Blowing Up at Infinity of Eigenfunctions on Riemannian Manifolds

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We study eigenfunctions of a complete, noncompact Riemannian manifold and investigate how they blow up at infinity. Our estimate is also related to the size of the manifold at infinity when the Ricci curvature is nonnegative.

KEYWORDS: Riemannian manifold, Laplacian, eigenfunction, Ricci curvature, comparison theorem

Introduction

Let M be a m -dimensional complete noncompact Riemannian manifold, and denote the Laplacian of M by Δ_M , where $\Delta_M = -\operatorname{div} \operatorname{grad}$. In [1] and [2], Borbély studies positive eigenfunctions on M under the condition that the Ricci curvature of M is bounded below by $-(m-1)k^2$, ($k \geq 0$), and shows among other things that a positive eigenfunction u of nonpositive eigenvalue $-\mu$, that is, a positive solution u of equation: $\Delta_M u = -\mu u$ satisfies

$$-\sqrt{\lambda_k} + \sqrt{\lambda_k + \mu} \leq \sup_M |\nabla \log u| \leq \sqrt{\lambda_k} + \sqrt{\lambda_k + \mu}, \quad (1)$$

where $\lambda_k = \frac{(m-1)^2 k^2}{4}$.

The aim of this paper is to investigate how eigenfunctions of negative eigenvalues blow up at infinity.

To state our main results, we fix a point o of M and denote by $B_r(o)$ the metric ball of radius r around o .

Theorem 1. *Let M be a complete noncompact Riemannian manifold of dimension m with Ricci curvature bounded below by $-(m-1)k^2$, ($k \geq 0$).*

Let u be an eigenfunction on M with negative eigenvalue $-\mu$. Then

$$\liminf_{r \rightarrow \infty} \frac{\log \max_{\partial B_r(o)} |u|}{r} \geq -\sqrt{\lambda_k} + \sqrt{\lambda_k + \mu},$$

where $\lambda_k = \frac{(m-1)^2 k^2}{4}$.

In the case where the Ricci curvature of M is nonnegative, we can see a certain relation between the growth of positive eigenfunctions on M and the size of M at infinity. In fact, we prove the following

Theorem 2. *Let M be a complete noncompact Riemannian manifold of dimension m with nonnegative Ricci curvature. Suppose that the diameter of the metric sphere $\partial B_r(o)$ of radius r around a fixed point $o \in M$ is bounded above by ar for some positive constant a and any large r . Let u be a positive eigenfunction of M with negative eigenvalue $-\mu$. Then*

$$\liminf_{r \rightarrow \infty} \frac{\log \min_{\partial B_r(o)} u}{r} \geq (1-a)\sqrt{\mu}.$$

In particular, every positive eigenfunction of negative eigenvalue exponentially blows up at infinity if $a < 1$.

The outline of the paper is as follows. Section 1 is devoted giving some preliminary results. In Section 2, we will find Theorem 1 still holds true for eigensections of certain Hermitian vector bundles over M (see Theorem 7). Theorem 2 will be proved at the end of Section 2. In Appendix, we review the Sturm–Liouville comparison theorem.

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1. Preliminaries

Let $f(r)$ be a monotone increasing smooth function defined on $[0, +\infty)$ with $f(0) = 0$ and $f'(0) = 1$. Given an integer $m \geq 2$ and a number $\mu > 0$, consider the ordinary differential equation:

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$$\phi'' + (m-1)\frac{f'}{f}\phi' = \mu\phi, \quad (2)$$

subject to the initial condition:

$$\phi(0) = 1, \quad \phi'(0) = 0. \quad (3)$$

By introducing a differential operator $E_f = f^{m-1}(r)\frac{d}{dr}$, the ordinary differential Eq. (2) is equivalent to the following

$$E_f^2\phi = \mu f^{2(m-1)}\phi. \quad (2')$$

We denote by $\phi_{\mu,f}$ the unique solution of (2) with (3). Then it is easy to see that $\phi_{\mu,f}$ is positive and increasing in $[0, +\infty)$ by the change of variables. In addition, we have

Lemma 3.

