Spectra of the Discrete and Continuous Laplacians on Graphs and Riemannian Manifolds

Hajime URAKAWA

*Mathematics Laboratories, Graduate School of Information Sciences,
Tohoku University, Katahira, 2-1-1, Sendai, 980-77, Japan*

Received December 16, 1996; final version accepted March 31, 1997

I. Introduction and generalities

This is a survey article concerning spectral problems for the Laplacian on Riemannian manifolds and the discrete Laplacian of graphs. This is also a supplement of my survey paper [U 8] published in 1993. Both spectral problems for Riemannian manifolds and graphs have strong similarities and seem to be connected each other. We first give a survey on the spectra of the Laplacian on Riemannian manifolds and then the one of the discrete Laplacians for graphs.

Namely, the contents are as follows:

- I. Introduction and generalities.
- II. Non-compact Riemannian manifolds.
 - II.1. *The heat kernel and Green kernel.*
 - II.2. *The spectrum.*
 - II.3. *Harmonic functions.*
- III. Discrete spectra.
 - III.1. *Setting of the problems.*
 - III.2. *Boundary value problems.*
 - III.3. *The first eigenvalue of a closed manifold.*
 - III.4. *Isospectral problems.*
- IV. Discrete Laplacians in graph theory.
 - IV.1. *Setting of the eigenvalue problems for graphs.*
 - IV.2. *Infinite graphs.*
 - IV.3. *Finite graphs.*

Acknowledgement

This work was done in preparing for my talk at a symposium “Inverse Problems Related to Geometry” held at September 25–27, 1996, at Tokyo Metropolitan University. I would like to express my gratitude to Professors H. Soga and M. Kawashita for giving an opportunity to give a talk there and to Professors A. Ikeda, M. Ikehata, Y. Isozaki, A. Katsuda, Y. Kurylev, G. Nakamura and T. Sunada for giving helpful comments and suggestions.

(1.1) For an n -dimensional Riemannian manifold (M, g) , let us consider the Laplacian Δ defined by

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x_j} \right), \quad f \in C^\infty(M),$$

where $\sqrt{g} = \sqrt{\det(g_{ij})}$ and $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ for a local coordinate $(x_1, \dots, x_n)$ of M .

(1.2) It is known that the Laplacian Δ is essentially self-adjoint (cf. [Ga 1 ~ 3]) and the heat semi-group $e^{t\Delta}$ has a kernel $p(t, x, y)$ which is symmetric in the variables x and y if (M, g) is complete (cf. [It], [Do 1], [St]).

(1.3) **Problem 1.** (1) *The first question is how about essential self-adjointness of Δ when (M, g) is not complete, for example, $M = \mathbf{R}^n - \Sigma$, where Σ is a fractal set.*

(2) *Estimate the heat kernel $p(t, x, y)$ in terms of the distance $d(x, y)$ between x and y , and t , and clarify when the following conservation property hold (cf. [L.Y 2], [Az], [Ku 1]):*

$$\int_M p(t, x, y) dy = 1, \quad \forall x \in M.$$

(1.4) Indeed, Li and Tian (cf. [L.Ti]) showed that it is the case for (1) and (2) when M is the complement of a real codimension 2 manifold in a compact algebraic variety. There are several results about the estimation of the heat kernel (see for example, [Gr 1 ~ 5], [Cou 1 ~ 3], [C.L], [C.S]) For (2), it is closely related to the behavior of the Ricci curvature (see [Ku 1]).

II. Non-compact Riemannian manifolds

II.1 The heat kernel and Green kernel

Here we assume (M, g) is a non-compact, complete Riemannian manifold.

(2.1) One can define the Green kernel $G(x, y)$ of (M, g) by

$$G(x, y) = \int_0^\infty p(t, x, y) dt$$

when this integral is convergent. In this case, (M, g) is called to be *hyperbolic*, otherwise *parabolic*. It is known (cf. [Ao 1]) that a non-compact complete Riemannian manifold (M, g) is hyperbolic if (1) $\dim M \geq 3$ and it has the non-positive sectional curvature, or (2) $\dim M = 2$ and it has negative curvature (see also [Ks 1], [L.Ta 1 ~ 4]).

(2.2) Kanai showed a complete Riemannian manifold with some bounded condition which is *rough isometric* to a hyperbolic manifold is also hyperbolic (cf. [Kn 1 ~ 3]). Cheeger and Yau, and Kasue replaced (cf. [Cr.Y], [Ks 1]) the sectional curvature condition into the Ricci curvature one in the estimations of the Green kernel due to Debiard, Gaveau and Mazet (cf. [D.G.M]). The original estimate are as follows: (1) if the sectional curvature K satisfies $K \leq a \leq 0$, then

$$p(t, x, y) \leq p^a(t, \tilde{x}, \tilde{y}), \quad G(x, y) \leq G^a(\tilde{x}, \tilde{y}),$$

for all $x, y \in M$, $\tilde{x}, \tilde{y} \in M^a$ with $d(x, y) = d(\tilde{x}, \tilde{y})$, where p^a and G^a are the heat and Green kernels of a simply connected manifold M^a of the constant curvature a .

(2) Moreover, if $a \leq K \leq 0$, then

$$p^a(t, \tilde{x}, \tilde{y}) \leq p(t, x, y), \quad G^a(\tilde{x}, \tilde{y}) \leq G(x, y),$$

for all $x, y \in M$, $\tilde{x}, \tilde{y} \in M^a$ with $d(x, y) = d(\tilde{x}, \tilde{y})$.

