An Introduction to Complex Networks

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We review some studies on complex networks. The complex networks often have structural characteristics: the power-law degree distribution; the logarithmic size dependence of the average path length; and the high clustering coefficient. Several network models with such properties are introduced. We also consider the percolation models on the scale-free networks to show how the critical probability and critical behavior change with network topology.

KEYWORDS: complex network, network models, percolation, phase transition

1. Introduction

Networks are intended for topological description of real-world systems from internet or WWW to social relationship among human societies, or species predation in food webs. A network consists of nodes and connectivities between them called edges, as shown in Fig. 1. In real networks, nodes correspond to many different real-world components such as web pages, Internet providers, electric providers, human, ecological species, etc. Edges describe connections between the nodes such as hyperlinks, Internet traffics, electricity supply service, friendships or sexual partnerships, predator-prey relationship, etc. Since the late 1990’s, many studies on such complex networks have been done in very diverse fields, e.g., physics, mathematics, biology, computer science, engineering, social sciences, etc. [1, 6, 12, 22, 23, 26, 44, 45]. In these lecture notes, we briefly review some studies on complex networks.

These lecture notes are composed as follows. In Section 2, we review some structural characteristics of real networks. In Section 3, we introduce several network models which realize some of common properties with real networks. In Section 4, we consider percolation on complex networks to show how network topology affect dynamics thereon. Section 5 is devoted to the concluding remarks.

2. Properties of Complex Networks

2.1 Definition of network

A network (or graph) consists of a set of nodes (or vertices) \( V = \{v_1, v_2, \ldots, v_N\} \) and a set of edges (or links) \( E = \{e_1, e_2, \ldots, e_M\} \). The number of nodes and edges are denoted by \( N = |V| \) and \( M = |E| \), respectively. A network with \( N = 5 \) nodes and \( M = 6 \) edges is shown in Fig. 2. Networks can be either directed or undirected. In undirected networks, each edge is simply defined by a couple of nodes \( v_i \) and \( v_j \), denoted by \( (i,j) \). In directed networks, each edge has an orientation: \( (i,j) \) means an edge from node \( v_i \) to \( v_j \), but \( (j,i) \) does not exist. In these lecture notes, we deal with simple undirected networks. Here “simple” means that a network does not contain self-loops, i.e., edges from a node to itself, nor multiple edges, i.e., couples of nodes connected by more than one edge. When nodes \( v_i \) and \( v_j \) are connected with an undirected edge, node \( v_i \) (\( v_j \)) is called the neighbor of \( v_j \) (\( v_i \)). A walk is a finite sequence of nodes such that each of its nodes is the neighbor of the next node in the sequence. If the nodes of a walk are distinct, then the walk is called a path. A connected component is defined by a subgraph such that any two nodes are connected to each other by paths.

A network is completely described by the adjacency matrix \( A \). The adjacency matrix is a \( N \times N \) matrix, whose elements \( a_{ij} \) are defined by

\[
a_{ij} = \begin{cases} 
1 & \text{if node } i \text{ and } j \text{ are connected,} \\
0 & \text{otherwise.}
\end{cases}
\]

If the edges have no weight, the components of the matrix have only two values, 0 and 1. In simple undirected networks, the matrix is symmetric and the diagonal elements \( a_{ii} \) are 0. For the network shown in Fig. 2, the adjacency matrix is

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The adjacency matrix contains all information about the connectivities between nodes, but requires a tedious treatment for large scale networks because the amount of storage required approaches $N^2$. In the following, we introduce some basic network measures for the characterization of complex networks.

### 2.2 Degree distribution

The degree of a node is defined to be the number of edges connecting to it. The degree $k_i$ of node $i$ is given in terms of the adjacency matrix:

$$k_i = \sum_{j=1}^{N} a_{ij}.$$  \hspace{1cm} (2)

Then, the mean degree $\langle k \rangle$ of the network is given by

$$\langle k \rangle = \frac{1}{N} \sum_{i=1}^{N} k_i = \frac{2M}{N}.$$  \hspace{1cm} (3)

For the network shown in Fig. 2, the degree sequence, a list of the degrees of $N$ nodes, is \{4, 3, 2, 2, 1\} and the mean degree is $\langle k \rangle = 12/5$. The network density is defined by

$$\rho \equiv \frac{\langle k \rangle}{N - 1}.$$  \hspace{1cm} (4)

We say, the network is dense when $M = O(N^2)$, i.e., $\rho$ remains constant for $N \gg 1$, and sparse when $M \ll N^2$, i.e., $\rho$ goes zero for $N \gg 1$. In these lecture notes, we deal mainly with sparse networks with $M = O(N)$, or equivalently, with $\langle k \rangle$ being constant.

The degree distribution is very important for the characterization of complex networks. $P(k)$ is the degree distribution when the number of nodes with degree $k$ is $NP(k)$ in a network with $N$ nodes. In other words, $P(k)$ is the probability that a randomly chosen node has degree $k$. Here $P(k)$ satisfies $0 \leq P(k) \leq 1$ for any $k$, and $\sum_{k=0}^{N-1} P(k) = 1$ (where $N - 1$ is the possible maximum degree of a simple graph with $N$ nodes). The degree distribution for the network shown in Fig. 2 is $P(1) = 1/5$, $P(2) = 2/5$, $P(3) = 1/5$, and $P(4) = 1/5$. Note that any network has only one degree distribution, but we cannot reconstruct a network uniquely from a given degree distribution. The $n$th moment of $P(k)$ is given by

$$\langle k^n \rangle = \sum_{k=0}^{N-1} k^n P(k).$$  \hspace{1cm} (5)
In finite dimensional regular lattices such as the square lattice and the cubic lattice, every node is connected to a fixed number \( z \) of other nodes and its degree distribution is described by the delta distribution, \( P(k) = \delta(k - z) \). On the other hand, the degree distributions of many real networks obey a power-law [3]:

\[
P(k) = \mathcal{N} k^{-\gamma},
\]

where \( \mathcal{N} \) is the normalization constant and \( \gamma \) is called the degree exponent. The degree exponent of real networks often lies between 2 and 3 [3]. A network is called scale-free (SF) when its degree distribution is a power-law. An example of the SF network is shown in Fig. 3. For small \( \gamma \), due to its heterogeneity, some nodes have extremely high degrees, which are called hubs, while almost all nodes have low degrees.

We estimate here the maximum degree \( k_{\text{max}} \), \( \langle k \rangle \), and \( \langle k^2 \rangle \) when a network with finite but large \( N \) nodes has a power-law degree distribution (6). For simplicity, we approximate degree \( k \) by a continuous variable.

