Quasi-Symmetry for the Nonlinear Green Function on a Tree

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Received March 26, 2007; final version accepted July 23, 2007

It is well known that the linear Green function is symmetric as a two variable function. However it is not known whether the nonlinear Green function is symmetric or not. Moreover it is not known even whether that is quasi-symmetric or not, which means that the symmetric ratio of the nonlinear Green function is bounded or not. In this article we show that, for every tree, there exists a resistance such that the nonlinear Green function is quasi-symmetric; also we show that, for every tree, there exists a resistance such that the nonlinear Green function is not quasi-symmetric.

KEYWORDS: quasi-symmetric, nonlinear Green function, discrete potential theory

1. Introduction

Let $1 < p < \infty$. We consider p -harmonic functions on a tree. Let $\mathcal{T} = (V, E, r)$ be a locally finite connected tree with a resistance r , where $V = V(\mathcal{T})$ is the vertex set and $E = E(\mathcal{T})$ is the edge set. An edge $(x, y) \in E$ is an ordered pair of vertices such that $(x, y) \in E$ if and only if $(y, x) \in E$. If $(x, y) \in E$, then we say that x is adjacent to y and write $x \sim y$. A resistance r is a positive function on E such that $r(y, x) = r(x, y)$. We define the discrete derivative ∇u and the discrete p -Laplacian $\Delta_p u$ for a function u on V by

$$\begin{aligned}\nabla u(x, y) &= r(x, y)^{-1}(u(y) - u(x)), \\ \Delta_p u(x) &= \sum_{\substack{y \in V \\ y \sim x}} |\nabla u(x, y)|^{p-2} \nabla u(x, y).\end{aligned}$$

Let $\varphi_p(t) = |t|^{p-2}t$. Then we can write

$$\Delta_p u(x) = \sum_{\substack{y \in V \\ y \sim x}} \varphi_p(\nabla u(x, y)) = - \sum_{\substack{y \in V \\ y \sim x}} \varphi_p(\nabla u(y, x)).$$

Let $D \subset V$. If $\Delta_p u = 0$ in D , then we say that u is p -harmonic in D . For these accounts see [4–6].

Let $x, y \in V$. A path joining x to y is a sequence $\{x = x_0, x_1, \dots, x_{l-1}, x_l = y\}$ of distinct vertices such that $x_0 \sim x_1 \sim \dots \sim x_{l-1} \sim x_l$. Since $\mathcal{T}$ is a tree, it is uniquely determined and denoted by $\overline{xy}$. For $x \in V$ let $\deg(x)$ be the degree of x , i.e., $\deg(x) = \#\{y \in V; x \sim y\}$. Let $A \subset E$ and $x \in V$. We remove A from E , then we obtain some components. We denote by $\mathcal{B}(\mathcal{T}, A, x)$ the component which contains x .

We define the Dirichlet sum $D_p[u]$ of order p by

$$D_p[u] = \frac{1}{2} \sum_{(x,y) \in E} r(x, y) |\nabla u(x, y)|^p.$$

Denote by $\mathbf{D}^{(p)}(\mathcal{T})$ the set of functions on V with finite Dirichlet sum of order p . Then $\mathbf{D}^{(p)}(\mathcal{T})$ is a Banach space with the norm $\|u\|_p = (D_p[u] + |u(x_0)|)^{1/p}$, where x_0 is a fixed vertex. Let $L_0(\mathcal{T})$ be the set of functions on V with finite support. Also let $\mathbf{D}_0^{(p)}(\mathcal{T})$ be the closure of $L_0(\mathcal{T})$ in $\mathbf{D}^{(p)}(\mathcal{T})$ with respect to the norm $\|\cdot\|_p$. A tree $\mathcal{T}$ is said to be of hyperbolic type of order p if $1 \notin \mathbf{D}_0^{(p)}(\mathcal{T})$; a tree $\mathcal{T}$ is said to be of parabolic type of order p otherwise. Consider the discrete boundary value problem

$$\Delta_p u = -\delta_a, \quad u \in \mathbf{D}_0^{(p)}(\mathcal{T}), \tag{1}$$

where δ_a is the characteristic function of $\{a\}$, i.e., $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. The solution u to (1) uniquely exists if and only if the tree is of hyperbolic type of order p . We call the solution u the p -Green function with pole at a and denote it by g_a . For details see Yamasaki [5, 6].

Let $\mathcal{T}$ be a tree of hyperbolic type of order p . Let

$$\begin{aligned}H(x, y) &= \frac{g_x(y)}{g_y(x)} \quad \text{for } x, y \in V, \\ M(\mathcal{T}) &= \sup_{x, y \in V} H(x, y).\end{aligned}$$

It is well known that $M(\mathcal{T}) = 1$ for any tree $\mathcal{T}$ if $p = 2$. However, if $p \neq 2$, then it is not known that $M(\mathcal{T}) = 1$, even whether $M(\mathcal{T})$ is finite or not. We consider the problem whether $M(\mathcal{T})$ is finite or not for $p \neq 2$. A tree $\mathcal{T}$ is said to have a symmetric p -Green function if $M(\mathcal{T}) = 1$; a tree $\mathcal{T}$ is said to have a quasi-symmetric p -Green function if $M(\mathcal{T})$ is finite.

Theorem 1. *Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p . Let $(a_1, a_2) \in E$. Let $\mathcal{T}_1 = \mathcal{J}(\mathcal{T}, \{(a_1, a_2)\}, a_1)$ and $\mathcal{T}_2 = \mathcal{J}(\mathcal{T}, \{(a_1, a_2)\}, a_2)$. Then $\mathcal{T}$ has a quasi-symmetric p -Green function if and only if each of $\mathcal{T}_1$ and $\mathcal{T}_2$ has a quasi-symmetric p -Green function.*

Theorem 2. *Let $p \neq 2$. Let (V, E, r) be a tree.*

- (i) *Suppose that there are only finitely many $x \in V$ such that $\deg(x) \geq 3$. Then (V, E, r) has a quasi-symmetric p -Green function whenever (V, E, r) is of hyperbolic type of order p .*
- (ii) *Suppose that there are infinitely many $x \in V$ such that $\deg(x) \geq 3$. Then we find two resistances r_1 and r_2 with the following conditions.*
  - (a) *The tree (V, E, r_1) is of hyperbolic type of order p and has a quasi-symmetric p -Green function.*
  - (b) *The tree (V, E, r_2) is of hyperbolic type of order p and, however, does not have a quasi-symmetric p -Green function.*

2. Proof of Theorem 1

First note that $cu|_{V(\mathcal{T}')} \in \mathbf{D}_0^{(p)}(\mathcal{T}')$ for any $u \in \mathbf{D}_0^{(p)}(\mathcal{T})$, any subtree $\mathcal{T}'$ of $\mathcal{T}$ and any constant c . This fact is applied repeatedly.

Lemma 1. *Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p . Let $(a, b) \in E$. Let $\mathcal{T}' = \mathcal{J}(\mathcal{T}, \{(a, b)\}, a)$ and $\mathcal{T}'' = \mathcal{J}(\mathcal{T}, \{(a, b)\}, b)$. Let $x \in V(\mathcal{T}') \cup \{b\}$.*

- (i) *If $\mathcal{T}''$ is of hyperbolic type of order p , then $g_x|_{V(\mathcal{T}'')}$ is constant times of $g_b^{\mathcal{T}''}$;*
- (ii) *if $\mathcal{T}''$ is of parabolic type of order p , then $g_x|_{V(\mathcal{T}'')}$ is a constant function in $V(\mathcal{T}'')$.*

Proof. We denote the p -Laplacian (resp. p -Green function, and so on) with respect to $\mathcal{T}'$ by $\Delta_p^{\mathcal{T}'}$ (resp. $g_x^{\mathcal{T}'}$, and so on).

