Equilibrium States and Their Macroscopic Uniformity in Fermion Lattice Systems

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We study equilibrium states and their macroscopic uniformity in the fermion lattice system satisfying the assumptions presented by Araki and Moriya [5]. We show that every translation-invariant KMS state of the even part of the fermion algebra extends uniquely to a translation-invariant KMS state of the whole fermion algebra. The macroscopic uniformity is established for an extremal translation-invariant equilibrium state of the fermion algebra.

KEYWORDS: fermion algebra, potentials, KMS states, Gibbs condition, macroscopic uniformity

1. Introduction

The fermion lattice system is obtained as the C^* -algebra generated by annihilation and creation operators on lattice sites of $\mathbb{Z}^\nu$, ν being a positive integer. We assume throughout this paper that $\mathcal{A}$ is the fermion algebra and α is a given strongly continuous one-parameter automorphism group of $\mathcal{A}$ satisfying the following assumptions presented in [5]:

- (I) $\alpha_t \Theta = \Theta \alpha_t$ for all $t \in \mathbb{R}$, where Θ is the parity automorphism of $\mathcal{A}$.
- (II) $\mathcal{A}_0 \subset \mathcal{D}(\delta_\alpha)$, where $\mathcal{A}_0$ is the union of all local algebras and δ_α is the generator of α_t with domain $\mathcal{D}(\delta_\alpha)$.
- (III) $\mathcal{A}_0$ is a core for δ_α .
- (IV) $\alpha_t \tau_k = \tau_k \alpha_t$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}^\nu$, where τ is the lattice translation automorphism group of $\mathcal{A}$.

In Section 3, using von Neumann's measurable cross-section principle and the chemical potential theory [4], we show that every τ -invariant $\alpha|_{\mathcal{A}_+}$ -KMS state of $\mathcal{A}_+$ extends uniquely to a τ -invariant α -KMS state of $\mathcal{A}$, where $\mathcal{A}_+$ is the even part of $\mathcal{A}$. There are several corollaries of the above result. First, the extreme boundary of the simplex of τ -invariant α -KMS states of $\mathcal{A}$ is contained in that of the simplex of τ -invariant states of $\mathcal{A}$. Next, restricting to the case $\nu = 1$, we assume that Φ is a τ -invariant potential in $\mathcal{A}_+$ satisfying the following conditions:

$$\sum_{K \ni 0} \|\Phi(K)\| < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|W_\Phi([1, n])\| < \infty,$$

where $W_\Phi([1, n])$ denotes the surface energy for the interval $[1, n] := \{1, \dots, n\}$ of $\mathbb{Z}$. Under the above conditions, Φ generates the one-parameter dynamics α^Φ on $\mathcal{A}$ and it is well-known [2, 12] that $(\mathcal{A}, \alpha^\Phi)$ has no phase transition. We see that $\mathcal{A}_+$ has a unique $\alpha^\Phi|_{\mathcal{A}_+}$ -KMS state. This seems meaningful because the local algebras are always full matrix algebras in the existing literatures showing no phase transition while local algebras of $\mathcal{A}_+$ are not.

Hiai and Petz studied the macroscopic uniformity for extremal equilibrium states in a spin lattice system [10]. Macroscopic uniformity is a basic feature of statistical mechanical systems (see [10, Sect. 3] for details). In Section 4 we establish the macroscopic uniformity for such states in the fermion lattice system as well.

Appendix is a simple remark concerning the correspondence between τ -invariant potentials having short range and their standard variants introduced in [5].

2. Preliminaries

The definitions and results in this section are entirely based on [5].

2.1 Fermion algebras

By the *fermion algebra* $\mathcal{A}$, we mean a unital C^* -algebra generated by a_i and a_i^* , $i \in \mathbb{Z}^\nu$, satisfying the *canonical anticommutation relations*:

$$\{a_i^*, a_j\} = \delta_{ij} \mathbf{1}, \quad \{a_i^*, a_j^*\} = \{a_i, a_j\} = 0, \quad i, j \in \mathbb{Z}^\nu,$$

where a_i^* is the adjoint of a_i , $\{x, y\} := xy + yx$, $\delta_{ij} := 1$ if $i = j$ and $\delta_{ij} := 0$ otherwise. The elements a_i and a_i^* are called *annihilation* and *creation operators*, respectively. Let $\mathcal{A}_0$ be the $*$ -algebra generated by all a_i and a_i^* , $i \in \mathbb{Z}^\nu$. For each $I \subset \mathbb{Z}^\nu$, the C^* -subalgebra of $\mathcal{A}$ generated by a_i and a_i^* , $i \in I$, is denoted by $\mathcal{A}(I)$. If the cardinality $|I|$ of I is finite, then $\mathcal{A}(I)$ is referred to as the *local algebra* for I . We set $\mathcal{A}(\emptyset) := \mathbb{C}\mathbf{1}$. The *parity automorphism* Θ of $\mathcal{A}$ is defined by $\Theta(a_i) := -a_i$ for all $i \in \mathbb{Z}^\nu$. The *even* and *odd parts* of $\mathcal{A}$ are defined as

$$\mathcal{A}_+ := \{a \in \mathcal{A} : \Theta(a) = a\}, \quad \mathcal{A}_- := \{a \in \mathcal{A} : \Theta(a) = -a\},$$

respectively.

The *translation* τ_k by $k \in \mathbb{Z}^v$ is an automorphism of $\mathcal{A}$ defined by $\tau_k(a_i) := a_{i+k}$ for all $i \in \mathbb{Z}^v$. A state ω of $\mathcal{A}$ is said to be *even* if $\omega \circ \Theta = \omega$. We denote by $\mathfrak{S}(\mathcal{A})$ the state space of $\mathcal{A}$ and by $\mathfrak{S}_\tau(\mathcal{A})$ the set of all τ -invariant states of $\mathcal{A}$. If X is a convex set in $\mathfrak{S}(\mathcal{A})$, we call the set of extreme points of X the *extreme boundary* of X and denote it by $\mathcal{E}X$.

The even part $(\mathcal{A}_+, \tau|_{\mathcal{A}_+})$ is asymptotically abelian in the norm sense while $(\mathcal{A}, \tau)$ is not. In fact, we note that $a \in \mathcal{A}_+$ if and only if the following asymptotically central property holds:

$$\lim_{|k| \rightarrow \infty} \|[\tau_k(a), b]\| = 0 \quad \text{for all } b \in \mathcal{A}, \quad (2.1)$$

where $[x, y] := xy - yx$. If $\omega \in \mathfrak{S}_\tau(\mathcal{A})$, then ω is even [6, 5.2.21] (see also [13]).

