end at w .

Switching P makes β a common missing color at v and w , and hence Case 1 happens.

Case 3: P has two vertices with a common missing color.

Let two vertices u_1, u_2 in P have a common missing color γ . Choose γ, u_1 and u_2 so that the subpath of P between u_1 and u_2 is shortest. Since Case 2 does not happen, $\gamma \neq \alpha, \beta$. One may assume that v, u_1, u_2 appear in this order in P . If u_1 and u_2 are consecutive in P , then coloring edge u_1u_2 in P with γ makes Case 1 or Case 2 occur. If u_1 and u_2 are not consecutive in P , then let u'_1 be the vertex succeeding u_1 on P and let $\gamma' \neq \alpha, \beta$ be a missing color at u'_1 . Then $\gamma' \neq \gamma$ due to the selection of γ, u_1 and u_2 . Switch the $\gamma'\gamma$ -alternating path P' starting at u'_1 . Since $\gamma, \gamma' \neq \alpha, \beta$, P remains an $\alpha\beta$ -alternating path. If P' did not end at u_1 , then the consecutive vertices u_1, u'_1 would have a common missing color γ . If P' ended at u_1 , then u'_1 and u_2 would have a common missing color γ and the subpath of P between u'_1 and u_2 on P would be shorter than the subpath between u_1 and u_2 .

Since none of Cases 1, 2 and 3 does not happen, P has at most eight vertices as follows. Let S be the set of vertices in P . Since $\sum_{x \in S} m(x, \alpha) \leq 1$ for every color α ,

$$u(G) \geq \sum_{x \in S} \sum_{\alpha \in C} m(x, \alpha).$$

Every vertex $x \in V$, $x \neq v, w$, satisfies

$$\sum_{\alpha \in C} m(x, \alpha) = u(G) - d(x) \geq u(G) - \Delta(G).$$

Since an uncolored edge e is incident to vertices v and w ,

$$\begin{aligned} \sum_{\alpha \in C} m(v, \alpha) &\geq (u(G) - \Delta(G)) + 1 \quad \text{and} \\ \sum_{\alpha \in C} m(w, \alpha) &\geq (u(G) - \Delta(G)) + 1. \end{aligned}$$

Thus

$$u(G) \geq \sum_{\alpha \in C} \sum_{x \in S} m(x, \alpha) \geq (u(G) - \Delta(G))|S| + 2.$$

One can immediately have $|S| < 9$, that is, $|S| \leq 8$ from the following four equations: the equation above, $u(G) \geq \lfloor (9\Delta(G) + 6)/8 \rfloor$, $u(G) \geq \Delta(G) + 1$ and $\Delta(G) \geq 2$.

This way, if Cases 1, 2 and 3 do not happen, then P ends at w and one of the three cases in which $|P| = 2, 4$ or 6 happens. Thus it is sufficient to consider only these three cases, but we omit these cases. (see Goldberg (1984) for the details.)

Similarly the proof of Nishizeki and Kashiwagi's (1990) upper bound

$$\chi'(G) \leq \max \left\{ l(G), \left\lfloor \frac{11\Delta(G) + 8}{10} \right\rfloor \right\}$$

shows that $|S| \leq 10$ and hence $|P| = 2, 4, 6$ or 8 . These proofs yield $O(|E|(\Delta(G) + |V|))$ time approximate algorithms for edge-coloring graphs with a number of colors not exceeding these upper bounds.

Since

$$\chi'(G) \leq \max_{v, w \in V} \{d(w) + p(vw)\} \leq 2\chi'(G),$$

the algorithm using colors no more than Vizing's upper bound has the approximation ratio 2. The algorithm using colors no more than Shannon's upper bound has the approximation ratio $3/2$ since

$$\chi'(G) \leq \left\lfloor \frac{3}{2} \Delta(G) \right\rfloor \leq \frac{3}{2} \chi'(G).$$

Furthermore Nishizeki and Kashiwagi's algorithm has the asymptotic approximation ratio 1.1 since

$$\chi'(G) \leq \max \left\{ l(G), \left\lfloor \frac{11\Delta(G) + 8}{10} \right\rfloor \right\} \leq \frac{11\chi'(G) + 8}{10}.$$

Goldberg (1984) posed a conjecture that for every odd integer $k \geq 5$

$$\chi'(G) \leq \max \left\{ l(G), \left\lfloor \frac{k\Delta(G) + (k-3)}{k-1} \right\rfloor \right\}.$$

Furthermore Goldberg (1984) and Seymour (1979) posed a conjecture that every multigraph G satisfies

$$\chi'(G) \leq \max \{ l(G), \Delta(G) + 1 \}.$$

Especially for series-parallel multigraph G it is known [Seymour (1990)] that

$$\chi'(G) = \max \{ l(G), \Delta(G) \},$$

and there exists an algorithm to decide $\chi'(G)$ and find an edge-coloring of G with $\chi'(G)$ colors in linear time [Zhou *et al.* (1993a)].