First suppose $x \in V(\mathcal{T}')$. Let $\overline{xb} = \{x = x_0, x_1, \dots, x_{l-1}, x_l = b\}$. Let

$$\begin{aligned} \mathcal{T}_j &= \mathcal{J}(\mathcal{T}, \{(x_{j-1}, x_j)\}, x_j) & \text{for } j = 1, \dots, l, \\ \mathcal{J}_j &= \mathcal{J}(\mathcal{T}, \{(x_{j-1}, x_j), (x_j, x_{j+1})\}, x_j) & \text{for } j = 1, \dots, l-1, \\ \mathcal{J}_0 &= \mathcal{J}(\mathcal{T}, \{(x_0, x_1)\}, x_0). \end{aligned}$$

If $\mathcal{T}_l$ is of hyperbolic type of order p , then we let $u = g_{x_l}^{\mathcal{T}_l}$ in $V(\mathcal{T}_l)$; if $\mathcal{T}_l$ is of parabolic type of order p , then we let $u = 1$ in $V(\mathcal{T}_l)$. Then u is p -harmonic in $V(\mathcal{T}_l) \setminus \{x_l\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T}_l)$. Let

$$u(x_{l-1}) = u(x_l) - r(x_{l-1}, x_l) \varphi_q(\Delta_p^{\mathcal{T}_l} u(x_l)),$$

where q is the number with $1/p + 1/q = 1$. Since φ_q is the inverse function of φ_p , we have that u is p -harmonic at x_l .

If $\mathcal{J}_{l-1}$ is of hyperbolic type of order p , then we let

$$u = \frac{u(x_{l-1})}{g_{x_{l-1}}^{\mathcal{J}_{l-1}}(x_{l-1})} g_{x_{l-1}}^{\mathcal{J}_{l-1}} \quad \text{in } V(\mathcal{J}_{l-1});$$

if $\mathcal{J}_{l-1}$ is of parabolic type of order p , then we let $u = u(x_{l-1})$ in $V(\mathcal{J}_{l-1})$. Then u is p -harmonic in $V(\mathcal{T}_{l-1}) \setminus \{x_{l-1}\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T}_{l-1})$.

Repeat this argument and obtain a function u which is p -harmonic in $V \setminus \{x\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T})$. Therefore u is a constant times of g_x . Since $\mathcal{T}_l = \mathcal{T}''$ and $x_l = b$, we have the result in this case.

Next suppose $x = b$. If $\mathcal{T}'$ is of hyperbolic type of order p , then we let $u = g_a^{\mathcal{T}'}$ in $V(\mathcal{T}')$; if $\mathcal{T}'$ is of parabolic type of order p , then we let $u = 1$ in $V(\mathcal{T}')$. Then u is p -harmonic in $V(\mathcal{T}') \setminus \{a\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T}')$. Let

$$u(b) = u(a) - r(a, b) \varphi_q(\Delta_p^{\mathcal{T}'} u(a)).$$

Then u is p -harmonic at a .

If $\mathcal{T}''$ is of hyperbolic type of order p , then we let

$$u = \frac{u(b)}{g_b^{\mathcal{T}''}(b)} g_b^{\mathcal{T}''} \quad \text{in } V(\mathcal{T}'');$$

if $\mathcal{T}''$ is of parabolic type of order p , then we let $u = u(b)$ in $V(\mathcal{T}'')$. Then u is p -harmonic in $V \setminus \{b\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T})$. Therefore u is a constant times of g_x , and the result follows. $\square$

Lemma 2. *For any $x, y, z \in V$ we have*

Fig. 1. The intersection vertex.

$$H(x, z) = H(x, y)H(y, z).$$

Proof. First assume that y is a vertex on the path $\overline{xz}$. Let y' be the vertex adjacent to y and on the path $\overline{yx}$. Let $\mathcal{T}' = \mathcal{J}(\mathcal{T}, \{(y, y')\}, y)$. By Lemma 1 both $g_x|_{V(\mathcal{T}')}$ and $g_y|_{V(\mathcal{T}')}$ are constant times of $g_y^{\mathcal{T}'}$ if $\mathcal{T}'$ is of hyperbolic type of order p , or both are constant functions on $V(\mathcal{T}')$ if $\mathcal{T}'$ is of parabolic type. Therefore

$$g_x|_{V(\mathcal{T}')} = \frac{g_x(y)}{g_y(y)} g_y|_{V(\mathcal{T}')}.$$

Especially

$$g_x(z) = \frac{g_x(y)}{g_y(y)} g_y(z).$$

Similarly we have

$$g_z(x) = \frac{g_z(y)}{g_y(y)} g_y(x).$$

Hence the result follows in this case.

Next we consider the general case. Let w be the intersection vertex among x , y and z , i.e., the vertex which is simultaneously on the three paths $\overline{xy}$, $\overline{yz}$ and $\overline{zx}$ (see Fig. 1). Then the first part implies that

$$\begin{aligned} H(x, z) &= H(x, w)H(w, z), \\ H(x, y) &= H(x, w)H(w, y), \\ H(y, z) &= H(y, w)H(w, z). \end{aligned}$$

Since $H(w, y)$ is the reciprocal of $H(y, w)$, we have the result in the general case. $\square$

Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p . Let $(y, z) \in E$ and $\mathcal{J} = \mathcal{J}(\mathcal{T}, \{(y, z)\}, z)$. If $\mathcal{J}$ is of parabolic type, then we call $\mathcal{J}$ a parabolic end of $\mathcal{T}$.

Lemma 3. *Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p . Let $\{\mathcal{J}_j\}_j$ be the set of maximal parabolic ends of $\mathcal{T}$. Let $\mathcal{T}' = (V', E', r')$ be the subtree which is obtained by removing $\bigcup_j \mathcal{J}_j$ from $\mathcal{T}$. Then*

$$M(\mathcal{T}) = M(\mathcal{T}').$$

Proof. For each j we can take an edge $(y_j, z_j) \in E$ such that $\mathcal{J}_j = \mathcal{J}(\mathcal{T}, \{(y_j, z_j)\}, z_j)$. Let $x \in V$. We denote the p -Green function with respect to $\mathcal{T}'$ by $g_x^{\mathcal{T}'}$. Then it is easy to verify that the p -Green function g_x is represented as

$$\begin{aligned} g_x(y) &= g_x^{\mathcal{T}'}(y) && \text{if } x \in V' \text{ and } y \in V', \\ g_x(y) &= g_x^{\mathcal{T}'}(y_j) && \text{if } x \in V' \text{ and } y \in V(\mathcal{J}_j), \\ g_x(y) &= g_{y_j}^{\mathcal{T}'}(y) && \text{if } x \in V(\mathcal{J}_j) \text{ and } y \in V', \\ g_x(y) &= g_{y_j}^{\mathcal{T}'}(y_i) && \text{if } x \in V(\mathcal{J}_j) \text{ and } y \in V(\mathcal{J}_i) \text{ with } i \neq j, \\ g_x(y) &= g_w(w) && \text{if } x \in V(\mathcal{J}_j) \text{ and } y \in V(\mathcal{J}_j), \end{aligned}$$

where w is the intersection vertex among x , y and z_j . Therefore

$$\begin{aligned}
H(x, y) &= H^{\mathcal{T}'}(x, y) && \text{if } x \in V' \text{ and } y \in V', \\
H(x, y) &= H^{\mathcal{T}'}(x, y_j) && \text{if } x \in V' \text{ and } y \in V(\mathcal{J}_j), \\
H(x, y) &= H^{\mathcal{T}'}(y_j, y) && \text{if } x \in V(\mathcal{J}_j) \text{ and } y \in V', \\
H(x, y) &= H^{\mathcal{T}'}(y_j, y_i) && \text{if } x \in V(\mathcal{J}_j) \text{ and } y \in V(\mathcal{J}_i) \text{ with } i \neq j, \\
H(x, y) &= 1 && \text{if } x \in V(\mathcal{J}_j) \text{ and } y \in V(\mathcal{J}_j).
\end{aligned}$$

This means $M(\mathcal{T}) = M(\mathcal{T}')$. □

Proof of Theorem 1. Let $\mathcal{T}_1 = (V_1, E_1, r_1)$ and $\mathcal{T}_2 = (V_2, E_2, r_2)$. By means of Lemma 3 we may assume that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are of hyperbolic type of order p .

