Optimization for Lattices and Diophantine Approximations

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In this article, we optimize a growth condition such that an analytic function being almost integer-valued at integers turns out to be a polynomial. Our argument relies on a computational optimization combined with Diophantine approximations on lattices.

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1. Introduction

Denote by $\mathbb{N}$ the set of strictly positive rational integers. Let us investigate the nature of an entire function $F(z)$ in one complex variable such that $F(\mathbb{N}) \subset \mathbb{Z}$, or $F(z)$ taking values close to integers at any $z \in \mathbb{N}$.

Our natural questions which arise are as follows.

(1) Is $F(z)$ a polynomial?

The answer is no, because of the existence of the functions such that $F(z) = 2^z$ or 3^z , etc. Then the second question arises:

(2) Does there exist a suitable condition such that $F(z)$ becomes a polynomial?

Denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Consider a complex function $F(z)$ in $z \in \mathbb{C}$ which satisfies $F(u) \in \overline{\mathbb{Q}}$ for any $u \in \overline{\mathbb{Q}}$. We may now ask the following more general question:

(3) Is $F(z)$ a polynomial or an algebraic function?

Indeed, there exist examples due to P. Stäckel [9] in 1895 of transcendental functions taking algebraic values at all algebraic points. Hence the answer to the third question is again, no, namely it is true that such $F(z)$ supposed to verify $F(u) \in \overline{\mathbb{Q}}$ for any $u \in \overline{\mathbb{Q}}$, is not necessarily algebraic function.

On the other hand, the Hermite–Lindemann theorem in transcendental number theory shows $\exp(\alpha) \notin \overline{\mathbb{Q}}$ for any $\alpha \in \overline{\mathbb{Q}}$, $\alpha \neq 0$, which says, there exists a transcendental function always taking transcendental values at any non-trivial algebraic point. Moreover, the Gel'fond–Schneider theorem notices $2^{\sqrt{3}} \notin \overline{\mathbb{Q}}$, then even an entire function $F(z) = 2^z$ with $F(\mathbb{N}) \subset \mathbb{Z}$ does not satisfy $F(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$.

Now let us restrict ourselves to consider an entire function $F(z)$ such that $F(\mathbb{N}) \subset \mathbb{Z}$. For our question (2), we refer to a fundamental result of G. Pólya [8] concerning with such functions.

Recall the definition of the order of a complex function.

Definition 1. Let $F(z)$ be an entire function in $\mathbb{C}$. Write $|F|_r = \sup_{|z| \leq r} |F(z)|$ and define $\alpha(F)$ the order of exponential type of $F(z)$ by

$$\alpha(F) = \limsup_{r \rightarrow +\infty} \frac{\log |F|_r}{r}.$$

Theorem A (Pólya). Let $F(z)$ be an entire function in $\mathbb{C}$ with $F(\mathbb{N}) \subset \mathbb{Z}$. Suppose $\alpha(F) < \log 2$. Then $F(z)$ is a polynomial.

Pólya also observed the case $F(\mathbb{Z}) \subset \mathbb{Z}$ as follows.

Theorem B (Pólya). Let $F(z)$ be an entire function in $\mathbb{C}$ with $F(\mathbb{Z}) \subset \mathbb{Z}$. Suppose $\alpha(F) < \log\left(\frac{3 + \sqrt{5}}{2}\right)$. Then $F(z)$ is a polynomial.

We see that $F(z)$ in theorems of Pólya is necessarily a polynomial in $\mathbb{Q}[z]$, but not in $\mathbb{Z}[z]$ [consider for example, $\frac{1}{2}z(z+1)$]. The bounds of the order $\log 2$ and $\log(\frac{3+\sqrt{5}}{2})$ are optimal because of the existence of the functions 2^z and $(\frac{3+\sqrt{5}}{2})^z + (\frac{3-\sqrt{5}}{2})^z$.

