On the First Eigenvalue of the Combinatorial Laplacian for a Graph

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The eigenvalues of the combinatorial Laplacian of graphs with boundaries and infinite graphs without boundary are studied. For a graph with boundary $G=(V \cup \partial V, E \cup \partial E)$, a sharp lower bound of the first eigenvalue $\lambda_1(G)$ is given provided G satisfies a general condition, the so called non-separation property. For an infinite graph G without boundary, the bottom of the spectrum, i.e., the infimum of the spectrum of the combinatorial Laplacian of G , denoted $\lambda_0(G)$, is estimated as

$$\lambda_0(G) \leq \frac{1}{4} \mu(G)^2 \exp(\mu(G)),$$

where $\mu(G)$ is the exponential growth of G . As a corollary, if G is subexponential, $\lambda_0(G) = 0$. On the contrary, $\lambda_0(G) > 0$ is shown for a simply connected infinite graph G with degree ≥ 4 at each vertex.

KEYWORDS: graph, combinatorial Laplacian, eigenvalue, exponential growth

§1. Introduction

In analysis on Riemannian manifolds, there are many studies on the Laplacian (cf. [Bd], [Bs1], [U], [N.U]). Some analogues are known in graph theory concerning eigenvalues of the adjacency matrix or the associated Laplacian (see [Bs2], [D], [D.K], [Fn], [Fa], [M.W], [S] and see [C.D.S] for a general reference). In this respect, our aim is to obtain more analogues in graph theory regarding graphs as Riemannian manifolds.

A graph is a collection of vertices together with a collection of edges joining pairs of vertices. Dodziuk [D] and Friedman [Fn] considered the *combinatorial Laplacian* Δ on a graph $G = (V \cup \partial V, E \cup \partial E)$ with *boundary* and studied the Dirichlet eigenvalue problem. In this paper, we deal with the first eigenvalue $\lambda_1(G)$ of this Dirichlet problem.

The celebrated Faber-Krahn inequality for the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian for a bounded domain Ω in the Euclidean space $\mathbf{R}^n$ is that:

$$\lambda_1(\Omega) \text{Vol}(\Omega)^{2/n} \geq \lambda_1(\Omega^*) \text{Vol}(\Omega^*)^{2/n},$$

and the equality holds only for the case Ω is a ball Ω^* in $\mathbf{R}^n$. This is proved by the so-called symmetrization technique (cf. [Bd]). In this paper, we first consider a symmetrization for a graph and show a sharp lower bound of the first eigenvalue for the combinatorial Laplacian of a graph satisfying the *non-separation property* (cf. §2 and Theorem 6.1):

$$\lambda_1(G) \geq \frac{6\pi^2}{\pi^2 + 6m^2},$$

where m is the number of edges in $E \cup \partial E$. In this connection, we make the following conjecture

Conjecture 1. *The first eigenvalue $\lambda_1(G)$ of the Dirichlet eigenvalue problem of a graph $G = (E \cup \partial E, V \cup \partial V)$ with the non-separation property satisfies the following inequality:*

$$\lambda_1(G) \geq 2 - 2 \cos\left(\frac{\pi}{m}\right),$$

where m is the number of edges of $E \cup \partial E$. Moreover, the equality holds if and only if G is the graph of type A_{m+1} (cf. Example 1 in §2).

Note that our estimate is asymptotically optimal in the sense that it has the same behavior of $2 - 2 \cos(\pi/m)$, the first eigenvalue of the graph of type A_{m+1} when m tends to ∞ . So we propose that the above conjecture might be a natural analogue, in graph theory, of the Faber-Krahn inequality in Riemannian geometry.

Next we study the asymptotic behavior of an infinite graph at infinity. Brooks [Bs1] defined the exponential growth $\mu(M, g)$ of volume of a metric ball $B_p(r)$ of radius r centered at some point p in a complete Riemannian

manifold (M, g) when r tends to infinity. He showed the bottom $\lambda_0(M, g)$ of the spectrum of the Laplacian can be estimated as $\lambda_0(M, g) \leq (1/4)\mu(M, g)^2$ if the volume of (M, g) is infinite. As its corollary, if (M, g) is subexponential, i.e., $\mu(M, g) = 0$, then $\lambda_0(M, g) = 0$. They imply that the analytic quantity can be estimated above by the volume growth. Moreover, McKean [M] showed that if (M, g) is an n -dimensional simply connected Riemannian manifold whose curvature is bounded below by a negative constant k , then $\lambda_0(M, g) \geq ((n-1)^2/4)k$.

We aim to show the analogue in graph theory of the above in Riemannian geometry. We define in §7, two quantities of an infinite graph G without boundary. One is the *bottom of the spectrum* of the combinatorial Laplacian Δ , say $\lambda_0(G)$, which is by definition the infimum of $\lambda_1(L)$, where $L \subset G$ vary in all finite subgraphs of G with boundaries. The other is the *exponential growth*, $\mu(G)$, measures how the graph G spreads into branches at infinity. Then we shall show (cf. Theorem 7.10) that

$$\lambda_0(G) \leq \frac{1}{4} \mu(G)^2 \exp(\mu(G)).$$

As its corollary (cf. Corollary 7.11), if G is subexponential, i.e., $\mu(G) = 0$, then $\lambda_0(G) = 0$. This result means $\lambda_0(G) = 0$ if G spreads slowly of polynomial order at infinity and contrasts entirely with Theorem 7.9, which is a discrete analogue of McKean's theorem generalizing Sunada's result [S] in the case of a tree T_{d+1} of degree $d+1 \geq 3$. Indeed, it holds that $\lambda_0(T_{d+1}) > 0$ if $d+1 \geq 3$ (cf. Corollary 7.7). Our result in Corollary 7.11 has the following intuitive meaning in a population growth model: *Any active society can gain at least one new member within a fixed period of time*, if one regards λ_0 as the base of activity of the society. This is because, if one regards $V(r)$ as the number of members of a society at time r , one may regard μ as the growth ratio of the number of its members.

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§2. Preliminaries

Following Dodziuk [D], Brooks [Bs2] and Friedman [Fn], we prepare some notations concerning the combinatorial Laplacian on graphs with boundaries.

