A Note on Galois Cohomology of Algebraic Integers

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Received January 7, 2009; final version accepted March 6, 2009

We consider an admissible system $(\mathfrak{g}, (G, M))$ where $\mathfrak{g}$ is the group of a Galois extension $k/\mathbf{Q}$ of number fields, G the ring $\mathcal{O}_k$ of integers of k and $M = (\mathcal{O}_k)^2$. The cohomology group $H^1(\mathfrak{g}, \mathcal{O}_k)$ is nicely described by a naive submodule Ξ_k of $\mathcal{O}_k$ and the Poincaré index $i_\gamma(\mathfrak{g}, M)$, $\gamma \in H^1(\mathfrak{g}, \mathcal{O}_k)$, for the system $(\mathfrak{g}, (G, M))$, can be expressed in terms of Ξ_k .

KEYWORDS: Galois extension, number fields, ring of integers, cohomology, system

1. The module Ξ_k

Let $k/\mathbf{Q}$ be a finite Galois extension. Denote by $\mathfrak{g}$ the Galois group $\text{Gal}(k/\mathbf{Q})$ and by d the degree $|\mathfrak{g}| = [k : \mathbf{Q}]$. Let $\mathcal{O}_k$ be the ring of integers in k . We set

$$\Xi_k = \{\xi \in \mathcal{O}_k; \sigma\xi \equiv \xi \pmod{d}, \forall \sigma \in \mathfrak{g}\} \quad (1)$$

This is a $\mathbf{Z}$ -module in $\mathcal{O}_k$. It contains $\mathbf{Z}$ and $d\mathcal{O}_k$, and is $\mathfrak{g}$ -stable as is easily seen. The definition implies that

$$\Xi_k/d\mathcal{O}_k = (\mathcal{O}_k/d\mathcal{O}_k)^{\mathfrak{g}}, \quad (2)$$

where, for any $\mathfrak{g}$ -module M , $M^{\mathfrak{g}}$ denotes the subset of elements of M invariant under the action of $\mathfrak{g}$.

For each $\xi \in \Xi_k$ and $\sigma \in \mathfrak{g}$, we can define an element $t(\xi)_\sigma$ in $\mathcal{O}_k$ by

$$t_\sigma = t(\xi)_\sigma = (\xi - \sigma\xi)/d. \quad (3)$$

As the group $\mathfrak{g}$ acts on the additive group of the ring $\mathcal{O}_k$, we may speak of the first cohomology group $H^1(\mathfrak{g}, \mathcal{O}_k) = Z^1(\mathfrak{g}, \mathcal{O}_k)/B^1(\mathfrak{g}, \mathcal{O}_k)$. In view of (3), we have

$$t_{\sigma\tau} = t_\sigma + {}^\sigma t_\tau \quad (4)$$

and hence a homomorphism

$$t : \Xi_k \rightarrow Z^1(\mathfrak{g}, \mathcal{O}_k), \text{ defined by } t(\xi)_\sigma = (\xi - \sigma\xi)/d. \quad (5)$$

Proposition 1. *The map t in (5) induces an isomorphism*

$$\Xi_k/\mathbf{Z} \approx Z^1(\mathfrak{g}, \mathcal{O}_k). \quad (6)$$

Proof. (i) $\text{Ker } t = \mathbf{Z}$. In fact,

$$\xi \in \text{Ker } t \iff t(\xi) = 0 \iff \xi - \sigma\xi = 0, \sigma \in \mathfrak{g} \iff \xi \in \mathcal{O}_k \cap \mathbf{Q} = \mathbf{Z}.$$

(ii) t is *surjective*. Let u be any cocycle in $Z^1(\mathfrak{g}, \mathcal{O}_k)$. We will find a $\xi \in \Xi_k$ so that $u = t(\xi)$. Let us try the following element

$$\xi = \sum_{\tau \in \mathfrak{g}} u_\tau \in \mathcal{O}_k.$$

It is not obvious at this moment that $\xi \in \Xi_k$ but the following argument implies it. In fact, we have

$$(\xi - \sigma\xi)/d = \left(\sum_{\tau} u_\tau - \sum_{\tau} {}^\sigma u_\tau \right) / d = \left(\sum_{\tau} u_{\sigma\tau} - \sum_{\tau} {}^\sigma u_\tau \right) / d = \left(\sum_{\tau} u_\sigma \right) / d = u_\sigma$$

which shows that our $\xi \in \Xi_k$ and $u = t(\xi)$. □

From Proposition 1 we have the relation:

$$\mathcal{O}_k/\mathbf{Z} \supset \Xi_k/\mathbf{Z} \approx Z^1(\mathfrak{g}, \mathcal{O}_k). \quad (7)$$