First, we calculate normalization factor \( \mathcal{N} \). From the condition \( \sum_{k=k_{\text{min}}}^{\infty} P(k) = 1 \) (where \( k_{\text{min}} \) is the minimum degree of the network), we have

\[
1 = \mathcal{N} \int_{k_{\text{min}}}^{\infty} k^{-\gamma} \, dk = \mathcal{N} \frac{k_{\text{min}}^{1-\gamma}}{\gamma - 1}, \quad \gamma > 1,
\]

which yields

\[
\mathcal{N} \simeq (\gamma - 1) k_{\text{min}}^{\gamma - 1}, \quad \gamma > 1.
\]

To obtain an estimate of \( k_{\text{max}} \), we consider a natural cutoff [29] in the following way. In a network with \( N \) nodes, the number of nodes with degrees larger than \( k_{\text{max}} \) should be at most one:

\[
\sum_{k=k_{\text{max}}}^{\infty} P(k) \simeq 1.
\]

Substituting \( P(k) = \mathcal{N} k^{-\gamma} \) into (9),

\[
N \sum_{k=k_{\text{max}}}^{\infty} P(k) \simeq N \mathcal{N} \int_{k_{\text{max}}}^{\infty} k^{-\gamma} \, dk = N \mathcal{N} \frac{k_{\text{max}}^{1-\gamma}}{\gamma - 1} = N \left( \frac{k_{\text{min}}}{k_{\text{max}}} \right)^{\gamma - 1} \simeq 1, \quad \gamma > 1.
\]

Then, we obtain

\[
k_{\text{max}} \simeq k_{\text{min}} N^{\frac{1}{\gamma - 1}}, \quad \gamma > 2.
\]

Equation (10) is unreasonable for \( 1 < \gamma < 2 \) because \( k_{\text{max}} \) is larger than \( O(N) \). We assume that \( k_{\text{max}} \) is proportional to \( N \) for \( 1 < \gamma < 2 \):

\[
k_{\text{max}} \propto N, \quad 1 < \gamma < 2.
\]

Now we evaluate \( \langle k \rangle \) and \( \langle k^2 \rangle \) using (8), (11), and (12). Because

\[
\langle k \rangle \simeq \mathcal{N} \int_{k_{\text{min}}}^{k_{\text{max}}} k^{1-\gamma} \, dk = \frac{\mathcal{N}}{2-\gamma} \left( k_{\text{max}}^{2-\gamma} - k_{\text{min}}^{2-\gamma} \right),
\]

we obtain
\[
\langle k \rangle \propto \begin{cases} 
N^{-\gamma/2}, & 1 < \gamma < 2, \\
1, & \gamma > 2.
\end{cases}
\] (14)

Similarly, \( \langle k^2 \rangle \) is
\[
\langle k^2 \rangle \simeq N \int_{k_{\text{min}}}^{k_{\text{max}}} k^2 k^{-\gamma} dk = \frac{N}{3-\gamma} (k_{\text{max}}^{3-\gamma} - k_{\text{min}}^{3-\gamma}),
\] (15)
and we obtain \( \langle k^2 \rangle \) with (8), (11), and (12) as
\[
\langle k^2 \rangle \propto \begin{cases} 
N^{-\gamma+3}, & 1 < \gamma < 2, \\
N^{\gamma-3}, & 2 < \gamma < 3, \\
1, & \gamma > 3.
\end{cases}
\] (16)

Equations (14) and (16) indicate that for \( 2 < \gamma \leq 3 \) \( \langle k^2 \rangle \) diverges in the large size limit \( N \to \infty \) while \( \langle k \rangle \) remains constant, i.e., the network is sparse. In Section 4, we will see the percolation model on the SF network shows extremal behavior in this region \( 2 < \gamma \leq 3 \).

2.3 Average path length

The distance \( d_{ij} \) between nodes \( i \) and \( j \) is the smallest number of edges one needs to travel to get from \( i \) to \( j \) \( (d_{ij} = \infty \) if nodes \( i \) and \( j \) are not in the same component):
\[
d_{ij} = \min_{\text{path}(i,j)} \sum_{k:\text{path}(i,j)} a_{kl},
\] (17)
where \( \text{path}(i,j) \) is a path connecting nodes \( i \) and \( j \). The largest distance from a node \( i \) to other nodes is called the eccentricity \( \text{ecc}_i \):
\[
\text{ecc}_i = \max_{\{j\}} d_{ij}.
\] (19)
For the network shown in Fig. 2, the distance \( d_{12} \) between node 1 and node 2 is one, and that between node 3 and node 5 is \( d_{35} = 2 \). The eccentricity of each node is \( \text{ecc}_1 = 1 \), and \( \text{ecc}_2 = \text{ecc}_3 = \text{ecc}_4 = \text{ecc}_5 = 2 \).

To characterize the network in terms of length, the following three measures are often used: radius, diameter, and average path length. The radius \( \text{rad} \) of the network is defined to be the minimum of the eccentricity:
\[
\text{rad} = \min_{\{i\}} \text{ecc}_i,
\] (19)
the diameter \( \text{diam} \) of the network is defined to be the maximum of the eccentricity:
\[
\text{diam} = \max_{\{i\}} \text{ecc}_i = \max_{\{i\}} d_{ij},
\] (20)
and the average path length \( \ell \) of the network is defined to be the mean path length over all pairs in the network:
\[
\ell = \frac{2}{N(N-1)} \sum_{i<j} d_{ij}.
\] (21)
For the network shown in Fig. 2, we obtain \( \text{rad} = 1 \), \( \text{diam} = 2 \), and \( \ell = 7/5 \).

In the \( d \)-dimensional regular lattices, the average path length grows with \( N \) as \( N^{1/d} \). On the other hand, the average path lengths of many real networks are relatively small: the average path length \( \ell \) depends at most logarithmically on \( N \). Such a network is called a small-world network.

2.4 Clustering coefficient

In many real networks, particularly in social networks, the neighbors of a node tend to connect each other. To measure such property, we introduce the notion of clustering coefficient. Two definitions of clustering coefficient are widely used.

The first definition of clustering coefficient \( C \) is given by the average of the local clustering coefficient. The local clustering coefficient \( C_i \) of node \( i \) is given by the fraction of pairs of its neighbors that are also neighbors of each other:
\[
C_i = \frac{2}{k_i(k_i - 1)} \times \text{number of edges between the neighbors of node } i,
\] (22)
or in terms of the adjacency matrix,
\[
C_i = \frac{\sum_{j<k} a_{ij}a_{ik}a_{jk}}{k_i(k_i - 1)/2},
\] (23)
where \( 0 \leq C_i \leq 1 \). We put \( C_i = 0 \) for nodes with \( k_i = 0 \) or 1 because both numerator and denominator in (22) are zero. Then, the clustering coefficient \( C \) of the whole network is
An alternative definition of the clustering coefficient, denoted by $C'$, is given by

$$C' = \frac{3 \times \text{number of triangles in the network}}{\text{number of connected triples of nodes in the network}}.$$  

For the network shown in Fig. 2, the local clustering coefficient of each node is $C_1 = 2/6, C_2 = 2/3, C_3 = 1, C_4 = 1,$ and $C_5 = 0$. The clustering coefficient $C$ is $C = 3/5$, and the alternative clustering coefficient $C'$ is $C' = 6/11$. As above mentioned, empirical data often show that many real networks have high clustering coefficients $C$ and $C'$.

To summarize, many real networks often have the following three structural characteristics: (i) a power-law degree distribution $P(k) \propto k^{-\gamma}$; (ii) the logarithmic size dependence of the average path length $\ell \propto \log N$; and (iii) the high clustering coefficient $C > 0$. We did not mention other important network measures, such as degree correlation [42], betweenness centrality [32], and network motif [40]. For further information on the network characteristics, see the reviews [23, 44].

