A Remark on Oriented Graphs

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Let G be an oriented graph. We introduce the notion of $(+)$ -subgraph, which comes naturally from the notion of $(+)$ -fcd in foliation theory. We give a property of oriented graphs having at least one $(+)$ -subgraph, which corresponds to the property of mean curvature functions in foliation theory.

KEYWORDS: oriented graph, $(+)$ -subgraph, weight

1. Introduction

In previous papers Oshikiri (1990, 1991), the author studied some problems concerning mean curvature functions of codimension-one foliations. It is possible to assign an oriented graph to each codimension-one foliation (see Oshikiri 1996), and one of the problems concerning mean curvature functions of codimension-one foliations is closely related to a property of oriented graphs. In fact, the author is able to give an affirmative answer to the problem on mean curvature functions of codimension-one foliations posed by P. Walczak (see Oshikiri 1995). The present paper deals with this property of oriented graphs.

Definitions and results are given in the next section, and proofs will be given in Section 3. The author is grateful for the hospitality of the Centre de Recerca Matemàtica during his sojourn in Barcelona.

2. Preliminaries and Results

We begin with some definitions on graphs. For the definition of cellular complexes, see the book of Spanier 1966, and for generalities on graph theory, see the book of Bollobas 1979.

DEFINITION 1. G is called a graph if G is a finite 1-dimensional cellular complex. We set $V(G) = \{v_i\} = \{\text{all 0-cells of } G\}$ and $E(G) = \{e_a\} = \{\text{all 1-cells of } G\}$. We call each $v \in V(G)$ a vertex, and $e \in E(G)$ an edge. For $e \in E(G)$ we also set $V(e) = \text{Cl}(e) - e = \{\text{endpoints of } e\}$, where the closure $\text{Cl}(e)$ of e is taken in G .

Remarks. (a) $V(e)$ may consist of only one point $\{v\}$. In this case, we call e a loop at v .

(b) $V(e_a) = V(e_b)$ may occur even if $a \neq b$. In this case, G is sometimes called a multigraph (see Bollobas 1979).

DEFINITION 2. $G' \subset G$ is called a full subgraph of G if

- (i) G' is a non-empty subcomplex of G , and
- (ii) any $e \in E(G)$ with $V(e) \subset G'$ is included in $E(G')$, i.e., $e \in E(G')$.

DEFINITION 3. Let V' be a non-empty subset of $V(G)$. The subgraph $g(V')$ induced by V' is the smallest full subgraph of G containing V' .

Let G be a graph. Then each edge e is homeomorphic to $(0, 1)$. If such a parametrization is fixed, e has a natural orientation induced from that of $(0, 1)$. In this case, we call e oriented. With this in mind, we give the following definition;

DEFINITION 4. G is called an oriented graph if G is a graph and each edge $e \in E(G)$ is oriented. For each edge $e \in E(G)$, we call $F(0) = I(e)$ the initial vertex, and $F(1) = T(e)$ the terminal vertex. Here $F: [0, 1] \rightarrow \text{Cl}(e) \subset G$ is the extended map of the given parametrization $f: (0, 1) \rightarrow e \in E(G)$.

Remark. If e is a loop at the vertex v , then $v = I(e) = T(e)$.

To represent our result, we need the following definitions. These definitions correspond to those of $(+)$ and $(-)$ -fcd's in foliation theory (see Oshikiri 1990).

DEFINITION 5. Let G be an oriented graph. A vertex $v \in V(G)$ is called $(+)$ -vertex (resp. $(-)$ -vertex) if any $e \in E(G)$ with $v \in V(e)$ implies $I(e) = v$ and $T(e) \neq v$ (resp. $T(e) = v$ and $I(e) \neq v$).

If e is a loop at v , then v is neither $(+)$ - nor $(-)$ -vertex. Thus we need further definition in order to admit this situation.

DEFINITION 6. Let G be an oriented graph. A vertex $v \in V(G)$ is called semi $(+)$ -vertex (resp. semi $(-)$ -vertex) if v is a $(+)$ -vertex (resp. $(-)$ -vertex) after removing all loops at v .