Let H be the subgroup of $\Xi_k/\mathbf{Z}$ corresponding to the coboundary group:

$$B^1(\mathfrak{g}, \mathcal{O}_k) = \{v \in Z^1(\mathfrak{g}, \mathcal{O}_k); v_\sigma = \eta - {}^\sigma\eta, \eta \in \mathcal{O}_k\}. \quad (8)$$

Denoting by $[\xi]$ the class of $\xi \in \mathcal{O}_k$ modulo $\mathbf{Z}$, we have, for $\xi \in \Xi_k$,

$$[\xi] \in H \iff t(\xi)_\sigma = (\xi - {}^\sigma\xi)/d = \eta - {}^\sigma\eta, \eta \in \mathcal{O}_k \iff \xi - d\eta = {}^\sigma(\xi - d\eta) \iff \xi - d\eta \in \mathbf{Z}.$$

In other words,

$$H = d(\mathcal{O}_k/\mathbf{Z}) = (\mathbf{Z} + d\mathcal{O}_k)/\mathbf{Z}. \quad (9)$$

By (6), (7), (8), (9), we obtain

Proposition 2. *Let $k/\mathbf{Q}$ be a finite Galois extension of degree d with the Galois group $\mathfrak{g}$ and let Ξ_k be the module given by (1). Then we have*

$$H^1(\mathfrak{g}, \mathcal{O}_k) \approx \Xi_k/(\mathbf{Z} + d\mathcal{O}_k).$$

We note here an obvious relation:

$$\mathcal{O}_k/d\mathcal{O}_k \approx (\mathbf{Z}/d\mathbf{Z})^d. \quad (10)$$

In connection with the cohomology group $H^1(\mathfrak{g}, \mathcal{O}_k)$, we will study the $\mathfrak{g}$ -fixed points of the finite module $\mathcal{O}_k/d\mathcal{O}_k$. Let us look at the sequence of modules:

$$\mathcal{O}_k \supset \Xi_k \supset \mathbf{Z} + d\mathcal{O}_k \supset d\mathcal{O}_k \quad (11)$$

We already know structures of several portions of (11). By (10), we have

$$(\mathcal{O}_k : d\mathcal{O}_k) = d^d, \quad (12)$$

by (2)

$$(\Xi_k : d\mathcal{O}_k) = |(\mathcal{O}_k/d\mathcal{O}_k)^\mathfrak{g}|, \quad (13)$$

by Proposition 2

$$(\Xi_k : \mathbf{Z} + d\mathcal{O}_k) = |H^1(\mathfrak{g}, \mathcal{O}_k)|. \quad (14)$$

Combining (12), (13), (14) with the obvious isomorphism

$$(\mathbf{Z} + d\mathcal{O}_k)/d\mathcal{O}_k \approx \mathbf{Z}/d\mathbf{Z},$$

we obtain

Theorem 1. *Let $k/\mathbf{Q}$ be a finite Galois extension of degree d with the Galois group $\mathfrak{g}$. Then we have*

$$|H^1(\mathfrak{g}, \mathcal{O}_k)| = |(\mathcal{O}_k/d\mathcal{O}_k)^\mathfrak{g}|/d.$$

2. The module $(\mathcal{O}_k/d\mathcal{O}_k)^\mathfrak{g}$

Notations being as before, let us express an element $\xi \in \mathcal{O}_k$ as

$$\xi = x_1\omega_1 + \cdots + x_d\omega_d = \Omega x, \quad \Omega = (\omega_1, \dots, \omega_d), \quad (15)$$

where $\{\omega_i\}$ being a $\mathbf{Z}$ -basis of $\mathcal{O}_k$ with

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathbf{Z}^d.$$

Then

$${}^\sigma\xi = {}^\sigma\Omega x, \quad \sigma \in \mathfrak{g}.$$

If we define a unimodular matrix A_σ by

$${}^\sigma\Omega = \Omega A_\sigma, \quad (16)$$

then, we have

$$A_{\sigma\tau} = A_\sigma A_\tau, \quad \sigma, \tau \in \mathfrak{g}.$$

In other words,

$$A : \sigma \mapsto A_\sigma \in GL_d(\mathbf{Z}) \quad (17)$$

is an integral representation of the Galois group $\mathfrak{g}$ of degree $d = [k : \mathbf{Q}]$.