### 3. Modeling of Complex Networks

In the previous section, we reviewed some structural characteristics in real networks. In this section, we introduce several network models which realize (some of) those network properties.

#### 3.1 Three seminal network models

In this subsection, we introduce three network models, the Erdős–Rényi model, the Watts–Strogatz model, and the Barabási–Albert model. The Erdős–Rényi model is given by a statistical ensemble of graphs [30, 31]. Due to its simplicity, the model has been exhaustively studied (see [13]), although the model has neither a power-law degree distribution nor a high clustering coefficient. The Watts–Strogatz model and the Barabási–Albert model were proposed in two seminal papers [53] and [3], which enlightened the small-world property and the scale-free property for the first time in the complex network literature, respectively. The Watts–Strogatz model realizes both small-world and high clustering property at the same time, and the Barabási–Albert model realizes a power-law degree distribution.

##### 3.1.1 Erdős–Rényi model

The Erdős–Rényi model, or called the random graph, is one of the best studied networks [13]. The construction of the Erdős–Rényi model is very simple:

**Construction** [Erdős–Rényi model]

- Take $N$ nodes and connect each node pair with probability $p$ (Fig. 4).
Let us consider the degree distribution of the Erdős–Rényi model. Since a node connects each of $N - 1$ nodes with probability $p$, the node has $k$ neighbors with probability

$$P(k) = \frac{(N-1)!}{k!(N-1-k)!} p^k (1-p)^{N-1-k}. \quad (26)$$

Now we focus on sparse networks. When we take the large size limit $N \to \infty$ holding the mean degree $\langle k \rangle = (N-1)p$ constant (or equivalently, $p \to 0$ keeping $\langle k \rangle$ constant), $P(k)$ approaches a Poisson distribution:

$$P(k) \approx \frac{e^{-\langle k \rangle}}{k!}. \quad (27)$$

The typical structure of random graph depends on $p$ (Fig. 5). We may consider whether a generated network with given $N$ and $\langle k \rangle$ has almost surely a giant connected component, whose size is of $O(N)$. As shown in the next section, the condition for the emergence of a giant connected component is

$$\langle k \rangle > 1,$$ 

while there exist only finite size components for $\langle k \rangle < 1$.

![Fig. 5. Erdős–Rényi model with (a) $p = 0$, (b) $p = 0.2$, and (c) $p = 0.4$. Here we set $N = 20$.](image)

The clustering coefficient $C$ is zero in the large size limit $N \to \infty$: since the probability of connection of two nodes is $p$ regardless of whether they have common neighbor,

$$C = p \simeq \frac{\langle k \rangle}{N}, \quad (29)$$

which tends to zero when $N$ increases.

Equation (29) reflects the following important property of the Erdős–Rényi model: the Erdős–Rényi model is locally tree-like in the large size limit, i.e., finite cyclic paths can be neglected if $N \gg 1$. Then, the mean number of neighbors at distance $l$ away from a randomly chosen node is $\langle k \rangle^l$. If such situation continues until neighbors cover the whole network, $\langle k \rangle^l \simeq N$, the typical distance $\ell$ of the network is

$$\ell \simeq \log N/\log \langle k \rangle. \quad (30)$$

Equation (30) indicates that the Erdős–Rényi model is small-world (see [13] for more details).

### 3.1.2 Watts–Strogatz model

Watts and Strogatz proposed a network model, called the Watts–Strogatz model, which has both the small-world property and a high clustering coefficient [53]. The Watts–Strogatz model interpolates between a finite dimensional lattice which is highly clustering and a random graph which is small-world. The Watts–Strogatz model is constructed as follows:

**Construction** [Watts–Strogatz model]

1. Consider a ring with $N$ nodes in which each node connects to its $\langle k \rangle$ nearest neighbors [Fig. 6(a)].
2. Take a fraction $p$ of edges and rewire them, i.e., moving one end of a edge to a new randomly-chosen node, such that multiple edges and self-loops are prohibited [Fig. 6(b)].

The rewiring process introduces on average $N\langle k \rangle p/2$ long range edges. The case with $p = 0$ corresponds to the ring [Fig. 6(a)], and the case with $p = 1$ corresponds to the random graph [Fig. 6(c)].

We summarize the network properties of the Watts–Strogatz model:

- Barrat and Weigt [8] derived the degree distribution of the Watts–Strogatz model. We easily find that $P(k)$ is a delta function $P(k) = \delta(k - \langle k \rangle)$ for $p = 0$, while it obeys a Poisson distribution (27) for $p = 1$. For intermediate $p$, noting that the distribution has a lower cutoff at $\langle k \rangle/2$,
\[
P(k) = \begin{cases} 
0, & k < \langle k \rangle / 2, \\
\frac{\min\left(k, \frac{\langle k \rangle}{2} \right)}{N} \left(1 - p\right)^{N - n} \frac{\left(p \frac{\langle k \rangle}{2}\right)^{k - \frac{\langle k \rangle}{2} - n} (k - \frac{\langle k \rangle}{2} - n)!}{\left(k - \frac{\langle k \rangle}{2} - n\right)!} e^{-p\langle k \rangle}, & k > \langle k \rangle / 2.
\end{cases}
\]

(31)

- Newman et al. [46] (see also [5]) calculated the average path length:

\[
\ell \approx \begin{cases} 
\frac{N}{2\langle k \rangle}, & N\langle k \rangle p / 2 \ll 1, \\
\log(N\langle k \rangle p) / (\langle k \rangle^2 p), & N\langle k \rangle p / 2 \gg 1.
\end{cases}
\]

(32)

Equation (32) means that the model is small world if the number of rewiring edges in the whole network is larger than \(O(1)\).

The clustering coefficient \(C\) is given in the following way. The Watts–Strogatz model with \(p = 0\), i.e., the ring with degree \(\langle k \rangle\), has

\[
C = \frac{3\langle k \rangle - 6}{4\langle k \rangle - 4}.
\]

(33)

Because each edge is not rewired with probability \(1 - p\), each triangle in the ring remains with probability \((1 - p)^3\). Thus, we obtain [8]

\[
C(p) = \frac{3\langle k \rangle - 6}{4\langle k \rangle - 4} (1 - p)^3.
\]

(34)

Equation (34) means the clustering coefficient \(C\) decreases with \(p\). But, it does slower than the decrease of \(\ell\). From (32) and (34) we find that there exists a finite region of \(p\), where the model has both a small average path length and a high clustering coefficient.

3.1.3 Barabási–Albert model

Real networks often grow with time, i.e., add new nodes and edges with time and change their topology. Barabási and Albert [3] proposed a simple growing network model, called Barabási–Albert model, which realizes a power-law degree distribution (Fig. 7). The model consists of the two rules, the network growth and the preferential attachment. The Barabási–Albert model is generated as follows:

**Construction** [Barabási–Albert model]

1. At time \(t = 0\), we start with a complete graph, in which all nodes connect each other, with \(m_0\) nodes.
2. At each time step, a new node joins the network and attaches \(m\) pre-existing nodes (the network growth rule). The probability that a new node attaches to node \(i\) with degree \(k_i\) is

\[
\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N} k_j},
\]

(35)

which means the nodes with high degrees get preferentially new edges than nodes with low degrees, as shown in Fig. 8 (the preferential attachment rule).
3. Wait until the number of nodes becomes \(N\), i.e., \(t = N - m_0\).