Uniqueness of solution of the heat equation holds under some curvature condition of (M, g) (cf. [Do 1], [Dn 7], [L 1]).

(2.3) **Problem 2.** *Study the heat kernel, hyperbolicity and uniqueness of the heat equation without the curvature conditions. Test cases are the (generalized) Heisenberg groups and solvable Lie groups with left invariant metrics.*

(See also books [V.S.C], [B.T.V].)

II.2. The spectrum

(2.4) For a study of the spectrum of the Laplacian of a complete Riemannian manifold, the first step is:

Problem 3. (1) *Determine the spectrum $\text{Spec}(M, g)$ of the Laplacian on a complete Riemannian manifold (M, g) . Generalize works on the spectra of Li and Kumura (cf. [L.J 2], [Ku 2]) in which they showed the essential spectrum coincides with $[0, \infty)$ under non-negativity of the Ricci curvature of (M, g) .*

(2) *Even many works (cf. [Br 1 ~ 4], [Du 1 ~ 6], [Su 5], [K.O.S], [N.U]), there remain many unsolved problems on the bottoms of the spectrum or essential spectrum of the Laplacian on a complete Riemannian manifold (M, g) , which are defined by*

$$\begin{aligned} \lambda_0(M, g) &:= \inf \text{Spec}(M, g) \\ \lambda_0^{\text{ess}}(M, g) &:= \inf \text{Ess-Spec}(M, g) \\ &= \sup \{ \lambda_0(M \setminus K); K \subset M, \text{compact} \}, \end{aligned}$$

where $\lambda_0(M \setminus K)$ is the infimum of the spectrum for $(M \setminus K, g)$.

(3) (Gromov's zero-in-the-spectrum question) *Is zero always in the spectrum of the Laplacian acting on some p -forms? (See a survey in [Lt].)*

(2.5) More precisely,

Problem 4.

(i) *Can scalar curvature controll $\lambda_0(M, g)$ or $\lambda_0^{\text{ess}}(M, g)$?*

(ii) *Does there exist a universal constant $C > 0$ such that the following inequalities hold: if (M, g) and (N, h) are rough isometric each other,*

$$\frac{1}{C} \lambda_0(M, g) \leq \lambda_0(N, h) \leq C \lambda_0(M, g)?$$

(iii) *For a solvable Lie group (G, g) with a left invariant Riemannian metric g , prove that*

$$\lambda_0(G, g) = 0 \quad \text{if and only if } G \text{ is nilpotent.}$$

Determine $\lambda_0(G, g)$ for a semisimple Lie group G with a left invariant Riemannian metric g (cf. [K.O.S]).

II.3. Harmonic functions

(2.6) Harmonic functions on a non-compact complete Riemannian manifold are particularly interesting. If (M, g) is hyperbolic, any positive harmonic functions f can be expressed by an integral with some Borel measure μ on the Martin kernel $M(\infty)$ as follows:

$$f(x) = \int_{Y \in M(\infty)} K(x, Y) d\mu(Y),$$

where $K(x, Y)$, $x \in M$, $Y \in M(\infty)$ is the Martin kernel.

(2.7) **Problem 5.** Kasue asked the following: if (M, g) is a simple connected Riemannian manifold whose sectional curvature K satisfies

$$K \leq -a < 0,$$

for some positive constant a , show that $\#(M(\infty)) \geq 2$, i.e., there exists a non-constant positive harmonic function on M . Greene and Wu (cf. [G.W]) proposed if there exists a positive constant C such that the sectional curvature K_x at $x \in M$ satisfies

$$K_x \leq -Cr(x)^{-2},$$

for all x in the outside of a compact subset in M , where $r(x)$ is the distance from some fixed point, then (M, g) has a nonconstant bounded harmonic function.

(2.8) Hsu and March (cf. [H.M]) proved Green and Wu's conjecture in the case that $C > 2$ and

$$-a \leq K_x \leq -Cr(x)^{-2},$$

for some positive constant a .

(2.9) Recall here Yau's theorem (cf. [Y 1, 2]) that every positive harmonic function on a complete Riemannian manifold with non-negative Ricci curvature is a constant.

(2.10) On the other hand, Anderson-Schoen's theorem (cf. [A], [SI], [A.S]) says that if M is an n -dimensional complete simply connected manifold of sectional curvature K satisfying

$$-b \leq K \leq -a < 0$$

for some positive constants a and b , then the Martin boundary of M is homeomorphic with the unit sphere S^{n-1} .

(2.11) Recently, Colding and Minicozzi (cf. [C.M 3]) settled Yau's conjecture which is the following: for a noncompact Riemannian manifold with nonnegative Ricci curvature, the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional.

