Limit Theorems for the Average Distance and the Degree Distribution of the Threshold Network Model

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The threshold network model is a type of finite random graph. For this model, we obtain an almost-sure convergence theorem for the average distance and derive a variant of Poisson approximation of the degree distribution by using extreme value theory.

KEYWORDS: complex networks, average distance, Poisson approximation

1. Introduction

Many real-world graphs (networks) are complex and often characterized by small diameters, high clustering, and power-law (scale-free) degree distributions [1, 3, 16]. We analyze the threshold network model defined as follows. Consider n vertices labeled $1, 2, \dots, n$ and independent and identically distributed (i.i.d.) real-valued random variables $X_1, \dots, X_n$ with a common distribution function F . We connect vertices i and j with $i \neq j$ by an edge when $X_i + X_j > \theta$ for a given threshold $\theta \in \mathbb{R}$. This model is a subclass of so called hidden variable models [6, 18], and their mean behavior [4, 6, 7, 9, 13, 17, 18] and limit theorems [10–12] for the degree, the clustering coefficients, and the number of subgraphs have been analyzed. See [10–12, 14, 15] for generalizations of the model.

The rest of the paper is organized as follows. In Sect. 2, we state Theorems 1 and 2 which are main results of this paper. We prove Theorem 1 in Sect. 3 and Theorem 2 in Sect. 4.

2. Results

2.1 Average distance

Limit theorems for the degree and the clustering coefficients were obtained in previous literature [10–12]. In this paper, we prove an almost-sure convergence theorem for the average distance defined by

$$L_n = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} d(i, j),$$

where $d(i, j)$ is the graph distance between vertices i and j . It is easy to show that $d(i, j) = 1, 2$, or ∞ for the threshold network model (see Sect. 3). If $d(i, j) = \infty$ for some i and j , then $L_n = \infty$, irrespective of the number of vertex pairs that realize $d(i, j) = 1$ or 2 . Thus we use an indeterminate w corresponding to a pair of vertices with distance ∞ . Let D_n be the number of edges and $N_n^{(\infty)}$ be the number of vertex pairs with distance ∞ . We obtain

$$\begin{aligned} L_n &= \frac{1}{\binom{n}{2}} \left[1 \times D_n + 2 \times \left\{ \binom{n}{2} - D_n - N_n^{(\infty)} \right\} + w \times N_n^{(\infty)} \right] \\ &= 2 - \frac{D_n}{\binom{n}{2}} - (2 - w) \frac{N_n^{(\infty)}}{\binom{n}{2}}. \end{aligned} \quad (1)$$

Using this fact, we obtain

Theorem 1. Let $x^* = \sup\{x \in \mathbb{R} : F(x) < 1\}$. Then we have

$$\lim_{n \rightarrow \infty} L_n = 2 - \mathbb{P}(X_1 + X_2 > \theta) - (2 - w)F(\theta - x^*)\{2 - F(\theta - x^*)\}, \quad a.s.$$

2.2 Degree distribution

An almost-sure convergence theorem for the degree of the model has been proved [10–12]. In this paper, we derive a weak convergence theorem for the degree distribution of a slightly modified model. When $X_1 = x$, the probability that vertex 1 is connected to another vertex j , i.e.,

$$\mathbb{P}(X_2 > \theta - x) = 1 - F(\theta - x),$$

does not depend on n . In order to make this probability $\mathcal{O}(1/n)$ to avoid the average degree to explode as $n \rightarrow \infty$, we take a sequence of thresholds $\{\theta_n\}$ such that

$$\lim_{n \rightarrow \infty} (n - 1)\mathbb{P}(X_2 > \theta_n - x) = c(x),$$

where $c(x) \in (0, \infty)$ depends only on x . This is an analog of the Poisson approximation of the degree distribution of the Erdős-Rényi random graph [5]. However, we can not necessarily take such a sequence of thresholds.