Let us derive the degree distribution of the Barabási–Albert model. We denote by \(P(k, t, t)\) the probability that node \(i\), which joined the network at time \(t_i\), has degree \(k\) at time \(t\). The master equation of \(P(k, t, t)\) is as follows:
Substituting \( P \left( \frac{k}{C_0} ; t_i ; t \right) \) into (36), we obtain
\[
P(k, t_i, t + 1) = m \Pi(k - 1) P(k - 1, t_i, t) + (1 - m \Pi(k)) P(k, t_i, t).
\]
Substituting \( \sum_{j=1}^{N} k_j = 2mN \approx 2mt \),
\[
P(k, t_i, t + 1) = \frac{k - 1}{2t} P(k - 1, t_i, t) + \left( 1 - \frac{k}{2t} \right) P(k, t_i, t).
\]
Now we consider the stationary degree distribution:
\[
P(k) = \lim_{t \to \infty} \frac{\sum_{i=1}^{t} P(k, t_i, t)}{t},
\]
for \( t \gg 1 \) \((N \gg 1)\). Summing (37) over \( t_i \), we obtain
\[
\sum_{i=1}^{t+1} P(k, t_i, t + 1) = \frac{k - 1}{2t} \sum_{i=1}^{t} P(k - 1, t_i, t) + \left( 1 - \frac{k}{2t} \right) \sum_{i=1}^{t} P(k, t_i, t).
\]
Then \( P(k) \) satisfies
\[
(t + 1)P(k) = \frac{k - 1}{2t} tP(k - 1) + \left( 1 - \frac{k}{2t} \right) tP(k) = \frac{k - 1}{2} P(k - 1) + \left( t - \frac{k}{2} \right) P(k),
\]
which yields
\[
P(k) = \frac{k - 1}{k + 2} P(k - 1).
\]
Because (41) is written as
\[
P(k) = \frac{k - 1 - 2k - 3k - 4 \ldots m}{k + 2 \ k + 1 \ k - 1 \ m + 3} P(m) = \frac{m + 2 \ m + 1 \ m}{k + 2 \ k + 1 \ k} P(m),
\]
so we obtain for \( k \gg 1 \).
which means that the Barabási–Albert model has the power-law degree distribution with degree exponent $\gamma = 3$. The mean degree $\langle k \rangle$ is $\langle k \rangle = 2m$ by substituting $M = m_0(m_0 - 1)/2 + (N - m_0)m \simeq Nm$ into (3).

We summarize other network properties of the Barabási–Albert model:

- The average path length is
  \[ \ell \propto \begin{cases} 
  \log N, & m = 1, \\
  \log N / \log \log N, & m \geq 2, 
  \end{cases} \]  
  which means that $\ell$ is smaller than $O(\log N)$ when $m \geq 2$. Such a network is said to have the ultra-small world property [21].
- The clustering coefficient $C$ is
  \[ C = \frac{m - 1}{8} \left( \frac{\log N}{N} \right)^2, \]  
  which tends to zero with $N$, although it does so slower than that of the Erdös–Rényi model, $C \simeq \langle k \rangle / N$. 

3.2 Three scale-free network models

In Section 3.1.3, we showed that the Barabási–Albert model has a power-law degree distribution. But, the Barabási–Albert model is limited in the sense that the degree exponent $\gamma$ takes only 3. Except the Barabási–Albert model, many SF network models with tunable $\gamma$ have been proposed (e.g., see Section 2 of [12] and reference therein). In this subsection, we introduce three SF network models: the configuration model, the Dorogovtsev–Mendes–Samukhin model, and the $(u,v)$-flower.

3.2.1 Configuration model

The configuration model, introduced by Bender and Canfield [10], is a generalization of the Erdös–Rényi model. The configuration model with an arbitrary degree distribution is generated as follows:

**Construction** [Configuration model]

1. Set a desired degree distribution $P(k)$.
2. Choose a degree sequence $\{k_1, k_2, \ldots, k_N\}$ according to $P(k)$. Attach $k_i$ stubs (half edges), which are the ends of edges-to-be to node $i$ [Fig. 9(a)].
3. Choose pairs of stubs at random from the set of stubs and connect them together [Fig. 9(b)]. Since stubs of the same nodes may be paired together or same pairs may be selected in many times, the configuration model allows a number of self-loops or multiple edges. But, these may be neglected in many cases.

The configuration model is the statistical ensemble of possible graphs with a given degree sequence, where each member is realized with equal probability. We care about whether the network with given $P(k)$ forms a giant component. As shown in the next section, a giant component exists almost surely when

\[ \frac{\langle k^2 \rangle}{\langle k \rangle} > 2, \]  
which is known as the Molloy–Reed criterion [41]. Note that (46) reduces to (28) when $P(k)$ is a Poisson distribution (27), which satisfies $\langle k^2 \rangle = \langle k \rangle^2 + \langle k \rangle$. 

![Fig. 9. Schematic of the configuration model: (a) the allocation of the stubs and (b) a realization of the network.](image)

The network properties of the configuration model are as follows:

- The configuration model is locally tree-like in the large size limit. Then, the average path length approximately satisfies (see the next section):
  \[ \ell \simeq 1 + \frac{\log \frac{N}{\langle k \rangle}}{\log \left( \frac{\langle k^2 \rangle}{\langle k \rangle} \right) \langle k \rangle}. \]  
  When $P(k) \propto k^{-\gamma}$, (47) reduces to
we obtain
\[
\ell \propto \begin{cases} 
\log \log N, & 2 < \gamma < 3, \\
\log N / \log \log N, & \gamma = 3, \\
\log N, & \gamma > 3.
\end{cases} 
\] (48)

- The clustering coefficient $C$ approximately satisfies
\[
C \simeq \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{k_i P(k_i) k_j P(k_j) (k_i - 1)(k_j - 1)}{N(k)} \simeq \frac{(k^2 - \langle k \rangle)^2}{N(k)^3},
\] (49)
which tends to zero with increasing $N$ if $\gamma$ is not too small ($\gamma > 7/3$).

### 3.2.2 Dorogovtsev–Mendes–Samukhin model

The Dorogovtsev–Mendes–Samukhin (DMS) model is a generalization of the Barabási–Albert model [27]. To construct the DMS model, we add initial attractiveness $k_0$ to the Barabási–Albert model:

**Construction** [Dorogovtsev–Mendes–Samukhin model]

1. At time $t = 0$, we start with a complete graph with $m_0$ nodes.
2. At each time step, a new node joins the network with attaching $m$ pre-existing nodes. The probability that a new edge attaches to node $i$ with degree $k_i$ is
\[
\Pi(k_i) = \frac{k_i + k_0}{\sum_{j=1}^{N}(k_j + k_0)}.
\] (50)
3. Wait until the number of nodes becomes $N$, i.e., $t = N - m_0$.

The constant $k_0$ ($k_0 > -m$) controls the effect of the preferential attachment: the resulting degree exponent $\gamma$ increases when $k_0$ increases (see below). The case of $k_0 = 0$ is reduced to the Barabási–Albert model.