First we observe that the function $g_{a_1}|_{V_1}$ is p -harmonic in $V_1 \setminus \{a_1\}$ and

$$\Delta_p^{\mathcal{T}_1}(g_{a_1}|_{V_1})(a_1) = \Delta_p g_{a_1}(a_1) - \varphi_p(\nabla g_{a_1}(a_1, a_2)) = -1 + \varphi_p(\nabla g_{a_1}(a_2, a_1)).$$

Let q be the number with $1/p + 1/q = 1$. Then we have that φ_q is the inverse function of φ_p . Therefore

$$\frac{1}{c_0} g_{a_1}|_{V_1} = g_{a_1}^{\mathcal{T}_1},$$

where $c_0 = \varphi_q(1 - \varphi_p(\nabla g_{a_1}(a_2, a_1)))$. Especially

$$g_{a_1}(x) = c_0 g_{a_1}^{\mathcal{T}_1}(x) \quad \text{for } x \in V_1. \quad (2)$$

Note that c_0 is independent of x .

Let $\{b_1, \dots, b_{d-1}\}$ be the neighbors of a_1 in $\mathcal{T}_1$, where $d = \deg(a_1)$. Let $\mathcal{T}_{1i} = \mathcal{J}(\mathcal{T}_1, \{(a_1, b_i)\}, b_i)$. Let $x \in V_1 \setminus \{a_1\}$ and $\{a_1 = x_0, x_1, \dots, x_{l-1}, x_l = x\}$ be the path $\overline{a_1 x}$. Then $x_1 = b_{i_0}$ for some i_0 . For $i \neq i_0$ the function $g_{x_1}|_{V(\mathcal{T}_{1i})}$ is p -harmonic in $V(\mathcal{T}_{1i}) \setminus \{b_i\}$ and

$$\Delta_p^{\mathcal{T}_{1i}}(g_{x_1}|_{V(\mathcal{T}_{1i})})(b_i) = \Delta_p g_{x_1}(b_i) - \varphi_p(\nabla g_{x_1}(b_i, a_1)) = -\varphi_p(\nabla g_{x_1}(b_i, a_1)).$$

Therefore

$$\frac{1}{\nabla g_{x_1}(b_i, a_1)} g_{x_1}|_{V(\mathcal{T}_{1i})} = g_{b_i}^{\mathcal{T}_{1i}},$$

and hence

$$\begin{aligned}
g_{x_1}(a_1) &= g_{x_1}(b_i) + r(a_1, b_i) \nabla g_{x_1}(b_i, a_1) \\
&= \nabla g_{x_1}(b_i, a_1) (g_{b_i}^{\mathcal{T}_{1i}}(b_i) + r(a_1, b_i)).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
g_x(a_1) &= \nabla g_x(b_i, a_1) (g_{b_i}^{\mathcal{T}_{1i}}(b_i) + r(a_1, b_i)), \\
g_{x_1}^{\mathcal{T}_1}(a_1) &= \nabla g_{x_1}^{\mathcal{T}_1}(b_i, a_1) (g_{b_i}^{\mathcal{T}_{1i}}(b_i) + r(a_1, b_i)), \\
g_x^{\mathcal{T}_1}(a_1) &= \nabla g_x^{\mathcal{T}_1}(b_i, a_1) (g_{b_i}^{\mathcal{T}_{1i}}(b_i) + r(a_1, b_i)).
\end{aligned}$$

Therefore

$$\frac{g_{x_1}(a_1)}{\nabla g_{x_1}(b_i, a_1)} = \frac{g_x(a_1)}{\nabla g_x(b_i, a_1)} = \frac{g_{x_1}^{\mathcal{T}_1}(a_1)}{\nabla g_{x_1}^{\mathcal{T}_1}(b_i, a_1)} = \frac{g_x^{\mathcal{T}_1}(a_1)}{\nabla g_x^{\mathcal{T}_1}(b_i, a_1)}. \quad (3)$$

Also we have

$$\frac{1}{\nabla g_{x_1}(a_2, a_1)} g_{x_1}|_{V_2} = g_{a_2}^{\mathcal{T}_2},$$

and hence

$$g_{x_1}(a_1) = \nabla g_{x_1}(a_2, a_1) (g_{a_2}^{\mathcal{T}_2}(a_2) + r(a_1, a_2)).$$

Similarly we have

$$g_x(a_1) = \nabla g_x(a_2, a_1) (g_{a_2}^{\mathcal{T}_2}(a_2) + r(a_1, a_2)).$$

Therefore

$$\frac{g_{x_1}(a_1)}{\nabla g_{x_1}(a_2, a_1)} = \frac{g_x(a_1)}{\nabla g_x(a_2, a_1)}. \quad (4)$$

Since $\Delta_p g_x(a_1) = 0$ and $\Delta_p g_{x_1}(a_1) = 0$, we have

$$\begin{aligned}\varphi_p(\nabla g_x(a_1, x_1)) &= \sum_{i \neq i_0} \varphi_p(\nabla g_x(b_i, a_1)) + \varphi_p(\nabla g_x(a_2, a_1)), \\ \varphi_p(\nabla g_{x_1}(a_1, x_1)) &= \sum_{i \neq i_0} \varphi_p(\nabla g_{x_1}(b_i, a_1)) + \varphi_p(\nabla g_{x_1}(a_2, a_1)).\end{aligned}$$

Using (3) and (4), we have

$$\begin{aligned}\nabla g_x(b_i, a_1) &= \frac{g_x(a_1)}{g_{x_1}(a_1)} \nabla g_{x_1}(b_i, a_1), \\ \nabla g_x(a_2, a_1) &= \frac{g_x(a_1)}{g_{x_1}(a_1)} \nabla g_{x_1}(a_2, a_1),\end{aligned}$$

and hence

$$\frac{\nabla g_x(a_1, x_1)}{\nabla g_{x_1}(a_1, x_1)} = \frac{g_x(a_1)}{g_{x_1}(a_1)}. \quad (5)$$

Similarly, since $\Delta_p^{\mathcal{T}_1} g_x^{\mathcal{T}_1}(a_1) = 0$ and $\Delta_p^{\mathcal{T}_1} g_{x_1}^{\mathcal{T}_1}(a_1) = 0$, we have

$$\begin{aligned}\varphi_p(\nabla g_x^{\mathcal{T}_1}(a_1, x_1)) &= \sum_{i \neq i_0} \varphi_p(\nabla g_x^{\mathcal{T}_1}(b_i, a_1)), \\ \varphi_p(\nabla g_{x_1}^{\mathcal{T}_1}(a_1, x_1)) &= \sum_{i \neq i_0} \varphi_p(\nabla g_{x_1}^{\mathcal{T}_1}(b_i, a_1)).\end{aligned}$$

Using (3), we have

$$\frac{\nabla g_x^{\mathcal{T}_1}(a_1, x_1)}{\nabla g_{x_1}^{\mathcal{T}_1}(a_1, x_1)} = \frac{g_x^{\mathcal{T}_1}(a_1)}{g_{x_1}^{\mathcal{T}_1}(a_1)}. \quad (6)$$