Let $D_n(1; x) = \sum_{2 \leq j \leq n} I_{(\theta_n, \infty)}(x + X_j)$ be the degree of vertex 1 with $X_1 = x$, where $I_A(y)$ denotes the indicator function of set $A \subset \mathbb{R}$, i.e., $I_A(y) = 1$ for $y \in A$ and $I_A(y) = 0$ otherwise, and $D_n(1) \equiv D_n(1; X_1)$ be the random variable corresponding to the degree of vertex 1. We obtain the following sufficient condition:

Theorem 2. Assume that there exist a sequence $\{a_n\}$ of positive numbers and a sequence of thresholds $\{\theta_n\}$ such that $\lim_{n \rightarrow \infty} F(a_n x + \theta_n)^n = \exp(-e^{-x})$, for any $x \in \mathbb{R}$, and $\lim_{n \rightarrow \infty} a_n = 1/\lambda \in (0, \infty)$.

1. If $x^* < \infty$ and $x_* > 0$, then $D_n(1)$ does not converge weakly.
2. If $x^* < \infty$ and $x_* \leq 0$, then $D_n(1)$ converges weakly to $D(1)$ such that

$$\mathbb{P}(D(1) = 0) = F_-(0) + \int_0^{x^*} \exp(-e^{\lambda x}) F(dx),$$

$$\mathbb{P}(D(1) = k) = \int_0^{x^*} \exp(-e^{\lambda x}) \frac{(e^{\lambda x})^k}{k!} F(dx), \quad (k = 1, 2, \dots),$$

where $F_-(0) = \lim_{x \uparrow 0} F(x)$, $x^* = \sup\{x \in \mathbb{R} : F(x) < 1\}$, and $x_* = \inf\{x \in \mathbb{R} : F(x) > 0\}$.

3. If $x^* = \infty$, then $D_n(1)$ converges weakly to $D(1) \equiv \text{Poisson}(e^{\lambda X_1})$, i.e.,

$$\mathbb{P}(D(1) = k) = \int_{x_*}^{\infty} \exp(-e^{\lambda x}) \frac{(e^{\lambda x})^k}{k!} F(dx), \quad (k = 0, 1, \dots).$$

Corollary 1. Let X_1 be absolutely continuous such that it has a probability density function f . Then f satisfying the assumption of Theorem 2 must have the following form:

$$\lim_{x \rightarrow \infty} \frac{f(x+y)}{f(x)} = e^{-\lambda y}, \quad (2)$$

for any $y \in \mathbb{R}$. In this case, $D_n(1)$ converges weakly to $D(1) \equiv \text{Poisson}(e^{\lambda X_1})$.

An example of f that satisfies Eq. (2) is the exponential distribution. In this case, we obtain $\lim_{k \rightarrow \infty} \mathbb{P}(D(1) = k)/k^{-2} = 1$. This result is consistent with a previously derived approximate result [7].

3. Proof of Theorem 1

In this section, we prove Theorem 1. Let x_i ($1 \leq i \leq n$) be a realization of X_i and $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(k)}$ ($2 \leq k \leq n$) be a reordered subsequence of the realizations. If $x_{(1)} + x_{(k)} > \theta$, the vertex corresponding to $x_{(k)}$ is connected to the other $k - 1$ vertices. If $x_{(1)} + x_{(k)} \leq \theta$, then the vertex corresponding to $x_{(1)}$ is isolated. Therefore, there exists a vertex connecting all other vertices in the subgraph comprising all nonisolated vertices. Thus $d(i, j) = 1, 2, \infty$ for any $1 \leq i < j \leq n$.