3. f -coloring

The *vertex-capacity* f is any function from the vertex set V to the natural numbers. An f -coloring of a graph G is a coloring of the edges of G such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f -color G is called the f -chromatic index of G , and is denoted by $\chi'_f(G)$. Figure 3 depicts an f -coloring with four colors, and $\chi'_f(G) = 4$ since there exists a vertex such that the vertex-capacity is 1 and the degree is 4. Hakimi and Kariv obtained the following upper bound on $\chi'_f(G)$.

$$\chi'_f(G) \leq \max_{v, w \in V} \left\lceil \frac{d(v) + p(vw)}{f(v)} \right\rceil.$$

This upper bound is a generalization of Vizing's bound for the ordinary edge-coloring because an ordinary edge-coloring is a special case of an f -coloring in which $f(v) = 1$ for every vertex $v \in V$. Their proof uses an extended version of switching an alternating path and shifting a fan. The proof immediately yields and $O(|E|^2)$ time algorithm to find an f -coloring of G with a number of colors not exceeding the upper bound. Furthermore the time complexity was improved to $O(|E| \sqrt{|E| \log |E|})$ [Nakano and Nishizeki (1993b)]. The following upper bound on $\chi'_f(G)$ is also known [Nakano *et al.* (1988)]:

$$\chi'_f(G) \leq \max \left\{ l_f(G), \left\lfloor \frac{9\Delta_f(G) + 6}{8} \right\rfloor \right\}$$

where

$$l_f(G) = \max_{H \subseteq G, |V(H)| \geq 3} \left\lceil \frac{|E(H)|}{\left\lfloor \frac{\sum_{v \in V(H)} f(v)}{2} \right\rfloor} \right\rceil$$

and

$$\Delta_f(G) = \max_{v \in V} \left\lceil \frac{d(v)}{f(v)} \right\rceil.$$

Note that $l_f(G)$ and $\Delta_f(G)$ are trivial lower bounds on $\chi'_f(G)$. Furthermore an algorithm is known to find an f -coloring of G with a number of colors not exceeding the upper bound above in $O(|E|^2)$ time [Nakano *et al.* (1988)].

4. fg -coloring

4.1 fg -coloring

The *edge-capacity* g is any function from the pair of vertices $V \times V$ to the natural numbers. An fg -coloring of G is a coloring of the edges of G such that at most $f(v)$ edges incident to v are colored with the same color for each vertex $v \in V$ and at most $g(vw)$ multiple edges joining v and w are colored with the same color for each pair of vertices v and w . The minimum number of colors needed to fg -color G is called the fg -chromatic index of G , and is denoted by $\chi'_{fg}(G)$. The fg -chromatic index of G in Fig. 4 is four. The following upper bound on $\chi'_{fg}(G)$ has been known [Nakano *et al.* (1991)]:

$$\chi'_{fg}(G) \leq \left\lfloor \frac{3}{2} \Delta_{fg}(G) \right\rfloor,$$

where

$$\Delta_{fg}(G) = \max \{ \Delta_f(G), \Delta_g(G) \},$$

$$\Delta_f(G) = \max_{v \in V} \left\lceil \frac{d(v)}{f(v)} \right\rceil,$$

and

$$\Delta_g(G) = \max_{vw \in E} \left\lceil \frac{p(vw)}{g(vw)} \right\rceil.$$

Note that $\Delta_f(G)$, $\Delta_g(G)$ and $\Delta_{fg}(G)$ are trivial lower bounds on $\chi'_{fg}(G)$. This upper bound is a generalization of Shannon's bound (1949) for the ordinary edge-coloring. An $O(|E|^2)$ time algorithm to fg -color G with a number of colors not exceeding $\left\lfloor \frac{3}{2} \Delta_{fg}(G) \right\rfloor$ has been known [Nakano *et al.* (1991)]. Since

$$\left\lfloor \frac{3}{2} \Delta_{fg}(G) \right\rfloor \leq \frac{3}{2} \chi'_{fg}(G),$$

this algorithm has the approximation ratio $3/2$.