DEFINITION 7. A non-empty proper full subgraph G' of an oriented graph G is called $(+)$ -subgraph (resp. $(-)$ -subgraph) if $e \in E(G)$ with $V(e) \cap G' \neq \emptyset$ and $V(e) \cap (V(G) - V(G')) \neq \emptyset$ implies $I(e) \in V(G')$ (resp. $T(e) \in V(G')$).

Remarks. (a) A semi $(+)$ -vertex (resp. $(-)$ -vertex) v with all loops at v is a $(+)$ -subgraph (resp. $(-)$ -subgraph).

(b) A $(+)$ -subgraph (resp. $(-)$ -subgraph) G' may not contain any semi $(+)$ -vertex (resp. $(-)$ -vertex).

LEMMA 1. Every $(+)$ -subgraph (resp. $(-)$ -subgraph) contains a minimal $(+)$ -subgraph (resp. $(-)$ -subgraph). Here “minimal” is meant in the usual set theoretical sense, that is, non-empty and containing no non-empty proper $(+)$ -subgraphs (resp. $(-)$ -subgraphs).

Proof. As the intersection of two $(+)$ -subgraphs (resp. $(-)$ -subgraphs) is empty or a $(+)$ -subgraph (resp. $(-)$ -subgraph), the existence of the minimal $(+)$ -subgraph (resp. $(-)$ -subgraph) follows from the standard argument in Set Theory.

LEMMA 2. Let G be an oriented graph and G' be a $(+)$ -subgraph (resp. $(-)$ -subgraph) of G . Then the induced subgraph $g(V(G) - V(G'))$ by $V(G) - V(G')$ is a $(-)$ -subgraph (resp. $(+)$ -subgraph) of G .

Proof. As the graph $g(V(G) - V(G'))$ is by definition a full subgraph of G , it is easy to see that $g(V(G) - V(G'))$ satisfies the conditions of Definition 7.

To state our result, we need the following definitions.

DEFINITION 8. Let G be a graph. We call a function $a: V(G) \rightarrow \mathbf{R}_+$ a weight for G .

The following definition corresponds to admissible functions for foliations (see Oshikiri 1990).

DEFINITION 9. Let G be an oriented graph having $(+)$ -subgraphs. We call a function $f: V(G) \rightarrow \mathbf{R}$ admissible if every minimal $(+)$ -subgraph contains a vertex v with $f(v) > 0$, and every minimal $(-)$ -subgraph contains a vertex w with $f(w) < 0$.

Now we state the main theorem.

THEOREM. Let G be a connected oriented graph having at least one $(+)$ -subgraph, and $f: V(G) \rightarrow \mathbf{R}$ be an admissible function. Then there is a weight $a: V(G) \rightarrow \mathbf{R}_+$ satisfying the following two conditions;

- (i) $\sum_{v \in V(G)} a(v)f(v) = 0$, and
- (ii) $\sum_{v \in V(H'')} a(v)f(v) > 0$ for any $(+)$ -subgraphs H'' .

3. Proof of Theorem

Let G be an oriented graph having at least one $(+)$ -subgraph. Note that, by Lemma 2, G also contains at least one minimal $(-)$ -subgraph.

Corresponding to the notion of “trail” in graph theory (see Bollobas 1979), here we introduce the notion of an “oriented trail”.

DEFINITION 10. An ordered set $P = \{e_1, \dots, e_k\}$ of edges of G is called a positive trail (resp. negative trail) if every edge of P is mutually distinct and $T(e_{j-1}) = I(e_j)$ (resp. $I(e_{j-1}) = T(e_j)$) for $j = 2, \dots, k$. We call P a positive trail from $I(P) = I(e_1)$ to $T(P) = T(e_k)$ (resp. negative trail from $I(P) = T(e_1)$ to $T(P) = I(e_k)$).

Remark. The vertex $I(P)$ (resp. $T(P)$) may not be contained in a minimal $(+)$ -subgraph (resp. $(-)$ -subgraph) even if P is a maximal positive trail.

LEMMA 3. Let v_1 and v_2 be two distinct vertices of a minimal $(+)$ -subgraph H . Then there is a positive trail in H from v_1 to v_2 .