- (i) If $\int_R^\infty (\int_R^z f^{m-1}(y)dy/f^{m-1}(z))dz = \infty$ for some $R > 0$, then $\phi_{\mu,f}(r)$ diverges to infinity as $r \rightarrow \infty$.
- (ii) If $\int_R^\infty (\int_R^z f^{m-1}(y)dy/f^{m-1}(z))dz < \infty$ and $\int_R^\infty (1/f^{m-1}(s))ds < \infty$ for some $R > 0$, then $\phi_{\mu,f}(r)$ is bounded as $r \rightarrow \infty$.

Proof. By integrating (2') from R to r , we have

$$E_f\phi_{\mu,f}(r) - E_f\phi_{\mu,f}(R) = \mu \int_R^r \phi_{\mu,f} f^{m-1} ds.$$

Since $E_f\phi_{\mu,f}(R) > 0$ and $\phi_{\mu,f}(R) \leq \phi_{\mu,f}(s)$ for $s \geq R$, we get

$$E_f\phi_{\mu,f}(r) \geq \mu\phi_{\mu,f}(R) \int_R^r f^{m-1} ds,$$

which shows

$$\phi'_{\mu,f}(r) \geq \mu\phi_{\mu,f}(R) \frac{\int_R^r f^{m-1} ds}{f^{m-1}(r)}.$$

Thus integrating this, we obtain

$$\phi_{\mu,f}(r) \geq \phi_{\mu,f}(R) + \mu\phi_{\mu,f}(R) \int_R^r \frac{\int_R^z f^{m-1}(s)ds}{f^{m-1}(z)} dz \quad \text{for } r > R.$$

This implies the first assertion (i).

On the other hand, since $\phi_{\mu,f}(R) \leq \phi_{\mu,f}(r)$, we have

$$E_f\phi_{\mu,f}(r) - E_f\phi_{\mu,f}(R) \leq \mu\phi_{\mu,f}(r) \int_R^r f^{m-1} ds,$$

and hence

$$\frac{\phi'_{\mu,f}(r)}{\phi_{\mu,f}(r)} \leq \frac{E_f\phi_{\mu,f}(R)}{\phi_{\mu,f}(R)f^{m-1}(r)} + \mu \frac{\int_R^r f^{m-1} ds}{f^{m-1}(r)}.$$

Integrating this, we obtain

$$\log \frac{\phi_{\mu,f}(r)}{\phi_{\mu,f}(R)} \leq \frac{E_f\phi_{\mu,f}(R)}{\phi_{\mu,f}(R)} \int_R^r \frac{1}{f^{m-1}(s)} ds + \mu \int_R^r \frac{\int_R^z f^{m-1}(s)ds}{f^{m-1}(z)} dz,$$

from which the second assertion (ii) follows. $\square$

Given a monotone increasing smooth function f on $[0, +\infty)$ satisfying $f(0) = 0$ and $f'(0) = 1$, we denote by M_f the warped product $\mathbf{R}_{\geq 0} \times_f S^{m-1}$ with a Riemannian metric $dr^2 + f^2(r)d\theta^2$, where $d\theta^2$ is the standard metric of the Euclidean unit sphere S^{m-1} . Let r be the distance in M_f to the pole o . Then for a positive number μ , the function $\phi_{\mu,f}(r)$ is an eigenfunction of M_f with negative eigenvalue $-\mu$, that is, $\Delta_{M_f}\phi_{\mu,f}(r) = -\mu\phi_{\mu,f}(r)$, such that $\phi_{\mu,f}(r) = 1$ at o . We recall here that the radial curvature (relative to the pole o) is given by $-f''(r)/f(r)$. Given a nonnegative constant k , let $f_k(r) = r$ for $k = 0$ and $f_k(r) = k^{-1} \sinh kr$ for $k > 0$. Then the warped product M_{f_k} of the warping function f_k is nothing but the simply connected space form $M(-k^2)$ of constant curvature $-k^2$.

Lemma 4. Let r be the distance function to a point o of $M(-k^2)$ and $\phi_{\mu,f_k}(r)$ the (unique) positive eigenfunction of $M(-k^2)$ (which is rotationally symmetric around o and satisfies $\phi_{\mu,f_k}(r) = 1$ at o). Then

$$\liminf_{r \rightarrow \infty} \frac{\log \phi_{\mu, f_k}(r)}{r} \geq -\sqrt{\lambda_k} + \sqrt{\lambda_k + \mu},$$

where $\lambda_k = \frac{(m-1)^2 k^2}{4}$.