Let $N_n^{(0)}$ be the number of isolated vertices. Using Eq. (1) and

$$N_n^{(\infty)} = \binom{N_n^{(0)}}{2} + N_n^{(0)}(n - N_n^{(0)}) = \frac{N_n^{(0)}(2n - N_n^{(0)} - 1)}{2},$$

we obtain

$$L_n = 2 - \frac{D_n}{\binom{n}{2}} - (2 - w) \frac{N_n^{(0)}(2n - N_n^{(0)} - 1)}{n(n-1)}. \quad (3)$$

The following limit theorem is known:

Fact 1 (Theorem 1 of [10], Fact 1 of [11]).

$$\lim_{n \rightarrow \infty} \frac{D_n}{\binom{n}{2}} = \mathbb{P}(X_1 + X_2 > \theta), \quad a.s.$$

So it is sufficient to know the asymptotic behavior of $N_n^{(0)}$. Let $N_n^{(0)}(i; x)$ be the indicator function for a vertex $i \in \{1, \dots, n\}$ with $X_i = x$, so that $N_n^{(0)}(i; x) = 1$ if vertex i is isolated and $N_n^{(0)}(i; x) = 0$ if vertex i is not isolated, i.e.,

$$N_n^{(0)}(i; x) = \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \{1 - I_{(\theta, \infty)}(x + X_j)\}.$$

Note that $N_n^{(0)} = \sum_{i=1}^n N_n^{(0)}(i)$, where $N_n^{(0)}(i) \equiv N_n^{(0)}(i; X_i)$. Then we have the following lemma:

Lemma 1. Let $x^* = \sup\{x \in \mathbb{R} : F(x) < 1\}$.

1. For any $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} N_n^{(0)}(1; x) = N^{(0)}(1; x) \equiv I_{(-\infty, \theta - x^*]}(x), \quad a.s. \quad (4)$$

2. In particular,

$$\lim_{n \rightarrow \infty} N_n^{(0)}(1) = N^{(0)}(1; X_1) \equiv I_{(-\infty, \theta - x^*]}(X_1), \quad a.s. \quad (5)$$

Proof. Since the sequence of events $\{N_n^{(0)}(1; x) = 1\}$ is nondecreasing in n , we obtain

$$\begin{aligned} \mathbb{P}\left(\lim_{n \rightarrow \infty} N_n^{(0)}(1; x) = 1\right) &= \lim_{n \rightarrow \infty} \mathbb{P}(N_n^{(0)}(1; x) = 1) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{2 \leq j \leq n} \{x + X_j \leq \theta\}\right) \\ &= \lim_{n \rightarrow \infty} F(\theta - x)^{n-1}. \end{aligned}$$

Thus we have

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} N_n^{(0)}(1; x) = 1\right) = \begin{cases} 1, & \text{if } F(\theta - x) = 1, \\ 0, & \text{if } F(\theta - x) < 1. \end{cases}$$

A similar argument gives

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} N_n^{(0)}(1; x) = 0\right) = \begin{cases} 1, & \text{if } F(\theta - x) < 1, \\ 0, & \text{if } F(\theta - x) = 1. \end{cases}$$

Thus

$$\lim_{n \rightarrow \infty} N_n^{(0)}(1; x) = \begin{cases} 1, & \text{if } F(\theta - x) = 1, \\ 0, & \text{if } F(\theta - x) < 1, \end{cases} \quad (6)$$

with probability 1. Combining Eq. (6) and

$$\{x \in \mathbb{R} : F(\theta - x) = 1\} = \{x \in \mathbb{R} : x^* \leq \theta - x\} = (-\infty, \theta - x^*]$$

leads to Eq. (4). Using Fubini's theorem and Eq. (4), we have Eq. (5) as follows:

$$\begin{aligned} &\mathbb{P}\left(\lim_{n \rightarrow \infty} N_n^{(0)}(1; X_1) = N^{(0)}(1; X_1)\right) \\ &= \int_{\mathbb{R}} \mathbb{P}\left(\lim_{n \rightarrow \infty} N_n^{(0)}(1; x) = N^{(0)}(1; x) \mid X_1 = x\right) F(dx) \\ &= \int_{\mathbb{R}} 1 \cdot F(dx) = 1. \end{aligned} \quad \square$$

We obtain the following proposition by combining Lemma 1 and the argument used for proving Theorem 3 of [11]:

Proposition 1.

$$\lim_{n \rightarrow \infty} \frac{N_n^{(0)}}{n} = \mathbb{E}[N^{(0)}(1; X_1)] = F(\theta - x^*), \quad a.s.$$

We obtain Theorem 1 by applying Fact 1 and Proposition 1 to Eq. (3).