2.1 Graphs with boundaries

A *graph with boundary* means an undirected graph $G = (V \cup \partial V, E \cup \partial E)$ such that (i) each edge in E has both endpoints in V , (ii) each edge in ∂E has exactly one endpoint in V and one in ∂V and (iii) any vertex which has exactly one edge is in ∂V . We call vertices in V (resp. ∂V) the *interior* (resp. *boundary*) vertices, and similarly for the edges. In the figures illustrating the following examples, black circles indicate the vertices in ∂V , and white circles those in V .

Example 1. (The graph of type A_{m+1}) This is a graph consisting of $m+1$ vertices connecting each other by m edges as a segment. See Figure 1.

Example 2. (A ball of radius k in a tree T_{d+1} of degree $d+1$ ($d \geq 2$)) Let T_{d+1} be a tree of degree $d+1$. We fix any vertex v_0 in T_{d+1} . For a positive integer k , the graph consisting of all vertices of distance k from v_0 is called a *ball of radius k* . See Figure 2.

2.2 The combinatorial Laplacian

Let $C_0^0(G)$ be the set of all real valued functions f on $V \cup \partial V$ satisfying $f(x) = 0$ for $x \in \partial V$. Let $C^1(G)$ be the space of all functions φ defined on the set of all directed edges of G and satisfying

$$\varphi([x, y]) = -\varphi([y, x]),$$

where $[x, y]$, $x, y \in V \cup \partial V$ denotes a directed edge in $E \cup \partial E$ beginning at x and ending at y . Here we fix once and for all a direction for each edge of G .

We define the inner products as follows:

$$\begin{cases} (f_1, f_2) := \frac{1}{2} \sum_{x \in V} m(x) f_1(x) f_2(x), \\ (\varphi_1, \varphi_2) := \sum_{\sigma \in E \cup \partial E} \varphi_1(\sigma) \varphi_2(\sigma), \end{cases} \quad (2.1)$$

for $f_1, f_2 \in C_0^0(G)$ and $\varphi_1, \varphi_2 \in C^1(G)$. Here $m(x)$, $x \in V \cup \partial V$ is the number of edges in $E \cup \partial E$ emanating from x . The coboundary operator

$$df([x, y]) := f(y) - f(x),$$

Fig. 1

Fig. 2

maps $C_0^0(G)$ into $C^1(G)$.

The *combinatorial Laplacian* is defined by

$$\Delta f = d^* df, \quad f \in C_0^0(G),$$

where d^* is the adjoint of d with respect to the above inner products. We get by definition

$$(\Delta f_1, f_2) = (df_1, df_2), \quad f_1, f_2 \in C_0^0(G). \quad (2.2)$$

Moreover we obtain:

Lemma 2.3. *For all $f \in C_0^0(G)$, we have*

$$\Delta f(x) = -\frac{2}{m(x)} \sum_{y \sim x} f(y) + 2f(x), \quad x \in V,$$

where $y \sim x$ means that x and y are connected by an edge in $E \cup \partial E$.

The proof follows from a simple calculation using (2.1), (2.2) and the definition of df . $\square$

The *eigenvalue* λ of Δ on $C_0^0(G)$ means that there exists a non-vanishing function $f \in C_0^0(G)$ such that $\Delta f(x) = \lambda f(x)$, $x \in V$, where f is called the *eigenfunction* with the eigenvalue λ . This means that f and λ satisfy the Dirichlet eigenvalue problem:

$$\begin{cases} \Delta f(x) = \lambda f(x), & x \in V, \\ f(x) = 0, & x \in \partial V. \end{cases}$$

Note that each eigenvalue of the Dirichlet eigenvalue problem is positive. Because each $f \in C_0^0(G)$ satisfies

$$(\Delta f, f) = (df, df) \geq 0,$$

and 0 cannot be an eigenvalue of Δ on $C_0^0(G)$ since $df([x, y]) = 0$ for all $[x, y] \in E \cup \partial E$, f is constant on $V \cup \partial V$.

Let us denote the eigenvalues of Δ on $C_0^0(G)$ as

$$0 < \lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_k(G),$$

where k is the number of V .

2.3 The geometric realization of a graph

Following Friedman [Fn], we define the *geometric realization* $\mathcal{G}$ of a graph $G = (V \cup \partial V, E \cup \partial E)$ with boundary by gluing an interval of length 1 between two vertices x and y in $V \cup \partial V$ connected by an edge $e = [x, y] \in E \cup \partial E$. We get then a metric space which is a 1-dimensional manifold except for vertices and we can define the notion of differentiability of a function excepting vertices.

Let $C_0^1(\mathcal{G})$ be the set of all real valued continuous functions on $\mathcal{G}$ which are differentiable on $\mathcal{G} - (V \cup \partial V)$ and have compact supports in $\mathcal{G} - (V \cup \partial V)$. The space $C_0^0(G)$ can be embedded in $C_0^1(\mathcal{G})$ by extending each $f \in C_0^0(G)$ as an edgewise linear function on $\mathcal{G}$ denoted by the same letter f . In the sequel, we always identify $C_0^0(G)$ with its image in $C_0^1(\mathcal{G})$. Then, for $f \in C_0^0(G)$, it holds that

$$(df, df) = \int_{\mathcal{G}} |\nabla f|^2 d\mu, \quad (2.4)$$

where $\nabla f(x)$ is the usual differential of f at $x \in \mathcal{G} - (V \cup \partial V)$ and $d\mu$ is the usual Lebesgue measure on $\mathcal{G}$. Because,

$$(df, df) = \sum_{[x,y] \in E \cup \partial E} (f(y) - f(x))^2 = \sum_{[x,y] \in E \cup \partial E} \int_{[x,y]} |\nabla f|^2 d\mu = \int_{\mathcal{G}} |\nabla f|^2 d\mu,$$

for all $f \in C_0^0(G)$.