Proof. Let $\tilde{\phi}(r) = e^{\sqrt{\mu_\epsilon} r}$, where $\sqrt{\mu_\epsilon} = (-\sqrt{\lambda_k} + \sqrt{\lambda_k + (1+\epsilon)\mu})/(1+\epsilon)$ and ϵ is a positive constant. Let $\Psi(r) = \phi'_{\mu, f_k}(r)/\phi_{\mu, f_k}(r)$. The function $\Psi(r)$ satisfies the following equation:

$$\Psi''(r) + 2\Psi(r)\Psi'(r) + (m-1)\frac{f'_k(r)}{f_k(r)}\Psi'(r) + (m-1)\left(\frac{f'_k(r)}{f_k(r)}\right)'\Psi(r) = 0.$$

By the maximal principle, there is no point attaining the maximum of the function $\Psi(r)$. So, comparing with (1), there exists a divergent sequence $\{r_n\}$ satisfying $\lim_{n \rightarrow \infty} \Psi(r_n) = \sup_{r \geq 0} \Psi(r)$. Thus for any $\epsilon > 0$ the following inequality holds:

$$\phi'_{\mu, f_k}(r_0)/\phi_{\mu, f_k}(r_0) \geq \sqrt{\mu_\epsilon} \left(= \tilde{\phi}'(r_0)/\tilde{\phi}(r_0) \right)$$

for sufficient large $r_0 > 0$.

For any $\epsilon > 0$ one can take a sufficient large positive number $r_0 > 0$ in such a way that

$$-\epsilon\sqrt{\mu_\epsilon} < 2\sqrt{\lambda_k} - (m-1)f'_k(r)/f_k(r) < \epsilon\sqrt{\mu_\epsilon}$$

for any $r \geq r_0$. Then direct computations show that

$$\begin{aligned} E_{f_k}^2 \tilde{\phi}(r) &= \mu_\epsilon \left(1 + (m-1) \frac{f'_k(r)}{\sqrt{\mu_\epsilon} f_k(r)} \right) f_k^{2(m-1)}(r) \tilde{\phi}(r) \\ &\leq \mu_\epsilon \left(1 + 2\sqrt{\frac{\lambda_k}{\mu_\epsilon}} + \epsilon \right) f_k^{2(m-1)}(r) \tilde{\phi}(r) = \mu f_k^{2(m-1)}(r) \tilde{\phi}(r) \end{aligned}$$

for all $r \geq r_0$.

These inequalities allow us to apply the Sturm–Liouville comparison theorem (see Lemma 10 in Appendix) to $\phi_{\mu, f_k}(r)$ and $\tilde{\phi}(r)$, and obtain $\phi_{\mu, f_k}(r)/\phi_{\mu, f_k}(r_0) \geq \tilde{\phi}(r)/\tilde{\phi}(r_0)$ for all $r \geq r_0$. As a result, we have $\liminf_{r \rightarrow \infty} (\log \phi_{\mu, f_k}(r))/r \geq \sqrt{\mu_\epsilon}$, and letting ϵ tend to 0, we thus attain Lemma 4. $\square$

Lemma 5. *Let M be a complete noncompact Riemannian manifold of dimension m and r the distance function to a fixed point o of M . Suppose that the Ricci curvature of M is bounded below by a function $-(m-1)f''(r)/f(r)$, where $f(r)$ is a positive smooth function on $[0, +\infty)$ such that $f(0) = 0$, $f'(0) = 1$ and $f'(r) \geq 0$ on $[0, +\infty)$. Given a positive number μ , let $\phi_{\mu, f}$ be the solution of Eq. (2) subject to the initial condition (3). Then the function $\phi_{\mu, f}(r)$ satisfies*

$$\Delta_M \phi_{\mu, f}(r) \geq -\mu \phi_{\mu, f}(r)$$

on M as a distribution. Moreover if a continuous function u on M satisfies $\Delta_M |u| \leq -\mu |u|$ as a distribution, then

$$\max_{x \in \partial B_r(o)} |u|(x) \geq |u(o)| \phi_{\mu, f}(r)$$

for all $r > 0$.