4. Proofs of Theorem 2 and Corollary 1

In this section, we prove Theorem 2 and Corollary 1 by using extreme value theory. The theory is concerned with limit behavior of the maximum $M_n = \max\{X_1, \dots, X_n\}$ of i.i.d. random variables $\{X_i\}$. First, we review the theory briefly [8]. Suppose there exist a sequence of positive numbers $\{a_n\}$ and a sequence of real numbers $\{b_n\}$ such that $(M_n - b_n)/a_n$ has a nondegenerate limit distribution G called extreme value distribution, i.e., $\lim_{n \rightarrow \infty} F(a_n x + b_n)^n = G(x)$ for every continuity point x of G .

Fact 2 (Theorem 1.1.3 of [8]). *The class of extreme value distributions is $G_\gamma(ax + b)$ with $a > 0$ and $b \in \mathbb{R}$, where*

$$G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}), \quad 1 + \gamma x > 0,$$

with $\gamma \in \mathbb{R}$. Here if $\gamma = 0$ then the right-hand side is interpreted as $\exp(-e^{-x})$.

These distributions corresponding to $\gamma = 0$, $\gamma > 0$, and $\gamma < 0$ are often called Gumbel, Fréchet, and Weibull distribution, respectively. We can determine a_n and b_n based on the following fact:

Fact 3 (Theorem 1.2.1 and Corollary 1.2.4 of [8]). *If F is in the class of domain of attraction of G_γ , i.e., there exist a sequence of positive numbers $\{a_n\}$ and a sequence of real numbers $\{b_n\}$ such that*

$$\lim_{n \rightarrow \infty} F(a_n x + b_n)^n = G_\gamma(x), \quad (7)$$

for every continuity point x of G_γ , then

1. for $\gamma > 0$: $x^* = \infty$, $a_n = U(n)$ and $b_n = 0$,
2. for $\gamma < 0$: $x^* < \infty$, $a_n = x^* - U(n)$ and $b_n = x^*$,
3. for $\gamma = 0$: $a_n = g(U(n))$ and $b_n = U(n)$, where

$$g(t) = \frac{\int_t^{x^*} [1 - F(s)] ds}{1 - F(t)},$$

where $U(n) = \inf\{y \in \mathbb{R} : F(y) \geq 1 - 1/n\}$.

To prove Theorem 2, we need the following lemma:

Lemma 2. *If F satisfies Eq. (7) with $\gamma = 0$ and $\lim_{n \rightarrow \infty} a_n = 1/\lambda \in (0, \infty)$, then*

$$\lim_{n \rightarrow \infty} n\mathbb{P}(X_1 > b_n - x) = e^{\lambda x}, \quad \text{for any } x \in \mathbb{R}. \quad (8)$$

Proof. For $0 < G_\gamma(x) < 1$, Eq. (7) with x replaced by $-x$ yields

$$\lim_{n \rightarrow \infty} n\mathbb{P}(X_1 > -a_n x + b_n) = -\log G_\gamma(-x). \quad (9)$$

In fact, for every n , the following relation holds:

$$F(a_n x + b_n)^n = \mathbb{P}(X_1 \leq a_n x + b_n)^n = \left[1 - \frac{n\mathbb{P}(X_1 > a_n x + b_n)}{n} \right]^n.$$

If Eq. (7) holds and $\lim_{n \rightarrow \infty} a_n = 1/\lambda \in (0, \infty)$, then

$$\lim_{n \rightarrow \infty} F(x + b_n)^n = \lim_{n \rightarrow \infty} \mathbb{P}\left(a_n \cdot \frac{M_n - b_n}{a_n} \leq x\right) = G_\gamma(\lambda x), \quad (10)$$

by the same argument as the one used for proving Lemma 1 in Section 14 of [2].