2.2 Potentials and derivations

For any $I \subset \mathbb{Z}^v$, there exists a unique conditional expectation E_I from $\mathcal{A}$ onto $\mathcal{A}(I)$ such that $\text{tr}(ab) = \text{tr}(E_I(a)b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{A}(I)$, where tr denotes the unique tracial state of $\mathcal{A}$. We denote by $\mathcal{F}$ the family of all finite subsets of $\mathbb{Z}^v$. A mapping Φ from $\mathcal{F}$ into $\mathcal{A}$ is called a *standard potential* if it satisfies the following conditions:

- (P1) $\Phi(I) \in \mathcal{A}(I)$, $\Phi(\emptyset) = 0$.
- (P2) $\Phi(I)^* = \Phi(I)$.
- (P3) $\Theta(\Phi(I)) = \Phi(I)$.
- (P4) $E_J(\Phi(I)) = 0$ if $J \subset I$ and $J \neq I$.
- (P5) For each $I \in \mathcal{F}$, the net $H_\Phi^J(I) := \sum_{K \cap I \neq \emptyset, K \subset J} \Phi(K)$ is a Cauchy net in the norm topology of $\mathcal{A}$ for $J \nearrow \mathbb{Z}^v$.

The real vector space of all standard potentials is denoted by $\mathcal{P}$. We denote by $\mathcal{P}_\tau$ the set of all $\Phi \in \mathcal{P}$ satisfying the following condition:

- (P6) $\tau_k(\Phi(I)) = \Phi(I + k)$ for all $I \in \mathcal{F}$ and $k \in \mathbb{Z}^v$.

For each $\Phi \in \mathcal{P}$ and $I \in \mathcal{F}$, define the *standard Hamiltonian* $H_\Phi(I)$ by

$$H_\Phi(I) := \lim_{J \nearrow \mathbb{Z}^v} H_\Phi^J(I) = \sum_{K \cap I \neq \emptyset} \Phi(K), \quad (2.2)$$

the *internal energy* $U_\Phi(I)$ by

$$U_\Phi(I) := \sum_{K \subset I} \Phi(K),$$

and the *surface energy* $W_\Phi(I)$ by

$$W_\Phi(I) := \sum_K \{\Phi(K) : K \cap I \neq \emptyset, K \cap I^c \neq \emptyset\}.$$

It follows that $\mathcal{P}_\tau$ is a separable real Banach space with the norm $\|\Phi\| := \|H_\Phi(\{0\})\|$. If $\Phi \in \mathcal{P}_\tau$, then

$$\|U_\Phi(I)\| \leq \|H_\Phi(I)\| \leq \|\Phi\| \cdot |I|. \quad (2.3)$$

For each $I \in \mathcal{F}$, we call the largest distance between two points in I the *diameter* of I . A standard potential Φ has *finite range* if $\Phi(I) = 0$ for all I of sufficiently large diameter. We denote by $\mathcal{P}_\tau^f$ the subspace of all $\Phi \in \mathcal{P}_\tau$ having finite range. Then $\mathcal{P}_\tau^f$ is dense in $\mathcal{P}_\tau$.

The real vector space of all $*$ -derivations with domain $\mathcal{A}_0$ and commuting with Θ on $\mathcal{A}_0$ is denoted by $\Delta(\mathcal{A}_0)$. We denote by $\Delta_\tau(\mathcal{A}_0)$ the real vector space of all $\delta \in \Delta(\mathcal{A}_0)$ commuting with τ . For every $\Phi \in \mathcal{P}$, define a $*$ -derivation δ_Φ with domain $\mathcal{A}_0$ by

$$\delta_\Phi(a) := i[H_\Phi(I), a], \quad a \in \mathcal{A}(I). \quad (2.4)$$

The equations (2.2) and (2.4) between $\Phi \in \mathcal{P}_\tau$ and $\delta_\Phi \in \Delta_\tau(\mathcal{A}_0)$ gives a one-to-one real linear mapping from $\mathcal{P}_\tau$ onto $\Delta_\tau(\mathcal{A}_0)$.

For a dynamics α given in Introduction, the $*$ -derivation $\delta \in \Delta_\tau(\mathcal{A}_0)$ is determined uniquely by $\delta := \delta_\alpha|_{\mathcal{A}_0}$, δ_α being the generator of α_t . Moreover, for the $*$ -derivation δ , the τ -invariant standard potential Φ is determined uniquely by the one-to-one correspondence mentioned in the preceding paragraph.

2.3 Variational principle

Let $I \in \mathcal{F}$ and $\omega \in \mathfrak{S}(\mathcal{A})$. We shall denote the restriction of ω to $\mathcal{A}(I)$ by ω_I . The *von Neumann entropy* $S(\omega_I)$ of ω_I is defined by

$$S(\omega_I) := -\omega \left(\log \frac{d\omega_I}{d\text{Tr}_I} \right),$$

where Tr_I denotes the canonical trace on $\mathcal{A}(I)$ such that $\text{Tr}_I(e) = 1$ for all minimal projections e in $\mathcal{A}(I)$. If $\omega \in \mathfrak{S}_\tau(\mathcal{A})$ and $\Phi \in \mathcal{P}_\tau$, then the following van Hove limits exist:

$$\begin{aligned} s(\omega) &:= \text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} S(\omega_I), \\ P(\Phi) &:= \text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \log \text{Tr}_I(e^{-H_\Phi(I)}) = \text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \log \text{Tr}_I(e^{-U_\Phi(I)}), \\ e_\Phi(\omega) &:= \text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \omega(H_\Phi(I)) = \text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \omega(U_\Phi(I)). \end{aligned}$$

We refer to $s(\omega)$, $P(\Phi)$ and $e_\Phi(\omega)$ as the *mean entropy*, *pressure* and *mean energy*, respectively. Define

$$\Lambda(\Phi) := \{\omega \in \mathfrak{S}_\tau(\mathcal{A}) : P(\Phi) = s(\omega) - e_\Phi(\omega)\}.$$

It follows that $\Lambda(\Phi)$ is non-empty. When $\omega \in \Lambda(\Phi)$, we say that ω satisfies the *variational principle* with respect to Φ .