Therefore, if $f \in C_0^0(G)$ is the eigenfunction of Δ with the eigenvalue λ , then

$$\lambda = \frac{\int_{\mathcal{G}} |\nabla f|^2 d\mu}{(f, f)}. \quad (2.5)$$

Indeed, by (2.2) and (2.4), it holds that

$$\lambda = \frac{(\Delta f, f)}{(f, f)} = \frac{(df, df)}{(f, f)} = \frac{\int_{\mathcal{G}} |\nabla f|^2 d\mu}{(f, f)}.$$

For $f \in C_0^0(\mathcal{G})$, the ratio

$$\mathcal{R}(f, \mathcal{G}) := \frac{\int_{\mathcal{G}} |\nabla f|^2 d\mu}{(f, f)}$$

is called the *Rayleigh quotient* of f .

2.4 The non-separation property

In this subsection, we introduce the following notion:

Definition 2.6. It is said a graph with boundary $G = (V \cup \partial V, E \cup \partial E)$ has the *non-separation property* if each connected component of the complement, say $\mathcal{G} - \{v\}$, of each vertex $v \in V$ contains at least one boundary vertex in ∂V .

Fig. 3

Example 2.7. A simply connected graph G , i.e., containing no cycle, has the non-separation property because of property (iii) in the definition of a graph with boundary (cf. §2.1). The examples above (Fig. 3) illustrate the non-separation property: the first example satisfies the non-separation property but the second does not.

By definition of the non-separation property, we get immediately:

Lemma 2.8. *Let $G = (V \cup \partial V, E \cup \partial E)$ be a graph with the non-separation property. Let f be a continuous function on the geometric realization $\mathcal{G}$ vanishing at the boundary $\partial \mathcal{G}$. Then the level set of f , $\{p \in \mathcal{G}; f(p) = t\}$, contains at least two points.*

3. Some formulas

We show that:

Proposition 3.1. *For an edgewise linear function f on $\mathcal{G}$ satisfying $f = 0$ on ∂V for a graph $G = (V \cup \partial V, E \cup \partial E)$, we have*

$$\begin{aligned} 3 \int_{\mathcal{G}} f^2 d\mu &= 2(f, f) + \sum_{x \sim y} f(x)f(y) \\ &= 3(f, f) - \frac{1}{2}(f, \Delta f). \end{aligned}$$

Proof. For an edgewise linear function f on $\mathcal{G}$ satisfying $f = 0$ on ∂V , we have

$$\begin{aligned} 3 \int_{\mathcal{G}} f^2 d\mu &= \sum_{[x,y] \in E \cup \partial E} 3 \int_{[x,y]} f^2 d\mu \\ &= \sum_{[x,y] \in E \cup \partial E} (f(x)^2 + f(y)^2 + f(x)f(y)) \\ &= \sum_{x \in V \cup \partial V} m(x)f(x)^2 + \sum_{x \sim y} f(x)f(y), \end{aligned}$$

which is the first equality. For the second equality, notice that

$$\left(\frac{1}{2}\Delta - I\right)f(x) = -\frac{1}{m(x)} \sum_{y \sim x} f(y), \quad x \in V,$$

where I is the identity operator. Then we have

$$\begin{aligned} \sum_{x \sim y} f(x)f(y) &= \frac{1}{2} \sum_x f(x) \sum_{y \sim x} f(y) \\ &= -\frac{1}{2} \sum_x f(x)m(x) \left(\frac{1}{2}\Delta - I\right)f(x) \\ &= \frac{1}{2} \sum_x m(x)f^2(x) - \frac{1}{2} \sum_x m(x)f(x) \frac{1}{2}\Delta f(x) \\ &= (f, f) - \frac{1}{2}(f, \Delta f). \end{aligned}$$

This implies the second equality by means of the first equality. $\square$

Immediately, we obtain:

Corollary 3.2. *Let $\lambda_i(G)$, $1 \leq i \leq k$ be the eigenvalue of the Dirichlet eigenvalue problem for a graph $G = (V \cup \partial V, E \cup \partial E)$:*

$$\begin{cases} \Delta f = \lambda f & \text{on } V, \\ f = 0 & \text{on } \partial V. \end{cases}$$

If we extend the eigenfunction f with the eigenvalue $\lambda_i(G)$ on $V \cup \partial V$ to an edgewise linear function on $\mathcal{G}$ denoted by the same letter, then we have

$$3 \int_{\mathcal{G}} f^2 d\mu = \left(3 - \frac{1}{2}\lambda_i(G)\right)(f, f). \quad (3.3)$$

Thus the eigenvalues $\lambda_i(G)$, $1 \leq i \leq k$, satisfy $0 < \lambda_i(G) < 6$, really $0 < \lambda_i(G) < 4$.

4. Co-area formula

To estimate the first eigenvalue $\lambda_1(G)$ of the Dirichlet eigenvalue problem for a graph $G = (V \cup \partial V, E \cup \partial E)$, we need the following co-area formula:

Proposition 4.1. Let $f \in C_0^1(\mathcal{G})$ and h a measurable function on $\mathcal{G}$.

(i) Then we have the co-area formula:

$$\int_{\mathcal{G}} h |\nabla f| d\mu = \int_{-\infty}^{\infty} \sum_{p \in \mathcal{G}, f(p)=t} h(p) dt.$$

(ii) If we put $\mathcal{G}_t = \{p \in \mathcal{G}; f(p) > t\}$, then $\text{Vol}(\mathcal{G}_t)$, the length of $\mathcal{G}_t$, is differentiable in t almost everywhere and at any regular point $t = f(x)$, i.e., $|\nabla f|(x) > 0$,

$$\frac{d}{dt} \text{Vol}(\mathcal{G}_t) = - \sum_{p \in \mathcal{G}, f(p)=t} |\nabla f|^{-1}(p).$$

Proof. Since $\mathcal{G}$ is a finite union of edges in $E \cup \partial E$, one may prove it on each edge. But this is a special case of the usual co-area formula for a Riemannian manifold with boundary. Indeed, for a regular value $t = f(x)$, it holds that

$$|\nabla f| d\mu = \begin{cases} f'(x) dx = dt, & \text{if } f'(x) > 0, \\ -f'(x) dx = -dt, & \text{if } f'(x) < 0. \end{cases}$$

Therefore we have, for an interval $[a, b]$,

$$\int_a^b h(x) |f'(x)| dx = \int_{-\infty}^{\infty} \sum_{f(p)=t} h(p) dt,$$

which yields (i). Putting $h = 1/|\nabla f|$ at a regular value $t = f(x)$,

$$\text{Vol}(\mathcal{G}_{t+s}) - \text{Vol}(\mathcal{G}_t) = - \int_t^{t+s} \sum_{p \in \mathcal{G}, f(p)=r} |\nabla f|^{-1}(p) dr,$$

by (i). Therefore we obtain (ii). $\square$

5. Symmetrization

For a continuous function f on the geometric realization $\mathcal{G}$ of a connected finite graph with boundary $G = (V \cup \partial V, E \cup \partial E)$, we consider a symmetrization f^* on the interval $\mathcal{G}^*$ of the real line $\mathbf{R}$ with length $\#(E \cup \partial E)$ (the number of edges in $E \cup \partial E$) as follows.