Proof. Since $\phi_{\mu, f}(r)$ is monotone increasing, we can deduce from the Laplacian comparison theorem (see [7], Proposition 2.15) that

$$\Delta_M \phi_{\mu, f}(r) \geq \Delta_{M_f} \phi_{\mu, f}(r) = -\mu \phi_{\mu, f}(r).$$

In fact this holds everywhere on M as a distribution (cf. [4], [5]).

Let u be as above and set $w = |u| - |u(o)|\phi_{\mu, f}(r)$. Then we have $\Delta_M w \leq -\mu w$ as a distribution on M . Hence the strong maximal principle for a weak sub-solution (see for example [6] and its final remark) allows us to conclude that $\max_{x \in \partial B_r(o)} w = \sup_{x \in B_r(o)} w \geq w(o) = 0$. This shows that

$$\max_{x \in \partial B_r(o)} |u|(x) \geq |u(o)| \phi_{\mu, f}(r)$$

for all $r > 0$. $\square$

2. A generalization of Theorem 1 and Proof of Theorem 2

In this section, we first prove Theorem 1 in a more general situation and add some observations. Theorem 2 will be verified at the end of the section.

Let us consider a Hermitian vector bundle E of rank r over M endowed with a metric connection ∇ . We denote by $\Gamma(E)$ the set of smooth sections of E over M and by $\langle \cdot, \cdot \rangle$ the fiberwise Hermitian inner product. And we write

$|s|(x) = \sqrt{\langle s, s \rangle(x)}$, ($x \in M$) for sections $s \in \Gamma(E)$. Let ∇^* be the formal adjoint operator of ∇ . Given a symmetric endomorphism R of E , we define an operator acting on $\Gamma(E)$ by $L = \nabla^* \nabla + R$.

Recall Kato's inequality for sections of E due to Braverman, Milatovich and Shubin ([3], Theorem 5.7) in the following:

Lemma 6. *Every section $s \in \Gamma(E)$ satisfies*

$$\Delta_M |s| \leq \text{Re} \langle \nabla^* \nabla s, \text{Sign}(s) \rangle$$

as a distribution on M , where $\text{Sign}(s)(x)$ is $s(x)/|s(x)|$ for $s(x) \neq 0$ and otherwise 0.

Now Theorem 1 is extended to the following:

Theorem 7. *Let M be a complete noncompact Riemannian manifold of dimension m and E a Hermitian vector bundle with a metric connection ∇ and a symmetric endomorphism R of E . Suppose that the Ricci curvature of M is bounded below by $-(m-1)k^2$ for some constant $k \geq 0$, and R satisfies $\langle Rs, s \rangle \geq K|s|^2$ for some constant K and any $s \in \Gamma(E)$. Let s be a nontrivial eigensection of E with eigenvalue $-\mu$, that is, $Ls = -\mu s$. If $\mu + K > 0$, then*

$$\liminf_{r \rightarrow \infty} \frac{\log \max_{\partial B_r(o)} |s|}{r} \geq -\sqrt{\lambda_k} + \sqrt{\lambda_k + (\mu + K)},$$

where o is a fixed point of M .

Proof. Let r be the distance to a fixed point o of M and set $w = |s| - |s(o)|\phi_{\mu+K, f_k}(r)$, where we may assume that $s(o) \neq 0$. Then in view of Lemma 6, we see that $\Delta_M w \leq -(\mu + K)w$ as a distribution on M , and hence the assertion follows from Lemmas 4 and 5. $\square$

Remark. We consider the case where E is the trivial bundle $M \times \mathbb{C}$ over M . Let A be a real-valued 1-form on M and V a real-valued function on M . Define a magnetic Laplacian by $\Delta_{M,A} = d_A^* d_A$, where $d_A = d - \sqrt{-1}A$ and d_A^* is its formal adjoint. For any nontrivial solution u of equation $\Delta_{M,A} u + Vu = -\mu u$ with a constant μ such that $\mu + \inf_M V > 0$, we have

$$\liminf_{r \rightarrow \infty} \frac{\log \max_{\partial B_r(o)} |u|}{r} \geq -\sqrt{\lambda_k} + \sqrt{\lambda_k + \mu + \inf_M V},$$

where o is a fixed point of M . Theorem 1 corresponds to the case where $A = 0$ and $V = 0$.