Combining (5) and (6), since $x_1 = b_{i_0}$, we have

$$\frac{\nabla g_x(a_1, x_1)}{\nabla g_{x_1}^{\mathcal{T}_1}(a_1, x_1)} = c_{i_0} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}, \quad (7)$$

where

$$c_j = \frac{g_{b_j}^{\mathcal{T}_1}(a_1)}{g_{b_j}(a_1)} \frac{\nabla g_{b_j}(a_1, b_j)}{\nabla g_{b_j}^{\mathcal{T}_1}(a_1, b_j)}.$$

If we put $c = \max(c_1, \dots, c_{d-1}, c_1^{-1}, \dots, c_{d-1}^{-1})$, then c is independent of x . Now we obtain by (7)

$$\begin{aligned}g_x(x_1) &= g_x(a_1) + r(a_1, x_1) \nabla g_x(a_1, x_1) \\ &= g_x(a_1) + r(a_1, x_1) c_{i_0} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \nabla g_x^{\mathcal{T}_1}(a_1, x_1) \\ &\leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} (g_x^{\mathcal{T}_1}(a_1) + r(a_1, x_1) \nabla g_x^{\mathcal{T}_1}(a_1, x_1)) = c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} g_x^{\mathcal{T}_1}(x_1).\end{aligned}$$

Therefore

$$\frac{g_x(x_1)}{g_x^{\mathcal{T}_1}(x_1)} \leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}. \quad (8)$$

Similarly we have

$$\frac{g_x(x_1)}{g_x^{\mathcal{T}_1}(x_1)} \geq c^{-1} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}. \quad (9)$$

Next let $y \in V_1$ with $y \sim x_1$ and $y \neq a_1, x_2$. Let $\mathcal{T}_{1y} = \mathcal{A}(\mathcal{T}_1, \{(x_1, y)\}, y)$. Then we have

$$\frac{1}{\nabla g_x(y, x_1)} g_x|_{V(\mathcal{T}_{1y})} = g_y^{\mathcal{T}_{1y}},$$

and therefore

$$g_x(x_1) = \nabla g_x(y, x_1) (g_y^{\mathcal{T}_{1y}}(y) + r(y, x_1)).$$

Similarly we have

$$g_x^{\mathcal{T}_1}(x_1) = \nabla g_x^{\mathcal{T}_1}(y, x_1)(g_y^{\mathcal{T}_1}(y) + r(y, x_1)).$$

Combined with (8) and (9), we have

$$c^{-1} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq \frac{\nabla g_x(y, x_1)}{\nabla g_x^{\mathcal{T}_1}(y, x_1)} \leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}. \quad (10)$$

Since $\Delta_p g_x(x_1) = 0$ and $\Delta_p^{\mathcal{T}_1} g_x^{\mathcal{T}_1}(x_1) = 0$, we have

$$\begin{aligned} \varphi_p(\nabla g_x(x_1, x_2)) &= \sum_{\substack{y \sim x_1 \\ y \neq a_1, x_2}} \varphi_p(\nabla g_x(y, x_1)) + \varphi_p(\nabla g_x(a_1, x_1)), \\ \varphi_p(\nabla g_x^{\mathcal{T}_1}(x_1, x_2)) &= \sum_{\substack{y \sim x_1 \\ y \neq a_1, x_2}} \varphi_p(\nabla g_x^{\mathcal{T}_1}(y, x_1)) + \varphi_p(\nabla g_x^{\mathcal{T}_1}(a_1, x_1)). \end{aligned}$$

Formulas (7) and (10) imply that

$$c^{-1} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq \frac{\nabla g_x(x_1, x_2)}{\nabla g_x^{\mathcal{T}_1}(x_1, x_2)} \leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}.$$

Using (8), we have

$$\begin{aligned} g_x(x_2) &= g_x(x_1) + r(x_1, x_2) \nabla g_x(x_1, x_2) \\ &\leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} (g_x^{\mathcal{T}_1}(x_1) + r(x_1, x_2) \nabla g_x^{\mathcal{T}_1}(x_1, x_2)) \\ &= c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} g_x^{\mathcal{T}_1}(x_2), \end{aligned}$$

and therefore

$$\frac{g_x(x_2)}{g_x^{\mathcal{T}_1}(x_2)} \leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}.$$

Also we have by (9)

$$\frac{g_x(x_2)}{g_x^{\mathcal{T}_1}(x_2)} \geq c^{-1} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}.$$

We repeat these arguments and obtain

$$c^{-1} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq \frac{\nabla g_x(x_{l-1}, x)}{\nabla g_x^{\mathcal{T}_1}(x_{l-1}, x)} \leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}, \quad (11)$$

$$\begin{aligned} c^{-1} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} &\leq \frac{g_x(x)}{g_x^{\mathcal{T}_1}(x)} \leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}, \\ c^{-1} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} &\leq \frac{\nabla g_x(z, x)}{\nabla g_x^{\mathcal{T}_1}(z, x)} \leq c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \quad \text{for } z \sim x \text{ with } z \neq x_{l-1}. \end{aligned} \quad (12)$$

We have

$$\begin{aligned} \sum_{\substack{z \sim x \\ z \neq x_{l-1}}} \varphi_p(\nabla g_x(z, x)) + \varphi_p(\nabla g_x(x_{l-1}, x)) &= -\Delta_p g_x(x) = 1, \\ \sum_{\substack{z \sim x \\ z \neq x_{l-1}}} \varphi_p(\nabla g_x^{\mathcal{T}_1}(z, x)) + \varphi_p(\nabla g_x^{\mathcal{T}_1}(x_{l-1}, x)) &= -\Delta_p^{\mathcal{T}_1} g_x^{\mathcal{T}_1}(x) = 1. \end{aligned}$$

Equations (11) and (12) imply that

$$\varphi_p\left(c^{-1} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}\right) \leq 1 \leq \varphi_p\left(c \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}\right).$$

This means that

$$c^{-1} \leq \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq c.$$

Hence, combining with (2), we have

$$c^{-1} c_0^{-1} H^{\mathcal{T}_1}(x, a_1) \leq H(x, a_1) \leq c c_0^{-1} H^{\mathcal{T}_1}(x, a_1).$$

We obtain similarly that there are constants c' and c'_0 such that

$$c'^{-1}c'_0{}^{-1}H^{\mathcal{T}_2}(x, a_2) \leq H(x, a_2) \leq c'c'_0{}^{-1}H^{\mathcal{T}_2}(x, a_2) \quad \text{for } x \in V_2.$$

Therefore Lemma 2 implies that, if $x, y \in V_1$, then

$$\begin{aligned} H(x, y) &= H(x, a_1)H(a_1, y) \leq cc_0^{-1}H^{\mathcal{T}_1}(x, a_1)cc_0H^{\mathcal{T}_1}(a_1, y) \\ &= c^2H^{\mathcal{T}_1}(x, y), \end{aligned}$$

and similarly

$$H(x, y) \geq c^{-2}H^{\mathcal{T}_1}(x, y);$$

if $x \in V_1$ and $y \in V_2$, then

$$\begin{aligned} H(x, y) &= H(x, a_1)H(a_1, a_2)H(a_2, y) \\ &\leq cc_0^{-1}H^{\mathcal{T}_1}(x, a_1) \times H(a_1, a_2) \times c'c'_0H^{\mathcal{T}_2}(a_2, y), \end{aligned}$$

and similarly

$$H(x, y) \geq c^{-1}c'^{-1}c_0^{-1}c'_0H^{\mathcal{T}_1}(x, a_1)H(a_1, a_2)H^{\mathcal{T}_2}(a_2, y).$$