III. Discrete spectra

III.1. Setting of the problems

(3.1) Now we assume (M, g) is complete, and let Ω is a bounded domain in M . Let us consider the following Dirichlet and Neumann eigenvalue problems:

$$\begin{cases} \Delta f + \lambda f = 0 & (\text{on } \Omega), \\ f = 0 & (\text{on } \partial\Omega), \\ \Delta f + \mu f = 0 & (\text{on } \Omega), \\ \frac{\partial f}{\partial \mathbf{n}} = 0 & (\text{a.e. } \partial\Omega), \end{cases}$$

and also the eigenvalue problem

$$\Delta f + \nu f = 0 \quad (\text{on } M),$$

when M is compact. Let us denote their spectra, respectively, by

$$\text{Spec}_D(\Omega) = \{\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots\},$$

$$\text{Spec}_N(\Omega) = \{0 = \mu_0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_i \leq \cdots\},$$

and

$$\text{Spec}(M) = \{0 = \nu_0 < \nu_1 \leq \nu_2 \leq \cdots \leq \nu_i \leq \cdots\}.$$

III.2. Boundary value problems

(3.2) The celebrated Faber-Krahn inequality in the standard Euclidean space $\mathbf{R}^n$ asserts that the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet eigenvalue problem for a bounded domain Ω always satisfies

$$\lambda_1(\Omega) \geq \lambda_1(\mathbf{B}),$$

where $\mathbf{B}$ is a ball in $\mathbf{R}^n$ with volume $\text{Vol}(\mathbf{B}) = \text{Vol}(\Omega)$. Equality holds if and only if Ω is isometric to $\mathbf{B}$.

(3.3) On the other hand, for the Neumann eigenvalue problem, Szegő (cf. [Sz]) showed that for any simply connected bounded plane domain $\Omega \subset \mathbf{R}^2$,

$$\mu_1(\Omega) \leq \mu_1(\mathbf{B}),$$

where $\mathbf{B}$ is the disk with $\text{Vol}(\mathbf{B}) = \text{Vol}(\Omega)$.

(3.4) Ashbaugh and Benguria (cf. [A.B 1]) settled the following Payne-Pólya-Weinberger conjecture: For a bounded domain $\Omega \subset \mathbf{R}^2$, ratio of the first and second eigenvalues of the Dirichlet problem satisfies

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\mathbf{B})}{\lambda_1(\mathbf{B})} \approx 2.539,$$

and equality holds if and only if Ω is a disk $\mathbf{B}$.

(3.5) Both Pólya's and Weyl's conjectures are still open (see [L.Y 1], [U 4], [Iv], [Me 1], [Bd 1]):

Problem 6. Prove (1) and (2) for any bounded domain Ω in $\mathbf{R}^n$:

(1) (Pólya) The k -th eigenvalues of the Dirichlet and Neumann problems satisfies that

$$\lambda_k(\Omega) \leq C_n \text{Vol}(\Omega)^{-2/n} k^{2/k} \leq \mu_k(\Omega), \quad \forall k = 1, 2, \dots,$$

where Weyl's theorem says

$$\lambda_k(\Omega) \sim \mu_k(\Omega) \sim C_n \text{Vol}(\Omega)^{-2/n} k^{2/k}, \quad \text{as } k \rightarrow \infty.$$

(2) (Weyl) The counting numbers $N_D(\lambda)$, $N_N(\lambda)$ of the Dirichlet, Neumann problems satisfy, as $\lambda \rightarrow \infty$,

$$N_D(\lambda) = C_n \text{Vol}(\Omega) \lambda^{n/2} - C_{n-1} \text{Vol}_{n-1}(\partial\Omega) \lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}),$$

and

$$N_N(\lambda) = C_n \text{Vol}(\Omega) \lambda^{n/2} + C_{n-1} \text{Vol}_{n-1}(\partial\Omega) \lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}),$$

respectively.

III.3. The first eigenvalue of a closed manifold

(3.6) For a compact Riemannian manifold (M, g) , Hersh's theorem says (cf. [He]) that for every Riemannian metric g on 2-sphere S^2 , the first eigenvalue, $v_1(S^2, g)$ satisfies

$$v_1(S^2, g) \text{Vol}(S^2, g) \leq v(S^2, g_0) \text{Vol}(S^2, g_0) = 8\pi,$$

where g_0 is the standard Riemannian metric on S^2 with curvature 1. Equality holds if and only if (S^2, g) is isometric to (S^2, g_0) . Both Szegő and Hersh's proofs above used in the essential way the uniformization or Riemann mapping theorem.

(3.7) In general, Yang and Yau (cf. [Y.Y]) showed that for an orientable compact Riemann surface M of genus γ and any Riemannian metric g on M ,

$$v_1(M, g) \text{Vol}(M, g) \leq 8\pi(\gamma + 1).$$

(3.8) Furthermore, if $\dim M = 2$, the multiplicity of $v_1(M, g)$, denoted by $m_1(M, g)$, satisfies (cf. [Ch 1, 3], see also [Bs 1, 2])

$$m_1(M, g) \leq \frac{1}{2} (2\gamma + 2)(2\gamma + 3),$$

where γ is the genus of M . Note that if M is 2-sphere S^2 , this inequality is

$$m_1(S^2, g) \leq 3 = m_1(S^2, g_0).$$

(3.9) To generalize the above to higher dimensional compact Riemannian manifold (M, g) , Berger (cf. [Br 2]) proposed to prove that there exists a positive constant $C(M)$ depending only on M such that

$$v_1(M, g) \text{Vol}(M, g) \leq C(M).$$

And it is also reasonable to believe that the more symmetries a Riemannian metric has, the more multiple its

eigenvalues are. The standard metric g_0 on S^n was expected to be the metric with the largest multiplicity of v_1 .

