Some Conditions for the Strong Starlikeness of Holomorphic Functions

Nozomu MOCHIZUKI* and Takashi SANO**

*Graduate School of Information Sciences, Tohoku University,
Katahira, Aobaku, Sendai 980-77, Japan

**Meitec Corporation, 2-11 Kandasuda-cho, Chiyodaku, Tokyo 101, Japan

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We shall obtain two conditions that guarantee the strong starlikeness of holomorphic functions on the unit disc.

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1. Introduction

The class of holomorphic functions f on the unit disc U that satisfy $f(0) = f'(0) - 1 = 0$ is denoted by A . Let $0 < \alpha \leq 1$. If $f \in A$ and f satisfies the condition

$$-\frac{\pi\alpha}{2} < \arg \frac{zf'(z)}{f(z)} < \frac{\pi\alpha}{2} \quad (z \in U),$$

then f is called strongly starlike of order α ([1], [8]). The class of all such functions is denoted by $S^*(\alpha)$. If $\alpha = 1$, then $S^*(1) = S^*$, the class of starlike functions.

We shall deduce two conditions that guarantee the strong starlikeness of $f \in A$. These will be seen to be best possible in some sense.

2. A Sufficient Condition in Terms of Coefficients

The following theorem generalizes a well known result for functions in S^* ([2], [3]). To begin with, we need a simple fact that is clear from a figure. Let w be a complex number.

(1) If $|w - 1| < r$, $0 < r \leq 1$, then $|\arg w| < \sin^{-1} r$.

Theorem 1. Let $0 < \alpha \leq 1$. Suppose $f \in A$, with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ($z \in U$). If the coefficients satisfy

$$(2) \quad \sum_{n=2}^{\infty} \left((n-1) \operatorname{cosec} \frac{\pi\alpha}{2} + 1 \right) |a_n| \leq 1,$$

then $f \in S^*(\alpha)$. The right-hand side cannot be replaced by any larger constants.

Proof. Suppose f is not the function $f(z) = z$. Put $c = \sin(\pi\alpha/2)$. Then, for $0 < |z| < 1$,

$$|zf'(z) - f(z)| - c|f(z)| < |z| \left(\sum_{n=2}^{\infty} (n-1) |a_n| - c \left(1 - \sum_{n=2}^{\infty} |a_n| \right) \right) \leq 0.$$

It follows that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < c \quad (z \in U),$$

hence $|\arg(zf'(z)/f(z))| < \pi\alpha/2$, by (1).

To prove the latter half, we suppose that $0 < \alpha < 1$, the case $\alpha = 1$ being well known. Put $a = \operatorname{cosec}(\pi\alpha/2)$ and take ε , $0 < \varepsilon < a - 1$. We shall show that there is a polynomial f that satisfies (2) with the right-hand side replaced by $1 + \varepsilon$ and $f \notin S^*(\alpha)$. Let

$$t_n = \frac{1 + \varepsilon}{(n-1)a + 1} \quad \text{and} \quad \theta_n = \cos^{-1}(-(n-1)t_n),$$

and define f_n by

$$f_n(z) = z + t_n e^{i\theta_n} z^n, \quad n \geq 2.$$

Letting $z = 1$ in $zf'_n(z)/f_n(z)$ and letting $n \rightarrow \infty$, we see that

$$\zeta_n = \frac{f'_n(1)}{f_n(1)} \rightarrow \zeta = \frac{1}{a^2} (a^2 - (1 + \varepsilon)^2 + i(1 + \varepsilon)\sqrt{a^2 - (1 + \varepsilon)^2}).$$

Since $0 < \arg \zeta < \pi/2$ and

$$\tan(\arg \zeta) > \frac{1}{\sqrt{a^2 - 1}} = \tan \frac{\pi\alpha}{2},$$

we can take N so that $\pi\alpha/2 < \arg \zeta_N < \pi/2$. This shows that $f_N \notin S^*(\alpha)$.