One may assume without loss of generality that $g(vw) \leq \max \{f(v), f(w)\}$ for any pair of vertices $v, w \in V$. Then the following upper bound holds for the fg -chromatic index $\chi'_{fg}(G)$ [Nakano *et al.* (1990)]:

$$\chi'_{fg}(G) \leq \max_{v, w \in V} \left\lceil \frac{d(v)}{f(v)} + \frac{p(vw)}{g(vw)} \right\rceil.$$

Although one may assume that $g(vw) \leq \min \{f(v), f(w)\}$, we assume as above since the bound would increase when g decreases. This upper bound is a generalization of Vizing's upper bound (1964) for the ordinary edge-coloring and Hakimi and Kariv's (1986) upper bound for the f -coloring. The proof for this upper bound is constructive, and immediately yields an $O(|E|^2)$ time algorithm to fg -color a given graph with a number of colors not exceeding the upper bound above. Since

$$\max_{v, w \in V} \left\lceil \frac{d(v)}{f(v)} + \frac{p(vw)}{g(vw)} \right\rceil \leq 2 \chi'_{fg}(G),$$

this algorithm has the approximation ratio 2. In the remaining of this section we present a proof for this upper bound. We present in Section 4.2 two techniques of fg -colorings, and prove the bound using these techniques in Section 4.3.

4.2 Techniques of fg -colorings

In this section we present some notations and two coloring techniques: "switching an alternating walk" and "shifting a fan."

Denote by $u(G)$ the upper bound which we wish to verify, that is

$$u(G) = \max_{v, w \in V} \left\lceil \frac{d(v)}{f(v)} + \frac{p(vw)}{g(vw)} \right\rceil.$$

For $S \subseteq V$, define $f(S) = \sum_{v \in S} f(v)$. Let C be the set of $u(G)$ colors available for an fg -coloring of G . The number of α -edges incident to vertex v is denoted by $d(v, \alpha)$, while the number of α -edges in $E(vw)$ is denoted by $p(vw, \alpha)$. Define $m(v, \alpha) = f(v) - d(v, \alpha)$ and $m(vw, \alpha) = g(vw) - p(vw, \alpha)$. Then G is fg -colored if and only if every color α satisfies $m(v, \alpha) \geq 0$ for every vertex $v \in V$ and $m(vw, \alpha) \geq 0$ for every pair of vertices $v, w \in V$. Color α is *available at* v if $m(v, \alpha) \geq 1$. Similarly color α is *available at* $E(vw)$ if $m(vw, \alpha) \geq 1$. We define

$$M(v) = \{ \alpha \in C \mid m(v, \alpha) \geq 1 \}$$

and

$$M(vw) = \{ \alpha \in C \mid m(vw, \alpha) \geq 1 \}.$$

Thus $M(v)$ is the set of colors available at vertex v , while $M(vw)$ is the set of colors available at multiple edges $E(vw)$.

Switching an alternating path is one of the standard techniques for an ordinary edge-coloring [Fiorini and Wilson (1977)]. We also use it with some modifications. A walk is used instead of a path. A *walk* W is a sequence of distinct edges $v_0v_1, v_1v_2, \dots, v_{k-1}v_k$, where the vertices $v_0, v_1, \dots, v_k$ are not necessarily distinct. When the edges of a walk W are colored with two colors α and β alternately, switching W means to interchange the

colors α and β of the edges in W . We define an “ $\alpha\beta$ -alternating walk” so that its switch would preserve an fg -coloring of G , as follows.

Let $G(\alpha, \beta)$ be the subgraph of G induced by all α - and β -edges. Delete successively all pairs of an α -edges and a β -edges joining the same two vertices whenever there exists, and let $G^*(\alpha, \beta)$ be the resulting graph which does not have any such pair. Denote by $E^*(vw)$ the set of multiple edges joining vertices v and w in $G^*(\alpha, \beta)$. Obviously each set $E^*(vw)$ contains either α -edges or β -edges exclusively.

An $\alpha\beta$ -alternating walk $W = v_0v_1, v_1v_2, \dots, v_{k-1}v_k$ is a walk of length one or more in $G^*(\alpha, \beta)$ such that
 (i) the edges in W are colored alternately with α and β (the i th edge e_i of W is colored β if i is an odd number; otherwise e_i is colored α);
 (ii) if W is not a cycle then $m(v_0, \alpha) \geq 1$, while if W is a cycle of odd length then $m(v_0, \alpha) \geq 2$; and
 (iii) if W is not a cycle and is of even length then $m(v_k, \beta) \geq 1$, while if W is not a cycle and is of odd length then $m(v_k, \alpha) \geq 1$.