For every real $t \in \mathbf{R}$, let us consider

$$\mathcal{G}_t := \{x \in \mathcal{G}; f(x) > t\}.$$

Let us consider an interval

$$\mathcal{G}_t^* := \{x \in \mathbf{R}; |x| < r\},$$

around the origin in the real line $\mathbf{R}$ with which radius r is given by

$$r = \frac{1}{2} \mu(\mathcal{G}_t).$$

Here μ is the Lebesgue measure of $\mathcal{G}$. Also, let $\mathcal{G}^*$ be $\mathcal{G}_t^*$ for $t < \min f$.

Let us define a continuous function f^* on $\mathcal{G}^*$ by

$$f^*(x) = t, \quad x \in \partial \mathcal{G}_t^*,$$

where $\partial \mathcal{G}_t^* = \{x \in \mathbf{R}; |x| = r\}$. We call $(f^*, \mathcal{G}^*)$, the *symmetrization* of $(f, \mathcal{G})$.

Then we have:

Proposition 5.1. Let $(f^*, \mathcal{G}^*)$ be a symmetrization of $(f, \mathcal{G})$ for a continuous function f on the geometric realization for a graph $G = (V \cup \partial V, E \cup \partial E)$.

(i) Then we have

$$\int_{\mathcal{G}} f^2 d\mu = \int_{\mathcal{G}^*} f^{*2} d\mu,$$

where $d\mu$ are the Lebesgue measures on $\mathcal{G}$ and $\mathcal{G}^*$, respectively.

(ii) If $f = 0$ on $\partial \mathcal{G} = \partial V$, then $f^* = 0$ on $\partial \mathcal{G}^*$.

(iii) Assume that G satisfies the non-separation property. If a continuous function f on $\mathcal{G}$ is differentiable on $\mathcal{G} - (V \cup \partial V)$, it holds that

$$\int_{\mathcal{G}} |\nabla f|^2 d\mu \geq \int_{\mathcal{G}^*} |\nabla f^*|^2 d\mu.$$

Remark. Proposition 5.1 does not mean that

$$\mathcal{R}(f, \mathcal{G}) \geq \mathcal{R}(f^*, \mathcal{G}^*),$$

because (f^*, f^*) and $\mathcal{R}(f^*, \mathcal{G}^*)$ are not well defined.

Proof. We only give a proof of (iii) since (i) and (ii) are clear. For f on $\mathcal{G}$, applying the co-area formula (Proposition 4.1) by putting $h = |\nabla f|$,

$$\int_{\mathcal{G}} |\nabla f|^2 d\mu = \int_{-\infty}^{\infty} \left\{ \sum_{p \in \partial \mathcal{G}_t} |\nabla f|(p) \right\} dt, \quad (5.2)$$

where we put $\partial \mathcal{G}_t = \{p \in \mathcal{G}; f(p) = t\}$. For a regular value $t \in \text{Range}(f)$, by the Cauchy-Schwarz inequality,

$$\sum_{p \in \partial \mathcal{G}_t} |\nabla f|(p) \geq \frac{(\sum_{p \in \partial \mathcal{G}_t} 1)^2}{\sum_{p \in \partial \mathcal{G}_t} |\nabla f|^{-1}(p)}. \quad (5.3)$$

Here, by Lemma 2.8, we have

$$\sum_{p \in \partial \mathcal{G}_t} 1 \geq \sum_{p \in \partial \mathcal{G}_t^*} 1 = 2. \quad (5.4)$$

Applying the co-area formula for the function h defined by

$$h(x) = \begin{cases} 1, & f(x) \geq t, \\ 0, & f(x) < t, \end{cases}$$

we obtain

$$\text{Vol}(\mathcal{G}_t) = \int_{\mathcal{G}_t} h(x) d\mu = \int_t^{\infty} \left(\sum_{p \in \partial \mathcal{G}_s} |\nabla f|^{-1}(p) \right) ds.$$

Differentiating both sides of the above in t ,

$$\frac{d}{dt} \text{Vol}(\mathcal{G}_t) = - \sum_{p \in \partial \mathcal{G}_t} |\nabla f|^{-1}(p) \quad (5.5)$$

at a regular value t . Making use of (5.5) and the equality $\text{Vol}(\mathcal{G}_t) = \text{Vol}(\mathcal{G}_t^*)$ which follows by definition,

$$\sum_{p \in \partial \mathcal{G}_t} |\nabla f|^{-1}(p) = \sum_{p \in \partial \mathcal{G}_t^*} |\nabla f^*|^{-1}(p). \quad (5.6)$$

Moreover, since $|\nabla f^*|$ is constant on $\partial \mathcal{G}_t^*$, the Cauchy-Schwarz inequality is in turn equality so we obtain

$$\sum_{p \in \partial \mathcal{G}_t^*} |\nabla f^*|(p) = \frac{(\sum_{p \in \partial \mathcal{G}_t^*} 1)^2}{\sum_{p \in \partial \mathcal{G}_t^*} |\nabla f^*|^{-1}(p)}. \quad (5.7)$$

Substituting (5.4), (5.6) and (5.7) into (5.3), we have

$$\sum_{p \in \partial \mathcal{G}_t} |\nabla f|(p) \geq \sum_{p \in \partial \mathcal{G}_t^*} |\nabla f^*|(p),$$