It is equivalent to say that $F(z)$ is a polynomial with coefficients in $\mathbb{Q}$, and that the functions $z^h F(z)^k$ ($h, k \in \mathbb{N} \cup \{0\}$) are linearly dependent over $\mathbb{Q}$. Then a natural generalization of works of Pólya is to seek a sufficient condition such that several functions, say, $f_1(z), \dots, f_L(z)$, are linearly dependent over $\mathbb{Q}$.

For $\zeta_1, \zeta_2, \dots \in \mathbb{C}$ denote by $r(N) := \max_{1 \leq n \leq N} |\zeta_n|$. We proved in [5]:

Theorem C ([5]). *Let L and N_0 be integers with $1 < N_0 < L$. There are constants $C_1 > 0$ and $C_2 > 0$ depending only on L, N_0 satisfying the following. Let $\zeta_1, \zeta_2, \dots \in \mathbb{C}$ be infinitely many pairwise distinct complex points. Let $f_1(z), \dots, f_L(z)$ be entire functions in $\mathbb{C}$. Suppose $f_j(\zeta_n) \in \mathbb{Z}$ for any j, n with $1 \leq j \leq L$ and $n \geq 1$. If we have $\max_{1 \leq j \leq L} \log |f_j|_{C_1 r(N)} \leq C_2 N$ for any $N \geq N_0$, then the functions $f_1(z), \dots, f_L(z)$ are linearly dependent over $\mathbb{Q}$.*

A consequence of Theorem C is for instance, a weaker version of Theorem A:

Corollary D ([5]). *Let $F(z)$ be an entire function in $\mathbb{C}$ with $F(\mathbb{N}) \subset \mathbb{Z}$. Suppose $\alpha(F) \leq \frac{1}{40}$. Then $F(z)$ is a polynomial over $\mathbb{Q}$.*

Proof. Consider $\frac{z(z-1)\cdots(z-h+1)}{h!} \cdot F(z)^k$ as $f_1(z), \dots, f_L(z)$ in Theorem C ($h, k \in \mathbb{N} \cup \{0\}$). □

2. Results for an Almost Integer-Valued Function

Our aim in the present article is to relax the hypothesis of theorem of Pólya from the viewpoint of Diophantine approximations.

Definition 2. We write $\|z\| := \min_{m \in \mathbb{Z}} |z - m|$, the distance between $z \in \mathbb{C}$ and the nearest integer.

In this section, by means of computational optimization joined with Diophantine approximation method, we prove the following results. These theorems show that an almost integer-valued entire function is a polynomial over $\mathbb{Q}$, provided that the growth of the function is sufficiently bounded.

Theorem 1. *Let $f(z)$ be an entire function in $\mathbb{C}$ with $\alpha(f) \leq 0.00437155$. Suppose*

$$\|f(n)\| < e^{-663143n} \quad (1)$$

for all sufficiently large $n \in \mathbb{N}$. Then $f(z) \in \mathbb{Q}[z]$.

A slight modification of the proof enables us to obtain:

Theorem 2. *Let $f(z)$ be an entire function in $\mathbb{C}$ with $\alpha(f) \leq 8.775 \times 10^{-6}$. Suppose*

$$\|f(n)\| < e^{-3n} \quad (2)$$

for all sufficiently large $n \in \mathbb{N}$. Then $f(z) \in \mathbb{Q}[z]$.

We note $0.00437155 > \frac{1}{229}$ and $8.775 \times 10^{-6} > \frac{1}{113961}$.

Theorems 1 and 2 show, not only an integer-valued entire function but also an almost integer-valued function, namely a function taking values very close to integers, may turn out to be a polynomial. This is an improvement of a result in [5] concerning with a growth condition or an assumption for the distance from the nearest integer. Theorems 1 and 2 give explicit versions of a theorem due to Ch. Pisot dealing with an almost integer-valued function (Théorème 2 in [7]). We prove both theorems by so-called Schneider's method (*refer* its applications found in [3, 10]) of Diophantine approximations, that differs from Pisot's original proof.

3. Computational Lemmata

We collect here a few classical estimates. Throughout the article, we denote $\varphi(x) = x \log x$.