Thus a cycle in $G^*(\alpha, \beta)$ is an $\alpha\beta$ -alternating walk (cycle) whenever it has even length and its edges are colored α and β alternately. The following lemma holds.

Lemma 1. Let G be fg -colored, and let W be any $\alpha\beta$ -alternating walk in G . Furthermore, let m' represent the function m with respect to the new coloring after switching W . Then the following (a)–(c) hold.

- (a) Switching W preserves an fg -coloring of G , that is, every color c satisfies $m'(v, c) \geq 0$ for any $v \in V$ and $m'(vw, c) \geq 0$ for any $vw \in E$.
- (b) For any $vw \in E$

$$m'(vw, \alpha) \geq \min \{m(vw, \alpha), m(vw, \beta)\}$$

and

$$m'(vw, \beta) \geq \min \{m(vw, \alpha), m(vw, \beta)\}.$$

- (c) If W passes through an α -edge in $E(vw)$ then $m'(vw, \alpha) > 0$, while if W passes through a β -edge in $E(vw)$ then $m'(vw, \beta) > 0$.

Proof. (a) Clearly switching W does not change $m(v, c)$ or $m(vw, c)$ for any color $c \neq \alpha, \beta$. Also switching W does not change $m(vw, \alpha)$ and $m(vw, \beta)$ unless W passes through an edge in $E(vw)$. Thus we shall verify that $m'(v, \alpha) \geq 0$ and $m'(v, \beta) \geq 0$ for every vertex v on W and that $m'(vw, \alpha) \geq 0$ and $m'(vw, \beta) \geq 0$ for every edge vw on W .

Switching W may change $m(v, c)$, $c = \alpha$ or β , only at the start vertex v_0 or the end vertex v_k of W . However conditions (ii) and (iii) above implies that after switching W , $m'(v, c) \geq 0$ for $c = \alpha, \beta$ and $v = v_0, v_k$.

Let vw be an edge in W . One may assume without loss of generality that vw is an α -edge and hence $E^*(vw)$ contains only α -edges. Then

$$m(vw, \beta) = m(vw, \alpha) + |E^*(vw)|$$

and

$$m(vw, \alpha) = \min \{m(vw, \alpha), m(vw, \beta)\}.$$

Since switching W decreases the number of α -edges in $E(vw)$ at least one,

$$m'(vw, \alpha) > m(vw, \alpha) \geq 0.$$

Since switching W increases the number of β -edges in $E(vw)$ at most $|E^*(vw)|$,

$$m'(vw, \beta) \geq m(vw, \beta) - |E^*(vw)| = m(vw, \alpha) \geq 0.$$

This completes the proof of (a). The proof of (b) and (c) is also implicit in the proof above.

Q.E.D.

Lemma 1(b) implies that if $m(vw, \alpha), m(vw, \beta) \geq 1$ then $m'(vw, \alpha), m'(vw, \beta) \geq 1$ after switching any $\alpha\beta$ -alternating walk.

We denote by $W(\alpha, \beta, v_0)$ an $\alpha\beta$ -alternating walk which starts at vertex v_0 and is not a cycle of even length. Switching $W(\alpha, \beta, v_0)$ may change $m(v, \alpha)$ or $m(v, \beta)$ only if v is the start or end vertex. On the other hand, switching an $\alpha\beta$ -alternating cycle of even length changes neither $m(v, \alpha)$ nor $m(v, \beta)$ for any $v \in V$.

Lemma 2. Let $v_0 \in V$, $\alpha, \beta \in C$, and $\alpha \in M(v_0)$ in an fg -coloring of a graph $G = (V, E)$. Then the following (a) and (b) hold.

- (a) If $G^*(\alpha, \beta)$ has a β -edge e_1 incident to vertex v_0 , then G has an $\alpha\beta$ -alternating walk W starting at v_0 and passing through e_1 .

- (b) If $\beta \notin M(v_0)$, then G has a walk $W(\alpha, \beta, v_0)$.