3. Translation-Invariant KMS States

For simplicity we consider only (α, β) -KMS states with $\beta = 1$ and refer to those states as α -KMS states; obviously all our results hold for any β without change. Let $K_\tau(\mathcal{A})$ be the set of all τ -invariant α -KMS states of $\mathcal{A}$. By (IV) and (2.1), $\mathcal{A}_+$ is invariant under α , so that we can consider an $\alpha|_{\mathcal{A}_+}$ -KMS state of $\mathcal{A}_+$. We denote by $K_\tau(\mathcal{A}_+)$ the set of all τ -invariant $\alpha|_{\mathcal{A}_+}$ -KMS states of $\mathcal{A}_+$ and by $\mathfrak{S}_\tau(\mathcal{A}_+)$ the simplex of all τ -invariant states of $\mathcal{A}_+$. For each $\omega \in K_\tau(\mathcal{A}_+)$, let $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ be the cyclic representation of $\mathcal{A}_+$ induced by ω and u_ω a unitary representation of $\mathbb{Z}^v$ implementing τ on $\mathcal{H}_\omega$. Since $\mathcal{A}_+$ is asymptotically abelian with respect to τ due to (2.1), it follows from [6, 4.3.14] that

$$\pi_\omega(\mathcal{A}_+)' \cap u_\omega(\mathbb{Z}^v)' \subset \pi_\omega(\mathcal{A}_+)' \cap \pi_\omega(\mathcal{A}_+)''. \quad (3.1)$$

Hence $K_\tau(\mathcal{A}_+)$ is a face of $\mathfrak{S}_\tau(\mathcal{A}_+)$, and so it is a simplex. Moreover, it is obvious from (3.1) and [17, Lemma 4.7] that

$$\mathcal{E}K_\tau(\mathcal{A}_+) \subset \mathcal{E}\mathfrak{S}_\tau(\mathcal{A}_+). \quad (3.2)$$

For simplicity we shall denote the restriction of a state ρ of $\mathcal{A}$ to $\mathcal{A}_+$ by ρ_+ .

Theorem 3.1. *If $\omega \in K_\tau(\mathcal{A}_+)$, then there exists a unique $\bar{\omega} \in K_\tau(\mathcal{A})$ such that $\bar{\omega}_+ = \omega$.*

Proof. The proof is divided into four steps.

Step 1. We consider the extremal decomposition of ω . Since $K_\tau(\mathcal{A}_+)$ is a simplex, it follows from the Choquet–Meyer theorem [15] that there exists a unique probability measure μ concentrated on $\mathcal{E}K_\tau(\mathcal{A}_+)$ such that

$$\omega(a) = \int_{\mathcal{E}K_\tau(\mathcal{A}_+)} \psi(a) d\mu(\psi), \quad a \in \mathcal{A}_+. \quad (3.3)$$

Step 2. We show that every $\psi \in \mathcal{E}K_\tau(\mathcal{A}_+)$ extends to a weakly τ -clustering state of $\mathcal{A}$. For each $\psi \in \mathcal{E}K_\tau(\mathcal{A}_+)$, define

$$X_\psi := \{\rho \in \mathfrak{S}_\tau(\mathcal{A}) : \rho_+ = \psi\}.$$

We prove that X_ψ is non-empty. First, there is a state $\bar{\psi}$ of $\mathcal{A}$ such that $\bar{\psi}_+ = \psi$. Let M be a Banach limit for bounded sequences. Since $\{\bar{\psi}(\tau^n(a))\}$ is a bounded sequence for all $a \in \mathcal{A}$, we can define the τ -invariant state $\tilde{\psi}$ of $\mathcal{A}$ by

$$\tilde{\psi}(a) := M(\{\bar{\psi}(\tau^n(a))\}), \quad a \in \mathcal{A}.$$

If $a \in \mathcal{A}_+$, then $\tilde{\psi}(a) = \psi(a)$ due to $\tau^n(a) \in \mathcal{A}_+$.

Since X_ψ is a non-empty weak* compact convex set in $\mathfrak{S}(\mathcal{A})$, X_ψ has an extreme point ϕ due to the Krein–Milman theorem. Let $\phi = \lambda\phi_1 + (1 - \lambda)\phi_2$ with $\phi_1, \phi_2 \in \mathfrak{S}_\tau(\mathcal{A})$ and $0 < \lambda < 1$. Since $\phi_{1+}, \phi_{2+} \in \mathfrak{S}_\tau(\mathcal{A}_+)$ while $\psi \in \mathcal{E}\mathfrak{S}_\tau(\mathcal{A}_+)$ due to (3.2) and $\psi = \phi_+ = \lambda\phi_{1+} + (1 - \lambda)\phi_{2+}$, we obtain $\phi_{1+} = \phi_{2+} = \psi$ and so $\phi_1, \phi_2 \in X_\psi$. Since $\phi \in \mathcal{E}X_\psi$, we obtain $\phi_1 = \phi_2 = \phi$; hence $\phi \in \mathcal{E}\mathfrak{S}_\tau(\mathcal{A})$. Since an extremal τ -invariant state of the fermion algebra is weakly τ -clustering due to [4, Sect. II.8], ϕ is a weakly τ -clustering state of $\mathcal{A}$.

Step 3. We show that ϕ constructed in Step 2 is an α -KMS state of $\mathcal{A}$. Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. The sextuple $(\mathcal{A}, \mathcal{A}_+, \mathbb{Z}_2, \alpha, \Theta, \tau)$ forms the field system in the chemical potential theory. By the evenness of ϕ , its stabilizer

$$G_\phi := \{m \in \mathbb{Z}_2 : \phi \circ \Theta^m = \phi\}$$

coincides with $\mathbb{Z}_2$. Since its asymmetry subgroup

$$N_\phi := \{m \in \mathbb{Z}_2 : \phi(a\Theta^m(b)) = \phi(ab) \text{ for all } a, b \in \mathcal{A}\}$$

is $\{0\}$, the N_ϕ -fixed point subalgebra $\mathcal{A}^{N_\phi}$ coincides with $\mathcal{A}$. From [4, Sect. II.8], ϕ is an α -KMS state of $\mathcal{A}$.

Step 4. We show that ω extends uniquely to a τ -invariant α -KMS state of $\mathcal{A}$. Since $\mathcal{A}$ is separable, $K_\tau(\mathcal{A})$ and $K_\tau(\mathcal{A}_+)$ are metrizable weak* compact convex sets in $\mathcal{A}^*$ and $\mathcal{A}_+^*$, $\mathcal{A}^*$ and $\mathcal{A}_+^*$ being the dual spaces of $\mathcal{A}$ and $\mathcal{A}_+$, respectively. By [15], $\mathcal{E}K_\tau(\mathcal{A})$ and $\mathcal{E}K_\tau(\mathcal{A}_+)$ are G_δ -sets. Let f be a continuous linear mapping from $K_\tau(\mathcal{A})$ into $K_\tau(\mathcal{A}_+)$ defined by $f(\rho) := \rho_+$ for all $\rho \in K_\tau(\mathcal{A})$. By Step 3, $\mathcal{E}K_\tau(\mathcal{A}_+)$ is contained in $Y := f(K_\tau(\mathcal{A}))$. From [18], we can choose a measurable mapping g from Y into $K_\tau(\mathcal{A})$ such that $f \circ g(\psi) = \psi$ for all $\psi \in Y$. Let

$$\bar{\omega}(a) := \int_{\mathcal{E}K_\tau(\mathcal{A}_+)} g(\psi)(a) d\mu(\psi), \quad a \in \mathcal{A}.$$

Then $\bar{\omega} \in K_\tau(\mathcal{A})$. For each $a \in \mathcal{A}_+$, we obtain

$$\begin{aligned} \bar{\omega}_+(a) &= \int_{\mathcal{E}K_\tau(\mathcal{A}_+)} f \circ g(\psi)(a) d\mu(\psi) \\ &= \int_{\mathcal{E}K_\tau(\mathcal{A}_+)} \psi(a) d\mu(\psi) = \omega(a), \end{aligned}$$

where the second equality is due to $f \circ g = \text{id}$ on Y and the last equality is due to (3.3). The uniqueness of $\bar{\omega}$ follows from its evenness. $\square$

The following corollary says that [17, Lemma 4.7] is valid in the fermion case as well.