Lemma 3. *Let N_0 be a sufficiently large integer. Let $h_0 \in \mathbb{Q}$ such that $h_0 N_0 \in \mathbb{Z}$, $0 < h_0 < \frac{1}{2}$. Let n, h be integers with $0 \leq n \leq N_0 - 1$, $1 \leq h \leq h_0 N_0$. Then we have*

$$\log \left| \binom{n}{h} \right| \leq -N_0 \{ \varphi(h_0) + \varphi(1 - h_0) \} + o(N_0).$$

Proof. Since $h_0 < \frac{1}{2}$, we have

$$\left| \binom{n}{h} \right| \leq \binom{N_0}{h} \leq \binom{N_0}{h_0 N_0}$$

thus the Stirling formula (*see* chapter XII, §33, page 251 of [12]) for $\frac{N_0!}{(N_0(1-h_0))! \cdot (h_0 N_0)!}$ gives the required estimate. $\square$

Indeed this is a reason why we use binomial polynomials instead of usual monomials as a basis of ring of polynomials.

Lemma 4. *Let $\lambda > 1$ be a parameter $\in \mathbb{R}$ and $N \geq 2$ be an integer. Put $\Lambda = \lambda^{2\lambda^2-1}(\lambda^2-1)^{1-\lambda^2} > 1$. There exists a constant C with $\frac{1}{2} < C < 4$ satisfying the following.*

Let $F(z)$ be an entire function in $\mathbb{C}$ with $F(n) = 0$ for all $n = 0, 1, \dots, N-1$. Then we have

$$\log |F(N)| \leq \log |F|_{\lambda N} - N \log \Lambda + C \log N. \quad (3)$$

If $N \geq 10^5$ then we have a more precised estimate:

$$\log |F(N)| \leq \log |F|_{\lambda N} - N \log \Lambda + \frac{1}{2} \log(N+1) + \frac{1}{2} \log \frac{2\pi(\lambda^2-1)}{\lambda^2} + \theta \quad (4)$$

with

$$-\frac{73}{36} < \theta < \frac{10}{9} + \frac{1}{2} \log \frac{\lambda^2}{\lambda^2-1}.$$

Proof. We put

$$Q(z) = Q_{(N)}(z) = \prod_{n=0}^{N-1} \frac{\lambda N(z-n)}{\lambda^2 N^2 - zn}.$$

The function Q and $\frac{F}{Q}$ are analytic in the disk $|z| \leq \lambda N$.

Since $|Q(z)| = 1$, we have for $|z| = \lambda N$:

$$\left| \frac{F(N)}{Q(N)} \right| \leq \left| \frac{F}{Q} \right|_{\lambda N} \leq |F|_{\lambda N}.$$

Therefore we obtain $|F(N)| \leq Q(N)|F|_{\lambda N}$.

On the other hand, by an asymptotic expansion of the logarithm of the Gamma function (called Stirling's series, *see* [12]), we have for $x > 0$,

$$\Gamma(x) = x^{x-\frac{1}{2}} e^{-x} (2\pi)^{\frac{1}{2}} e^{\frac{\theta_0}{12x}}$$

with certain θ_0 , $0 < \theta_0 < 1$.

It is well-known that Euler's formula implies for z which is not negative integer (chapter XII, §12, page 237 of [12]):

$$\Gamma(z+1) = z\Gamma(z),$$

consequently we have

$$\Gamma(\lambda^2 N + 1) = \lambda^2 N (\lambda^2 N - 1) \cdots (\lambda^2 N - N + 1) \cdot \Gamma(\lambda^2 N - N + 1)$$

from which we get

$$\prod_{n=0}^{N-1} (\lambda^2 N - n) = \frac{\Gamma(\lambda^2 N + 1)}{\Gamma(\lambda^2 N - N + 1)}.$$

Hence we obtain

$$\begin{aligned} |Q(N)| &= \frac{(\lambda N)^N \cdot \Gamma(N+1) \cdot \Gamma(\lambda^2 N - N + 1)}{N^N \cdot \Gamma(\lambda^2 N + 1)} \\ &= \frac{\sqrt{2\pi} \cdot \lambda^N \cdot (N+1)^{N+\frac{1}{2}} \cdot (\lambda^2 N - N + 1)^{\lambda^2 N - N + \frac{1}{2}} \cdot e^{\frac{\theta_1}{12(N+1)}} e^{\frac{\theta_2}{12(\lambda^2 N - N + 1)}}}{(\lambda^2 N + 1)^{\lambda^2 N + \frac{1}{2}} \cdot e \cdot e^{\frac{\theta_3}{12(\lambda^2 N + 1)}}} \end{aligned}$$

with $0 < \theta_i < 1$ ($1 \leq i \leq 3$).