(3.10) In 1978, $v_1(g)$ of a left invariant Riemannian metric g on a compact Lie group G has been studied by the author. It turns out that *all the above guesses are wrong!*.

Indeed, the only compact Lie group which has a positive answer of Berger's problem is abelian, i.e., a torus (cf. [U 1]). That is, if G is not abelian, there exists a one-parameter family of left invariant Riemannian metrics on G , denoted by g_t ($0 < t < \infty$), such that

$$\lim_{t \rightarrow \infty} v_1(G, g_t) = \infty, \quad \text{and} \quad \lim_{t \rightarrow \infty} v_1(G, g_t) = 0,$$

but $\text{Vol}(G, g_t)$ is constant in t . Therefore, if $G = SU(2) = S^3$, this implies Berger's problem is negative. Moreover, there exists $t_0 > 0$ such that

$$m_1(S^3, g_{t_0}) = 7.$$

Tanno [Tn 2] and Muto [Mu 1] extended these results to S^n ($n \geq 3$). In [BB.B], [M.U], it was also extended to Riemannian submersions and homogeneous spaces.

(3.11) In 1986, Colin de Verdiere [Cl 1] showed that if M is a compact manifold of dimension $n \geq 3$, then for any positive integer N , there exists a Riemannian metric g on M such that

$$m_1(M, g) = N.$$

(3.12) Moreover, he also showed [Cl 2] that if $\dim M \geq 3$, every given finite sequence

$$0 < v_1 \leq v_2 \leq \dots \leq v_k$$

is the sequence of the first $k + 1$ eigenvalues of the Laplacian for some Riemannian metric g on M .

(3.13) Furthermore, Dodziuk [Do 3] showed that if M is a compact manifold of dimension $n \geq 3$, then for any positive real number A , there exists a Riemannian metric g on M such that

$$\text{Vol}(M, g) = 1, \quad v_1(M, g) \geq A.$$

(3.14) **Problem 7.** (1) (cf. [Do 3]) *In the theorem in [Cl 2], can the additional requirement that $\text{Vol}(M, g) \geq 1$ be satisfied?*

(2) *Can one generalize Szegő's result to arbitrary plane domain or higher dimensional domain?*

(3) *Can one expect a similar to Colin de Verdiere's result in case of the Dirichlet problem for a bounded domain in $\mathbf{R}^n$.*

(4) (Yau) *Prove that the first eigenvalue of embedded minimal hypersurfaces (M, g) of the standard unit sphere (S^{n+1}, g_0) satisfies*

$$v_1(M, g) = n.$$

For (4), partial answers are known (see [Ci.W], [M.O.U], [Kt], [Mu 2]).

III.4. Isospectral problems

(3.15) In 1965, Kac (cf. [Kc]) asked an inverse spectral problem whether the spectra determine the shape of a bounded domain or a closed manifold. Namely, for two closed Riemannian manifolds (M, g) and (N, h) ,

“does $\text{Spec}(M, g) = \text{Spec}(N, h)$ imply that (M, g) and (N, h) are isometric?”

For the Dirichlet or Neumann problem of bounded domains, for two bounded domains Ω and Ω' ,

“do $\text{Spec}_D(\Omega) = \text{Spec}_D(\Omega')$ or, $\text{Spec}_N(\Omega) = \text{Spec}_N(\Omega')$ imply that Ω and Ω' are isometric?”

(3.16) More detailed references are in [Bd 3, 4] [Bd.B]. We only give here some comments. For a closed manifold, the asymptotic expansion of the trace (cf. [M.P], [M.S], [Br 1], [Sa], [B.G])

$$\begin{aligned} Z_M(t) &= \sum_{i=0}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k t^k, \\ Z_D(t) &= \sum_{i=0}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} b_{k/2} t^{k/2}, \\ Z_N(t) &= \sum_{i=0}^{\infty} e^{-t\mu_i} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} c_{k/2} t^{k/2}, \end{aligned}$$

where all coefficients $a_k, b_{k/2}, c_{k/2}$ ($k = 0, 1, \dots$) have the universal expressions determined by the curvatures and their covariant derivatives of (M, g) and (Ω, g) , and n is dimension of M or Ω . The first order terms are

$$a_0 = \text{Vol}(M, g), \quad b_0 = c_0 = \text{Vol}(\Omega, g),$$

respectively. Using these data and the determinant of the Laplacian, $\text{Det}(\Delta)$, the following compactness theorem is obtained (cf. [O.P.S 1 ~ 4], [Cg.Y], [B.P.y], [C.X], [Bu], [Me 2]. (See also [M.M 1, 2], [G.M].)

(3.17) **Compactness Theorem.** (i) *For a given orientable closed Riemann surface (M, g) , the set of Riemannian metrics g_t such that $\text{Vol}(M, g_t) = 1$ and $v_1(M, g_t) \geq \epsilon$ is compact.*

nian metrics on M isospectral to (M, g) is compact with respect to C^∞ topology on Riemannian metrics.

(ii) For a given compact Riemannian manifold (M, g) with $\dim M \geq 3$, the set of C^∞ functions $\varphi > 0$ of which e^g is isospectral is compact in $C^\infty(M)$.

(iii) Let Ω be a fixed bounded domain in $\mathbf{R}^2$. The set of plane domains isospectral of the Dirichlet problem is compact in C^∞ topology.