Let us derive the degree distribution of the DMS model. The master equation of $P(k, t, t)$ is
\[
P(k, t, t+1) = m \Pi(k-1) P(k-1, t, t) + (1 - m \Pi(k)) P(k, t, t)
\]
\[
= m \frac{k - 1 + k_0}{(2m + k_0)t} P(k-1, t, t) + \left(1 - m \frac{k + k_0}{(2m + k_0)t}\right) P(k, t, t),
\] (51)
where we used
\[
\sum_{j=1}^{N}(k_j + k_0) = 2mN + k_0N \simeq (2m + k_0)t.
\] (52)

Summing (51) over $t$, we obtain the master equation of the stationary degree distribution $P(k)$ (38) as
\[
(t + 1)P(k) = m \frac{k - 1 + k_0}{2m + k_0} P(k-1) + \left(t - m \frac{k + k_0}{2m + k_0}\right) P(k).
\] (53)

After arrangement, we obtain
\[
P(k) = \frac{k + k_0 - 1}{k + k_0 + 2 + k_0/m} P(k - 1).
\] (54)

Using the well known formula of the gamma function:
\[
\Gamma(z + 1) = z\Gamma(z),
\] (55)
we obtain
\[
P(k) = \frac{\Gamma(k + k_0 + 2 + k_0/m)}{\Gamma(k + k_0 + 3 + k_0/m)} \frac{\Gamma(k + k_0)}{\Gamma(k + k_0 - 1)} P(k - 1)
\]
\[
= \frac{\Gamma(k + k_0 + 2 + k_0/m)}{\Gamma(k + k_0)} P(k - 1) + \left(k + k_0 + 1 + k_0/m\right) \frac{\Gamma(k + k_0 - 1)}{\Gamma(k + k_0 + 2 + k_0/m)} P(k - 2)
\]
\[
= \cdots
\]
\[
= \frac{\Gamma(k + k_0)}{\Gamma(k + k_0 + 3 + k_0/m)} P(m).
\] (56)

Because $\Gamma(k)$ can be approximated for $k \gg 1$ as
\[
\Gamma(k) \simeq \sqrt{2\pi} k^{k-1} e^{-k},
\] (57)
equation (56) reduces to
Thus, the DMS model has a power-law degree distribution,
\[ P(k) \propto k^{3-\gamma} \quad \text{where} \quad \gamma = 3 + \frac{k_0}{m}. \]

As the initial attractiveness \( k_0 \) increases from \(-m\) to \( \infty \), the degree exponent \( \gamma \) increases from 2 to \( \infty \).

### 3.2.3 \((u, v)\)-flower

One can produce SF networks in a deterministic way. Such networks often have a hierarchical structure, and therefore are called hierarchical SF networks. Hierarchical SF networks have a great advantage for analytical treatments because the structural properties of the networks and cooperative behaviors on the networks are often solved by the renormalization group technique. Many hierarchical SF networks have been proposed [4, 25, 48, 51]. We consider here a special class of hierarchical SF networks, called the \((u, v)\)-flower [50, 51]. The \((u, v)\)-flower is a generalization of the Dorogovtsev–Goltsev–Mendes (DGM) hierarchical network [25]. The \((u, v)\)-flower \( F_n \) with generation \( n \) is recursively constructed as follows:

**Construction** \([(u, v)\]-flower]

1. At generation \( n = 0 \), the flower \( F_0 \) consists of two nodes connected by a bond. We call these nodes roots.
2. The \((u, v)\)-flower \( F_n \) with generation \( n \) is obtained from \( F_{n-1} \), such that each existing bond in \( F_{n-1} \) is replaced by two parallel paths consisting of \( u \) edges and \( v \) edges each (Fig. 10). In other words, to produce \( F_n \) with generation \( n \), make \( w = u + v \) copies of \( F_{n-1} \) and join them at the roots (see Fig. 11). Without loss of generality, we assume \( u \leq v \).

The realizations of the \((1, 2)\)-flower and the \((2, 2)\)-flower with generation \( n = 1, 2, 3 \) are shown in Fig. 12(a) and (b), respectively. The \((1, 2)\)-flower corresponds to the DGM network. Also, the \((2, 2)\)-flower is often called the diamond lattice or Migdal-Kadanoff hierarchical lattice in the field of statistical physics.

![Fig. 10. Bond replacement rule of the \((u, v)\)-flower.](image)

![Fig. 11. \( F_{n+1} \) created by joining \( u + w \) copies of \( F_n \).](image)

Due its recursive procedure, we easily find the number of edges \( M_n \) of the \((u, v)\)-flower with generation \( n \) as
\[ M_n = (u + v)^n = w^n. \]

The number of nodes \( N_n \) is given recursively by
\[ N_n = wN_{n-1} - w, \quad \text{where} \quad N_1 = w, \]
which yields
Let us derive the degree distribution \[51\]. We denote by 
\[N_n(m)\] 
the number of nodes with degree \(k = 2^m\). \(N_n(m)\) is given recursively as
\[63\]
We easily find
\[64\]
Here we consider the cumulative degree distribution,
\[65\]
When \(P(k) \propto k^{-\gamma}\), the cumulative degree distribution is also a power-law with degree exponent \(1 - \gamma\): \(P_{\text{cum}}(k) \propto k^{1-\gamma}\). By calculating
\[66\]
we obtain the asymptotic form of \(P_{\text{cum}}(k)\) as
Thus, the degree distribution of the \((u, v)\)-flower is

\[
P(k) \propto k^{-\gamma}, \quad \text{where} \quad \gamma = 1 + \frac{\log(u+v)}{\log 2},
\]

which means that the \((u, v)\)-flower has a power-law degree distribution for any combination of \(u\) and \(v\). In particular, we have \(P(k) \propto k^{-1-\log 3/\log 2}\) for the \((1, 2)\)-flower and \(P(k) \propto k^{-3}\) for the \((2, 2)\)-flower.

We summarize the other network properties:

- Rozenfeld et al. [51] calculated the diameter \(diam_v\) of \(F_u\):
  \[
diam_v \simeq \begin{cases} 
(v-1)n, & u = 1, \\
n^u, & u > 1.
\end{cases}
\]

The flower is small-world only if \(u = 1\). When \(u > 1\), the diameter increases as a power of \(N\) like the finite dimensional Euclidean lattice.

- Clustering coefficient takes a nonzero value only if \(u = 1\) and \(v = 2\) [25], and zero otherwise:
  \[
  C = \begin{cases} 
  4/5 & \text{for } u = 1 \text{ and } v = 2, \\
  0 & \text{otherwise}.
  \end{cases}
\]

The \((u, v)\)-flower has neither the small-world property nor the high clustering property if \(u > 1\). However, a small-world hierarchical SF network is achieved by adding the shortcuts to the \((2, 2)\)-flower. Such a network is called the decorated \((2, 2)\)-flower. The decorated \((2, 2)\)-flower \(F_2\) with generation \(n\) is given by adding shortcuts to the \((2, 2)\)-flower \(F_2\) with same generation, as shown in Fig. 12(c). The decorated \((2, 2)\)-flower has both the small-world property \(l \sim \log N\), and a high clustering coefficient \(C \sim 0.82\), while also keeping a power-law degree distribution \(P(k) \propto k^{-3}\) [35, 51].