Thanks to a simple estimate for $x > 0$:

$$(x+1)^x = x^x \cdot e^a$$

with certain $0 < a < 1$, we have

$$\begin{aligned}
\log |Q(N)| &= -N \log \Lambda + \frac{1}{2} \log(N+1) + \frac{1}{2} \log \frac{\lambda^2 N - N + 1}{\lambda^2 N + 1} \\
&\quad + \frac{1}{2} \log 2\pi - 1 + a_1 + a_2 - a_3 \\
&\quad + \frac{\theta_1}{12(N+1)} + \frac{\theta_2}{12(\lambda^2 N - N + 1)} - \frac{\theta_3}{12(\lambda^2 N + 1)} \\
&= -N \log \Lambda + \frac{1}{2} \log(N+1) + \frac{1}{2} \log \left(2\pi \left(1 - \frac{1}{\lambda^2} \right) \right) + \theta
\end{aligned}$$

where $0 < a_i < 1$ ($1 \leq i \leq 3$), $0 < \theta_i < 1$ ($1 \leq i \leq 3$), $-\frac{73}{36} < \theta < \frac{10}{9} + \frac{1}{2} \log \frac{\lambda^2}{\lambda^2 - 1}$.

This completes the proof. $\square$

Lemma 5. *Let $f(z)$ be an analytic function on a disk $|z| \leq R$ in $\mathbb{C}$. Let $\zeta_0, \dots, \zeta_\ell$ be pairwise distinct points in $|z| < R$. Then we have*

$$|f(\zeta_0)| \leq E_1 + E_2$$

where

$$\begin{aligned}
E_1 &= |f|_R \cdot \frac{R}{R - |\zeta_0|} \prod_{n=1}^{\ell} \frac{|\zeta_0 - \zeta_n|}{R - |\zeta_n|}, \\
E_2 &= \prod_{k=1}^{\ell} |\zeta_0 - \zeta_k| \cdot \sum_{n=1}^{\ell} \left(\frac{|f(\zeta_n)|}{|\zeta_0 - \zeta_n|} \cdot \prod_{\substack{i=1 \\ i \neq n}}^{\ell} \frac{1}{|\zeta_i - \zeta_n|} \right).
\end{aligned}$$

Proof. Put

$$A(z) = \prod_{n=1}^{\ell} \frac{R - \zeta_n}{z - \zeta_n}$$

and for $1 \leq n \leq \ell$ write also

$$A_n(z) = A(z) \cdot (z - \zeta_n) = \prod_{n=1}^{\ell} (R - \zeta_n) \cdot \prod_{\substack{i=1 \\ i \neq n}}^{\ell} \frac{1}{z - \zeta_i}.$$

The residue formula gives

$$A(\zeta_0)f(\zeta_0) = \frac{1}{2i\pi} \int_{\Gamma} A(\zeta)f(\zeta) \frac{d\zeta}{\zeta - \zeta_0} + \sum_{n=1}^{\ell} A_n(\zeta_n)f(\zeta_n) \cdot (\zeta_0 - \zeta_n)^{-1}$$

where we denote $\Gamma = \{\zeta \in \mathbb{C}; |\zeta| = R\}$. For $\zeta \in \Gamma$, we get

$$|A(\zeta)| \leq \prod_{n=1}^{\ell} \frac{|R - \zeta_n|}{R - |\zeta_n|}$$

then the required statement appears. $\square$

Lemma 6. *Let $f(z)$ be an analytic function on a disk $|z| \leq R$ in $\mathbb{C}$. Let $\zeta_0, \dots, \zeta_\ell$ be pairwise distinct points in $|z| < R$. Then*