(3.18) For the inverse isospectral problem, the following results are known:

(1) Milnor (cf. [Mil]), Kneser (cf. [Kn]), Kitaoka (cf. [Ki]), and Schiemann (cf. [Sc], also [Cy.S]) found, respectively, 16-dim, 12-dim, 8-dim, 4-dim flat isospectral, but not isometric tori.

(2) Ikeda (cf. [Ik 1 ~ 4]) found isospectral but not isometric examples of lens spaces S^{2n+1}/Γ ($n \geq 2$).

(3) For Riemann surfaces with curvature -1 of genus $\gamma \geq 2$, Vignéras (cf. [Vg]) found examples of higher genus, and furthermore, Sunada (cf. [Su 2]) found a very useful and strong method to construct examples of isospectral but not isometric and constructed also many examples of isospectral Riemann surfaces of curvature -1 . Using Sunada's technique, Riemann surfaces of curvature -1 and all genus $\gamma \geq 4$ which are isospectral but not isometric are found (cf. [B.T]).

(4) (cf. [G.Wi 1], [De.G 2, 3], [D.G.G.W 3], [Pe 1, 2]) There exist nilpotent Lie groups N with compact quotients $\Gamma \backslash N$ on which there are determinations of isospectral, but not isometric Riemannian metrics g_i .

(5) In 1982, in [U 2], if $n \geq 4$, examples of bounded domains in $\mathbf{R}^n$, which are isospectral for both the Dirichlet and Neumann eigenvalue problems, but not isometric was found.

(6) Using Sunada's idea, in [G.W.W 1, 2], Gordon-Webb-Wolpert found isospectral for both the Dirichlet and Neumann problem, but not isometric domains in $\mathbf{R}^2$.

(3.19) **Problem 8.** (cf. [Bd 3]) (1) It is still open to find three dimensional isospectral flat tori.

(2) It is open to find examples of closed Riemann isospectral surfaces of curvature -1 and genus 2 or 3.

(3) Is the set of isospectral surfaces finite?

(4) Is the set of isospectral Riemannian manifolds compact?

(5) Find isospectral Riemannian manifolds of which have different Cheeger's isoperimetric constants. Can one find isospectral simply connected compact Riemannian manifolds?

(6) Find isospectral (plane) domains with C^∞ boundary.

(3.20) **Problem 9.** (1) Prove that a compact Riemannian manifold isospectral to the standard unit sphere (S^n, g_0) is isometric to (S^n, g_0) . Partial answers are known (cf. [Br 1], [Tn 1, 3], [Kw 1]).

(2) Prove the similar for the other compact rank one symmetric spaces (cf. [Ky]).

(3) How about spectral rigidity for compact negatively curved n -dimensional Riemannian manifolds with $n \geq 3$ (cf. [Gu.K 1, 2], [MO]).

(3.21) **Problem 10.** (cf. Sunada) For a compact Riemannian manifold (M, g) , let

$$\lambda'_0 = 0 < \lambda'_1 < \lambda'_2 < \cdots < \lambda'_k < \cdots$$

be the set of all distinct eigenvalues of (M, g) and n_k be the multiplicity of λ'_k $k = 0, 1, \dots$. Let us consider the following Poincaré series:

$$F(z) := \sum_{k=0}^{\infty} n_k z^k, \quad z \in \mathbf{C}.$$

It is easy to show that $F(z)$ converges absolutely on $|z| < 1$. Prove that $F(z)$ is a rational function.

IV. Discrete Laplacians in graph theory

IV.1. Setting of the eigenvalue problems for graphs

(4.1) From now on, we survey in the spectral analysis on the discrete Laplacian of a graph. A graph, denoted by $G = (V, E)$, is a collection of a set V of vertices and a set E of edges connecting two vertices. We denote by $x \sim y$ if two vertices x and y are connected by an edge. And we assume G is locally finite, that is, all the numbers of edges emanating from vertices x , denote by $m(x)$ and called the *degree*, are finite. Then the discrete Laplacians are defined by

$$\Delta_P f(x) = f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y),$$

$$\Delta_A f(x) = m(x)f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y), \quad f \in C(V),$$

acting on the space $C(V)$ of all functions on V .

Note that spectral properties of Δ_P and Δ_A are quite different each other in general.

For general references, see [M.W], [W], [Fr 1], [Bi], [U 11].

$$\begin{aligned}(f_1, f_2)_\# &:= \sum_{x \in V} m(x) f_1(x) f_2(x), \\ (f_1, f_2) &:= \sum_{x \in V} f_1(x) f_2(x), \quad f_1, f_2 \in C(V), \\ (\varphi, \psi) &:= \sum_{e \in E} \varphi(e) \psi(e), \quad \varphi, \psi \in C(E),\end{aligned}$$
$$df(e) = f(t(e)) - f(o(e)), \quad e = (o(e), t(e)) \in E,$$

Then Δ_A and Δ_P satisfy the so-called Green's identity:

(4.3) Notice that the following correspondence between Riemannian geometry and graph theory:

- (i) for each $e = (x, y) \in E$ with $x, y \in V$:

Then one can formulate the Dirichlet problem for $\Delta = \Delta_P$ or Δ_A :

and the Neumann problem:

where $du \in C(E)$ is defined above.