4. Percolation on Complex Networks

An important property of networks is robustness against the removal of nodes caused by failures or attacks. Albert et al. [2] studied the robustness of networks against two types of attacks: random failure, where nodes are sequentially removed with equal probability, and intentional attack, where hubs (i.e., nodes with large degrees) are preferentially removed. Random failure process is modeled by site percolation. Percolation is the simplest model which undergoes a phase transition [52]. In this section, we consider percolation on the SF networks. After a short introduction of percolation (Section 4.1), we calculate percolation on the several network models to show how the phase transition and the critical behavior are affected by the network topology (Section 4.2 and Section 4.3). In particular, we show that the SF network with small \(\gamma\) is very robust to random failure (Section 4.2.2 and Section 4.2.3). We also mention the fragility of the SF network against intentional attack (Section 4.2.4).

4.1 Percolation on euclidean lattices

Let us consider a square lattice (Fig. 13). Site percolation is as follows: each node is occupied (or undamaged) with probability \(p\), and unoccupied (or damaged to be removed) otherwise [Fig. 14(a)]. A connected component of occupied nodes is called cluster. The size of a cluster is given by the number of nodes belonging to it. In Fig. 14(a), the size of cluster to which node \(i\) belongs is \(5\).

When the state of each node is set to occupied or unoccupied, an important problem is whether a giant component, whose size is of \(O(N)\), exists. For small \(p\), a large number of unoccupied nodes divide the lattice into finite clusters, as shown in Fig. 14(a). However, when \(p\) exceeds a certain value \(p_c\), a giant component appears [Fig. 14(b)]. In the language of physics, a phase transition between the nonpercolating phase, where there exist finite size clusters, and the percolating phase, there exists a giant component and finite size clusters, occurs at the percolation threshold \(p = p_c\).

To characterize the phase transition, we consider the following order parameter. We denote by \(s_{\text{max}}(N)\) the mean size of the largest cluster for the percolation with site occupation probability \(p\) on the lattice with \(N\) nodes. The order parameter \(S\) is defined by

\[
S = \lim_{N \to \infty} s_{\text{max}}(N)/N,
\]

Then, \(S = 0\) for \(p \leq p_c\) because only finite clusters exist, while \(S > 0\) for \(p > p_c\) because a giant component almost surely exists (Fig. 15). Near the percolation threshold \(p_c\), the order parameter \(S\) exhibits a power-law behavior:

\[
S \propto (p - p_c)^\beta, \quad p > p_c.
\]
Other observables also show a power-law behavior near $p_c$. For example, the mean size of finite size clusters $\langle s \rangle$ behaves as

$$\langle s \rangle \propto |p - p_c|^{-\gamma},$$

which diverges at $p = p_c$. Just at $p_c$, the mean number $n_s$ of clusters with size $s$ per node, we call cluster size distribution, obeys a power-law:

$$n_s \propto s^{-z}, \quad p = p_c,$$

reflecting a fractal nature.

---

**Fig. 13.** Square lattice.

**Fig. 14.** Schematic of site percolation for (a) $p < p_c$ and (b) $p > p_c$. Occupied and unoccupied nodes are represented by blue and red circles, respectively.

**Fig. 15.** Schematic of the order parameter $S$ of percolation.
We also consider bond percolation on a lattice. In bond percolation, each edge is open with probability $p$, and closed otherwise. A cluster is a connected component of open edges. In the similar way as the site percolation, there exists a phase transition between the nonpercolating phase and the percolating phase at a certain value $p = p_c$. Note that the percolation threshold of the bond percolation is usually different from that of the site percolation.

4.2 Percolation on configuration model

Here we consider percolation on the configuration model. Because the configuration model is locally tree-like (when $N \gg 1$), we can calculate many physical properties and the percolation threshold using the generating function technique [17, 43, 47].

4.2.1 Generating functions

The generating function $G_0(x)$ for degree distribution $P(k)$ is defined by

$$G_0(x) = \sum_{k=0}^{\infty} P(k)x^k,$$

where

$$G_0(1) = \sum_{k=0}^{\infty} P(k) = 1.$$  

The generating function $G_0(x)$ contains all information about $P(k)$. We extract $P(k)$ for any $k$ by $G_0(x)$ as

$$P(k) = \frac{1}{k!}\frac{d^k}{dx^k}G_0(x)\bigg|_{x=0}.$$  

By differentiating $G_0(x)$ with respect to $x$, we obtain $\langle k \rangle$ as

$$G'_0(1) = \sum_{k=1}^{\infty} kP(k) = \langle k \rangle,$$

and similarly obtain the $n$-moment of $P(k)$ as

$$\langle k^n \rangle = \left( \frac{d}{dx} \right)^n G_0(x)\bigg|_{x=1}.$$  

In a heterogeneous network, the degree distribution of neighbors of randomly chosen nodes is not the same as $P(k)$. A node with degree $k$ is easily reached by following a randomly chosen edge $k$ times than a node with degree one. The probability that a randomly chosen edge reaches a node with degree $k$ is $kP(k)/\langle k \rangle$. Here we denote by $P_{ex}(k)$ the distribution of the excess degree which is the degree of the node reached by following a randomly chosen edge minus 1: $P_{ex}(k) = (k+1)P(k+1)/\langle k \rangle$ [see Fig. 16 for schematics of $P(k)$ and $P_{ex}(k)$]. The generating function $G_1(x)$ for the excess degree distribution is

$$G_1(x) = \sum_{k=0}^{\infty} P_{ex}(k)x^k = \sum_{k=0}^{\infty} \frac{(k+1)P(k+1)}{\langle k \rangle} \frac{x^k}{k!} = \frac{G_0(x)}{\langle k \rangle},$$

where $G_1(1) = 1$. When the degree distribution $P(k)$ is a Poisson distribution, $P(k) = e^{-\langle k \rangle}\langle k \rangle^k/k!$, we obtain

$$G_0(x) = e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{\langle k \rangle^k}{k!} x^k = \exp[\langle k \rangle(x-1)], \quad G_1(x) = \exp[\langle k \rangle(x-1)].$$

When the degree distribution $P(k)$ is a power-law, $P(k) = k^{-\gamma}/\sum_{k} k^{-\gamma}$, we obtain

$$G_0(x) = \sum_{k=1}^{\infty} P(k)x^k = \frac{L_{i\gamma}(x)}{L_{i\gamma}(1)}, \quad G_1(x) = \frac{\sum_{k=1}^{\infty} kP(k)x^{k-1} \frac{L_{i\gamma}(x)}{xL_{i\gamma}(1)},}{\sum_{k=1}^{\infty} kP(k)},$$

where $L_{i\gamma}(x)$ is the $n$th polylogarithm of $x$, i.e., $L_{i\gamma}(x) = \sum_{k=1}^{\infty} x^k k^{-\gamma}$.