Together with (5.2), we have

$$\int_{\mathcal{G}} |\nabla f|^2 d\mu \geq \int_{\mathcal{G}^*} |\nabla f^*|^2 d\mu.$$

Thus we obtain (iii). $\square$

6. Inequality of the first eigenvalue

Our main theorem is

Theorem 6.1. *Let $\lambda_1(G)$ be the first eigenvalue of the Dirichlet eigenvalue problem for a graph $G = (V \cup \partial V, E \cup \partial E)$ with the non-separation property. Then we have*

$$\lambda_1(G) \geq \frac{6\pi^2}{\pi^2 + 6\#(E \cup \partial E)^2}.$$

Proof. Let f be the eigenfunction with the eigenvalue $\lambda_1(G)$ of the Dirichlet eigenvalue problem for a graph $G = (V \cup \partial V, E \cup \partial E)$. Extend f to an edgewise linear function on $\mathcal{G}$, denoted by the same letter f , and take its

symmetrization $(f^*, \mathcal{G}^*)$. By Propositions 3.1 and 5.1, we obtain

$$\begin{aligned} \lambda_1(G) &= \frac{\int_{\mathcal{G}} |\nabla f|^2 d\mu}{(f, f)} \\ &= \frac{3 - \frac{1}{2} \lambda_1(G) \int_{\mathcal{G}} |\nabla f|^2 d\mu}{3 \int_{\mathcal{G}} f^2 d\mu} \\ &\geq \frac{3 - \frac{1}{2} \lambda_1(G) \int_{\mathcal{G}^*} |\nabla f^*|^2 d\mu}{3 \int_{\mathcal{G}^*} f^{*2} d\mu} \\ &\geq \frac{3 - \frac{1}{2} \lambda_1(G)}{3} \mu_1(\mathcal{G}^*), \end{aligned}$$

where $\mu_1(\mathcal{G}^*)$ is the first eigenvalue of the Dirichlet eigenvalue problem of the usual Laplacian $-(d^2/dx^2)$ for the interval $\mathcal{G}^*$ in $\mathbf{R}$:

$$\begin{cases} -\frac{d^2}{dx^2} f = \mu f & \text{on } \mathcal{G}^*, \\ f = 0 & \text{on } \partial \mathcal{G}^*. \end{cases}$$

Therefore, we have

$$\lambda_1(G) \geq \frac{6\mu_1(\mathcal{G}^*)}{6 + \mu_1(\mathcal{G}^*)}.$$

Since it is well known that

$$\mu_1(\mathcal{G}^*) = \frac{\pi^2}{\#(E \cup \partial E)^2},$$

we obtain

$$\lambda_1(G) \geq \frac{6\pi^2}{\pi^2 + 6\#(E \cup \partial E)^2},$$

which is our theorem. $\square$

Remark 6.2. For a graph of type A_{m+1} (cf. Example 1), the eigenvalues of Δ are $2(1 - \cos(j\pi/m))$, $j = 1, 2, \dots, m-1$. Then, the first eigenvalue $\lambda_1(G)$ is

$$\lambda_1(G) = 2 \left(1 - \cos \frac{\pi}{m} \right) \sim \frac{\pi^2}{m^2}, \quad \text{as } m \rightarrow \infty.$$

In this case, our lower bound $6\pi^2/(\pi^2 + 6m^2)$ is asymptotically π^2/m^2 as $m \rightarrow \infty$. Therefore our estimate is optimal asymptotically as $\#(E \cup \partial E) \rightarrow \infty$. This is a reason why we propose Conjecture 1 in the Introduction.

Remark 6.3. If G is a ball of radius k in a tree T_{d+1} of degree $d+1$ (cf. Example 2), the number of $E \cup \partial E$ is

$$\#(E \cup \partial E) = \frac{d+1}{d-1} (d^k - 1).$$

Therefore, by Theorem 6.1, we get

$$\lambda_1(G) \geq \frac{6\pi^2}{\pi^2 + 6 \left[\frac{d+1}{d-1} (d^k - 1) \right]^2}.$$

7. Asymptotic behavior of infinite graphs

Subsequently, we deal with infinite graphs and study asymptotic behaviors of several quantities.

Definition 7.1. For an infinite graph $G = (V, E)$ without boundary, let us define the *bottom of the spectrum* as follows:

$$\lambda_0(G) = \inf_{L \subset G} \lambda_1(L), \quad (7.2)$$

where $L \subset G$ runs over all finite subgraphs of G with boundaries and $\lambda_1(L)$ is the first eigenvalue of the Dirichlet eigenvalue of L . Let us define the *bottom of the essential spectrum* as follows:

$$\lambda_0^{\text{ess}}(G) = \lim_K \lambda_0(G - K), \quad (7.3)$$

where K runs over finite subgraphs of G without boundaries, tending to G , $G - K$ is a subgraph of G with boundary which is defined by the complement of K in G and $\lambda_0(G - K)$ is the above bottom of the spectrum for $G - K$.

Note that, by definition,

$$\lambda_0(G) \leq \lambda_0^{\text{ess}}(G).$$

Definition 7.4. Let the *exponential growth* of G be defined as

$$\mu(G) = \limsup_{r \rightarrow \infty} \frac{1}{r} \log V(r), \quad (7.5)$$

where for $G = (V, E)$, $V(r)$, $r > 0$, is the volume of a ball $B_p(r)$ with center $p \in V$ and radius r . That is, $B_p(r) = \{x \in V; d(p, x) < r\}$, $V(r)$ is the number of edges connecting two vertices in $B_p(r)$ and $d(x, y)$ is the distance of x and y in V which is defined by the length of the shortest path connecting x and y .

Note that $\mu(G)$ does not depend on choice of a point $p \in V$.