Now we consider a complete Riemannian manifold M of dimension m which is minimally immersed into a simply connected, complete Riemannian manifold H with sectional curvature bounded above by $-k_1^2$. Assume that the Ricci curvature of M is bounded below by $-(m-1)k_2^2$ ($k_2 \geq k_1 > 0$). Let b_γ be the Busemann function of H associated with a ray γ starting from a point $o \in H$, that is, $b_\gamma(x) = \lim_{t \rightarrow \infty} \{d_H(x, \gamma(t)) - t\}$, $x \in H$. Recall the fact that the Hessian of b_γ satisfies

$$\nabla db_\gamma \geq k_1(g_H - db_\gamma \otimes db_\gamma)$$

(cf. Theorem A in [7]), and hence we have

$$\nabla de^{k_1 b_\gamma} \geq k_1^2 e^{k_1 b_\gamma} g_H,$$

where g_H denotes the metric of H . Let $j : M \rightarrow H$ be a minimal immersion. Then $e^{k_1 b_\gamma \circ j}$ satisfies

$$\Delta_M e^{k_1 b_\gamma \circ j} \leq -k_1^2 m e^{k_1 b_\gamma \circ j}$$

in M . Thus by the same reason as in Theorem 7, we have

$$\liminf_{r \rightarrow \infty} \frac{\max_{x \in \partial B_r(o)} k_1 b_\gamma(j(x))}{r} \geq -\sqrt{\lambda_{k_2}} + \sqrt{\lambda_{k_2} + k_1^2 m} > 0.$$

Since $b_\gamma(p) \leq d_H(o, p)$ for all $p \in H$, we have thus proved the following:

Proposition 8. *Let $j : M \rightarrow H$ be a minimal immersion of a complete noncompact Riemannian manifold M of dimension m into a simply connected, complete Riemannian manifold H whose sectional curvature is bounded above by $-k_1^2$. If the Ricci curvature of M is bounded below by $-(m-1)k_2^2$ for some constant $k_2 (\geq k_1 > 0)$, then*

$$\liminf_{r \rightarrow \infty} \frac{\max_{x \in \partial B_r(o)} d_H(j(x), o)}{r} \geq \frac{-\sqrt{\lambda_{k_2}} + \sqrt{\lambda_{k_2} + k_1^2 m}}{k_1},$$

where $d_H(j(x), o)$ is the distance in H between $j(x)$ and a fixed point o of H .

We note that the function $e^{k_1 b_\gamma \circ j}$ as above is a positive eigenfunction of M with eigenvalue $-k_1^2 m$ in the case where $H = M(-k_1^2)$, that is, H is the simply connected hyperbolic space form of constant curvature $-k_1^2$. We also remark that if H has nonpositive sectional curvature, then

$$\max_{x \in \partial B_r(o)} d_H(j(x), o) \geq C\sqrt{r}$$

for some positive constant C and all sufficient large r . See Proposition 2 in [8] for the proof. In addition, relevantly to the subject of [8], we have the following:

Proposition 9. *Let M be a complete noncompact Riemannian manifold of dimension m and suppose that the Ricci curvature is bounded below by $-c(r^2 + 1)$ for some positive constant c , where r is the distance to a fixed point of M . Let u be a nontrivial continuous function on M satisfying $\Delta_M|u| \leq -\mu|u|$ with a negative constant $-\mu$ as a distribution. Then u is an unbounded function on M .*

Proof. Let f be the solution of equation: $f''(r) - c(r^2 + 1)f(r) = 0$, subject to the conditions: $f(0) = 0$ and $f'(0) = 1$. Then in view of Lemmas 3 and 5, it suffices to verify that

$$\frac{\int_R^r f^{m-1}(s)ds}{f^{m-1}(r)} \geq \frac{C}{r} \quad (\text{on } (R, +\infty)) \quad (4)$$

for some constant C and R .