The above generating functions become powerful tools when the network is locally tree-like. We count here the number of $l$th nearest neighbors of a randomly chosen node. The generating function for the number of nearest neighbors is $G_0(x)$, because the number of nearest neighbors of a randomly chosen node is equal to the number of edges connecting to it [Fig. 17(a)]. The generating function for the number of second nearest neighbors is $G_0(G_1(x))$. The number of second nearest neighbors of a randomly chosen node is equal to the sum of the excess degrees of the first nearest neighbors [Fig. 17(b)]. When a node has $k_0(\geq 1)$ neighbors with probability $P(k_0)$, each neighbor $i (i = 1, 2, \ldots, k_0)$ has the excess degree $k_i$ with probability $P_{ex}(k_i)$, and then the sum of the excess degrees of the neighbors is $k = \sum_{i=1}^{k_0} k_i$. Noting that a node has no neighbor with probability $P(0)$, the generating function for the number of second nearest neighbors of a randomly chosen node is given by
The corresponding generating functions for the third nearest neighbors and the fourth nearest neighbors are
\[ G_0(G_1(G_1(x))) \text{ and } G_0(G_1(G_1(G_1(x)))) \]
respectively, and so on. The average number \( N_{l\text{th}} \) of \( l \)th nearest neighbors of a randomly chosen node is given by differentiating the corresponding generating function with respect to \( x \) and setting \( x = 1 \):
\[ N_{l\text{th}} = G_0(1)G_1^{l-1}(1) = \langle k \rangle \left( \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \right)^{l-1}. \]
When we assume that the situation continues until neighbors cover the whole network, (84) gives an estimate (47) of \( \ell \).

Equation (84) also estimates the condition for the emergence of a giant component. When the network forms a giant component, \( N_{\text{th}} \) does not decrease with \( l \). The condition \( N_{l+1\text{th}} > N_{\text{th}} \) immediately gives the Molloy–Reed criterion (46).

### 4.2.2 Site percolation

Now, we consider site percolation with occupation probability \( p \) on the configuration model. We derive the percolation threshold by using generating functions [17] (see [19] for an alternative derivation). We denote by \( H_0(x) \) the generating function for the size distribution of finite clusters to which a randomly chosen node belongs. We also denote by \( H_1(x) \) the generating function for the size distribution of finite clusters to which the node reached by following a randomly chosen edge belongs. When the network is infinitely large, \( H_0(x) \) and \( H_1(x) \) satisfy the following recursive equations (Fig. 18):
\[ H_0(x) = (1 - p) + pxG_0[H_1(x)], \]
\[ H_1(x) = (1 - p) + pxG_1[H_2(x)]. \]
The mean cluster size \( \langle s \rangle \) is given in terms of \( H_0(x) \) by
\[ \langle s \rangle = H_0'(1). \]
From (85) and (86) we have
Thus, we obtain

$$\langle s \rangle = H'_0(1) = p + pg'_0(1)H'_0(1),$$  

(88)

and

$$H'_1(1) = p + pg'_1(1)H'_1(1) = \frac{p}{1 - pg'_1(1)}.  

(89)$$

Thus, we obtain

$$\langle s \rangle = p \left( 1 + \frac{pg'_0(1)}{1 - pg'_1(1)} \right).  

(90)$$

As mentioned in the previous subsection, $\langle s \rangle$ diverges at the percolation threshold $p = p_c$. From (90) we find that the percolation threshold of this model is

$$p_c = \frac{1}{G'_1(1)} = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}.$$  

(91)

Equation (91) holds for an arbitrary $P(k)$ as long as the network is locally tree-like. When $P(k)$ is a Poisson distribution, we have

$$p_c = \frac{1}{\langle k \rangle}.$$  

(92)

When $P(k) \propto k^{-\gamma}$ with $2 < \gamma \leq 3$, $\langle k^2 \rangle$ diverges while $\langle k \rangle$ remains constant [see (14) and (16)]. Thus, we find that the percolation threshold is zero for the infinite SF network with $2 < \gamma \leq 3$, i.e., the network always has a giant component. This indicates that the SF network with small $\gamma$ is highly robust to random failure, because almost all nodes have to be removed to disintegrate the network.

Above the percolation threshold $p = p_c$, there exists a single giant component and then $H_0(1) < 1$. The fraction of the giant component $S = 1 - H_0(1)$ is given by

$$1 - S = 1 - p + pg_0(u),$$  

(93)

where $u = H_1(1)$ is the probability that the node reached by following a random chosen edge does not belong to the giant component. From (86) $u$ satisfies

$$u = 1 - p + pg_1(u).$$  

(94)

Cohen et al. [18] derived the critical behavior of the order parameter $S$ for the configuration model with $P(k) \propto k^{-\gamma}$:

$$S \propto \begin{cases} 
  p - p_c, & \gamma \geq 4, \\
  (p - p_c)^{1/(\gamma - 3)}, & 3 < \gamma < 4, \\
  p \exp(-2/p(k)), & \gamma = 3, \\
  p^{1/(3 - \gamma)}, & 2 < \gamma < 3.
\end{cases}$$  

(95)

Equation (95) indicates that the critical exponents depends crucially on $\gamma$. 

---

**Fig. 18.** Schematic of (a) $H_0(x)$ and (b) $H_1(x)$. 

(a) 

$$H_0(x) = \begin{cases} 
  1 - p, & \\
  \frac{pg_0(1)}{1 - pg_1(1)}, & \\
  \frac{pg_0(2)}{1 - pg_1(2)}, & \\
  \frac{pg_0(3)}{1 - pg_1(3)}, & \\
  \frac{pg_0(4)}{1 - pg_1(4)}, & \\
  \vdots
\end{cases}$$

(b) 

$$H_1(x) = \begin{cases} 
  1 - p, & \\
  \frac{pg_0(1)}{1 - pg_1(1)}, & \\
  \frac{pg_0(2)}{1 - pg_1(2)}, & \\
  \frac{pg_0(3)}{1 - pg_1(3)}, & \\
  \frac{pg_0(4)}{1 - pg_1(4)}, & \\
  \vdots
\end{cases}$$
4.2.3 Bond percolation

Similarly, we derive the percolation threshold of bond percolation on the configuration model [17]. We introduce \( H_0(x) \) and \( H_1(x) \) for the bond percolation with open bond probability \( p \). Then, the recursion equation for \( H_1(x) \) is

\[
H_1(x) = (1 - p) + p x G_1[H_1(x)],
\]

which is the same as (86), and that for \( H_0(x) \) is

\[
H_0(x) = x G_0[H_1(x)].
\]

From (96) and (97) we obtain the mean cluster size \( \langle s \rangle \) as

\[
\langle s \rangle = H_0(1) = 1 + \frac{p G_0(1)}{1 - p G_1(1)},
\]

which diverges at \( p_c = \langle k \rangle / ((\langle k^2 \rangle - \langle k \rangle)) \), which is the same as that of the site percolation. Thus, the percolation threshold is zero for the bond percolation on the infinite SF network with \( 2 < \gamma \leq 3 \).

4.2.4 Intentional attack

Finally, we consider the intentional attack on the configuration model, where \( N(1 - p) \) nodes with the largest degrees are damaged to be removed. The minimum degree of removed nodes \( k_{\text{cut}}(p) \) is determined by

\[
\sum_{k = k_{\text{cut}}(p)}^{\infty} P(k) = 1 - p.
\]

After the intentional attack, the probability that a randomly chosen node is removed is \( 1 - p \), while the probability that the node reached by following a randomly chosen edge is removed is not \( 1 - p \), but \( 1 - p' \), where

\[
1 - p' = \sum_{k = k_{\text{cut}}(p)}^{\infty} \frac{k P(k)}{\langle k \rangle},
\]

The probability \( p' \) that a neighbor of a randomly chosen node is not removed is extremely small compared to \( p \) when the network is SF with small \( \gamma \).