The exponential growth $\mu(G)$ measures how the graph G spreads into branches at infinity. If $G = T_{d+1}$ is a tree of degree $d + 1$, then

$$V(r) = \frac{d+1}{d-1} (d^r - 1),$$

and

$$\mu(G) = \log d > 0, \quad d \geq 2.$$

T. Sunada determined (cf. [S]) the spectrum $\text{Spect}(A)$ of the adjacency operator A , which is defined by

$$Af(x) = \sum_{y \sim x} f(y), \quad x \in V,$$

for a function f on V , as follows:

Theorem 7.6. (T. Sunada) Let $G = T_{d+1}$ be a tree of degree $d + 1$ ($d \geq 2$). Then the spectrum of A is

$$\text{Spect}(A) = [-2\sqrt{d}, 2\sqrt{d}].$$

Since our Laplacian Δ is

$$\Delta = 2 \left(I - \frac{1}{d+1} A \right),$$

in this case, we have:

Corollary 7.7. The bottom of the spectrum of a tree $G = T_{d+1}$ of degree $d + 1$, $d \geq 2$, $\lambda_0(G)$, is

$$\lambda_0(G) = 2 \left(1 - \frac{2\sqrt{d}}{d+1} \right) \geq 2 \left(1 - \frac{2\sqrt{2}}{3} \right) = 0.1144 \dots > 0.$$

More generally, we wish to obtain a discrete analogue of McKean's theorem of the lower estimate of the bottom of the spectrum of the Laplacian of a simply connected Riemannian manifold of negative curvature (cf. [M]):

Theorem 7.8. (H. P. McKean) Let (M, g) be an n -dimensional simply connected Riemannian manifold with sectional curvature bounded above by a negative constant $-k$ ($k > 0$). Then the bottom $\lambda_0(M, g)$ of the spectrum of the Laplacian Δ of (M, g) satisfies

$$\lambda_0(M, g) \geq \frac{(n-1)^2}{4} k.$$

For a graph G , one may regard

$$K(x) = 2 - m(x),$$

as “curvature” of G at vertex x . For an infinite graph G , we define two numbers $k(G)$ and $l(G)$ as follows:

$$k(G) := \min \{m(x); x \in V\},$$

$$l(G) := \sup \{m(x); x \in V\},$$

where $l(G)$ may be infinity. So the “curvature” K of any G satisfies

$$2 - l(G) \leq K(x) \leq 2 - k(G),$$

for each vertex $x \in V$. Now we obtain:

Theorem 7.9. *Let G be a simply connected (i.e., admitting no closed path) infinite graph. Then*

$$\text{Spect}(\Delta) \subset \left[2 \left(1 - 2\sqrt{\frac{1}{k(G)}} \sqrt{1 - \frac{1}{l(G)}} \right), 2 \left(1 + 2\sqrt{\frac{1}{k(G)}} \sqrt{1 - \frac{1}{l(G)}} \right) \right].$$

In particular,

$$\lambda_0(G) \geq 2 \left(1 - 2\sqrt{\frac{1}{k(G)}} \sqrt{1 - \frac{1}{l(G)}} \right).$$

Also, if either (i) $k(G) \geq 5$, (ii) $k(G) = 4$ and $l(G) < \infty$, or (iii) $k(G) = l(G) = 3$, then $\lambda_0(G) > 0$.

We may propose the following conjecture:

Conjecture 2. *Let G be a simply connected infinite graph satisfying $m(x) \geq k \geq 3$ for each vertex $x \in V$. Then*

$$\lambda_0(G) > 0.$$

Moreover, $\lambda_0(G)$ can be estimated below by some positive constant depending only on k .

However we shall also show

Theorem 7.10. *For any infinite graph G ,*

$$\lambda_0^{\text{ess}}(G) \leq \frac{1}{4} \mu(G)^2 \exp \mu(G).$$

Corollary 7.11. *If G is subexponential, i.e., $\mu(G) = 0$, then $\lambda_0^{\text{ess}}(G) = 0$.*

Remark 7.12. That G is subexponential when $\mu(G) = 0$ means that G spreads slowly of polynomial order into branches at infinity.

The next example was provided by A. Katsuda:

Example 3. (Ladder with infinite steps) Let an infinite graph G be a ladder with infinite steps, as in Figure 4. Then $m(x) = 3$ for each vertex $x \in V$, but $\mu(G) = 0$, so by Corollary 7.11, $\lambda_0(G) = 0$. This example shows that one needs simply connectedness of G in Theorem 7.9 and Conjecture 2.

8. Proof of Theorem 7.9

Sunada's idea proving Theorem 7.6 works well in general to prove Theorem 7.9. For any function f on V vanishing outside a finite subset of V , we have also

$$\Delta f = 2(I - P)f,$$

where $Pf(x) := (1/m(x)) \sum_{y \sim x} f(y)$, $x \in V$ (cf. Lemma 2.3). Thus it suffices to show:

Lemma 8.1.

$$|(Pf, f)| \leq 2\sqrt{\frac{1}{k(G)}} \sqrt{1 - \frac{1}{l(G)}} (f, f),$$

for all f on V vanishing outside a finite subset of V .

Proof. For an oriented edge $e = [x, y]$, let the origin of e be $o(e) = x$ and the terminal of e , $t(e) = y$. Since G admits no cycle (i.e. no closed path), we may take an orientation on each edge of E in such a way that for each x

Fig. 4

$\in V$ there exists a unique $e \in E$, such that $x = o(e)$: choose an infinite branch of G and arrange G as in Figure 5.
Then

- (i) the correspondence $e \in E \mapsto o(e) = x \in V$ is one-to-one, and
- (ii) for each $x \in V$,

$$\#\{e \in E; t(e) = x\} = m(x) - 1.$$

Therefore, we obtain

$$\begin{aligned} |(Pf, f)| &= \frac{1}{2} \left| \sum_{x \in V} m(x) Pf(x) f(x) \right| \\ &= \frac{1}{2} \left| \sum_{x \in V} \sum_{y \sim x} f(y) f(x) \right| \\ &= \frac{1}{2} \left| \sum_{e \in E} \{f(t(e))f(o(e)) + f(o(e))f(t(e))\} \right| \\ &= \left| \sum_{e \in E} f(t(e))f(o(e)) \right| \\ &\leq \left(\sum_{e \in E} f(t(e))^2 \right)^{1/2} \left(\sum_{e \in E} f(o(e))^2 \right)^{1/2}. \end{aligned}$$