As for the base change $\Omega \rightarrow \Omega'$ afforded by $\Omega' = \Omega T, T \in GL_d(\mathbf{Z})$, let A'_σ be the matrix for the new basis. One then finds

$$A'_\sigma = T^{-1} A_\sigma T,$$

so A' is $\mathbf{Z}$ -equivalent to $A : A' \sim A$.

Let $\xi = \Omega x$ be the relation in (15). We want to translate the condition $\xi \in \Xi_k$ in terms of one for $x \in \mathbf{Z}^d$. Now follow the following sequence of equivalences

$$\xi \in \Xi_k \Leftrightarrow {}^\sigma \xi \equiv \xi \pmod{d} \Leftrightarrow {}^\sigma \Omega x \equiv \Omega x \pmod{d} \Leftrightarrow \Omega A_\sigma x \equiv \Omega x \pmod{d} \Leftrightarrow (A_\sigma - I)x \equiv 0 \pmod{d}.$$

Therefore we obtain the following theorem:

Theorem 2. *Notations being as above, we have*

$$|(\mathcal{O}_k/d\mathcal{O}_k)^\mathfrak{g}| = |\{x \in \mathbf{Z}^d/d\mathbf{Z}^d; (A_\sigma - I)x \equiv 0 \pmod{d}\}|.$$

3. Case $d = 2$

Let us study here a special case where $d = |\mathfrak{g}| = [k : \mathbf{Q}] = 2$. Let σ be the generator of $\mathfrak{g}$. As we know there is an Ω of the form $\Omega = (1, \omega)$. [In general, it is true that 1 can appear in a basis of $\mathcal{O}_k$ for any number field k .] Then the matrix $A_\sigma, \sigma \neq 1$, satisfying (16) must be of the form

$$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}. \quad (18)$$

Furthermore it is easy to check that the parity of b in (18) is unchanged by the base change $\Omega = (1, \omega) \rightarrow \Omega' = (1, \omega')$. We shall call the parity of b simply the parity of the quadratic field k . Let us set $p(k) = 1$ or 2 according as the parity is odd or even. Theorem 1 and Theorem 2, with elementary facts on quadratic fields, imply the following

Proposition 3. *Let $k/\mathbf{Q}$ be a quadratic field. Then*

$$p(k) = |H^1(\mathfrak{g}, \mathcal{O}_k)| = \frac{1}{2} |H^0(\mathfrak{g}, \mathcal{O}_k/2\mathcal{O}_k)|.$$

4. The system $(\mathfrak{g}, (G, M))$

To begin with let us introduce a simple general setting

$$(\mathfrak{g}, (G, M))$$

where G is a group, M a left G -module and $\mathfrak{g}$ is a finite group acting on (G, M) naturally. Namely, we assume that G is a left $\mathfrak{g}$ -group, M is a left $\mathfrak{g}$ -module so that

$${}^\sigma(sx) = {}^\sigma s {}^\sigma x, \quad \sigma \in \mathfrak{g}, s \in G, x \in M.$$

For a cocycle C of $\mathfrak{g}$ with values in G , we associate a subgroup M_C of M by

$$M_C = \{x \in M; C_\sigma {}^\sigma x = x, \quad \sigma \in \mathfrak{g}\}. \quad (19)$$

We consider also a subgroup of M_C given by

$$P_C = \{y = p_C(x), \quad x \in M\} \quad (20)$$

where

$$p_C(x) = \sum_{\tau \in \mathfrak{g}} C_\tau {}^\tau x. \quad (21)$$

One verifies that the structure of the module M_C/P_C depends only on the cohomology class $\gamma = [C] \in H^1(\mathfrak{g}, G)$. If we put $C = 1$ in (19), (20) then we have $M_1 = M^\mathfrak{g}$, $P_1 = N(M)$ hence $M_1/P_1 = \hat{H}^0(\mathfrak{g}, M)$, the Tate cohomology group. For a general $\gamma = [C] \in H^1(\mathfrak{g}, G)$, we have a right to make identification

$$M_C/P_C = \hat{H}^0(\mathfrak{g}, M)_\gamma$$

and call this the Tate group twisted by γ . Furthermore, we shall put

$$i_\gamma(\mathfrak{g}, M) = [M_C : P_C], \quad \gamma = [C] \in H^1(\mathfrak{g}, G). \quad (22)$$

The determination of the *index* $i_\gamma(\mathfrak{g}, M)$ is a theme inspired by Poincaré. (See [1], Appendix 3 (in English) and references there.)