Callaway et al. [17] derived the percolation threshold of the intentional attack using generating functions (see also [20] for an alternative derivation): the percolation threshold \( p_c \) is given by

\[
\sum_{k = k_{\text{cut}}(p)}^{k_{\text{cut}}(p)} k (k - 1) P(k) / \langle k \rangle = 1.
\]

Numerical solution of (101) shows that the percolation threshold \( p_c \) is very close to one when the network is SF with small \( \gamma \), i.e., the removal of a few nodes with large degrees immediately disintegrates the network. This indicates that the SF network is very fragile against the intentional attack.

The robustness of the Internet and the WWW against random failure and intentional attack was numerically investigated [2]. Numerical result shows that the Internet and the WWW are robust to random failure but fragile against intentional attack, as expected in the configuration model.

4.3 Percolations on other network models

In the last of this section, we briefly mention the bond percolation on other network models: the DMS model with \( k_0 \rightarrow \infty \) and the decorated \((2,2)\)-flower. These models show some abnormal behaviors which are not observed on the configuration model.

4.3.1 Bond percolation on growing networks

First, we consider the bond percolation on the DMS model with \( k_0 \rightarrow \infty \). For \( k_0 \rightarrow \infty \), new nodes attach \( m \) pre-existing nodes randomly and the degree distribution of the resulting network follows an exponential decay, \( P(k) \propto [(m + 1)/m]^{-3} \), rather than a power-law decay. The bond percolation on the DMS model with \( k_0 \rightarrow \infty \) has been solved in [15, 49], after [54]. The percolation threshold \( p_c \), above which a giant component whose size is of \( O(N) \) appears, is

\[
p_c = \frac{1}{m} \left( 1 - \sqrt{\frac{m - 1}{m}} \right).
\]

Near \( p_c \) the order parameter \( S \) follows

\[
S \propto \exp[-\alpha / (p - p_c)^{\beta'}], \quad p > p_c,
\]

where \( \alpha = \pi / [2(m(m - 1)]^{1/4} \) and \( \beta' = 1/2 \). Equation (103) means the singularity of the phase transition at \( p_c \) is infinitely weak (such transition is called infinite order transition). Krapivsky and Derrida [38] also derived a similar
result on a generalized model. Moreover, they showed that the cluster size distribution $n_s$ is a power-law in the whole region below $p_c$ (not just at $p_c$):

$$n_s \propto s^{-\tau(p)}, \quad p \leq p_c,$$

(104)

with $p$-dependent exponent $\tau(p)$. The same abnormal behavior has been observed for other growing network models with exponential and power-law degree distributions [16, 24, 28, 33, 39]. Also, the similar behavior has been observed on the Ising model and Potts model on growing networks [9, 35, 36].

### 4.3.2 Bond percolation on decorated (2, 2)-flower

Next, we consider the following bond percolation on the decorated (2, 2)-flower $\tilde{F}_n$: the open bond probability of consisting of $F_n$ is $p$ and that of the shortcuts is $\tilde{p}$ [11, 34]. The case for $\tilde{p} = p$ corresponds to the ordinary bond percolation on the decorated (2, 2)-flower and the case for $\tilde{p} = 0$ corresponds to the bond percolation on the (2, 2)-flower. The phase diagram is obtained by renormalization group (RG) technique [11, 34, 50]. We denote by $P^{(n)}$ the probability that two root nodes are in the same cluster after a bond percolation trial on $F_n$. We easily obtain the recursive equation of $P^{(n)}$ as

$$P^{(n+1)} = \tilde{p} + (1 - \tilde{p})[2(P^{(n)})^2 - (P^{(n)})^4],$$

(105)

where the initial value is set to $P^{(0)} = p$. The fixed point $P$ is given by $P = P^{(n+1)} = P^{(n)}$.

For the (2, 2)-flower ($\tilde{p} = 0$), $P^{(n)}$ starting at $0 < p < p_c$ (the nonpercolating phase) converges to a trivial stable fixed point $P = 0$ with increasing $n$, while $P^{(n)}$ starting at $p_c < p < 1$ (the percolating phase) converges to another trivial stable fixed point $P = 1$ (Fig. 19). And one unstable fixed point exists at $P = p_c = (\sqrt{5} - 1)/2$, giving the percolation threshold $p_c$ of bond percolation on the (2, 2)-flower.

![Flow of $P^{(n)}$ for the bond percolation on the (2, 2)-flower, obtained from (105) with $\tilde{p} = 0$.](image)

On the other hand, we have a quite different picture for the decorated (2, 2)-flower from (105). The flow diagram and the phase diagram are shown in Fig. 20. For a fixed $\tilde{p}$ ($0 < \tilde{p} < \tilde{p}_c = 5/32$), there are two stable fixed points, $P = 1$ and $0 < P = p^*(\tilde{p}) < 1$, corresponding to the percolating phase and another phase, called the critical phase, where RG flow converges onto the line of nontrivial stable fixed points, respectively. And there is one unstable fixed point between the two giving the phase boundary, $P = p_c(\tilde{p})$. For $\tilde{p} > \tilde{p}_c$, there is only one stable fixed point at $P = 1$, so that the system is always percolating. Thus, the bond percolation on the decorated (2, 2)-flower is either in the percolating phase or the critical phase, not in the nonpercolating phase, as long as $\tilde{p} \neq 0$. By using generating functions, we find that in the critical phase the mean size of the largest cluster $s_{\text{max}}(N)$ obeys $s_{\text{max}}(N) \propto N^\gamma(p; \tilde{p})$ where $0 < \gamma(p; \tilde{p}) < 1$ and the cluster size distribution is $n_s \propto s^{-\tau(p; \tilde{p})}$ where $\tau(p; \tilde{p}) = 1 + \gamma(p; \tilde{p})^{-1}$ [34].

We may say that this abnormal behavior is the same as that observed in the region below the percolation threshold on growing network models.

We also mention the followings for the case of $\tilde{p} = p$:

- The percolation threshold is nonzero, although the decorated (2, 2)-flower has a power-law degree distribution with degree exponent $\gamma = 3$.
- Above the percolation threshold, the order parameter $S$ follows (103), indicating that this transition is of infinite order.
These examples indicate that the phase transition on the complex networks is not determined only by the degree distribution (e.g., the configuration model), but also affected by other structural properties.

5. Concluding Remarks

We have reviewed some studies about complex networks. In Section 2, we have introduced some structural characteristics of real networks. In Section 3, we have introduced several network models which realize common properties with real networks. In Section 4, we have considered percolation on complex networks. We calculate percolation on the configuration model to show that (i) its phase transition changes crucially with the degree distribution, and (ii) in particular, the system is very robust to random failure but very fragile against intentional attack when the degree distribution is a power-law with small $\gamma$.

In Section 4.3, we have mentioned that the percolation on growing networks and hierarchical networks shows an abnormal phase transition. At present, the key to reveal necessary conditions for the occurrence of such abnormal behavior is still missing. Further study would be required to clarify the relation between network topology and dynamics thereon.

Many significant topics were omitted. For further information on the theory of complex networks, see the books [6, 22, 45] and reviews [1, 12, 26, 44].

REFERENCES

An Introduction to Complex Networks

195