Here by (i) and (ii), we obtain

$$\begin{aligned} \sum_{e \in E} f(o(e))^2 &= \sum_{x \in V} f(x)^2 = \frac{1}{2} \sum_{x \in V} m(x) \frac{2}{m(x)} f(x)^2 \\ &\leq \frac{2}{k(G)} (f, f), \end{aligned}$$

and

$$\begin{aligned} \sum_{e \in E} f(t(e))^2 &= \sum_{x \in V} (m(x) - 1) f(x)^2 = \frac{1}{2} \sum_{x \in V} m(x) \frac{2(m(x) - 1)}{m(x)} f(x)^2 \\ &\leq 2 \left(1 - \frac{1}{l(G)} \right) (f, f), \end{aligned}$$

since

$$\frac{2(m(x) - 1)}{m(x)} = 2 \left(1 - \frac{1}{m(x)} \right) \leq 2 \left(1 - \frac{1}{l(G)} \right).$$

Therefore, we obtain

$$|(Pf, f)| \leq 2 \sqrt{\frac{1}{k(G)}} \sqrt{1 - \frac{1}{l(G)}} (f, f). \quad \square$$

Fig. 5

9. Proof of Theorem 7.10

To prove Theorem 7.10, we follow Brooks' method [Bs1], which requires the geometric realization $\mathcal{G}$ of an infinite graph G realized in a Hilbert space, and we take the usual 1-dimensional Lebesgue measure $d\mu$ on $\mathcal{G}$. Then Theorem 7.10 is reduced to prove:

Theorem 9.1. *Assume that there exists a positive constant α such that*

$$\alpha^2 \exp(2\alpha) < \lambda_0(G - K) \quad \text{and} \quad \int_{\mathcal{G}-\mathcal{K}} \exp(-2\alpha\rho(x)) d\mu < \infty,$$

where $\rho(x) := d(p, x)$, $x \in \mathcal{G}$. Then, we obtain

$$\int_{\mathcal{G}-\mathcal{K}} \exp(2\alpha\rho(x)) d\mu < \infty.$$

Proof of Theorem 7.10 from Theorem 9.1. Assume Theorem 7.10 would not hold. Then there exists a finite subgraph K of G such that

$$\frac{1}{4} \mu(G)^2 \exp(\mu(G)) < \lambda_0(G - K).$$

Since a real valued function $f(x) = x \exp(x)$ on the positive real half-line is monotone increasing, there exists a positive number α such that $\mu(G)/2 < \alpha$ and $\alpha \exp(\alpha) < \sqrt{\lambda_0(G - K)}$. Then, we have

$$\begin{aligned} \int_{\mathcal{G}} \exp(-2\alpha\rho(x)) d\mu &\leq \sum_{r=1}^{\infty} \{V(r) - V(r-1)\} \exp(-2\alpha(r-1)) \\ &\leq \sum_{r=1}^{\infty} V(r) \exp(-2\alpha r) (\exp(2\alpha) - 1) < \infty, \end{aligned}$$

by definition of $\mu(G)$. By Theorem 9.1,

$$\int_{\mathcal{G}-\mathcal{K}} d\mu \leq \int_{\mathcal{G}-\mathcal{K}} \exp(2\alpha\rho(x)) d\mu < \infty,$$

which contradicts that $G - K$ is an infinite graph. $\square$

10. Proof of Theorem 9.1

Lemma 10.1. *For $\alpha > 0$ and $j = 1, 2, \dots$, define the functions h_j on $\mathcal{G}$ by*

$$h_j(x) := \min\{\alpha\rho(x), -\alpha\rho(x) + j\}, \quad x \in \mathcal{G}.$$

Then

- (i) $|\nabla h_j| \leq \alpha, j = 1, 2, \dots$,
- (ii) $\int_{\mathcal{G}-\mathcal{K}} \exp(2h_j) d\mu < \infty, j = 1, 2, \dots$,
- (iii) h_j increases pointwise to $\alpha\rho$ as $j \rightarrow \infty$, and
- (iv) $\int_e \exp(2h_j) d\mu \leq \frac{1}{2} \exp(2\alpha)(\exp(2h_j(u)) + \exp(2h_j(v)))$.

for each edge $e = [u, v]$.

Proof. For (i), it suffices to note that the function ρ is of the following form on each edge $e = [u, v]$: In the case $\rho(u) = \rho(v)$,

$$\rho(x) = \begin{cases} \rho(u) + d(u, x), & x \in e \text{ and } d(u, x) \leq \frac{1}{2}, \\ \rho(v) + d(v, x), & x \in e \text{ and } d(v, x) \leq \frac{1}{2}. \end{cases}$$

In the case $\rho(u) < \rho(v)$,

$$\rho(x) = \rho(u) + d(u, x), \quad x \in e = [u, v].$$

For (ii), note that $h_j(x) = -\alpha\rho(x) + j$ at $x \in \mathcal{G}$ with large $\rho(x)$. So we obtain

$$\int_{\mathcal{G}-\mathcal{K}} \exp(2h_j) d\mu \leq C + \int_{\mathcal{G}-\mathcal{K}} \exp(-2\alpha\rho(x) + 2j) d\mu < \infty,$$

because of the assumption $\int_{\mathcal{G}-\mathcal{K}} \exp(-2\alpha\rho(x)) d\mu < \infty$. The assertion (iii) follows from the definition of h_j .

For (iv), note that $\exp(2h_j)$ is convex in $\rho(x)$ except only at $\rho(x) = j/(2\alpha)$. If $e = [u, v]$ does not contain such exceptional points, then

$$\exp(2h_j(x)) \leq d(x, u)\rho(u) + (1 - d(x, u))\rho(v), \quad x \in e = [u, v].$$