Corollary 3.2. $\mathcal{E}K_\tau(\mathcal{A}) \subset \mathcal{E}\mathfrak{S}_\tau(\mathcal{A})$.

Proof. For each $\omega \in \mathcal{E}K_\tau(\mathcal{A})$, let $(\pi_{\omega_+}, \mathcal{H}_{\omega_+})$ be the cyclic representation of $\mathcal{A}_+$ induced by ω_+ . Let $\omega_+ = \lambda\omega_1 + (1 - \lambda)\omega_2$ with $\omega_1, \omega_2 \in K_\tau(\mathcal{A}_+)$ and $0 < \lambda < 1$. For $i = 1, 2$, from Theorem 3.1, there exists a unique $\bar{\omega}_i$ in $K_\tau(\mathcal{A})$ such that $\bar{\omega}_{i+} = \omega_i$. By the evenness of τ -invariant states, we have $\omega = \lambda\bar{\omega}_1 + (1 - \lambda)\bar{\omega}_2$. By the extremality of ω , we obtain $\omega = \bar{\omega}_1 = \bar{\omega}_2$, and hence $\omega_+ \in \mathcal{E}K_\tau(\mathcal{A}_+)$. Let u_{ω_+} be a unitary representation of $\mathbb{Z}^v$ implementing τ on $\mathcal{H}_{\omega_+}$. It then follows that

$$\pi_{\omega_+}(\mathcal{A}_+)' \cap \pi_{\omega_+}(\mathcal{A}_+)' \cap u_{\omega_+}(\mathbb{Z}^v)' = \mathbb{C}\mathbf{1}_{\omega_+}.$$

By (2.1) and [6, 4.3.17], we obtain $\omega_+ \in \mathcal{E}\mathfrak{S}_\tau(\mathcal{A}_+)$. Now, from the evenness of τ -invariant states, it is easy to see that $\omega \in \mathcal{E}\mathfrak{S}_\tau(\mathcal{A})$. $\square$

Corollary 3.3. *If $\omega \in K_\tau(\mathcal{A})$, then ω is extremal in $K_\tau(\mathcal{A})$ if and only if it is extremal in $\mathfrak{S}_\tau(\mathcal{A})$.*

Proof. By Corollary 3.2, it suffices to show that $\mathcal{E}\mathfrak{S}_\tau(\mathcal{A}) \cap K_\tau(\mathcal{A}) \subset \mathcal{E}K_\tau(\mathcal{A})$. But this is immediately seen. $\square$

In the remainder of this section let us confine ourselves to the one-dimensional case $v = 1$. For each $n \in \mathbb{N}$, let $I_n := \{-n, -n+1, \dots, n\}$. Let $\mathcal{B}_0$ denote the set of all potentials Φ satisfying (P1), (P2), (P3), (P6) and the following conditions:

$$\sum_{K \ni 0} \|\Phi(K)\| < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|W_\Phi([1, n])\| < \infty.$$

Then $\mathcal{B}_0$ is a real Banach space with the usual linear operations and the norm

$$\|\Phi\|_0 := \sum_{K \ni 0} \|\Phi(K)\| + \sup_{n \in \mathbb{N}} \|W_\Phi([1, n])\| < \infty.$$

Let $\Phi \in \mathcal{B}_0$. Since $\{W_\Phi([1, n])\}$ is uniformly bounded, it follows from [11, 14] that the closure of δ_Φ is a generator of a strongly continuous automorphism group α^Φ of $\mathcal{A}$. Moreover, we have

$$\lim_{n \rightarrow \infty} \|\alpha_t^\Phi(a) - e^{itU_\Phi(I_n)} a e^{-itU_\Phi(I_n)}\| = 0, \quad a \in \mathcal{A}, \quad (3.4)$$

where the convergence is uniform in t on every compact subset of $\mathbb{R}$. By (3.4), it is easily verified that $(\mathcal{A}, \alpha^\Phi)$ is a dynamics satisfying (I)–(IV). We denote by $K(\mathcal{A}_+, \Phi)$ the set of all $\alpha^\Phi|_{\mathcal{A}_+}$ -KMS states of $\mathcal{A}_+$.

For simplicity we denote by τ the translation τ_1 . For each $k \in \mathbb{Z}$, let $[k]$ denote its representative in I_n , namely $[k] \in I_n$ and $k \equiv [k] \pmod{2n+1}$, and let $\bar{\tau}$ be the periodic translation defined by $\bar{\tau}(a_i) := a_{[i+1]}$ for all $i \in I_n$.

Lemma 3.4. *If $\Phi \in \mathcal{B}_0$ and $\omega \in K(\mathcal{A}_+, \Phi)$, then ω is τ -invariant.*

Proof. Since $\mathcal{A}(I_n)$ is a factor of type I, there exists a unitary $u_n \in \mathcal{A}(I_n)$ such that $\bar{\tau}(a) = u_n a u_n^*$ for all $a \in \mathcal{A}(I_n)$. Since $\bar{\tau}$ commutes with Θ , we have

$$u_n \Theta(a) u_n^* = \bar{\tau}(\Theta(a)) = \Theta(\bar{\tau}(a)) = \Theta(u_n) \Theta(a) \Theta(u_n)^*, \quad a \in \mathcal{A}(I_n).$$

Thus $\Theta(u_n)^* u_n$ is a scalar multiple of $\mathbf{1}$, and hence there exists a complex number λ with modulus 1 such that

$\Theta(u_n) = \lambda u_n$. Since $u_n = \Theta(\Theta(u_n)) = \lambda \Theta(u_n) = \lambda^2 u_n$ due to $\Theta^2 = \text{id}$, we obtain $\lambda = \pm 1$. This implies that u_n is either even or odd. Suppose that u_n is odd. Let $w_{n+1} := a_{n+1} + a_{n+1}^*$ and $v_n := u_n w_{n+1}$. Then w_{n+1} is a self-adjoint unitary in $\mathcal{A}(\{n+1\})_-$ and v_n is a unitary in $\mathcal{A}([-n, n+1])_+$. Since $a \in \mathcal{A}(\mathbb{I}_n)_+$ commutes with w_{n+1} , we obtain

$$v_n a v_n^* = u_n w_{n+1} a w_{n+1}^* u_n^* = u_n a u_n^* = \bar{\tau}(a), \quad a \in \mathcal{A}(\mathbb{I}_n)_+.$$