Because $U(n)$ appearing in Fact 3 converges to x^* as $n \rightarrow \infty$, we obtain $\lim_{n \rightarrow \infty} a_n = \infty$ for $\gamma > 0$ and $\lim_{n \rightarrow \infty} a_n = 0$ for $\gamma < 0$. Thus $\lim_{n \rightarrow \infty} a_n = 1/\lambda \in (0, \infty)$ implies $\gamma = 0$. Therefore, we prove Eq. (8) by Eqs. (9) and (10). $\square$

Proof of Theorem 2. In order to obtain the limit distribution of $D_n(1)$, we calculate the characteristic function $\mathbb{E}[e^{itD_n(1)}]$ for $t \in \mathbb{R}$. Since $D_n(1; x) = \sum_{j=2}^n I_{(\theta_n, \infty)}(x + X_j)$ and $\mathbb{P}(I_{(\theta_n, \infty)}(x + X_2) = 1) = \mathbb{P}(X_2 > \theta_n - x)$ for $x \in \mathbb{R}$, $D_n(1; x)$ is binomially distributed. So we have

$$\begin{aligned}\mathbb{E}[e^{itD_n(1;x)}] &= \sum_{k=0}^{n-1} e^{itk} \cdot \binom{n-1}{k} \mathbb{P}(X_2 > \theta_n - x)^k [1 - \mathbb{P}(X_2 > \theta_n - x)]^{n-1-k} \\ &= \left[1 + \frac{(n-1)\mathbb{P}(X_2 > \theta_n - x)}{n-1} (e^{it} - 1) \right]^{n-1}.\end{aligned}$$

By using Fubini's theorem, we obtain

$$\begin{aligned}\mathbb{E}[e^{itD_n(1)}] &= \int_{x_*}^{x^*} \mathbb{E}[e^{itD_n(1;x)}] F(dx) \\ &= F(\theta_n - x^*) \cdot e^{it \cdot 0} + [1 - F(\theta_n - x_*)] \cdot e^{it(n-1)} \\ &\quad + \int_{\theta_n - x^*}^{\theta_n - x_*} \left[1 + \frac{(n-1)\mathbb{P}(X_2 > \theta_n - x)}{n-1} (e^{it} - 1) \right]^{n-1} F(dx).\end{aligned}\quad (11)$$

Because $\lim_{n \rightarrow \infty} \theta_n = x^*$ by the assumption and Fact 3, we have $\lim_{n \rightarrow \infty} F(\theta_n - x^*) = F_-(0)$. If $x_* > 0$, then $\lim_{n \rightarrow \infty} (\theta_n - x_*) = x^* - x_* < x^*$ which implies $\lim_{n \rightarrow \infty} [1 - F(\theta_n - x_*)] > 0$. In this case, the second term of the right-hand side of Eq. (11) does not converge. If $x_* \leq 0$, then $\lim_{n \rightarrow \infty} [1 - F(\theta_n - x_*)] = 0$ so that this term converges to 0. So we have, from Lemma 2,

$$\lim_{n \rightarrow \infty} \mathbb{E}[e^{itD_n(1)}] = \begin{cases} \text{does not exist,} & \text{for case 1 of Theorem 2,} \\ F_-(0) \cdot e^{it \cdot 0} + \int_0^{x^*} \exp(e^{\lambda x} (e^{it} - 1)) F(dx), & \text{for case 2 of Theorem 2,} \\ \int_{x_*}^{\infty} \exp(e^{\lambda x} (e^{it} - 1)) F(dx), & \text{for case 3 of Theorem 2.} \end{cases}$$

Noting that $\exp(e^{\lambda x} (e^{it} - 1))$ is the characteristic function of the Poisson distribution with parameter $e^{\lambda x}$, we obtain, for case 3,

$$\mathbb{P}(D(1) = k) = \int_{x_*}^{\infty} \mathbb{P}(\text{Poisson}(e^{\lambda x}) = k) F(dx) = \int_{x_*}^{\infty} \exp(-e^{\lambda x}) \frac{(e^{\lambda x})^k}{k!} F(dx).$$