These imply the result. $\square$

3. Proof of Theorem 2

Lemma 4. *Let $\mathcal{T} = (V, E, r)$ be a tree. Let $a, x \in V$ and $\{a = x_0, x_1, \dots, x_{l-1}, x_l = x\}$ the path $\overline{ax}$. Let $r_j = r(x_{j-1}, x_j)$ and*

$$\begin{aligned} \mathcal{T}_j^- &= \mathcal{S}(\mathcal{T}, \{(x_j, x_{j+1})\}, x_j) \quad \text{for } j = 0, \dots, l-1, \\ \mathcal{T}_j^+ &= \mathcal{S}(\mathcal{T}, \{(x_{j-1}, x_j)\}, x_j) \quad \text{for } j = 1, \dots, l. \end{aligned}$$

Suppose that $\mathcal{T}_j^-$ and $\mathcal{T}_j^+$ are of hyperbolic type of order p . Let

$$\lambda_j = g_{x_j}^{\mathcal{T}_j^-}(x_j), \quad \rho_j = g_{x_j}^{\mathcal{T}_j^+}(x_j).$$

Then

$$H(a, x)^{p-1} = \prod_{j=1}^l \frac{(\lambda_{j-1} + r_j)^{p-1} + \rho_j^{p-1}}{\lambda_{j-1}^{p-1} + (\rho_j + r_j)^{p-1}}.$$

Proof. Since

$$g_a|_{V(\mathcal{T}_j^+)} = \frac{g_a(x_j)}{g_{x_j}^{\mathcal{T}_j^+}(x_j)} g_{x_j}^{\mathcal{T}_j^+},$$

we have

$$\nabla g_a(x_j, y) = \frac{g_a(x_j)}{\rho_j} \nabla g_{x_j}^{\mathcal{T}_j^+}(x_j, y) \quad \text{for } y \in V(\mathcal{T}_j^+) \text{ with } y \sim x_j.$$

Since $\Delta_p g_a(x_j) = 0$ and $\Delta_p^{\mathcal{T}_j^+} g_{x_j}^{\mathcal{T}_j^+}(x_j) = -1$, we have

$$\varphi_p(\nabla g_a(x_j, x_{j-1})) + \sum_{\substack{y \in V(\mathcal{T}_j^+) \\ y \sim x_j}} \varphi_p(\nabla g_a(x_j, y)) = 0,$$

$$\sum_{\substack{y \in V(\mathcal{T}_j^+) \\ y \sim x_j}} \varphi_p(\nabla g_{x_j}^{\mathcal{T}_j^+}(x_j, y)) = -1.$$

Therefore

$$\varphi_p(\nabla g_a(x_j, x_{j-1})) - \varphi_p\left(\frac{g_a(x_j)}{\rho_j}\right) = 0,$$

or

$$\frac{g_a(x_{j-1}) - g_a(x_j)}{r_j} = \frac{g_a(x_j)}{\rho_j},$$

and hence

$$g_a(x_j) = \frac{\rho_j}{\rho_j + r_j} g_a(x_{j-1}).$$

Therefore

$$g_a(x) = \left(\prod_{j=1}^l \frac{\rho_j}{\rho_j + r_j} \right) g_a(a).$$

Similarly we have

$$g_x(a) = \left(\prod_{j=1}^l \frac{\lambda_{j-1}}{\lambda_{j-1} + r_j} \right) g_x(x).$$

Therefore

$$H(a, x) = \frac{g_a(a)}{\lambda_0(\rho_1 + r_1)} \left(\prod_{j=1}^{l-1} \frac{\rho_j(\lambda_{j-1} + r_j)}{\lambda_j(\rho_{j+1} + r_{j+1})} \right) \frac{(\lambda_{l-1} + r_l)\rho_l}{g_x(x)}. \quad (13)$$

Since

$$g_{x_j}|_{V(\mathcal{T}_{j-1}^-)} = \frac{g_{x_j}(x_{j-1})}{g_{x_{j-1}}(x_{j-1})} \frac{\mathcal{T}_{j-1}^-}{g_{x_{j-1}}(x_{j-1})},$$

we have

$$\nabla g_{x_j}(x_{j-1}, y) = \frac{g_{x_j}(x_{j-1})}{\lambda_{j-1}} \nabla g_{x_{j-1}}^{\mathcal{T}_{j-1}^-}(x_{j-1}, y)$$

for $y \in V(\mathcal{T}_{j-1}^-)$ with $y \sim x_{j-1}$. Therefore $\Delta_p g_{x_j}(x_{j-1}) = 0$ and $\Delta_p^{\mathcal{T}_{j-1}^-} g_{x_{j-1}}^{\mathcal{T}_{j-1}^-}(x_{j-1}) = -1$ imply that

$$\varphi_p(\nabla g_{x_j}(x_{j-1}, x_j)) - \varphi_p\left(\frac{g_{x_j}(x_{j-1})}{\lambda_{j-1}}\right) = 0,$$

that is

$$g_{x_j}(x_{j-1}) = \frac{\lambda_{j-1}}{\lambda_{j-1} + r_j} g_{x_j}(x_j).$$

Hence

$$\nabla g_{x_j}(x_j, x_{j-1}) = -\frac{1}{\lambda_{j-1} + r_j} g_{x_j}(x_j). \quad (14)$$

Next since

$$g_{x_j}|_{V(\mathcal{T}_j^+)} = \frac{g_{x_j}(x_j)}{g_{x_j}^{\mathcal{T}_j^+}(x_j)} \frac{\mathcal{T}_j^+}{g_{x_j}^{\mathcal{T}_j^+}(x_j)},$$

we have similarly

$$\nabla g_{x_j}(x_j, y) = \frac{g_{x_j}(x_j)}{\rho_j} \nabla g_{x_j}^{\mathcal{T}_j^+}(x_j, y) \quad \text{for } y \in V(\mathcal{T}_j^+) \text{ with } y \sim x_j. \quad (15)$$

Since $\Delta_p g_{x_j}(x_j) = -1$ and $\Delta_p^{\mathcal{T}_j^+} g_{x_j}^{\mathcal{T}_j^+}(x_j) = -1$, we have by (14) and (15)

$$-\varphi_p\left(\frac{g_{x_j}(x_j)}{\lambda_{j-1} + r_j}\right) - \varphi_p\left(\frac{g_{x_j}(x_j)}{\rho_j}\right) = -1.$$

Therefore

$$\frac{1}{g_{x_j}(x_j)^{p-1}} = \frac{1}{(\lambda_{j-1} + r_j)^{p-1}} + \frac{1}{\rho_j^{p-1}} \quad \text{for } j = 1, \dots, l.$$

Similarly we have

$$\frac{1}{g_{x_j}(x_j)^{p-1}} = \frac{1}{\lambda_j^{p-1}} + \frac{1}{(\rho_{j+1} + r_{j+1})^{p-1}} \quad \text{for } j = 0, \dots, l-1.$$

Hence

$$\begin{aligned} \left(\frac{g_a(a)}{\lambda_0(\rho_1 + r_1)} \right)^{p-1} &= \frac{1}{\lambda_0^{p-1} + (\rho_1 + r_1)^{p-1}}, \\ \left(\frac{\rho_j(\lambda_{j-1} + r_j)}{\lambda_j(\rho_{j+1} + r_{j+1})} \right)^{p-1} &= \frac{\rho_j^{p-1} + (\lambda_{j-1} + r_j)^{p-1}}{\lambda_j^{p-1} + (\rho_{j+1} + r_{j+1})^{p-1}}, \\ \left(\frac{(\lambda_{l-1} + r_l)\rho_l}{g_x(x)} \right)^{p-1} &= (\lambda_{l-1} + r_l)^{p-1} + \rho_l^{p-1}. \end{aligned}$$

Combining these and (13) we have the result. $\square$

Lemma 5. Let $V = \{x_j\}_{j=0}^\infty$, $E = \{(x_j, x_{j+1})\}_{j=0}^\infty$ and r a resistance.