Replacing u_n by v_n , we may assume that u_n is even. Since $\{\mathbb{I}_n\}$ tends to $\mathbb{Z}$ whenever $n \rightarrow \infty$, it follows that

$$\tau(a) = \lim_{n \rightarrow \infty} u_n a u_n^*, \quad a \in \mathcal{A}$$

in the norm topology. Here, notice that $\alpha_t^\Phi \tau = \tau \alpha_t^\Phi$ for all $t \in \mathbb{R}$. It is obvious that $\{u_n\}$ is contained in $\mathcal{D}(\delta_{\alpha^\Phi})$. Moreover, in the same way as in the proof of [1, Proposition 1.1], we can show that $\sup_{n \in \mathbb{N}} \|\delta_{\alpha^\Phi}(u_n)\| < \infty$. Now, we can apply [9, Corollary II.3] (or [6, 5.3.33A]) to conclude that if ω is an $\alpha^\Phi|_{\mathcal{A}_+}$ -KMS state of $\mathcal{A}_+$, then $\omega \circ \tau = \omega$. $\square$

Corollary 3.5. *If $\Phi \in \mathcal{B}_0$, then $\mathcal{A}_+$ has a unique $\alpha^\Phi|_{\mathcal{A}_+}$ -KMS state.*

Proof. Each $\omega \in K(\mathcal{A}_+, \Phi)$ extends uniquely to an α^Φ -KMS state $\bar{\omega}$ of $\mathcal{A}$ due to Lemma 3.4 and Theorem 3.1. Since $\Phi \in \mathcal{B}_0$, it follows from [6, 6.2.47] (see also [2, 12]) that $\mathcal{A}$ has a unique α^Φ -KMS state ψ . Hence we obtain $\omega = \psi_+$, and thus $K(\mathcal{A}_+, \Phi)$ is a one-point set $\{\psi_+\}$. $\square$

4. Macroscopic Uniformity

Let α be a one-parameter dynamics on the fermion algebra $\mathcal{A}$ over $\mathbb{Z}^v$ as mentioned in Introduction. In this section, let δ be the restriction of δ_α to $\mathcal{A}_0$, and let Φ denote the unique τ -invariant standard potential determined by δ [5, Corollary 8.5]. To introduce the Gibbs condition, we need the notion of perturbations of states of $\mathcal{A}$. Let $\psi, \omega \in \mathfrak{S}(\mathcal{A})$. For each $\mathbb{I} \in \mathcal{F}$, the *relative entropy* of $\psi_{\mathbb{I}}$ with respect to $\omega_{\mathbb{I}}$ is given by

$$S(\psi_{\mathbb{I}}, \omega_{\mathbb{I}}) := \text{Tr}_{\mathbb{I}} \left(\frac{d\psi_{\mathbb{I}}}{d\text{Tr}_{\mathbb{I}}} \left(\log \frac{d\psi_{\mathbb{I}}}{d\text{Tr}_{\mathbb{I}}} - \log \frac{d\omega_{\mathbb{I}}}{d\text{Tr}_{\mathbb{I}}} \right) \right).$$

Then the relative entropy $S(\psi, \omega)$ is defined by

$$S(\psi, \omega) := \sup_{\mathbb{I} \subset \mathbb{Z}^v} S(\psi_{\mathbb{I}}, \omega_{\mathbb{I}}) = \lim_{\mathbb{I} \nearrow \mathbb{Z}^v} S(\psi_{\mathbb{I}}, \omega_{\mathbb{I}}).$$

For each $\omega \in \mathfrak{S}(\mathcal{A})$ and $h = h^* \in \mathcal{A}$, since $\psi \mapsto S(\psi, \omega) + \psi(h)$ is weakly* lower semicontinuous and strictly convex on $\mathfrak{S}(\mathcal{A})$, the *perturbed state* $[\omega^h]$ by h is defined as a unique minimizer of this functional [8]. Recall [3, 8] that

$$|S(\psi, \omega) - S(\psi, [\omega^h])| \leq 2\|h\| \quad (4.1)$$

for every $\psi, \omega \in \mathfrak{S}(\mathcal{A})$ and $h = h^* \in \mathcal{A}$.

Let $\omega \in \mathfrak{S}(\mathcal{A})$ and $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ be the cyclic representation of $\mathcal{A}$ induced by ω . We denote by $[\omega^{-W_\Phi(\mathbb{I})}]^\sim$ the normal extension of $[\omega^{-W_\Phi(\mathbb{I})}]$ to $\pi_\omega(\mathcal{A})''$. A state ω of $\mathcal{A}$ is said to satisfy the δ -Gibbs condition, or alternatively the Φ -Gibbs condition if Ω_ω is separating for $\pi_\omega(\mathcal{A})''$ and if, for each $\mathbb{I} \in \mathcal{F}$,

$$\sigma_t^{[\omega^{-W_\Phi(\mathbb{I})}]^\sim}(\pi_\omega(a)) = \pi_\omega(e^{-itU_\Phi(\mathbb{I})} a e^{itU_\Phi(\mathbb{I})}), \quad a \in \mathcal{A}(\mathbb{I}).$$

Here, $\sigma_t^{[\omega^{-W_\Phi(\mathbb{I})}]^\sim}$ denotes the modular automorphism group for $[\omega^{-W_\Phi(\mathbb{I})}]^\sim$ (this definition is due to [5]).

If ω is a state of $\mathcal{A}$, then the following conditions are equivalent [5, Sect. 7.4]:

- (i) ω is an α -KMS state.
- (ii) ω satisfies the δ -Gibbs condition.

For each $\mathbb{I} \in \mathcal{F}$, the *local Gibbs state* $\phi_{\mathbb{I}}^G$ of $\mathcal{A}(\mathbb{I})$ with respect to δ is defined by

$$\phi_{\mathbb{I}}^G(a) := \frac{\text{Tr}_{\mathbb{I}}(e^{-U_\Phi(\mathbb{I})} a)}{\text{Tr}_{\mathbb{I}}(e^{-U_\Phi(\mathbb{I})})}, \quad a \in \mathcal{A}(\mathbb{I}). \quad (4.2)$$

Let $\mathbb{I} \in \mathcal{F}$, $\mathbb{I} \neq \emptyset$ and ω be a τ -invariant state of $\mathcal{A}$ satisfying the δ -Gibbs condition. Since ω is even, it follows from [5, Proposition 7.7] that

$$[\omega^{-W_\Phi(\mathbb{I})}](ab) = \phi_{\mathbb{I}}^G(a)[\omega^{-W_\Phi(\mathbb{I})}](b), \quad a \in \mathcal{A}(\mathbb{I}), \quad b \in \mathcal{A}(\mathbb{I}^c). \quad (4.3)$$

To establish the macroscopic uniformity of an extremal τ -invariant state of $\mathcal{A}$ satisfying the δ -Gibbs condition, we need some auxiliary results.