In this case, we get

$$\int_e \exp(2h_j) d\mu \leq \frac{1}{2} (\exp(2h_j(u)) + \exp(2h_j(v))).$$

If $e = [u, v]$ contains the exceptional points x with $\rho(x) = j/(2\alpha)$, by the definition of h_j , we obtain

$$\begin{aligned} \int_e \exp(2h_j) d\mu &\leq \frac{1}{2} \{ \exp(-2\alpha\rho(u) + 2j) + \exp(2\alpha\rho(v)) \} \\ &\leq \frac{1}{2} \{ \exp(-2\alpha(\rho(v) - 1) + 2j) + \exp(2\alpha(\rho(u) + 1)) \} \\ &= \frac{1}{2} \exp(2\alpha) \{ \exp(2h_j(u)) + \exp(2h_j(v)) \}, \end{aligned}$$

where we used the fact $|\rho(u) - \rho(v)| \leq 1$. $\square$

Let $\{K_i\}_{i=1}^\infty$ be an increasing sequence of finite subgraphs of $G - K$ such that $\cup_i K_i = \mathcal{G} - \mathcal{K}$. We set a sequence of functions on $\mathcal{G} - \mathcal{K}$, $\{\chi_i\}_{i=1}^\infty$, satisfying the following

- (i) $\text{supp}(\chi_i) \subset K_i \subset \mathcal{G} - \mathcal{K}$,
- (ii) $\chi_i(x) = 1$, if $d(x, \mathcal{G} - K_i) \geq 1$, $x \in \mathcal{G} - \mathcal{K}$,
- (iii) $\chi_i(x) = d(x, \mathcal{G} - K_i)$, if $0 \leq d(x, \mathcal{G} - K_i) \leq 1$.

Then we get

- (iv) $|\nabla \chi_i| \leq 1$,
- (v) $\text{supp}(\nabla \chi_i) \subset B_1(\partial K_i) = \{x \in K_i; d(x, \partial K_i) \leq 1\}$.

Moreover, we obtain

$$(vi) \quad \int_{K_i - B_1(\partial K_i)} \exp(2h_j) d\mu \leq \exp(2\alpha)(f, f)_{G-K},$$

where $(f, f)_{G-K}$ is the inner product relative to the graph $G - K$ of the function f which is defined as

$$f(x) := \exp(h_j(x))\chi_i(x), \quad x \in \mathcal{G} - \mathcal{K},$$

for each j and i , since from (iv) of Lemma 10.1,

$$\begin{aligned} \int_{K_i - B_1(\partial K_i)} \exp(2h_j) d\mu &= \int_{K_i - B_1(\partial K_i)} \exp(2h_j) \chi_i^2 d\mu \\ &\leq \frac{1}{2} \exp(2\alpha) \sum_{u \in V} m(u) \exp(2h_j(u)) \chi_i(u)^2 \\ &= \exp(2\alpha)(f, f)_{G-K}, \end{aligned}$$

where V is the set of vertices of G .

Now, we need to estimate

$$\int_{\mathcal{G}-\mathcal{K}} f^2 d\mu = \int_{K_i} f^2 d\mu = \int_{B_1(\partial K_i)} f^2 d\mu + \int_{K_i - B_1(\partial K_i)} f^2 d\mu. \quad (10.2)$$

The second term of the right hand side of (10.2) can be estimated as follows:

$$\int_{K_i - B_1(\partial K_i)} f^2 d\mu \leq C(f, f)_{G-K} \leq C\lambda_0(G - K)^{-1} \int_{\mathcal{G}-\mathcal{K}} |\nabla f|^2 d\mu, \quad (10.3)$$

where $C = \exp(2\alpha)$. Since $\nabla f = \nabla(\exp(h_j)\chi_i) = \exp(h_j)(\chi_i \nabla h_j + \nabla \chi_i)$,

$$\begin{aligned} \int_{\mathcal{G}-\mathcal{X}} |\nabla f|^2 d\mu &= \int_{\mathcal{G}-\mathcal{X}} \exp(2h_j) \{ \chi_i^2 |\nabla h_j|^2 + 2\chi_i \langle \nabla \chi_i, \nabla h_j \rangle + |\nabla \chi_i|^2 \} d\mu \\ &\leq \int_{\mathcal{G}-\mathcal{X}} f^2 |\nabla h_j|^2 d\mu + \int_{\mathcal{G}-\mathcal{X}} \exp(2h_j) \{ 2\chi_i \langle \nabla \chi_i, \nabla h_j \rangle + |\nabla \chi_i|^2 \} d\mu. \end{aligned}$$

Since $|2\chi_i \langle \nabla \chi_i, \nabla h_j \rangle + |\nabla \chi_i|^2| \leq 1 + 2\alpha$ by (i) of Lemma 10.1 and (iv) above, we have, together with (v) above,

$$\int_{\mathcal{G}-\mathcal{X}} |\nabla f|^2 d\mu \leq \int_{\mathcal{G}-\mathcal{X}} f^2 |\nabla h_j|^2 d\mu + (1 + 2\alpha) \int_{B_1(\partial \mathcal{X}_i)} \exp(2h_j) d\mu.$$

Therefore, we obtain

$$\begin{aligned} \int_{\mathcal{G}-\mathcal{X}} f^2 \{1 - C\lambda_0(G - K)^{-1}\alpha^2\} d\mu &\leq \int_{\mathcal{G}-\mathcal{X}} f^2 \{1 - C\lambda_0(G - K)^{-1}|\nabla h_j|^2\} d\mu \\ &\leq \{1 + (1 + 2\alpha)C\lambda_0(G - K)^{-1}\} \int_{B_1(\partial \mathcal{X}_i)} \exp(2h_j) d\mu. \end{aligned}$$

Here, as $i, j \rightarrow \infty$,

$$\int_{B_1(\partial \mathcal{X}_i)} \exp(2h_j) d\mu \rightarrow \int_{B_1(\partial \mathcal{X})} \exp(2\alpha\rho) d\mu,$$

which is finite because of the compactness of $B_1(\partial \mathcal{X})$. Thus, recalling $C = \exp(2\alpha)$, if $1 - C\lambda_0(G - K)^{-1}\alpha^2 > 0$, that is, $\lambda_0(G - K) > \alpha^2 \exp(2\alpha)$, then

$$\int_{\mathcal{G}-\mathcal{X}} \exp(2h_j) \chi_i^2 d\mu = \int_{\mathcal{G}-\mathcal{X}} f^2 d\mu$$

is bounded by a constant not depending on i and j . By (iii) of Lemma 10.1, as $i, j \rightarrow \infty$, we can conclude that $\int_{\mathcal{G}-\mathcal{X}} \exp(2\alpha\rho) d\mu$ is finite, which is what we set out to prove. $\square$

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