- (i) If $\sum_{j=0}^\infty r(x_j, x_{j+1}) = \infty$, then (V, E, r) is of parabolic type of order p ;
- (ii) If $\sum_{j=0}^\infty r(x_j, x_{j+1}) < \infty$, then (V, E, r) is of hyperbolic type of order p and has a symmetric p -Green function.

Proof. We shall show only (ii). It is easy to see that the p -Green function g_{x_m} is represented as

$$g_{x_m}(x_l) = \sum_{j=\max(l,m)}^\infty r(x_j, x_{j+1}).$$

Therefore $H(x_m, x_l) = 1$. $\square$

Lemma 6. For $0 \leq s, t \leq M$ we have

$$2^{-|p-2|} \leq \frac{t^{p-1} + (M-t)^{p-1}}{s^{p-1} + (M-s)^{p-1}} \leq 2^{|p-2|}.$$

Proof. It is clearly that, if $p < 2$, then we have $M^{p-1} \leq t^{p-1} + (M-t)^{p-1} \leq 2^{2-p}M^{p-1}$; if $p \geq 2$, then we have $2^{2-p}M^{p-1} \leq t^{p-1} + (M-t)^{p-1} \leq M^{p-1}$. This leads to the result. $\square$

Lemma 7. Let $V = \{x_j\}_{j=-\infty}^\infty$, $E = \{(x_j, x_{j+1})\}_{j=-\infty}^\infty$ and r a resistance. Let $S^+ = \sum_{j=0}^\infty r(x_j, x_{j+1})$ and $S^- = \sum_{j=-\infty}^{-1} r(x_j, x_{j+1})$.

- (i) If both S^+ and S^- diverge, then (V, E, r) is of parabolic type of order p ;
- (ii) If one of S^+ and S^- diverges and the other converges, then (V, E, r) is of hyperbolic type of order p and has a symmetric p -Green function;
- (iii) If both S^+ and S^- converge, then (V, E, r) is of hyperbolic type of order p and has a quasi-symmetric p -Green function.

Proof. Lemma 3 reduces (ii) to Lemma 5 (ii). We shall show (iii). We use the same notation as in Lemma 4. Then we easily have $\rho_m = \sum_{j=m}^\infty r(x_j, x_{j+1})$ and $\lambda_m = \sum_{j=-\infty}^{m-1} r(x_j, x_{j+1})$. Also we have that $\lambda_{m-1} + r_m = \lambda_m$ and $\rho_m + r_m = \rho_{m-1}$. Therefore Lemma 4 implies that, if $l > 0$, then

$$H(x_0, x_l)^{p-1} = \frac{\rho_l^{p-1} + \lambda_l^{p-1}}{\rho_0^{p-1} + \lambda_0^{p-1}}.$$

Since $M = \rho_m + \lambda_m$ is independent of m , we have by Lemma 6

$$2^{-|p-2|} \leq H(x_0, x_l)^{p-1} \leq 2^{|p-2|}.$$

The case $l < 0$ can be treated similarly. $\square$

Proof of Theorem 2 (i). Let $\mathcal{T}$ be a tree as in Theorem 2 (i). Using Lemma 3, we may assume that there are no parabolic ends. Then $\mathcal{T}$ is represented as the union of finitely many trees in Lemmas 5 and 7. Since they have quasi-symmetric p -Green functions for any resistances, $\mathcal{T}$ also has quasi-symmetric p -Green functions by Theorem 1. $\square$

Lemma 8. Let q be the number with $1/p + 1/q = 1$. Let $\mathcal{T} = (V, E, r)$ be a tree such that $\deg(x) \geq 3$ for each x and

$$r(x, y) = \frac{\psi(x)\psi(y) - 1}{(\psi(x) + 1)(\psi(y) + 1)},$$

where $\psi(x) = (\deg(x) - 1)^{q-1}$. Then $\mathcal{T}$ has a quasi-symmetric p -Green function.

Proof. Let x, y be distinct vertices and let $\{x = x_0, x_1, \dots, x_{l-1}, x_l = y\}$ be the path $\overline{xy}$. It is easy to see that the p -Green function g_x is represented as

$$g_x(y) = \frac{1}{\deg(x)^{q-1}} \frac{1}{\psi(x_1) \cdots \psi(x_{l-1})} \frac{1}{\psi(y) + 1},$$

$$g_x(x) = \frac{1}{\deg(x)^{q-1}} \frac{\psi(x)}{\psi(x) + 1}.$$

Therefore

$$H(x, y) = \frac{\deg(y)^{q-1}}{\deg(x)^{q-1}} \frac{\psi(x) + 1}{\psi(y) + 1} = \frac{(1 - \deg(x)^{-1})^{q-1} + \deg(x)^{1-q}}{(1 - \deg(y)^{-1})^{q-1} + \deg(y)^{1-q}}.$$

Using Lemma 6 for q instead of p , we have

$$2^{-|q-2|} \leq H(x, y) \leq 2^{|q-2|}.$$

Hence the result follows. $\square$

Lemma 9. Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p . Let $x_0, y_0 \in V$ and $\{x_0, x_1, \dots, x_{l-1}, x_l = y_0\}$ the path $\overline{x_0 y_0}$. Suppose that $\deg(x_0) \geq 3$, $\deg(y_0) \geq 3$ and $\deg(x_j) = 2$ for $j = 1, \dots, l-1$. Then

$$2^{-|q-2|} H(x_0, y_0) \leq H(x_0, x_j) \leq 2^{|q-2|} H(x_0, y_0)$$

for $j = 1, \dots, l-1$, where q is the number with $1/p + 1/q = 1$.

Proof. Let $\mathcal{T}_1 = \mathcal{S}(\mathcal{T}, \{(x_0, x_1)\}, x_0)$ and $\mathcal{T}_2 = \mathcal{S}(\mathcal{T}, \{(x_{l-1}, x_l)\}, y_0)$. We denote

$$u_1(j) = g_{x_0}^{\mathcal{T}_1}(x_0) + \sum_{i=1}^j r(x_{i-1}, x_i),$$

$$u_2(j) = g_{y_0}^{\mathcal{T}_2}(y_0) + \sum_{i=j+1}^l r(x_{i-1}, x_i).$$

Then it is easy to see that the p -Green function with pole at x_j is given by

$$g_{x_j}(x_j) = \frac{1}{(u_1(j)^{1-p} + u_2(j)^{1-p})^{q-1}},$$

$$g_{x_j}(x_k) = \frac{u_1(k)}{u_1(j)} g_{x_j}(x_j) \quad \text{if } 0 \leq k < j,$$

$$g_{x_j}(x) = \frac{g_{x_0}^{\mathcal{T}_1}(x)}{u_1(j)} g_{x_j}(x_j) \quad \text{if } x \in V(\mathcal{T}_1),$$

$$g_{x_j}(x_k) = \frac{u_2(k)}{u_2(j)} g_{x_j}(x_j) \quad \text{if } j < k \leq l,$$

$$g_{x_j}(x) = \frac{g_{y_0}^{\mathcal{T}_2}(x)}{u_2(j)} g_{x_j}(x_j) \quad \text{if } x \in V(\mathcal{T}_2).$$

Similarly we have

$$g_{x_0}(x_0) = \frac{1}{(u_1(0)^{1-p} + u_2(0)^{1-p})^{q-1}},$$

$$g_{x_0}(x_j) = \frac{u_2(j)}{u_2(0)} g_{x_0}(x_0).$$

Therefore

$$H(x_0, x_j)^{p-1} = \frac{u_1(j)^{p-1} + u_2(j)^{p-1}}{u_1(0)^{p-1} + u_2(0)^{p-1}}.$$

Similarly we have

$$H(x_0, y_0)^{p-1} = \frac{u_1(l)^{p-1} + u_2(l)^{p-1}}{u_1(0)^{p-1} + u_2(0)^{p-1}}.$$

Hence

$$\left(\frac{H(x_0, x_j)}{H(x_0, y_0)} \right)^{p-1} = \frac{u_1(j)^{p-1} + u_2(j)^{p-1}}{u_1(l)^{p-1} + u_2(l)^{p-1}}.$$

Since $u_1(j) + u_2(j)$ is independent of j , Lemma 6 implies that

$$2^{-|p-2|} \leq \left(\frac{H(x_0, x_j)}{H(x_0, y_0)} \right)^{p-1} \leq 2^{|p-2|}.$$

Hence the result follows. $\square$

Proof of Theorem 2 (ia). Let $\mathcal{T} = (V, E, r)$ be a tree which has infinitely many $x \in V$ such that $\deg(x) \geq 3$. If there is a subtree such that either

$$\deg(x_0) \geq 3, \deg(x_1) = 2, \dots, \deg(x_l) = 2, \deg(x_{l+1}) = 1$$

for some $l \geq 0$, or

$$\deg(x_0) \geq 3, \deg(x_1) = 2, \dots, \deg(x_l) = 2, \dots,$$

then we may remove it from $\mathcal{T}$ since we can make it a parabolic end.