Now Let $k/\mathbf{Q}$ be a finite Galois extension with the Galois group $\mathfrak{g}$. As for the group G and the module M , we set $G = \{g = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathcal{O}_k\}$, $M = \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1, x_2 \in \mathcal{O}_k\}$.

With the natural action of G on M and those of $\mathfrak{g}$ on each of g and M , we obtain an admissible system $(\mathfrak{g}, (G, M))$. The action of G on M is the matrix multiplication: $g \circ x = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + tx_2 \\ x_2 \end{pmatrix}$. For our Galois actions, the relation $^\sigma(t \circ x) = (^\sigma t) \circ (^\sigma x)$ is trivial. We can use matrices $C_\sigma = \begin{pmatrix} 1 & t_\sigma \\ 0 & 1 \end{pmatrix}$, $\sigma \in \mathfrak{g}$ as a cocycle $\in Z^1(\mathfrak{g}, G)$. Instead of additive relation (4), we have, this time, a multiplicative one: $C_{\sigma\tau} = C_\sigma {}^\sigma C_\tau$. Since any cocycle $t \in Z^1(\mathfrak{g}, \mathcal{O}_k)$ is of the form $t_\sigma = t(\xi)_\sigma = (\xi - {}^\sigma \xi)/d$ for some $\xi \in \Xi_k$ by (6), we can write

$$C_\sigma = A(\xi) \cdot {}^\sigma A(\xi)^{-1} \quad (23)$$

with the matrix

$$A(\xi) = \begin{bmatrix} 1 & \xi/d \\ 0 & 1 \end{bmatrix}.$$

Along with (19), (20), for the cocycle C , we associate a $\mathbf{Z}$ -module M_C by

$$M_C = \{x \in M; C_\sigma {}^\sigma x = x, \sigma \in \mathfrak{g}\}. \quad (24)$$

We have also a submodule of M_C given by

$$P_C = \{y = p_C(x), x \in M\} \quad (25)$$

where $p_C(x) = \sum_{\tau} C_\tau {}^\tau x$.

We know that the quotient M_C/P_C depends only on the cohomology class $\gamma = [t] = [t(\xi)] \in H^1(\mathfrak{g}, \mathcal{O}_k)$ and is identified with the module $\hat{H}^0(\mathfrak{g}, M)_\gamma$. Finally we set *index* $i_\gamma(\mathfrak{g}, M) = [M_C : P_C]$.

In what follows, when a $\xi \in \Xi_k$ is fixed, we set simply $A = A(\xi)$.

(i) M_C . For $x \in M$,

$$x \in M_C \iff C_\sigma {}^\sigma x = x \iff A {}^\sigma A^{-1} {}^\sigma x = x \iff A^{-1} x = {}^\sigma (A^{-1} x).$$

Hence we have

$$A^{-1} M_C = \{A^{-1} x \in A^{-1} M \cap (\mathbf{Q})^2\}.$$

In other words,

$$A^{-1} M_C = (A^{-1} M)^\mathfrak{g}. \quad (26)$$

(ii) P_C . By (20), we find

$$A^{-1} P_C = \left\{ A^{-1} \sum_{\tau} C_\tau {}^\tau x \right\} = \left\{ \sum_{\tau} {}^\tau (A^{-1} x) \right\} = \{Tr(A^{-1} x), x \in M\}.$$

In other words,

$$A^{-1} P_C = Tr(A^{-1} M). \quad (27)$$

By (24), (25) we obtain

Theorem 3. *Let $k/\mathbf{Q}$ be a finite Galois extension with the Galois group $\mathfrak{g}$. Let $M = (\mathcal{O}_k)^2$. For a cocycle class $\gamma = [C] \in H^1(\mathfrak{g}, \mathcal{O}_k)$ corresponding to an element $\xi \in \Xi_k$ by (6), we have $i_\gamma(\mathfrak{g}, M) = |\hat{H}^0(\mathfrak{g}, A(\xi)^{-1} M)|$.*

REFERENCES

[1] Ono, T., Gauss sums and Poincaré sums (in Japanese), Nippon Hyoronsha, Tokyo (2008).