Proof. (a) Note first that for every vertex v the difference $d(v, \alpha) - d(v, \beta)$ with respect to G is the same as that with respect to $G^*(\alpha, \beta)$. Then one can construct an $\alpha\beta$ -alternating walk W as follows. Choose the β -edge $e_1 \in E(v_0v_1)$ as the first edge of W . If $m(v_1, \alpha) \geq 1$, then the single edge e_1 is an $\alpha\beta$ -alternating walk. So suppose that

$m(v_1, \alpha) = 0$. Then $d(v_1, \alpha) - d(v_1, \beta) \geq 0$, and consequently $G^*(\alpha, \beta)$ has an α -edge $e_2 \in E^*(v_1 v_2)$ incident to v_1 . Add the α -edge e_2 to W as the second edge. Similarly repeat adding an edge to W , choosing alternately α - and β -edges which have not been included in W so far, until the condition (ii) and (iii) above are satisfied for the start and end vertices of W .

Especially when W returns to the start vertex v_0 , we proceed the construction of W as follows. If W returns to v_0 with an α -edge, then end the construction of W . In this case an $\alpha\beta$ -alternating cycle W of even length is obtained. Also if W returns to v_0 with a β -edges and $m(v_0, \alpha) \geq 2$, then end the construction of W . In this case an $\alpha\beta$ -alternating cycle of odd length is obtained. If W returns to v_0 with a β -edge but $m(v_0, \alpha) = 1$, then add to W an α -edge incident to v_0 and continue the construction of W ; since $d(v_0, \alpha) - d(v_0, \beta) \geq -1$, $G^*(\alpha, \beta)$ contains such an α -edge which has not been included in W so far.

(b) Since $m(v_0, \alpha) \geq 1$ and $m(v_0, \beta) = 0$, $d(v_0, \beta) - d(v_0, \alpha) \geq 1$ and hence $G^*(\alpha, \beta)$ has a β -edge incident to v_0 . Therefore (a) above implies that there is an $\alpha\beta$ -alternating walk W starting at v_0 . Since $d(v_0, \beta) - d(v_0, \alpha) \geq 1$, one can choose a β -edge incident to v_0 which has not been included in W so far whenever W returns to v_0 with an α -edge. Thus one can construct an $\alpha\beta$ -alternating walk $W(\alpha, \beta, v_0)$ which is not a cycle of even length. Q.E.D.

In the case of an ordinary or f -coloring, if the ends of an uncolored edge vw have a common available color c , that is, $c \in M(v) \cap M(w)$, then the coloring of G proceeds with coloring vw by c . It is not the case when fg -coloring a graph. For, if $m(vw, c) = 0$, then one cannot color an uncolored edge vw with a color $c \in M(v) \cap M(w)$. Thus an uncolored edge $e \in E(vw)$ can be colored with color c only if $c \in M(v) \cap M(w) \cap M(vw)$. Such a color does not always exist in a partial fg -coloring of G . [We will later show that any partial fg -coloring of G with $u(G)$ colors can always be altered so that there is a color $c \in M(v) \cap M(w) \cap M(vw)$ for the ends of an uncolored edge vw .] However, by a simple counting argument one can show that both $M(v) \cap M(vw)$ and $M(w) \cap M(vw)$ contain at least one color, as follows.

Lemma 3. Let G be fg -colored with at most $u(G)$ colors in C . Then for any edge $vw \in E$

$$\sum_{\alpha \in M(vw)} m(v, \alpha) \geq 1. \quad (1)$$

Especially if $g(vw) \leq f(v)$, then

$$\sum_{\alpha \in M(vw)} m(v, \alpha) \geq p(vw). \quad (2)$$

Proof. By the definition of $u(G)$

$$u(G) \geq \frac{d(v)}{f(v)} + \frac{p(vw)}{g(vw)}. \quad (3)$$

Let S be the set of colors not available at $E(vw)$, that is,

$$S = \{\alpha \in C \mid m(vw, \alpha) = 0\} = C - M(vw).$$

Let t be the number of edges in $E(vw)$ which are colored with colors in S , that is

$$t = |S| \cdot g(vw). \quad (4)$$

Then we have

$$\sum_{\alpha \in M(vw)} m(v, \alpha) = \sum_{\alpha \in M(vw)} \{f(v) - d(v, \alpha)\} \geq f(v)\{u(G) - |S|\} - \{d(v) - t\}. \quad (5)$$

From equations (3), (4) and (5) we have

$$\begin{aligned} \sum_{\alpha \in M(vw)} m(v, \alpha) &\geq d(v) + f(v) \cdot \frac{p(vw)}{g(vw)} - f(v) \cdot \frac{t}{g(vw)} - d(v) + t \\ &= (p(vw) - t) \frac{f(v)}{g(vw)} + t. \end{aligned}$$

Noting the fact that $p(vw) - t \geq 0$, one can easily verify equations (1) and (2) from the equation above.