Remark 1. Let $0 < \alpha < \beta \leq 1$ and let $b = \operatorname{cosec}(\pi\beta/2)$. If we take ε , $0 < \varepsilon < (n-1)(a-b)/((n-1)b + 1)$, then $((n-1)b + 1)t_n < 1$, so $f_n \in S^*(\beta)$ by the above theorem. We thus have $f_N \in S^*(\beta) \setminus S^*(\alpha)$ for a large N .

3. Another Condition

For $f \in A$, we shall write $P(z)$ and $Q(z)$, respectively, in place of $z^2 f'(z)/f(z)^2$ and $z/f(z)$. If $|P(z) - 1| \leq 1$ ($z \in U$), then f is univalent, by a result of [7]. If, in particular, $f(z) = z + a_2 z^2 + \dots$ and $|P(z) - 1| = 0$ ($z \in U$), then $f(z) = z/(1 - az)$ with $|a| \leq 1$, where $a = a_2$.

Lemma. Let $0 < \alpha \leq 1$. If $f(z) = z/(1 - az)$, $|a| \leq 1$ ($z \in U$), then the following three statements are equivalent.

- (3) $|a| \leq \sin \frac{\pi\alpha}{2}.$
- (4) $f \in S^*(\alpha).$
- (5) $|Q(z) - 1| < \sin \frac{\pi\alpha}{2} \quad (z \in U).$

Proof. The equivalence of (3) and (5) is obvious. Next, suppose (3) is satisfied. Then

$$\left| \arg \frac{zf'(z)}{f(z)} \right| = |\arg(1 - az)| < \frac{\pi\alpha}{2},$$

by (1); this shows (4). Finally, we show that (4) implies (3), under the assumption that $0 < \alpha < 1$. Suppose $\sin(\pi\alpha/2) < |a| \leq 1$, contrary to (3). Let $a = \rho e^{i\theta}$. Take r , $0 < r < 1$, and let $\lambda = \cos^{-1}(\rho r) - \theta$. Then, for $z_r = re^{i\lambda}$,

$$\zeta_r = \frac{z_r f'(z_r)}{f(z_r)} = \frac{1 - \rho^2 r^2 + i\rho r \sqrt{1 - \rho^2 r^2}}{|1 - az_r|^2}.$$

This shows that

$$\tan(\arg \zeta_r) = \frac{\rho r}{\sqrt{1 - \rho^2 r^2}}, \quad \text{with } 0 < \arg \zeta_r < \frac{\pi}{2}.$$

Suppose first that $\rho = 1$. Then we can choose r_0 , $0 < r_0 < 1$, so that $\tan(\arg \zeta_{r_0}) > \tan(\pi\alpha/2)$, hence $\pi\alpha/2 < \arg \zeta_{r_0} < \pi/2$. This means that $f \notin S^*(\alpha)$. Next suppose that $\sin(\pi\alpha/2) < \rho < 1$. Since $t(1 - t^2)^{-1/2}$ is an increasing function of t , we have $\rho(1 - \rho^2)^{-1/2} > \tan(\pi\alpha/2)$. Hence we can choose r_0 , $0 < r_0 < 1$, so that

$$\frac{\rho r_0}{\sqrt{1 - \rho^2 r_0^2}} > \tan \frac{\pi\alpha}{2}.$$

Theorem 2. Let $0 < \alpha \leq 1$.

(i) Suppose $f \in A$ satisfies the following conditions for some k , $0 \leq k \leq 1$.

- (6) $|P(z) - 1| \leq \sin(\alpha \sin^{-1} k) \quad (z \in U).$
- (7) $|Q(z) - 1| \leq \sin(\alpha \cos^{-1} k) \quad (z \in U).$

Then $f \in S^*(\alpha)$.