Let $\{z_0^i, z_1^i, \dots, z_{m_i-1}^i, z_{m_i}^i\}_i$ be all of paths such that

$$\deg(z_0^i) \geq 3, \deg(z_1^i) = 2, \dots, \deg(z_{m_i-1}^i) = 2, \deg(z_{m_i}^i) \geq 3$$

for some $m_i \geq 2$. Let $V' = V \setminus \{z_j^i\}_{i,j}$ and $E' = E \cup \{(z_0^i, z_1^i)\}_i \setminus \{(z_{j-1}^i, z_j^i)\}_{i,j}$. Then $\deg^{(V', E')}(x) \geq 3$ for all $x \in V'$. Therefore Lemma 8 shows that there is a resistance r' on E' such that $\mathcal{T}' = (V', E', r')$ has a quasi-symmetric p -Green function. Let r be a resistance on E such that $r = r'$ on $E \cap E'$ and

$$r'(z_0^i, z_{m_i}^i) = \sum_{j=1}^{m_i} r(z_{j-1}^i, z_j^i) \quad \text{for each } i.$$

Let $x \in V'$. Then it is easy to see that the p -Green function g_x is

$$g_x = g_x^{\mathcal{T}'} \quad \text{on } V',$$

$$g_x(z_k^i) = g_x^{\mathcal{T}'}(z_0^i) + \nabla g_x^{\mathcal{T}'}(z_0^i, z_{m_i}^i) \sum_{j=1}^k r(z_{j-1}^i, z_j^i).$$

Therefore

$$H(x, y) = H^{\mathcal{T}'}(x, y) \quad \text{for } x, y \in V'.$$

Hence Lemma 9 implies that, if $x \in V'$ and $y = z_j^i$, then

$$\begin{aligned} H(x, y) &= H(x, z_0^i)H(z_0^i, y) \leq H^{\mathcal{T}'}(x, z_0^i) \cdot 2^{|q-2|} H^{\mathcal{T}'}(z_0^i, z_{m_i}^i) \\ &= 2^{|q-2|} H^{\mathcal{T}'}(x, z_{m_i}^i); \end{aligned}$$

if $x = z_j^i$ and $y = z_k^i$, then

$$\begin{aligned} H(x, y) &= H(x, z_{m_i}^i)H(z_{m_i}^i, z_0^i)H(z_0^i, y) \\ &\leq 2^{|q-2|} H^{\mathcal{T}'}(z_0^i, z_{m_i}^i) \cdot H^{\mathcal{T}'}(z_{m_i}^i, z_0^i) \cdot 2^{|q-2|} H^{\mathcal{T}'}(z_0^i, z_{m_i}^i) \\ &= 2^{2|q-2|} H^{\mathcal{T}'}(z_0^i, z_{m_i}^i). \end{aligned}$$

Therefore

$$M(\mathcal{T}) \leq 2^{2|q-2|} M(\mathcal{T}').$$

This completes the proof. $\square$

Lemma 10. Let $\mathcal{T}_0 = (V_0, E_0, r_0)$, $\mathcal{T}_j = (V_j, E_j, r_j)$ and $\mathcal{T}'_j = (V'_j, E'_j, r'_j)$ for $j \geq 1$. Let $a_j \in V_0$, $b_j \in V_j$ and $b'_j \in V'_j$. Let ρ_j be positive numbers. Let $r = r_0$ on E_0 , $r = r_j$ on E_j and $r(a_j, b_j) = \rho_j$. Let $r' = r_0$ on E_0 , $r' = r'_j$ on E'_j and $r'(a_j, b'_j) = \rho_j$. Let

$$\mathcal{T} = \left(V_0 \cup \bigcup_j V_j, E_0 \cup \bigcup_j E_j \cup \{(a_j, b_j)\}_j, r \right),$$

$$\mathcal{T}' = \left(V_0 \cup \bigcup_j V'_j, E_0 \cup \bigcup_j E'_j \cup \{(a_j, b'_j)\}_j, r' \right).$$

If $g_{b_j}^{\mathcal{T}_j}(b_j) = g_{b'_j}^{\mathcal{T}'_j}(b'_j)$ for all j , then $g_x(y) = g_x^{\mathcal{T}'}(y)$ for $x, y \in V_0$.

Proof. For $x \in V_0$ we have

$$g_x|_{V_j} = \frac{g_x(b_j)}{g_{b_j}^{\mathcal{T}_j}(b_j)} g_{b_j}^{\mathcal{T}_j}.$$

Therefore

$$\nabla g_x(b_j, y) = \frac{g_x(b_j)}{g_{b_j}^{\mathcal{T}_j}(b_j)} \nabla g_{b_j}^{\mathcal{T}_j}(b_j, y) \quad \text{for } y \in V_j \text{ with } y \sim b_j.$$

Since $\Delta_p g_x(b_j) = 0$ and $\Delta_p^{\mathcal{T}_j} g_{b_j}^{\mathcal{T}_j}(b_j) = -1$, we have

$$\varphi_p(\nabla g_x(b_j, a_j)) - \varphi_p\left(\frac{g_x(b_j)}{g_{b_j}^{\mathcal{T}_j}(b_j)}\right) = 0,$$

and hence

$$\frac{g_x(a_j) - g_x(b_j)}{\rho_j} = \frac{g_x(b_j)}{g_{b_j}^{\mathcal{T}_j}(b_j)},$$

that is

$$g_x(b_j) = \frac{g_{b_j}^{\mathcal{T}_j}(b_j)}{\rho_j + g_{b_j}^{\mathcal{T}_j}(b_j)} g_x(a_j).$$

Therefore it is easy to see that the p -Green function $g_x^{\mathcal{T}'}$ is

$$\begin{aligned} g_x^{\mathcal{T}'} &= g_x && \text{in } V_0, \\ g_x^{\mathcal{T}'} &= \frac{g_x(b_j)}{g_{b_j}^{\mathcal{T}_j}(b_j)} g_{b_j}^{\mathcal{T}_j} && \text{in } V_j'. \end{aligned}$$

Hence the result follows. $\square$

Lemma 11. Let α , β , and γ be the numbers with $0 < \alpha, \beta, \gamma < 1$. Let a , b , and c be the positive numbers such that

$$\begin{aligned} a &= \left(\frac{(1-\beta)^{p-1} + \beta^{p-1}(1-\alpha)^{p-1}}{(1-\alpha)^{p-1} + \alpha^{p-1}(1-\beta)^{p-1}} \right)^{q-1} \frac{\alpha}{\beta}, \\ b &= \left(\frac{(1-\beta)^{p-1} + \beta^{p-1}(1-\alpha)^{p-1}}{1 - \alpha^{p-1}\beta^{p-1}} \right)^{q-1} \frac{(1-\gamma)\alpha}{(1-\beta)(1-\alpha)}, \\ c &= \frac{\alpha}{((1-\alpha)^{p-1} + \alpha^{p-1}(1-\beta)^{p-1})^{q-1}}, \end{aligned} \tag{16}$$

where q is the number with $1/p + 1/q = 1$.