Q.E.D.

Note that if $g(vw) \leq \max\{f(v), f(w)\}$ then either

$$\sum_{\alpha \in M(vw)} m(v, \alpha) \geq p(vw) \quad \text{or} \quad \sum_{\alpha \in M(vw)} m(w, \alpha) \geq p(vw).$$

“Shifting a fan” is another standard technique of an ordinary edge-coloring. For example, it is employed in the proof of Vizing’s theorem. We also use it with modifying the definition of a fan suitable for an fg -coloring. Let $e_0 = wv_0$ be an uncolored edge. Then a *fan* F is a sequence of distinct edges $e_0, e_1, \dots, e_k$ incident to vertex $w \in V$ such that there is a sequence of distinct colors $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ satisfying the following conditions (a) and

(b), where v_i , $0 \leq i \leq k$, is the end of e_i other than w :

(a) $\alpha_i \in M(v_i) \cap M(wv_i)$, $0 \leq i \leq k-1$; and

(b) e_i , $1 \leq i \leq k$, is colored with α_{i-1} .

Note that vertices $v_0, v_1, \dots, v_k$ are not always distinct. *Shifting a fan F* means to recolor e_i with α_i for each i , $0 \leq i \leq k-1$, and erase the color α_{k-1} of e_k . Shifting F yields another fg -coloring of G in which e_k instead of e_0 is uncolored.

4.3 An upper bound on fg -chromatic index

In this subsection we prove the following theorem.

Theorem 4. Assume without loss of generality that $g(vw) \leq \max \{f(v), f(w)\}$ for any edge $vw \in E$ of a graph $G = (V, E)$. Then

$$\chi'_{fg}(G) \leq \max_{v, w \in V} \left\lceil \frac{d(v)}{f(v)} + \frac{p(vw)}{g(vw)} \right\rceil.$$

Proof. We will prove the claim by induction on the number of edges. Let w be a vertex of degree one or more having the minimum vertex-capacity $f(w)$ among such vertices. Let $e_0 \in E(vw)$. By the induction hypothesis the graph $G - e_0$ obtained from G by deleting e_0 can be fg -colored with $u(G) (\geq u(G - e_0))$ colors. We shall prove that G also can be fg -colored with $u(G)$ colors.

It follows from the selection of w that $g(vw) \leq f(v)$ for any vertex v adjacent to w . Therefore, by Lemma 3, for any vertex v adjacent to w

$$\sum_{\alpha \in M(vw)} m(v, \alpha) \geq p(vw)$$

and

$$\sum_{\alpha \in M(wv)} m(w, \alpha) \geq 1.$$

Noting this fact, we construct a fan F as follows. Clearly a single edge e_0 is a fan. Assume in general that fan $F = e_0, e_1, \dots, e_k$ has been constructed so far. If there exists a color $\alpha_k \in M(wv_k)$ such that $m(w, \alpha_k) + m(v_k, \alpha_k) \geq 2$, then stop the construction of F here and let $F = e_0, e_1, \dots, e_k$. Otherwise, there exists a color $\alpha_k \in M(wv_k)$ such that

- (a) $m(v_k, \alpha_k) = 1$;
- (b) $m(w, \alpha_k) = 0$; and
- (c) if $v_i = v_k$, $0 \leq i \leq k-1$, then $\alpha_k \neq \alpha_i$.

This follows from

$$\sum_{\alpha \in M(wv_k)} m(v_k, \alpha) \geq p(wv_k).$$

If α_k is the same as $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$, then stop the construction of F here, and let $F = e_0, e_1, \dots, e_k$. Otherwise, continue the construction of F with adding to F an α_k -edge $e_{k+1} \in E(wv_{k+1})$ incident to w and adding the color α_k to the color sequence.

Let $F = e_0, e_1, \dots, e_k$ be the fan F constructed as above. Then $\alpha_k \in M(wv_k)$, and one of the following three cases happens.

Case 1: $m(w, \alpha_k) \geq 1$ and $m(v_k, \alpha_k) \geq 1$.