Proof. Let H' be the union of all vertices and edges of all positive trails in H from v_1 . H' satisfies a similar condition to being a $(-)$ -subgraph in H , that is, $e \in E(H)$ with $V(e) \cap V(H') \neq \emptyset$ and $V(e) \cap (V(H) - V(H')) \neq \emptyset$ implies $I(e) \in (V(H) - V(H'))$ and $T(e) \in V(H')$. Thus, if $V(H')$ is a proper subset of $V(H)$, then, by the same argument as in Lemma 2, $g(V(H) - V(H'))$ is a non-empty $(+)$ -subgraph of H , which contradicts the minimality of H . Therefore $V(H') = V(H)$, which implies the existence of a positive trail in H from v_1 to v_2 .

LEMMA 4. Let G be a graph as in the Theorem and H be a $(+)$ -subgraph of G . For any edge $e \in E(G)$ with $V(e) \cap V(H) \neq \emptyset$ and $V(e) \cap (V(G) - V(H)) \neq \emptyset$, there is a positive trail P containing e such that $I(P)$ is contained in a minimal $(+)$ -subgraph and $T(P)$ is contained in a minimal $(-)$ -subgraph.

Proof. Let H'' be the union of all vertices and edges of all positive trails with the last edge being e , and set $H' = H'' - \{e, T(e)\}$. Then, by a similar argument to the proof of Lemma 3, $g(V(H'))$ is a $(+)$ -subgraph of G . As $g(V(H'))$ contains, by Lemma 1, a minimal $(+)$ -subgraph, there is a positive trail P' with the last edge being e and $I(P')$ is contained in a minimal subgraph. Similarly, there is a positive trail P'' with the first edge being e and $T(P'')$ contained in a minimal $(-)$ -subgraph. As every positive trail with the last edge being e is contained in H , and every positive trail with the first edge being e is contained in $g(V(G) - V(H))$, the union P of P' and P'' gives the desired trail.

The following proposition is the key step to the proof of the theorem.

PROPOSITION. Let G be a graph as in the Theorem. For each minimal $(+)$ -subgraph H choose an arbitrary vertex $v_H \in V(H)$, and for each minimal $(-)$ -subgraph K choose an arbitrary vertex $w_K \in V(K)$. Then there is a function $F: V(G) \rightarrow \mathbf{R}$ satisfying the following three conditions:

- (i) $\sum_{v \in V(G)} F(v) = 0$
- (ii) $\sum_{v \in V(H')} F(v) > 0$ for every positive $(+)$ -subgraph H'
- (iii) $F(v_H) > 0$ for every H and $F(w_K) < 0$ for every K , and $F(v) = 0$ otherwise,

where v_H 's and w_K 's are the vertices chosen above.

Note firstly that, by Lemmas 3 and 4, we obtain the following.

LEMMA 5. Let G be a graph as in the Theorem and H be a $(+)$ -subgraph of G . For any edge $e \in E(G)$ with $V(e) \cap V(H) \neq \emptyset$ and $V(e) \cap (V(G) - V(H)) \neq \emptyset$, there is a positive trail P containing e such that $I(P)$ is one of the v_H 's and $T(P)$ is one of the w_K 's (v_H 's and w_K 's as given in the proposition).

Proof of Proposition. For each v_H define $F(v_H)$ as the number of w_K 's which can be joined from the v_H by a positive trail, and for each w_K define $-F(w_K)$ as the number of v_H 's which can be joined from the w_K by a negative trail. Set $F(v) = 0$ if v is neither a v_H nor a w_K . We now show that this F has the proposed value. If a v_H is joined to a w_K by a positive trail, then that w_K is joined to the v_H by a negative trail, and vice versa. Thus F satisfies condition (i). For a v_H , define a full subcomplex H' of G consisting of all vertices and edges of all positive trails from v_H . The subgraph H' is a $(-)$ -subcomplex, because if an edge e is oriented from H' to $G - H'$, then e must be in a positive trail from v_H and $e \in E(H')$, which is a contradiction. Thus, by Lemma 1, H' contains a minimal $(-)$ -subgraph. Therefore, there is a positive trail from v_H to one of the w_K 's, that is, $F(v_H) > 0$. For a w_K , by the same argument, we have $-F(w_K) > 0$, that is, $F(w_K) < 0$. This means F satisfies condition (iii). To show that F satisfies condition (ii), let H' be a $(+)$ -subgraph. If $w_K \in V(H')$, as every negative trail from w_K is contained in H' , it is easy to see $\sum_{v \in V(H')} F(v) \geq 0$. If $\sum_{v \in V(H')} F(v) = 0$, then every v_H is connected to only the w_K 's in H' . But this is impossible, because an edge from H' to $G - H'$ is contained in a positive trail P from one of the v_H 's to one of the w_K 's, by Lemma 5, and $I(P)$ must be in H' and $T(P)$ must be outside of H' as H' is a $(+)$ -subgraph. This completes the proof.