For a finite graph without boundary, the eigenvalue problem is:

$$\Delta u = vu \quad (\text{on } V).$$

(4.4) The first problem is that:

Problem 11. *Determine spectra*

$\text{Spec}_D(\Delta), \text{Spec}_N(\Delta), \text{ or } \text{Spec}(\Delta),$

Fig. 1

Fig. 2

for $\Delta = \Delta_P$ or Δ_A for finite or infinite graphs with or without boundary.

IV.2. Infinite graphs

(4.5) For the heat equation analogue to Riemannian manifolds, one can consider the discrete heat equation:

$$u(n+1, x) - u(n, x) + \Delta_x u(n, x) = 0, \quad n = 1, 2, \dots, \quad \text{and } x \in V.$$

The heat kernel $p(n, x, y)$ with a property that

$$u(n, x) = \sum_{y \in V} u(n, x, y) f(y), \quad x \in V, \quad y \in C(V),$$

is a solution of the heat equation satisfying $u(0, x) = f(x)$, $x \in V$, is obtained inductively as follows:

$$p(n+1, x, y) = \sum_{z \in V} p(x, z) p(n, z, y),$$

where

$$p(x, y) = \begin{cases} \frac{1}{m(x)}, & x \sim y, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p(0, x, y) = \begin{cases} 1, & x = y, \\ 0, & \text{otherwise.} \end{cases}$$

(4.6) The Green kernel $G(x, y)$ is defined by

$$G(x) = \sum_{n=0}^{\infty} p(n, x, y),$$

if the series of the right hand side converges. If $G(x, y) < \infty$ for all x and y , the graph is called to be *transient*, otherwise *recurrent*. See [W], [Hi], [Sh 1], [Be.S].

(4.7) **Problem 12.** (1) Determine transience or recurrence for a given infinite graph (cf. [W]).

(2) Determine or estimate the heat kernel and the Green kernel.

(4.8) It is well known (cf. [Ca]) that the Green kernel of the homogeneous regular graph (see Fig. 3), denoted by T_d of degree $d \geq 3$, is given by

$$G(x, y) = \frac{d-1}{d-2} \left(\frac{1}{d-1} \right)^{d(x, y)},$$

where $d(x, y)$ is the distance between x and y . In the following, we denote by G_d , the Green kernel for T_d .

(4.9) In [U 9], optimal estimates of the Green kernel and the heat kernel are given as follows:

(1) If $m(x) \leq d$ ($\forall x \in V$),

$$p(n, x, y) \geq p_d(n, \tilde{x}, \tilde{y}) \quad \text{and} \quad G(x, y) \geq G_d(\tilde{x}, \tilde{y}),$$

for $n = 1, 2, \dots$, and $d(x, y) = d(\tilde{x}, \tilde{y})$, where $p_d(n, \tilde{x}, \tilde{y})$ is the heat kernel of T_d .

(2) If $m(x) \geq d$ ($\forall x \in V$),

$$p(n, x, y) \leq p_d(n, \tilde{x}, \tilde{y}) \quad \text{and} \quad G(x, y) \leq G_d(\tilde{x}, \tilde{y}),$$

for $n = 1, 2, \dots$, and $d(x, y) = d(\tilde{x}, \tilde{y})$.

(4.10) **Problem 13.** (1) Determine explicitly the heat kernel, Green kernel and the Martin boundary for a given infinite graph.

(2) Clarify geometric interpretation of the Martin boundary for a Cayley graph the fundamental group of a complete Riemannian manifold. (See also [V.S.C]).

Fig. 3

Fig. 4

IV.3. Finite graphs

(4.11) For a finite graph, estimations of the first eigenvalue of Δ for finite graphs with or without boundary are fundamental and important.

The graph analogue of the Faber-Krahn inequality for a finite graph with boundary was obtained in [K.U]. For relations between the first eigenvalue and isoperimetric constant, see [Al], [Mo 2 ~ 4]. We only here propose general questions.

Problem 14. (1) *Determine the spectra of the Dirichlet and Neumann problem or the spectra of the eigenvalue problem for a given finite graph with or without boundary.*

(2) *Find isospectral graphs which are not isomorphic. Any example of pair of isospectral graphs which have different isoperimetric constant is much interesting.*

It is known that the complete regular graph K_n is spectrally rigid.

(4.12) Recently, Jin-Song Tan, a student of the author, showed that

(1) if a pair of isospectral graphs G_1 and G_2 without boundary, then a pair of graphs $\tilde{G}_1$ and $\tilde{G}_2$ with boundary which are obtained by adding a boundary vertex to each vertex of G_i (regarding interior vertex) are isospectral of the Dirichlet problem.

(2) Moreover, he classified all isospectral graphs of Δ of $\#(V) \leq 7$, and found a pair of regular graphs which are isospectral, but have different isoperimetric constant. The graphs of this example are degree 4 and 16 vertices.

REFERENCES

- [Al] N. Alon, *Eigenvalues and expanders*, Combinatorica **6** (1986), 83–96.
- [An 1] M. T. Anderson, *The Dirichlet problem at infinity for manifolds of negative curvature*, J. Differ. Geom. **18** (1983), 701–721.
- [An 2] ———, *Remarks on the compactness of isospectral sets in lower dimensions*, Duke Math. J. **63** (1991), 699–711.
- [A.S] ——— and R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. Math. **121** (1985), 429–461.
- [Ao] K. Aomoto, *L'analyse harmonique sur des espaces riemanniennes a courbure riemannienne négative 1*, J. Fac. Sci. Tokyo Sect. 1 **13** (1966), 85–105.
- [A.K] A. Aomoto and Y. Kato, *Green functions and spectra of free products of cyclic groups*, Ann. Inst. Fourier, Grenoble **38** (1988), 59–85.
- [A.B 1] M. S. Ashbaugh and R. D. Benguria, *Proof of the Payne-Pólya-Weinberg conjecture*, Bull. Amer. Math. Soc. **25** (1991), 19–29.
- [A.B 2] ———, *Isoperimetric bound for λ_3/λ_2 for the membrane problem*, Duke Math. J. **63** (1991), 333–341.