In this case $\alpha_k \in M(w) \cap M(v_k) \cap M(wv_k)$. Furthermore $v_i \neq v_k$, $0 \leq i \leq k-1$, since there is no color $\alpha_i \in M(wv_i)$ such that $m(w, \alpha_i) + m(v_i, \alpha_i) \geq 2$. After shifting F , $\alpha_k \in M(w) \cap M(v_k) \cap M(wv_k)$ as it was. Therefore, shifting F and coloring the new uncolored edge e_k with α_k would complete an fg -coloring of G .

Case 2: $m(v_k, \alpha_k) \geq 2$ or $m(w, \alpha_k) \geq 2$.

Also in this case $v_i \neq v_k$, $0 \leq i \leq k-1$. Assume $m(v_k, \alpha_k) \geq 2$. (The proof for the case $m(w, \alpha_k) \geq 2$ is similar.) By Lemma 3 there is a color $\beta \in M(wv_k)$ such that $m(w, \beta) \geq 1$. One may assume that Case 1 does not occur. Then $m(v_k, \beta) = 0$ and $\beta \neq \alpha_k$. Shift fan F , then $e_k = wv_k$ becomes the new uncolored edge, and $\alpha_k, \beta \in M(wv_k)$, $\alpha_k \in M(v_k)$ and $\beta \notin M(v_k)$. By Lemma 2(b) there exists an $\alpha_k\beta$ -alternating walk $W(\alpha_k, \beta, v_k)$ which starts with v_k and is not a cycle of even length. Switch $W(\alpha_k, \beta, v_k)$, then by Lemma 1(b) $\alpha_k, \beta \in M(wv_k)$ after switching it since $\alpha_k, \beta \in M(wv_k)$ before switching it. If $W(\alpha_k, \beta, v_k)$ ended at w , then $\alpha_k \in M(w) \cap M(v_k) \cap M(wv_k)$ and consequently coloring the uncolored edge $e_k = wv_k$ with color α_k would complete an fg -coloring of G . If $W(\alpha_k, \beta, v_k)$ did not end at w , then $\beta \in M(w) \cap M(v_k) \cap M(wv_k)$ and coloring e_k with β would complete an fg -coloring of G .

Case 3: $m(v_k, \alpha_k) = 1$, $m(w, \alpha_k) = 0$, and $\alpha_k = \alpha_t$ for some t , $0 \leq t < k$.

One may assume that neither Case 1 nor Case 2 occurs. By (c) above $v_t \neq v_k$. By Lemma 3 there exists a color $\beta \in M(w) \cap M(wv_t)$. If $\beta \notin M(v_k)$, then by Lemma 2(b) there is an $\alpha_k\beta$ -alternating walk $W(\alpha_k, \beta, v_k)$ which is not a cycle of even length. If $\beta \in M(v_k)$, then $\beta \notin M(wv_k)$ since Case 1 did not occur, and consequently

$G^*(\alpha_k, \beta)$ contains a β -edge $e' \in E^*(wv_k)$. Therefore by Lemma 2(a) there is an $\alpha_k\beta$ -alternating walk W starting at v_k and passing through e' . Since $m(v_k, \alpha_k) = 1$, W is not a cycle of odd length, but may be a cycle of even length. Thus, no matter whether $\beta \in M(v_k)$ or not, there is an $\alpha_k\beta$ -alternating walk W starting at v_k . Construct a walk W in the following way:

- (i) If $\beta \notin M(wv_k)$, then choose a β -edge in $E^*(wv_k)$ as the first edge of W ;
- (ii) If W reaches w with an α_k -edge, then terminate W there; and
- (iii) If W reaches w with a β -edge, then add to W an α_k -edge other than e_{t+1} , which is incident to w and has not been included in W . (There is such an edge since $m(w, \alpha_k) = 0$ and $m(w, \beta) = 1$ and consequently $d(w, \alpha_k) > d(w, \beta)$.)

The condition (i) above together with Lemmas 1(b) and (c) implies that switching W makes $\beta \in M(wv_k)$ no matter whether $\beta \in M(wv_k)$ or not before switching W . Here $\beta \neq \alpha_0, \alpha_1, \dots, \alpha_{k-1}$ because $m(w, \beta) = 1$ and $m(w, \alpha_i) = 0, 0 \leq i \leq k-1$. Since F is a fan, colors $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ are all distinct. Therefore W does not pass through any edges of F except the α_k -edge e_{t+1} . Furthermore (ii) and (iii) above imply that if W passes through the α_k -edge e_{t+1} , then W must end at w . We separate this case into the following three subcases depending on the end vertex of W .