Lemma 4.1. *If ω is a τ -invariant state of $\mathcal{A}$ satisfying the δ -Gibbs condition, then*

$$\log \frac{d\phi_{\mathbb{I}}^G}{d\text{Tr}_{\mathbb{I}}} - \log \frac{d\omega_{\mathbb{I}}}{d\text{Tr}_{\mathbb{I}}} \leq 2\|W_\Phi(\mathbb{I})\| \mathbf{1}$$

for every $\mathbb{I} \in \mathcal{F}$.

Proof. For every $\psi \in \mathfrak{S}(\mathcal{A})$, let $\tilde{\psi}_I$ be the state of $\pi_\omega(\mathcal{A}(I))$ defined by $\tilde{\psi}_I(\pi_\omega(a)) := \psi_I(a)$ for all $a \in \mathcal{A}(I)$. In particular, let $\tilde{\phi}_I^G$ be that for ϕ_I^G . From the δ -Gibbs condition, we obtain

$$\sigma_t^{[\omega^{-W_\Phi(I)}]^\sim}(\pi_\omega(\mathcal{A}(I))) = \pi_\omega(\mathcal{A}(I)), \quad t \in \mathbb{R}.$$

Hence [16] implies that there exists a conditional expectation ε_I from $\pi_\omega(\mathcal{A})''$ onto $\pi_\omega(\mathcal{A}(I))$ with respect to $[\omega^{-W_\Phi(I)}]^\sim$. Let $\tilde{\omega}$ be the normal extension of ω to $\pi_\omega(\mathcal{A})''$ and $\tilde{\omega}_I := \tilde{\omega}|_{\pi_\omega(\mathcal{A}(I))}$. Note that $[\omega^{-W_\Phi(I)}]^\sim = [\tilde{\omega}^{-\pi_\omega(W_\Phi(I))}]$. It follows from (4.3) that $[\omega^{-W_\Phi(I)}]^\sim = [\omega^{-W_\Phi(I)}]^\sim \circ \varepsilon_I = \tilde{\phi}_I^G \circ \varepsilon_I$. Then we have

$$\begin{aligned} S(\psi_I, \omega_I) &= S(\tilde{\psi}_I, \tilde{\omega}_I) \\ &\leq S(\tilde{\psi}_I \circ \varepsilon_I, \tilde{\omega}) \\ &\leq S(\tilde{\psi}_I \circ \varepsilon_I, [\omega^{-W_\Phi(I)}]^\sim) + 2\|W_\Phi(I)\| \\ &= S(\tilde{\psi}_I \circ \varepsilon_I, \tilde{\phi}_I^G \circ \varepsilon_I) + 2\|W_\Phi(I)\| \\ &= S(\psi_I, \phi_I^G) + 2\|W_\Phi(I)\|, \end{aligned}$$

where the first inequality is due to the monotonicity of relative entropy and the second inequality is due to (4.1). Hence

$$\psi \left(\log \frac{d\phi_I^G}{d\text{Tr}_I} - \log \frac{d\omega_I}{d\text{Tr}_I} \right) \leq 2\|W_\Phi(I)\|$$

for all $\psi \in \mathfrak{S}(\mathcal{A})$. This implies the conclusion. $\square$

If Ψ is a τ -invariant standard potential having relatively short range, *i.e.*,

$$\sum_{K \ni 0} \frac{\|\Psi(K)\|}{|K|} < \infty,$$

then we can define the *mean energy operator* A_Ψ of Ψ by

$$A_\Psi := \sum_{K \ni 0} \frac{\Psi(K)}{|K|}.$$

Since Ψ is Θ -invariant, so is A_Ψ .

Lemma 4.2. *If $\psi \in \mathfrak{S}_\tau(\mathcal{A})$ and $\Psi \in \mathcal{P}_\tau$ has relatively short range, then $e_\Psi(\psi) = \psi(A_\Psi)$.*

Proof. Let $I \in \mathcal{F}$. Since

$$\begin{aligned} \left| \frac{\psi(U_\Psi(I))}{|I|} - \psi(A_\Psi) \right| &\leq \frac{1}{|I|} \left\| \sum_{K \subset I} \Psi(K) - \sum_{k \in I} \tau_k(A_\Psi) \right\| \\ &\leq \frac{1}{|I|} \sum_{k \in I} \sum_{\substack{K \cap I^c \neq \emptyset \\ K \ni k}} \frac{\|\Psi(K)\|}{|K|}, \end{aligned}$$

we obtain $e_\Psi(\psi) = \psi(A_\Psi)$ in the same way as [6, 6.2.39]. $\square$

Lemma 4.3. *If $\psi \in \mathfrak{E}\mathfrak{S}_\tau(\mathcal{A})$ and $\Psi \in \mathcal{P}_\tau$ has relatively short range, then*

$$\text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \pi_\psi(U_\Psi(I)) = e_\Psi(\psi) \mathbf{1}$$

in the strong operator topology.

Proof. From the proof of Lemma 4.2, we have

$$\text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} U_\Psi(I) = \text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \sum_{k \in I} \tau_k(A_\Psi) \quad (4.4)$$

in the norm topology. Using the mean ergodic theorem (see *e.g.* [7]) and [5, Lemma A.5], we have

$$\text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \sum_{k \in I} \pi_\psi(\tau_k(A_\Psi)) = \psi(A_\Psi) \mathbf{1} \quad (4.5)$$

in the strong operator topology. By (4.4), (4.5) and Lemma 4.2, we obtain the conclusion. $\square$

We shall remove the assumption of relatively short range as follows.

Lemma 4.4. *If $\omega \in \mathfrak{E}\mathfrak{S}_\tau(\mathcal{A})$, then*

$$\text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \pi_\omega(U_\Phi(I)) = e_\Phi(\omega) \mathbf{1}$$

in the strong operator topology.