$$|f(\zeta_0)| \leq G_1 + G_2$$

where

$$G_1 = |f|_R \cdot \frac{R}{R - |\zeta_0|} \prod_{n=1}^{\ell} \frac{R|\zeta_0 - \zeta_n|}{|R^2 - \zeta_0 \bar{\zeta}_n|}$$

and

$$G_2 = \sum_{n=1}^{\ell} \left(|f(\zeta_n)| \cdot \prod_{i=1}^{\ell} \left| \frac{R^2 - \zeta_n \bar{\zeta}_i}{R^2 - \zeta_0 \bar{\zeta}_i} \right| \cdot \prod_{\substack{k=1 \\ k \neq n}}^{\ell} \left| \frac{\zeta_0 - \zeta_k}{\zeta_n - \zeta_k} \right| \right).$$

Proof. Write Blaschke product by

$$B(z) = \prod_{n=1}^{\ell} \frac{R^2 - z\bar{\zeta}_n}{R(z - \zeta_n)}.$$

For $1 \leq n \leq \ell$ we also put

$$B_n(z) = B(z)(z - \zeta_n) = \prod_{i=1}^{\ell} \frac{R^2 - z\bar{\zeta}_i}{R} \prod_{\substack{k=1 \\ k \neq n}}^{\ell} \frac{1}{z - \zeta_k}.$$

Again by the residue formula, we have

$$B(\zeta_0)f(\zeta_0) = \frac{1}{2i\pi} \int_{\Gamma} B(\zeta)f(\zeta) \frac{d\zeta}{\zeta - \zeta_0} + \sum_{n=1}^{\ell} B_n(\zeta_n)f(\zeta_n) \cdot (\zeta_0 - \zeta_n)^{-1}$$

where $\Gamma = \{\zeta \in \mathbb{C}; |\zeta| = R\}$.

For $\zeta \in \Gamma$, we have $|B(\zeta)| = 1$, then the proof of the lemma is achieved. $\square$

4. Proof of Theorem 1

Let N_0 be a sufficiently large integer. Let $h_0 \in \mathbb{Q}$ such that $h_0 N_0 \in \mathbb{Z}$, $0 < h_0 < \frac{1}{2}$. Let $k_0 \in \mathbb{Z}$ with $k_0 \geq 1$. Let $g > 0$ such that

$$\|f(n)\| < e^{-gn}.$$

Finally, let $\lambda > 1$ be a parameter $\in \mathbb{R}$.

[First step]: construction of an auxiliary function.

Write

$$f_j(z) = \binom{z}{h} f(z)^k$$

for $1 \leq h \leq h_0 N_0$, $0 \leq k \leq k_0 - 1$, $1 \leq j = h(k+1) \leq L = h_0 k_0 N_0 \in \mathbb{Z}$.

Suppose that the entire function $f(z)$ satisfies for sufficiently large $r > 0$:

$$\log |f|_r < \alpha r.$$

Let $b_n \in \mathbb{Z}$ such that $\|f(n)\| = |f(n) - b_n| \leq e^{-gn}$ for all sufficiently large n and put

$$a_{jn} = \binom{n}{h} b_n^k$$

for $1 \leq h \leq h_0 N_0$, $0 \leq k \leq k_0 - 1$, $1 \leq j = h(k+1) \leq L = h_0 k_0 N_0$.

We construct an auxiliary function

$$F(z) = \sum_{1 \leq j \leq L} p_j f_j(z)$$

with $p_j \in \mathbb{Z}$ not all zero such that

$$\sum_{1 \leq j \leq L} p_j a_{jn} = 0, \quad u(N_0 - 1) \leq n \leq N_0 - 1 \quad (5)$$

with a real parameter u satisfying $0 < u < 1$.