(ii) Take an arbitrary k , $0 < k < 1$. Then, for any ε , $0 < \varepsilon < k$, there exists $f \in A$ such that

- (8) $|P(z) - 1| < \sin(\alpha \sin^{-1} k) \quad (z \in U),$
- (9) $|Q(z) - 1| < \sin(\alpha \cos^{-1}(k - \varepsilon)) \quad (z \in U),$

and yet $f \notin S^*(\alpha)$.

Remark 2. Put

$$\phi(k) = \sin(\alpha \sin^{-1} k) \quad \text{and} \quad \psi(k) = \sin(\alpha \cos^{-1} k) \quad (0 \leq k \leq 1).$$

Then part (ii) is equivalent to the following:

For any ε , $0 < \varepsilon < 1 - k$, there is $f \in A$ such that

$$|P(z) - 1| < \phi(k + \varepsilon), \quad |Q(z) - 1| < \psi(k) \quad (z \in U)$$

and $f \notin S^*(\alpha)$.

Remark 3. Part (i) is closely related to Theorem 2 and Corollary 1 of [5]. Moreover, letting $\alpha = 1$ in (8), we can see that for each k , $0 < k < 1$, there is $f \in A$ such that $|P(z) - 1| < k$ ($z \in U$) and $f \notin S^*$. This is not settled for $0 < k \leq (\sqrt{10} - \sqrt{2})/2$ in [5].

Remark 4. We denote by R_0 the class of holomorphic functions f of the form

$$(10) \quad f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n} \quad (z \in U).$$

Clearly, $R_0 \subset A$. These functions were discussed in [4] and [5]. For $f \in R_0$,

$$P(z) - 1 = - \sum_{n=2}^{\infty} (n-1) b_n z^n.$$

Here we note that Theorem 1 of [6] implies the following:

If $f \in A$ satisfies $|P(z) - 1| < 1$ ($z \in U$), then f can be written in the form (10).

Accordingly, the functions treated in this section are those of the class R_0 .

Proof of Theorem 2. (i) Suppose first that $0 < k < 1$. In this case, equalities do not occur in (6) and (7). Since

$$|\arg P(z)| < \alpha \sin^{-1} k \quad \text{and} \quad |\arg Q(z)| < \alpha \cos^{-1} k,$$

by (1), we have

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq |\arg P(z)| + |\arg Q(z)| < \frac{\pi\alpha}{2} \quad (z \in U),$$

which implies that $f \in S^*(\alpha)$. If $k = 0$, then $|P(z) - 1| = 0$ and $|Q(z) - 1| < \sin(\pi\alpha/2)$, so $f \in S^*(\alpha)$, by Lemma. If $k = 1$, then $f(z) = z$. (ii) Let $0 < k < 1$, and take an arbitrary ε , $0 < \varepsilon < k$. Put $t = \phi(k)$, $s = \psi(k - \varepsilon)$ and let $\theta = \cos^{-1}(-s)$, $\lambda = \cos^{-1}t$. For a positive integer n such that $ns - t > 0$, we define f_n by

$$f_n(z) = \frac{z}{1 + \frac{ns - t}{n} e^{-i\theta} z + \frac{t}{n} e^{-i\lambda} z^{n+1}}.$$

This is holomorphic on $\bar{U}$ and

$$|P(z) - 1| < t, \quad |Q(z) - 1| < s \quad (z \in U);$$

the function f_n thus satisfies (8) and (9). Letting $z = 1$ in $zf'_n(z)/f_n(z)$, we can see that

$$\zeta_n := \frac{f'_n(1)}{f_n(1)} \rightarrow \zeta := \frac{1 - te^{-i\lambda}}{1 + se^{-i\theta}} \quad \text{as } n \rightarrow \infty.$$

Put $\xi = 1 - te^{-i\lambda}$ and $\eta = 1 + se^{-i\theta}$. Then

$$\begin{aligned} \tan(\arg \xi) &= \tan(\alpha \sin^{-1} k) \quad \text{with } 0 < \arg \xi < \pi/2, \\ \tan(\arg \eta) &= \tan(\alpha \cos^{-1}(k - \varepsilon)) \quad \text{with } 0 < \arg \eta < \pi/2, \end{aligned}$$

and $\arg \zeta = \arg \xi + \arg \eta$. Hence $\pi\alpha/2 < \arg \zeta < \pi$. We can take N so that $f_N \notin S^*(\alpha)$.