Proof. Since $\mathcal{P}_\tau^f$ is dense in $\mathcal{P}_\tau$, there exists a sequence $\{\Phi_n\}$ in $\mathcal{P}_\tau^f$ such that $\{\Phi_n\}$ is convergent to Φ in the norm topology. Given any $\varepsilon > 0$, there exists a positive integer N such that $\|\Phi_n - \Phi\| \leq \varepsilon$ whenever $n \geq N$. Let $\{\mathbf{I}_\alpha\}$ be a van Hove net tending to $\mathbb{Z}^v$. By (2.3), we have

$$\begin{aligned} \left\| \frac{1}{|\mathbf{I}_\alpha|} \pi_\omega(U_{\Phi_n}(\mathbf{I}_\alpha)) - \frac{1}{|\mathbf{I}_\alpha|} \pi_\omega(U_\Phi(\mathbf{I}_\alpha)) \right\| &= \frac{1}{|\mathbf{I}_\alpha|} \|U_{\Phi_n - \Phi}(\mathbf{I}_\alpha)\| \\ &\leq \|\Phi_n - \Phi\| \leq \varepsilon \end{aligned} \quad (4.6)$$

whenever $n \geq N$. Note that

$$|e_{\Phi_n}(\omega) - e_\Phi(\omega)| = |e_{\Phi_n - \Phi}(\omega)| \leq \|\Phi_n - \Phi\| \leq \varepsilon \quad (4.7)$$

for all $n \geq N$. By Lemma 4.3, we can choose an index $\alpha(N)$ of the net $\{\mathbf{I}_\alpha\}$ such that

$$\left\| \frac{1}{|\mathbf{I}_\alpha|} \pi_\omega(U_{\Phi_N}(\mathbf{I}_\alpha)) - e_{\Phi_N}(\omega) \mathbf{1} \right\| \leq \varepsilon \quad (4.8)$$

whenever $\alpha \geq \alpha(N)$. Hence, by (4.6), (4.7) and (4.8), for $\alpha \geq \alpha(N)$ we have

$$\begin{aligned} &\left\| \frac{1}{|\mathbf{I}_\alpha|} \pi_\omega(U_\Phi(\mathbf{I}_\alpha)) - e_\Phi(\omega) \mathbf{1} \right\| \\ &\leq \left\| \frac{1}{|\mathbf{I}_\alpha|} \pi_\omega(U_\Phi(\mathbf{I}_\alpha)) - \frac{1}{|\mathbf{I}_\alpha|} \pi_\omega(U_{\Phi_N}(\mathbf{I}_\alpha)) \right\| \\ &\quad + \left\| \frac{1}{|\mathbf{I}_\alpha|} \pi_\omega(U_{\Phi_N}(\mathbf{I}_\alpha)) - e_{\Phi_N}(\omega) \mathbf{1} \right\| + |e_{\Phi_N}(\omega) - e_\Phi(\omega)| \leq 3\varepsilon. \end{aligned}$$

This estimation implies the conclusion. $\square$

Proposition 4.5. *If ω is an extremal τ -invariant state of $\mathcal{A}$ satisfying the δ -Gibbs condition, then*

$$\text{v.H.} \lim_{\mathbf{I} \rightarrow \mathbb{Z}^v} \frac{1}{|\mathbf{I}|} \pi_\omega \left(-\log \frac{d\omega_{\mathbf{I}}}{d\text{Tr}_{\mathbf{I}}} \right) = \text{v.H.} \lim_{\mathbf{I} \rightarrow \mathbb{Z}^v} \frac{1}{|\mathbf{I}|} \pi_\omega \left(-\log \frac{d\phi_{\mathbf{I}}^G}{d\text{Tr}_{\mathbf{I}}} \right) = s(\omega) \mathbf{1}$$

in the strong operator topology.

Proof. By (4.2), we obtain

$$\pi_\omega \left(-\log \frac{d\phi_{\mathbf{I}}^G}{d\text{Tr}_{\mathbf{I}}} \right) = \pi_\omega(U_\Phi(\mathbf{I})) + \log \text{Tr}_{\mathbf{I}}(e^{-U_\Phi(\mathbf{I})}) \mathbf{1}.$$

By dividing the above equality by $|\mathbf{I}|$ and then taking the van Hove limit $\mathbf{I} \rightarrow \mathbb{Z}^v$, we obtain the following strong convergence

$$\text{v.H.} \lim_{\mathbf{I} \rightarrow \mathbb{Z}^v} \frac{1}{|\mathbf{I}|} \pi_\omega \left(-\log \frac{d\phi_{\mathbf{I}}^G}{d\text{Tr}_{\mathbf{I}}} \right) = e_\Phi(\omega) \mathbf{1} + P(\Phi) \mathbf{1} = s(\omega) \mathbf{1}, \quad (4.9)$$

where the first equality is due to Lemma 4.4 and the second equality follows from the variational principle of ω [5, Proposition 12.1].

Now let

$$a_{\mathbf{I}} := -\frac{1}{|\mathbf{I}|} \log \frac{d\omega_{\mathbf{I}}}{d\text{Tr}_{\mathbf{I}}}, \quad b_{\mathbf{I}} := -\frac{1}{|\mathbf{I}|} \log \frac{d\phi_{\mathbf{I}}^G}{d\text{Tr}_{\mathbf{I}}} + \frac{2\|W_\Phi(\mathbf{I})\|}{|\mathbf{I}|} \mathbf{1}.$$

It follows from [5, Lemma 9.1] that

$$\text{v.H.} \lim_{\mathbf{I} \rightarrow \mathbb{Z}^v} \frac{\|W_\Phi(\mathbf{I})\|}{|\mathbf{I}|} = 0.$$

From this fact and (4.9), $\pi_\omega(b_{\mathbf{I}}) \rightarrow s(\omega) \mathbf{1}$ in the strong operator topology. By Lemma 4.1, we have $0 \leq a_{\mathbf{I}} \leq b_{\mathbf{I}}$. Since $\{b_{\mathbf{I}}\}$ is uniformly bounded, so is $\{b_{\mathbf{I}} - a_{\mathbf{I}}\}$. Let $\tilde{\omega}$ be the normal extension of ω to $\pi_\omega(\mathcal{A})''$. Since ω satisfies the δ -Gibbs condition, Ω_ω is separating for $\pi_\omega(\mathcal{A})''$. Hence $\tilde{\omega}$ is automatically faithful. Then we have

$$\text{v.H.} \lim_{\mathbf{I} \rightarrow \mathbb{Z}^v} \tilde{\omega}(\pi_\omega(b_{\mathbf{I}} - a_{\mathbf{I}})) = \text{v.H.} \lim_{\mathbf{I} \rightarrow \mathbb{Z}^v} \omega(b_{\mathbf{I}} - a_{\mathbf{I}}) = 0.$$

By the faithfulness of $\tilde{\omega}$, we obtain

$$\text{v.H.} \lim_{\mathbf{I} \rightarrow \mathbb{Z}^v} \pi_\omega(a_{\mathbf{I}}) = \text{v.H.} \lim_{\mathbf{I} \rightarrow \mathbb{Z}^v} \pi_\omega(b_{\mathbf{I}}) = s(\omega) \mathbf{1}$$

in the strong operator topology. $\square$

To study the macroscopic uniformity of extremal equilibrium states in a quantum spin system, Hiai and Petz introduced the quantity defined as follows: Let $0 < \varepsilon < 1$. For each $I \in \mathcal{F}$ and each state ω of $\mathcal{A}(I)$, define

$$\beta_\varepsilon(\omega) := \min\{\text{Tr}_I(q) : q \in \mathcal{A}(I) \text{ is a projection with } \omega(q) \geq 1 - \varepsilon\}$$

(see [10]).