By definition of a_{jn} and our assumption on the growth of $f(z)$, if $h_0 < \min(u, 1/2)$, then we have

$$\log \max_{1 \leq j \leq L} |a_{jn}| \leq c_1 N_0 + o(N_0)$$

where

$$c_1 = \alpha(k_0 - 1) - \varphi(h_0) - \varphi(1 - h_0).$$

Suppose $h_0 k_0 > 1 - u$. Then by Siegel's lemma (confer for example [11]) there exist integers p_j , $1 \leq j \leq L$ which are not all zero such that

$$\max_{1 \leq j \leq L} \log |p_j| \leq \frac{1 - u}{h_0 k_0 + u - 1} \times c_1 N_0 + o(N_0).$$

For $N \geq N_0$ and $\lambda > 1$, it follows

$$\log |F|_{\lambda N} \leq \left(\frac{1 - u}{h_0 k_0 + u - 1} \times c_1 + c_2 \right) N + o(N) + o(N_0)$$

with

$$c_2 = \varphi(\lambda + h_0) - \varphi(\lambda) - \varphi(h_0) + \alpha\lambda(k_0 - 1).$$

[Second step]: extrapolation.

We consider for each $N \geq N_0$ the following two properties:

$$A(N): \quad \sum_{1 \leq j \leq L} p_j a_{jn} = 0, \quad u(N-1) \leq n \leq N-1,$$

$$B(N): \quad \log |F|_N < -\gamma N,$$

where $\gamma > 0$ is independent of N_0 and N .

We shall show $A(N) \implies B(N)$ and $B(N) \implies A(N+1)$.

Proof of $A(N) \implies B(N)$. For $u(N-1) \leq n \leq N-1$ and $0 \leq k \leq k_0 - 1$, we have:

$$\begin{aligned} \max_{1 \leq j \leq L} \log |f_j(n) - a_{jn}| &\leq \max_{1 \leq j \leq L} \log \left| \binom{n}{h} (f(n)^k - b_n^k) \right| \\ &\leq c_1 N - \alpha N - guN + o(N). \end{aligned}$$

By hypothesis $A(N)$, it follows

$$\log \sum_{u(N-1) \leq n \leq N-1} |F(n)| \leq \frac{h_0 k_0}{h_0 k_0 + u - 1} c_1 \times N - (\alpha + gu)N + o(N) + o(N_0)$$

By Lemma 6, we have

$$|F|_N \leq T_1 + T_2$$

with

$$T_1 = |F|_{\lambda N} \frac{\lambda}{\lambda - 1} \prod_n \frac{\lambda N(N+n)}{\lambda^2 N^2 + Nn}$$

and

$$T_2 = \left(\sum_n |F(n)| \right) \frac{1}{(\Gamma(\frac{1-u}{2}N - 1))^2} \prod_n \frac{(\lambda^2 N^2 - u(N-1)n)(N+n)}{\lambda^2 N^2 + Nn}$$

because $(z-n)(\lambda^2 N^2 - zn)^{-1}$ takes the maximal absolute value on the disk $\{|z| = N\}$ at the point $z = -N$. Therefore we obtain

$$\begin{aligned} \log T_1 &\leq \log |F|_{\lambda N} + N(1-u) \log \lambda + \log \frac{\Gamma(2N) \cdot \Gamma((\lambda^2 + u)N)}{\Gamma((1+u)N) \cdot \Gamma((\lambda^2 + 1)N)} + o(N) + o(N_0) \\ &\leq \left(\frac{1-u}{h_0 k_0 + u - 1} \times c_1 + c_2 + c_3 \right) N + o(N) + o(N_0) \end{aligned}$$

where

$$c_3 = (1-u) \log \lambda + \varphi(2) + \varphi(\lambda^2 + u) - \varphi(1+u) - \varphi(\lambda^2 + 1)$$