- [A.B 3] ———, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. Math. **135** (1992), 601–628.
- [A.B 4] ———, *Universal bounds for the low eigenvalues of Neumann laplacians in N dimensions*, SIAM J. Math. Anal. **24** (1993), 557–570.
- [A.B 5] ———, *On Rayleigh's conjecture for the clamped plate and its generalization to three dimension*, Duke Math. J. **78** (1995), 1–18.
- [Az] R. Azencott, *Behavior of diffusion semigroups at infinity*, Bull. Soc. Math. France **102** (1974), 193–240.
- [B.U] S. Bando and H. Urakawa, *Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds*, Tohoku Math. J. **35** (1983), 155–172.
- [B.C] R. Bañuelos and T. Carroll, *Brownian motion and the fundamental frequency of a drum*, Duke Math. J. **75**, **82** (1994, 1996), 575–602, 227.
- [Be.S] I. Benjamini and O. Schramm, *Harmonic functions on planar and almost planar graphs and manifolds, via circle packings*, Invent. Math. **126** (1996), 565–587.
- [Bd 1] P. H. Berard, *Remarques sur la conjecture de Weyl*, Compositio Math. **48** (1983), 35–53.
- [Bd 2] ———, *From vanishing theorems to estimating theorems: the Bochner technique revisited*, Bull. Amer. Math. Soc. **19** (1988), 371–406.
- [Bd 3] ———, *Variétés riemanniennes isospectrales non isométriques*, Sém. Bourbaki **705** (1988/89).
- [Bd 4] ———, *Spectral Geometry: Direct and Inverse Problems*, Lect. Notes in Math., vol. 1207, Springer, 1986.
- [Bd.B] ——— and M. Berger, *Le spectre d'une variété riemannienne en 1982*, Spectra of Riemannian manifolds, Kaigai Publ., 1983, pp. 139–194.
- [BB.B] L. Berard Bergery and J. P. Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibres*, Illinois J. Math. **26** (1982), 181–200.
- [Br 1] M. Berger, *Le spectre des variétés riemanniennes*, Rev. Roum. Math. Pure Appl. **13** (1968), 915–931.
- [Br 2] ———, *Sur les premières valeurs propres des variétés riemanniennes*, Compositio Math. **26** (1973), 129–149.
- [B.G.M] ———, P. Gauduchon and E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math., vol. 194, Springer, 1971.
- [B.T.V] J. Berndt, F. Tricerri and L. Vanhecke, *Generalized Heisenberg Groups and Demek-Ricci Harmonic Spaces*, Lecture Notes in Math., vol. 1598, Springer, 1995.
- [Bs 1] G. Besson, *Sur la multiplicité de la première valeur propres des surfaces riemanniennes*, Ann. Inst. Fourier, Grenoble **30** (1980), 109–128.
- [Bs 2] ———, *On the multiplicity of the eigenvalues of the Laplacian*, Lecture Notes in Math., Springer, vol. 1339, 1988, pp. 32–53.
- [Bs 3] ———, *Propriétés génériques des fonctions propres et multiplicité*, Comment. Math. Helvetici **64** (1989), 542–588.
- [Bi] F. Bien, *Constructions of telephone networks by group representations*, Notice Amer. Math. Soc. **36** (1989), 5–22.
- [B.L.Y] J. P. Bourguignon, P. Li and S. T. Yau, *Upper bound for the first eigenvalue of algebraic submanifolds*, Comment. Math. Helvetici **69** (1994), 199–207.
- [B.G] T. P. Branson and P. B. Gilkey, *The asymptotics of the Laplacian on a manifold with boundary*, Commun. Partial Diff. Eq. **15** (1990), 245–272.
- [Br 1] R. Brooks, *A relation between growth and the spectrum*, Math. Z. **178** (1981), 501–508.
- [Br 2] ———, *Exponential growth and the spectrum of the Laplacian*, Proc. Amer. Math. Soc. **82** (1981), 473–477.
- [Br 3] ———, *The fundamental group and the spectrum of the Laplacian*, Comment. Math. Helvetici **56** (1981), 581–598.
- [Br 4] ———, *On the spectrum of non-compact manifolds with finite volume*, Math. Z. **187** (1984), 425–432.
- [Br 5] ———, *On manifolds of negative curvature with isospectral potential*, Topology **26** (1987), 63–66.
- [Br 6] ———, *Constructing isospectral manifolds*, Amer. Math. Monthly **95** (1988), 823–839.
- [Br 7] ———, *The spectral geometry of k -regular graphs*, J. D'analyse Math. **57** (1991), 120–151.
- [Br 8] ———, *L^p spectral theory*, Contemp. Math. **173** (1994), 71–87.
- [B.G] ——— and P. Glezen, *An L^p spectral bootstrap theorem*, Contemp. Math. **173** (1994), 89–97.
- [B.P.Y] ———, P. Perry and P. Yang, *Isospectral sets of conformally equivalent metrics*, Duke Math. J. **58** (1989), 131–150.
- [B.G] ——— and C. Gordon, *Isospectral conformally equivalent Riemannian metrics*, Bull. Amer. Math. Soc. **23** (1990), 433–436.
- [B.P.Y] ———, P. Perry and P. Petersen, *Compactness and finiteness theorems for isospectral manifolds*, J. Reine Angew. Math. **426** (1992), 67–89.
- [B.T] ——— and R. Tse, *Isospectral surfaces of small genus*, Nagoya Math. J. **107** (1987), 13–24.
- [Bu] J. Brüning, *On the compactness of isospectral potentials*, Commun. Partial Diff. Eq. **9** (1984), 687–698.
- [Ca] P. Cartier, *Fonctions harmoniques sur un arbre*, Sympos. Math. **9** (1972), 203–270.
- [C.S.W] D. I. Cartwright, P. M. Soardi and W. Woess, *Martin and compactifications for nonlocally finite graphs*, Trans. Amer. Math. Soc. **338** (1993), 679–693.
- [Cg.Y] A. S. Y. Chang and P. Yang, *Compactness of isospectral conformal metrics on S^3* , Comment. Math. Helvetici **64** (1989), 363–374.
- [Cg.De] P. K. Chang and D. M. DeTurck, *On hearing the shape of a triangle* (to appear).
- [Cg.Y 2] ———, *Isospectral conformal metrics on 3-manifolds*, J. Amer. Math. Soc. **3** (1990), 117–145.
- [Cha] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.