Let $\mathcal{T} = (V, E, r)$ be a tree (as shown in Fig. 2) such that

$$\begin{aligned} V &= \{x_l, y_{l,k}\}_{l \in \mathbb{Z}, k \in \mathbb{N}}, \\ E &= \{(x_{l-1}, x_l), (x_l, y_{l,1}), (y_{l,k}, y_{l,k+1})\}_{l \in \mathbb{Z}, k \in \mathbb{N}}, \\ r(x_{l-1}, x_l) &= a^l, \quad r(x_l, y_{l,1}) = a^l b, \quad r(y_{l,k}, y_{l,k+1}) = a^l b \gamma^k. \end{aligned}$$

Fig. 2. The tree of Lemma 11.

If $p \neq 2$ and $\alpha \neq \beta$, then $\mathcal{T}$ does not have a quasi-symmetric p -Green function. More precisely, we have

$$\sup_{l \in \mathbb{Z}} H(x_0, x_l) = \infty.$$

Proof. First we observe that a , b , and c satisfy

$$\begin{aligned} (\alpha^{-1} - 1)^{p-1} - a^{1-p}(1 - \alpha)^{p-1} - b^{1-p}(1 - \gamma)^{p-1} &= 0, \\ -(1 - \beta)^{p-1} + a^{1-p}(\beta^{-1} - 1)^{p-1} - b^{1-p}(1 - \gamma)^{p-1} &= 0, \\ c^{p-1}(a^{1-p}(1 - \alpha)^{p-1} + (1 - \beta)^{p-1} + b^{1-p}(1 - \gamma)^{p-1}) &= 1. \end{aligned}$$

These imply that the p -Green function g_{x_m} satisfies

$$\begin{aligned} g_{x_m}(x_l) &= ca^m \alpha^{l-m} & \text{if } l \geq m, \\ g_{x_m}(x_l) &= ca^m \beta^{m-l} & \text{if } l < m, \\ g_{x_m}(y_{l,k}) &= \gamma^k g_{x_m}(x_l). \end{aligned}$$

Especially

$$\begin{aligned} g_{x_0}(x_l) &= c\alpha^l, & g_{x_l}(x_0) &= ca^l \beta^l & \text{if } l > 0, \\ g_{x_0}(x_l) &= c\beta^{-l}, & g_{x_l}(x_0) &= ca^l \alpha^{-l} & \text{if } l < 0. \end{aligned}$$

Therefore, using (16), we have

$$H(x_0, x_l) = \left(\frac{\alpha}{a\beta} \right)^l = \left(\frac{(1 - \alpha)^{p-1} + \alpha^{p-1}(1 - \beta)^{p-1}}{(1 - \beta)^{p-1} + \beta^{p-1}(1 - \alpha)^{p-1}} \right)^{l(q-1)}. \quad (17)$$

Suppose that

$$\frac{(1 - \alpha)^{p-1} + \alpha^{p-1}(1 - \beta)^{p-1}}{(1 - \beta)^{p-1} + \beta^{p-1}(1 - \alpha)^{p-1}} = 1. \quad (18)$$

Then we have

$$\frac{(1 - \alpha)^{p-1}}{1 - \alpha^{p-1}} = \frac{(1 - \beta)^{p-1}}{1 - \beta^{p-1}}.$$

The function $(1 - t)^{p-1}/(1 - t^{p-1})$ is strictly increasing for $0 < t < 1$ if $1 < p < 2$; that is strictly decreasing if $p > 2$. Since $p \neq 2$ and $\alpha \neq \beta$, it follows that (18) never holds. Therefore the right hand side of (17) diverges when $l \rightarrow \infty$ or $l \rightarrow -\infty$. $\square$

Proof of Theorem 2 (iib). Let $\mathcal{T} = (V, E, r)$ be a tree which has infinitely many $x \in V$ such that $\deg(x) \geq 3$. If there is a subtree such that either

$$\deg(x_0) \geq 3, \deg(x_1) = 2, \dots, \deg(x_l) = 2, \deg(x_{l+1}) = 1$$

for some $l \geq 0$, or

$$\deg(x_0) \geq 3, \deg(x_1) = 2, \dots, \deg(x_l) = 2, \dots,$$

then we may remove it from $\mathcal{T}$ since we can make it a parabolic end.

Let $\{z_0^i, z_1^i, \dots, z_{m_i-1}^i, z_{m_i}^i\}_i$ be all of paths such that

$$\deg(z_0^i) \geq 3, \deg(z_1^i) = 2, \dots, \deg(z_{m_i-1}^i) = 2, \deg(z_{m_i}^i) \geq 3$$

for some $m_i \geq 2$. Let $V' = V \setminus \{z_j^i\}_{i,j}$ and $E' = E \cup \{(z_0^i, z_{m_i}^i)\}_i \setminus \{(z_{j-1}^i, z_j^i)\}_{i,j}$. Then $\deg^{(V', E')}(x) \geq 3$ for all $x \in V'$. We choose $\{x_l\}_{l \in \mathbb{Z}} \subset V'$ be a two-sided infinite path, i.e., $\dots \sim x_{-2} \sim x_{-1} \sim x_0 \sim x_1 \sim \dots$.

Let $\mathcal{T}'' = (V'', E'', r'')$ be a tree such that

$$\begin{aligned} V &= \{x_l, y_{l,k}\}_{l \in \mathbb{Z}, k \in \mathbb{N}}, \\ E &= \{(x_{l-1}, x_l), (x_l, y_{l,1}), (y_{l,k}, y_{l,k+1})\}_{l \in \mathbb{Z}, k \in \mathbb{N}}, \\ r(x_{l-1}, x_l) &= a^l, \quad r(x_l, y_{l,1}) = a^l b, \quad r(y_{l,k}, y_{l,k+1}) = a^l b \gamma^k, \end{aligned}$$

where a , b and γ are as in Lemma 11. Then that lemma shows that

$$\sup_{l \in \mathbb{Z}} H^{\mathcal{T}''}(x_0, x_l) = \infty.$$

Let $\mathcal{S}'_l = \mathcal{S}((V', E'), \{(x_{l-1}, x_l), (x_l, x_{l+1})\}, x_l)$. We choose a resistance r'_l on $E(\mathcal{S}'_l)$ such that

$$g_{x_l}^{\mathcal{J}'_l}(x_l) = g_{x_l}^{\mathcal{J}''_l}(x_l),$$

where $\mathcal{J}''_l = \mathcal{J}(\mathcal{T}'', \{(x_{l-1}, x_l), (x_l, x_{l+1})\}, x_l)$. Then Lemma 10 shows that there exists a resistance r' on E' such that

$$H^{\mathcal{T}'}(x_0, x_l) = H^{\mathcal{T}''}(x_0, x_l),$$

and therefore

$$\sup_{l \in \mathbb{Z}} H^{\mathcal{T}'}(x_0, x_l) = \infty.$$

Next we choose a resistance r on E such that $r = r'$ on $E \cap E'$ and

$$r'(z_0^i, z_{m_i}^i) = \sum_{j=1}^{m_i} r(z_{j-1}^i, z_j^i) \quad \text{for each } i.$$

Then a similar argument to Proof of Theorem 2 (iia) implies

$$H^{\mathcal{T}}(x_0, x_l) = H^{\mathcal{T}'}(x_0, x_l),$$

and therefore

$$\sup_{l \in \mathbb{Z}} H^{\mathcal{T}}(x_0, x_l) = \infty.$$

This completes the proof. □

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