Theorem 4.6. *If ω is an extremal τ -invariant state of $\mathcal{A}$ satisfying the δ -Gibbs condition, then*

$$\text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \log \beta_\varepsilon(\omega_I) = s(\omega)$$

for every $0 < \varepsilon < 1$.

Proof. This follows from Proposition 4.5 and the same argument as in the proof of [10, Theorem 3.1]. $\square$

Remark 4.7. Assume that $\{W_\Phi(I_\alpha)\}$ is uniformly bounded for some van Hove net $\{I_\alpha\}$. Let ω be an even state of $\mathcal{A}$ satisfying the δ -Gibbs condition. Since such ω is unique, it is extremal in $\mathfrak{S}_\tau(\mathcal{A})$ by Corollary 3.2. Then the following equalities are valid:

$$\text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \log \beta_\varepsilon(\omega_I) = s(\omega) = \text{v.H.} \lim_{I \rightarrow \mathbb{Z}^v} \frac{1}{|I|} \log \beta_\varepsilon(\phi_I^G)$$

for every $0 < \varepsilon < 1$. The proof is the same as that of [10, Theorem 3.3].

Appendix A: General Potentials

A mapping Φ from $\mathcal{F}$ into $\mathcal{A}$ is called a *general potential* if it satisfies (P1), (P2), (P3) and (P5). For each general potential Φ , we can define $H_\Phi(I)$, $U_\Phi(I)$, $W_\Phi(I)$ and δ_Φ in the same way as the standard case. Two general potentials Φ and Ψ are said to be *physically equivalent* if $\delta_\Phi = \delta_\Psi$. If Φ is a τ -invariant general potential, then $\delta_\Phi \in \Delta_\tau(\mathcal{A}_0)$, and hence there is a unique $\Phi_s \in \mathcal{P}_\tau$ being physically equivalent to Φ [5, Corollary 8.5]. For each τ -invariant general potential Φ , let Φ_s denote the standard potential associated with Φ . Let $\mathcal{B}_1$ denote the set of all potentials satisfying (P1), (P2), (P3) and of short range in the sense that

$$\|\Phi\|_1 := \sum_{K \ni 0} \|\Phi(K)\| < \infty. \quad (\text{A.1})$$

Then $\mathcal{B}_1$ becomes a real Banach space with the usual linear operations and the norm $\|\cdot\|_1$. Each $\Phi \in \mathcal{B}_1$ is a τ -invariant general potential since (A.1) implies (P5) automatically.

Proposition A.1. *The equation $S\Phi := \Phi_s$ ($\Phi \in \mathcal{B}_1$) defines a bounded linear mapping S from $\mathcal{B}_1$ into $\mathcal{P}_\tau$.*

Proof. Since the linearity of S is obvious, we show that S is bounded. For this, suppose that a sequence $\{\Phi_n\}$ in $\mathcal{B}_1$ converges to $\Phi \in \mathcal{B}_1$ while $\{S\Phi_n\}$ converges to $\Psi \in \mathcal{P}_\tau$; we have to show that $S\Phi = \Psi$. Let $\Psi_n := S\Phi_n$ for all $n \in \mathbb{N}$, and let $I \in \mathcal{F}$. Then we have

$$\|H_{\Phi_n}(I) - H_\Phi(I)\| \leq \|\Phi_n - \Phi\|_1 \cdot |I| \rightarrow 0$$

as $n \rightarrow \infty$. By (2.3), we have

$$\|H_{\Psi_n}(I) - H_\Psi(I)\| \leq \|\Psi_n - \Psi\| \cdot |I| \rightarrow 0$$

as $n \rightarrow \infty$. Hence $\delta_\Psi = \delta_\Phi$; so $S\Phi = \Psi$. From the closed graph theorem, S is bounded. $\square$

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REFERENCES

- [1] Akiho, N., Hiai, F., and Petz, D., “Equilibrium states and their entropy densities in gauge-invariant C^* -systems,” *Rev. Math. Phys.*, **17**: 365–389 (2005).
- [2] Araki, H., “On uniqueness of KMS states of one-dimensional quantum lattice systems,” *Commun. Math. Phys.*, **44**: 1–7 (1975).
- [3] Araki, H., “Relative entropy for states of von Neumann algebras II,” *Publ. Res. Inst. Math. Sci.*, **13**: 173–192 (1977).
- [4] Araki, H., Haag, R., Kastler, D., and Takesaki, M., “Extension of KMS states and chemical potential,” *Commun. Math. Phys.*, **53**: 97–134 (1977).
- [5] Araki, H., and Moriya, H., “Equilibrium statistical mechanics of Fermion lattice systems,” *Rev. Math. Phys.*, **15**: 93–198 (2003).
- [6] Bratteli, O., and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics 1, 2*, 2nd edition, Springer-Verlag, 2002.

- [7] Conze, J. P., and Dang-Ngoc, N., “Ergodic theorems for noncommutative dynamical systems,” *Invent. Math.*, **46**: 1–15 (1978).
- [8] Donald, M. J., “Relative Hamiltonians which are not bounded from above,” *J. Funct. Anal.*, **91**: 143–173 (1990).
- [9] Fannes, M., Vanheuverzwijn, P., and Verbeure, A., “Quantum energy-entropy inequalities: A new method for proving the absence of symmetry breaking,” *J. Math. Phys.*, **25**: 76–78 (1984).
- [10] Hiai, F., and Petz, D., “Entropy densities for Gibbs states of quantum spin systems,” *Rev. Math. Phys.*, **5**: 693–712 (1993).
- [11] Kishimoto, A., “Dissipations and derivations,” *Commun. Math. Phys.*, **47**: 25–32 (1976).
- [12] Kishimoto, A., “On uniqueness of KMS states of one-dimensional quantum lattice systems,” *Commun. Math. Phys.*, **47**: 167–170 (1976).
- [13] Narnhofer, H., and Thirring, W., “Spontaneously broken symmetries,” *Ann. Inst. Henri Poincaré, Phys. Théor.*, **70**: 1–21 (1999).
- [14] Powers, R. T., and Sakai, S., “Unbounded derivations in operator algebras,” *J. Funct. Anal.*, **19**: 81–95 (1975).
- [15] Phelps, R. R., Lectures on Choquet’s Theorem, 2nd edition, *Lecture Notes in Mathematics*, **1757**, Springer-Verlag, 2001.
- [16] Takesaki, M., “Conditional expectations in von Neumann algebras,” *J. Funct. Anal.*, **9**: 306–321 (1972).
- [17] Takesaki, M., and Winnink, M., “Local normality in quantum statistical mechanics,” *Commun. Math. Phys.*, **30**: 129–152 (1973).
- [18] Takesaki, M., *Theory of Operator Algebras I*, Springer-Verlag, 1979.