and also

$$\begin{aligned} \log T_2 &\leq \log \left(\sum_n |F(n)| \right) + \log \frac{\Gamma((\frac{\lambda^2}{u} - u)N) \cdot \Gamma((\lambda^2 + u)N) \cdot \Gamma(2N)}{\Gamma((\frac{\lambda^2}{u} - 1)N) \cdot \Gamma((\lambda^2 + 1)N) \cdot \Gamma((1+u)N) \cdot (\Gamma(\frac{1-u}{2}N))^2} \\ &\quad + N(1-u) \log u + o(N) + o(N_0) \\ &\leq \left(\frac{h_0 k_0}{h_0 k_0 + u - 1} \times c_1 - (\alpha + gu) + c_4 \right) N + o(N_0) + o(N) \end{aligned}$$

where

$$c_4 = \varphi\left(\frac{\lambda^2}{u} - u\right) + \varphi(\lambda^2 + u) + \varphi(2) - \varphi\left(\frac{\lambda^2}{2} - 1\right) - \varphi(\lambda^2 + 1) - \varphi(1+u) - 2\varphi\left(\frac{1-u}{2}\right) + (1-u) \log u.$$

Hence the property $B(N)$ is true for N_0 sufficiently large, whenever the following inequalities are satisfied:

$$\frac{1-u}{h_0 k_0 + u - 1} \times c_1 + c_2 + c_3 < 0 \tag{6}$$

and

$$\frac{h_0 k_0}{h_0 k_0 + u - 1} \times c_1 - (\alpha + gu) + c_4 < 0. \quad (7)$$

□

Proof of $B(N) \implies A(N+1)$. We are going to show that the property $B(N)$ implies $A(N+1)$. We apply Liouville's theorem. Obviously:

$$\left| \sum_j p_j a_{jN} \right| \leq \left| \sum_j p_j (a_{jN} - f_j(N)) \right| + |F(N)|$$

where by assumption $B(N)$ we have,

$$\log |F(N)| \leq \log |F|_N < -\gamma N.$$

On the other hand, Lemma 3 shows

$$\log \left| \sum_j p_j (a_{jN} - f_j(N)) \right| \leq \left(\frac{1-u}{h_0 k_0 + u - 1} \times c_1 - \varphi(h_0) - \varphi(1-h_0) + (k_0-2)\alpha - g \right) N + o(N) + o(N_0).$$

Then the property $A(N)$ is satisfied when N_0 is sufficiently large and whenever the following condition holds:

$$\frac{1-u}{h_0 k_0 + u - 1} \times c_1 - \varphi(h_0) - \varphi(1-h_0) + (k_0-2)\alpha - g < 0. \quad (8)$$

Carrying out technical computational optimizations, we choose parameters such that the three conditions (6), (7), and (8) are simultaneously satisfied.

We set:

$$h_0 = 0.108496, \quad k_0 = 16, \quad u = 0.108496, \quad \lambda = 11.9194, \quad g = 663143,$$

then the inequalities (6), (7), and (8) are verified with

$$\alpha = 0.00437155 \left(> \frac{1}{229} \right).$$

[Third step]: conclusion.

By the construction of the auxiliary function $F(z)$, the property $A(N_0)$ is true. Since $A(N) \implies B(N)$ and $B(N) \implies A(N+1)$, both properties $A(N)$ and $B(N)$ hold for any $N \geq N_0$. If the function $F(z)$ is not identically zero, then by Liouville's theorem, we obtain for $N \rightarrow \infty$:

$$|F|_N \rightarrow \infty$$

which contradicts $B(N)$.

As a consequence, the function $F(z)$ is identically zero, namely the functions $f_j(z)$ are linearly dependent over $\mathbb{Q}$. Since it is equivalent to say that $f(z)$ is a polynomial with coefficients in $\mathbb{Q}$ and that the functions $f_j(z) = \binom{z}{h} f(z)^k$ ($h, k \in \mathbb{N} \cup \{0\}$) are linearly dependent over $\mathbb{Q}$, hence the proof of Theorem 1 is achieved. □

5. Proof of Theorem 2

Take

$$h_0 = 0.000991044, \quad k_0 = 640, \quad u = 0.78256, \quad \lambda = 1.5253, \quad g = 3,$$

then the conditions (6), (7), and (8) are valid with

$$\alpha = 8.775 \times 10^{-6}$$

which establishes Theorem 2. □

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