To see this, let $\psi_f(r) = f'(r)/f(r)$ and $\eta(r) = Ar$, where A is a positive constant chosen in such a way that $A \geq \max\{\sqrt{c}, c, \psi_f(R)/R\}$. Then obviously $\eta(R) \geq \psi_f(R)$ and also

$$\eta'(r) + \eta(r)^2 \geq A^2 r^2 + A \geq cr^2 + c = \psi_f'(r) + \psi_f(r)^2.$$

Therefore it follows from the classical comparison theorem on Riccati equation that

$$\psi_f(r) \leq \eta(r) = Ar \quad \text{for all } r \geq R. \quad (5)$$

On the other hand, direct calculus shows that

$$\frac{\int_R^r f^{m-1}(s)ds}{f^{m-1}(r)} = \frac{1}{(m-1)\psi_f(\xi_r)} \left(1 - \frac{\int_0^R f^{m-1}(s)ds}{\int_0^r f^{m-1}(s)ds} \right) \quad (6)$$

for some $0 < \xi_r < r$. Combining (5) and (6), we get the inequality (4) mentioned above. $\square$

Proof of Theorem 2. For any $r > 0$, take points $x, y \in \partial B_r(p)$ satisfying $u(x) = \max_{\partial B_r(p)} u$ and $u(y) = \min_{\partial B_r(p)} u$. Then by Borbély's estimate (1) and integrating the gradient of $\log u$ along a minimizing path joining x to y , we get $u(x) \leq e^{\sqrt{\mu}d(x,y)}u(y)$. By the assumption on the diameter of the metric sphere $\partial B_r(o)$, we see that $u(x) \leq e^{a\sqrt{\mu}r}u(y)$. On the other hand, in view of Theorem 1, for any $\epsilon > 0$, there exists a constant $C > 0$ such that $u(x) \geq Ce^{\sqrt{\mu}(1-\epsilon)r}$ for all sufficient large r . Hence we have $\min_{\partial B_r(p)} u \geq Ce^{(1-a-\epsilon)\sqrt{\mu}r}$, from which it follows that $\liminf_{r \rightarrow \infty} (\log \min_{\partial B_r(o)} u)/r \geq (1-a-\epsilon)\sqrt{\mu}$. Letting ϵ tend to 0, we obtain Theorem 2. $\square$

Appendix

We show the Sturm–Liouville comparison theorem with respect to the operator $E_f = f^{m-1}(r) \frac{\partial}{\partial r}$.

Lemma 10. *Let $f(r)$ be a monotone increasing C^2 -function on $[0, +\infty)$ with $f(0) = 0$. Let u, v be respectively positive C^2 -functions satisfying*

$$\begin{aligned} E_f^2 u(r) &\geq P(r)u(r), \\ E_f^2 v(r) &\leq Q(r)v(r) \end{aligned}$$

on $[r_0, +\infty)$ for some $r_0 > 0$. If $u'(r_0)/u(r_0) \geq v'(r_0)/v(r_0)$ and $P(r) \geq Q(r)$ for all $r \geq r_0$, then

$$\frac{u(r)}{u(r_0)} \geq \frac{v(r)}{v(r_0)} \quad \text{for } r \geq r_0.$$

Proof. Noting that $E_f(u/v) = \{E_f u \cdot v - u \cdot E_f v\}/v^2$, we have

$$\begin{aligned} E_f\{E_f u \cdot v - u \cdot E_f v\} &= E_f^2 u \cdot v + E_f u \cdot E_f v - E_f u \cdot E_f v - u \cdot E_f^2 v \\ &= (P - Q)uv \geq 0. \end{aligned}$$

As a result, $E_f u \cdot v - u \cdot E_f v$ is an increasing function, and hence $(E_f u \cdot v - u \cdot E_f v)(r) \geq (E_f u \cdot v - u \cdot E_f v)(r_0) \geq 0$ for all $r \geq r_0$. This shows that $E_f(u/v) \geq 0$ on $[r_0, +\infty)$. Thus we have $u(r)/v(r) \geq u(r_0)/v(r_0)$ if $r \geq r_0$. $\square$

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