Proof of Theorem. Let G be a graph as in the Theorem, and $f: V(G) \rightarrow \mathbf{R}$ be an arbitrarily given admissible function. For each minimal $(+)$ -subgraph H , choose a vertex v_H of H with $f(v_H) > 0$, and for each minimal $(-)$ -subgraph K , choose a vertex w_K of K with $f(w_K) < 0$. This is possible because f is admissible. Let F be the function constructed in the proof of the Proposition, with these v_H 's and w_K 's. Set $A = \#(V(G))$, where " $\#X$ " means the number of elements of the set X . Set also $B = \min \{ \sum_{v \in V(H')} F(v) : H' \text{ varies for all } (+)\text{-subgraphs} \}$. Then, as G is a finite graph, A is finite and $B > 0$, by Proposition (ii). Choose $c > 0$ so that $2Ac < B$.

For each vertex $v \in V(G)$, we give the following weight $b: V(G) \rightarrow \mathbf{R}_+$:

$$\begin{aligned} b(v)f(v) &= F(v) & \text{if } v &= v_H \text{ for some minimal } (+)\text{-subgraph } H \\ b(w)f(w) &= F(w) & \text{if } w &= w_K \text{ for some minimal } (-)\text{-subgraph } K \\ |b(v)f(v)| &< c & \text{if } v &\in L, \text{ where } L = V(G) - \{v_H\text{'s}\} - \{w_K\text{'s}\}. \end{aligned}$$

By Proposition (i), it follows that

$$|\sum_{v \in V(G)} b(v)f(v)| = |\sum_{v \in L} b(v)f(v)| < Ac.$$

In order to obtain a desired weight system $a(v)$, we change $b(v)$ as follows: Choose one $v_{H'}$ and set $a(v_{H'}) = b(v_{H'}) - (\sum_{v \in L} b(v)f(v))/f(v_{H'})$. For other vertices v , set $a(v) = b(v)$. Note that, as $a(v_{H'})f(v_{H'}) > B - Ac > 0$, it follows that $a(v_{H'}) > 0$. We show that this $a(v)$ is the desired weight. In fact

$$\begin{aligned} \sum_{v \in V(G)} a(v)f(v) &= a(v_{H'})f(v_{H'}) + \sum_{v \neq v_{H'}} a(v)f(v) = b(v_{H'})f(v_{H'}) - \sum_{v \in L} b(v)f(v) + \sum_{v \neq v_{H'}} b(v)f(v) \\ &= \sum_{v \in L} b(v)f(v) - \sum_{v \in L} b(v)f(v) = 0. \end{aligned}$$

This shows that $a(v)$ satisfies condition (i). Let H'' be a $(+)$ -subgraph. We must show that $\sum_{v \in V(H'')} a(v)f(v) > 0$. If $v_{H'} \notin V(H'')$, where $v_{H'}$ is the vertex chosen above, then

$$\sum_{v \in V(H'')} a(v)f(v) \geq \sum_{v \in V(H'')} F(v) - \sum_{v \in L} |a(v)f(v)| \geq B - Ac > 0.$$

If $v_{H'} \in V(H'')$, then

$$\sum_{v \in V(H'')} a(v)f(v) \geq \sum_{v \in V(H'')} F(v) - \sum_{v \in L} b(v)f(v) - \sum_{v \in L} |a(v)f(v)| \geq B - 2Ac > 0.$$

This completes the proof.

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