Similarly, we obtain for case 2. □

Proof of Corollary 1. Since F is continuous and nondecreasing, we obtain, by Fact 3, $\theta_n = F^{-1}(1 - 1/n)$ and

$$a_n = \frac{\int_{\theta_n}^{x^*} (1 - F(s)) ds}{1 - F(\theta_n)} = n \int_{\theta_n}^{x^*} (1 - F(s)) ds.$$

If we assume $x^* < \infty$, we have

$$0 \leq a_n \leq n[1 - F(\theta_n)] \int_{\theta_n}^{x^*} ds = x^* - \theta_n. \quad (12)$$

Because $\lim_{n \rightarrow \infty} \theta_n = F^{-1}(1) = x^*$, Eq. (12) results in $\lim_{n \rightarrow \infty} a_n = 0$ which is a contradiction. Thus we have $x^* = \infty$.

The assumption of Theorem 2 gives

$$\lim_{n \rightarrow \infty} n[1 - F(\theta_n + y)] = e^{-\lambda y}, \quad (13)$$

for all $y \in \mathbb{R}$ by Lemma 2. Because $\theta_n = F^{-1}(1 - 1/n)$, we can rewrite Eq. (13) as

$$\lim_{n \rightarrow \infty} \frac{1 - F(\theta_n + y)}{1 - F(\theta_n)} = e^{-\lambda y}.$$

For each $x \in \mathbb{R}$, there exists n with $\theta_{n-1} \leq x \leq \theta_n$ such that

$$\frac{1 - F(\theta_n + y)}{1 - F(\theta_{n-1})} \leq \frac{1 - F(x + y)}{1 - F(x)} \leq \frac{1 - F(\theta_{n-1} + y)}{1 - F(\theta_n)},$$

since $1 - F(x)$ is nonincreasing in x and θ_n is nondecreasing in n . Thus we obtain $\lim_{x \rightarrow \infty} [1 - F(x + y)] / [1 - F(x)] = e^{-\lambda y}$. This implies

$$\lim_{x \rightarrow \infty} \frac{f(x + y)}{f(x)} = e^{-\lambda y},$$

for any $y \in \mathbb{R}$ by L' Hôpital's rule.

Conversely, we assume $\lim_{x \rightarrow \infty} f(x + y)/f(x) = e^{-\lambda y}$, for any $y \in \mathbb{R}$. We can easily prove $\lim_{n \rightarrow \infty} n[1 - F(a_n y + \theta_n)] = e^{-\lambda y}$ by choosing $a_n = 1/\lambda$ and $\theta_n = F^{-1}(1 - 1/n)$. □

Example 1. Consider the exponential distribution with parameter $\lambda > 0$ whose distribution function is given by

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In this case, we take $a_n = 1/\lambda$ and $\theta_n = (1/\lambda) \log n$ so that

$$F(a_n x + \theta_n)^n = [1 - e^{-(x + \log n)}]^n = \left(1 - \frac{e^{-x}}{n}\right)^n \rightarrow \exp(-e^{-x}).$$

For $k = 0, 1, \dots$, we have

$$\mathbb{P}(D(1) = k) = \frac{1}{k!} \int_0^\infty \exp[k\lambda x - e^{\lambda x}] \lambda e^{-\lambda x} dx = \frac{1}{k!} \int_1^\infty e^{-x} x^{k-2} dx.$$

Using $\mathbb{P}(D(1) = 2) = 1/(2e)$ and

$$k(k-1)\mathbb{P}(D(1) = k) - (k-1)(k-2)\mathbb{P}(D(1) = k-1) = \frac{1}{e(k-2)!} \quad (k \geq 2),$$

we obtain

$$\mathbb{P}(D(1) = k) = \frac{1}{k(k-1)} \cdot \frac{1}{e} \sum_{l=2}^k \frac{1}{(l-2)!} \quad (k \geq 2).$$

This gives $\lim_{k \rightarrow \infty} \mathbb{P}(D(1) = k)/k^{-2} = 1$.

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