Case 3.1: W ends at neither v_t nor w .

In this case W passes through none of the edges of F . Switch W , then $m(wv_k, \beta) \geq 1$, and $m(w, \beta) \geq 1$ as it was. Furthermore we claim that F remains to be a fan as it was. Since W passes through none of the edges of F , switching W does not change the coloring of the edges in F at all. Although W may pass through edges in $E(wv_i)$, $1 \leq i \leq k-1$, switching W clearly preserves $\alpha_i \in M(v_i) \cap M(wv_i)$ for each i , $0 \leq i \leq k-1$, except t since colors $\beta, \alpha_0, \alpha_1, \dots, \alpha_{k-1}$ are distinct and $\alpha_i \neq \alpha_k$ if $i \neq t$. On the other hand, since $\alpha_t, \beta \in M(wv_t)$ before switching W , by Lemma 1(b) $\alpha_t, \beta \in M(wv_t)$ after switching W . Furthermore $\alpha_t \in M(v_t)$ as it was since W did not end at v_t . Hence $\alpha_t \in M(v_t) \cap M(wv_t)$ as it was. Thus we have shown that F remains to be a fan as it was.

It W did not end at v_k , that is, W was not a cycle of even length, then switching W makes $\beta \in M(v_k)$. If W ended at v_k , that is, W was a cycle of even length, then $\beta \in M(v_k)$ after switching W since $\beta \in M(v_k)$ before switching W . Thus in either case $\beta \in M(w) \cap M(v_k) \cap M(wv_k)$ after switching W . Therefore shifting F and coloring the new uncolored edge $e_k = wv_k$ with β would complete an fg -coloring of G .

Case 3.2: W ends at v_t .

W passes through none of the edges of F . Switch W , then $m(v_t, \beta) \geq 1$, and $m(w, \beta) \geq 1$ and $\alpha_t, \beta \in M(wv_t)$ as it was. Since colors $\beta, \alpha_0, \alpha_1, \dots, \alpha_{k-1}$ are all distinct, switching W does not destroy the subfan $e_0, e_1, \dots, e_t$ of F . Shift the subfan, then $e_t = wv_t$ becomes the new uncolored edge and $\beta \in M(w) \cap M(v_t) \cap M(wv_t)$. Therefore coloring e_t with β would complete an fg -coloring of G .

Case 3.3: W ends at w .

Since $m(w, \alpha_k) = m(v_t, \beta) = 0$ and $m(w, \beta) = m(v_t, \alpha_t) = 1$, $d(v_t, \alpha_t) < d(v_t, \beta)$ and $d(w, \alpha_t) = d(w, \beta)$ in the graph $G^*(\alpha_k, \beta) - W$ obtained from $G^*(\alpha_k, \beta)$ by deleting all the edges of W . Therefore there exists $W(\alpha_k, \beta, v_t)$ which does not end at w and is edge-disjoint with W . Switching $W(\alpha_k, \beta, v_t)$ makes $\beta \in M(w) \cap M(v_t) \cap M(wv_t)$ and does not destroy the subfan $e_0, e_1, \dots, e_t$ of F . Shifting the subfan and coloring the new uncolored edge e_t with β would complete an fg -coloring of G .

Q.E.D.

Hakimi and Kariv's upper bound for the f -coloring immediately follows from Theorem 4 as a corollary.

Corollary 5. [Hakimi and Kariv (1986)] The f -chromatic index $\chi'_f(G)$ of a graph G satisfies

$$\chi'_f(G) \leq \max_{v, w \in V} \left\lceil \frac{d(v) + p(vw)}{f(v)} \right\rceil.$$

Proof. Define an edge-capacity g in terms of the given vertex-capacity f as follows:

$$g(vw) = \max \{f(v), f(w)\} \text{ for every } vw \in E.$$

Then

$$u(G) = \max_{v, w \in V} \left\lceil \frac{d(v)}{f(v)} + \frac{p(vw)}{g(vw)} \right\rceil \leq \max_{v, w \in V} \left\lceil \frac{d(v) + p(vw)}{f(v)} \right\rceil.$$

Q.E.D.

Vizing's upper bound for the ordinary edge-coloring also follows from Theorem 4 as a corollary.

Corollary 6. [Vizing (1964)] The chromatic index $\chi'(G)$ of a graph G satisfies

$$\chi'(G) \leq \max_{v, w \in V} \{d(v) + p(vw)\}.$$

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