Corollary 1. Let $0 < \alpha \leq 1$. Suppose $f \in A$ satisfies the following conditions for some k , $0 \leq k \leq 1$.

$$|P(z) - 1| \leq \alpha k \quad \text{and} \quad |Q(z) - 1| \leq \alpha \sqrt{1 - k^2} \quad (z \in U).$$

Then $f \in S^*(\alpha)$.

Proof. The conclusion follows from $\alpha \sin x \leq \sin \alpha x$, valid for $0 \leq \alpha \leq 1$, $0 \leq x \leq \pi$.

Now, to have a better understanding of the meaning of Theorem 2, let us consider as follows. For $0 < \alpha \leq 1$, we denote by D_α the closed domain of the (x, y) -plane bounded by the line segments $\{(x, 0) | 0 \leq x \leq \sin(\pi\alpha/2)\}$, $\{(0, y) | 0 \leq y \leq \sin(\pi\alpha/2)\}$, and the curve C_α defined by $x = \phi(k)$, $y = \psi(k)$ ($0 \leq k \leq 1$). Let $E_\alpha = \{(x, 0) | \sin(\pi\alpha/2) < x < +\infty\}$, $0 < \alpha \leq 1$, and let $F_1 = \{(0, y) | 1 < y < +\infty\}$. Put $\Omega_\alpha = D_\alpha \cup E_\alpha$ for $0 < \alpha < 1$, and put $\Omega_1 = D_1 \cup E_1 \cup F_1$ for $\alpha = 1$. Theorem 2 is restated in the following form.

Corollary 2. *Let $0 < \alpha \leq 1$.*

(i) *Take $(x_0, y_0) \in \Omega_\alpha$. Then every $f \in A$ that satisfies*

$$(11) \quad |P(z) - 1| \leq x_0 \quad \text{and} \quad |Q(z) - 1| \leq y_0 \quad (z \in U)$$

belongs to $S^(\alpha)$.*

(ii) *Take $(x_0, y_0) \notin \Omega_\alpha$. Then there is $f \in A$ that satisfies (11) and $f \notin S^*(\alpha)$.*

Proof. (i) Suppose $f \in A$ satisfies (11) for $(x_0, y_0) \in \Omega_\alpha$. If $x_0 = 0$ or $y_0 = 0$, then it is easy to see that $f \in S^*(\alpha)$. Suppose $x_0 \neq 0, y_0 \neq 0$. Then the straight line joining the origin to the point (x_0, y_0) intersects the curve C_α exactly at one point $(\phi(k_0), \psi(k_0))$, $0 < k_0 < 1$. Since $x_0 \leq \phi(k_0)$ and $y_0 \leq \psi(k_0)$, Theorem 2 shows that $f \in S^*(\alpha)$. (ii) Suppose $(x_0, y_0) \notin \Omega_\alpha$, $0 < \alpha \leq 1$. The point that is the intersection of the above mentioned straight line and the curve C_α is $(\phi(k_0), \psi(k_0))$, where $0 \leq k_0 < 1$. If $0 < k_0 < 1$, then $\phi(k_0) < x_0$ and $\psi(k_0) < y_0$, so Theorem 2 shows that there is $f \in A$ that satisfies (11), with $f \notin S^*(\alpha)$. Let $0 < \alpha < 1$ and let $(x_0, y_0) = (0, y_0)$, where $y_0 > \sin(\pi\alpha/2)$. If we take $f(z) = z/(1 - az)$, with $\sin(\pi\alpha/2) < |a| < \min(1, y_0)$, then $f \notin S^*(\alpha)$, by Lemma.

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