- [Cr.Y] J. Cheeger and S. T. Yau, *A lower bound for the heat kernel*, Commun. Pure Appl. Math. **36** (1981), 465–480.
- [Cr.C.M] J. Cheeger, T. H. Colding and W. P. Minicozzi II *Linear growth harmonic functions on complete manifolds with nonnegative Ricci curvature*, Geom. and Funct. Anal. (GAFA) **5** (1995), 948–954.
- [C.X] R. Chen and X. Xu, *Compactness of isospectral conformal metrics and isospectral potentials on a 4-manifold*, Duke Math. J. **84** (1996), 131–154.
- [Ch 1] S. Y. Cheng, *Eigenfunctions and eigenvalues of Laplacian*, Proc. Sympos. Pure Math. **27** (1975), 185–193.
- [Ch 2] ———, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143** (1975), 289–297.
- [Ch 3] ———, *Eigenfuctions and nodal sets*, Comment. Math. Helvetici **51** (1976), 43–55.
- [C.L.Y 1] ———, P. Li and S. T. Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math. **103** (1981), 1021–1063.
- [C.L.Y 2] ———, *Heat equations on minimal submanifolds and their applications*, Amer. J. Math. **106** (1984), 1033–1065.
- [C.Z] Xu Cheng and D. Zhou, *First eigenvalue estimate on riemannian manifolds*, Hokkaido Math. J. **24** (1995), 564–472.
- [Ci.W] H. I. Choi and A. N. Wang, *A first eigenvalue estimate for minimal hypersurfaces*, J. Differ. Geom. **18** (1983), 559–562.
- [C.M 1] T. H. Colding and W. P. Minicozzi II, *On function theory on spaces with a lower Ricci curvature bound*, Math. research Letter **3** (1996), 241–246.
- [C.M 2] ———, *Harmonic functions with polynomial growth* (to appear).
- [C.M 3] ———, *Harmonic functions on manifolds* (to appear).
- [Cl 1] Y. Colin de Verdière, *Sur la multiplicité de la première valeur propre non nulle du Laplacien*, Comment. Math. Helvetici **61** (1986), 254–270.
- [Cl 2] ———, *Construction de Laplaciens dont une partie finie du spectre est donnée*, Ann. Scient. École Norm. Sup **20** (1987), 599–615.
- [Cy.S] J. H. Conway and N. J. A. Sloane, *Four-dimensional lattices with the same theta series*, Intern. Math. Res. Notices **4** (1992), 93–96.
- [Cou 1] T. Coulhon, *Noyau de la chaleur et discretisation d’une variété riemannienne*, Israel J. Math. **80** (1992), 289–300.
- [Cou 2] ———, *Sobolev inequalities on graphs and on manifolds*, Harmonic Analysis and Discrete Potential Theory (M. A. Picardello, ed.), Plenum Press, New York, 1992, pp. 207–214.
- [Cou 3] ———, *Iteration de Moser et estimation gaussienne du noyau de la chaleur*, J. Operator Theory **29** (1993), 157–165.
- [C.L] ——— and M. Ledoux, *Isopérimétrie, décroissance du noyau de la chaleur et transformations de Riesz: un contre-exemple*, Ark. Mat. **32** (1994), 63–77.
- [C.S] ——— and L. Saloff-Coste, *Isopérimétrie pour les groupes et les variétés*, Revista Math. Iberoamericana **9** (1993), 293–314.
- [D.W] X. Dai and G. Wei, *Finite part of spectrum and isospectrality*, Contemp. Math. **173** (1994), 99–107.
- [D.G.M] A. Debiard, B. Gaveau and E. Mazet, *Théorèmes de comparaison en géométrie riemannienne*, Publ. RIMS, Kyoto Univ. **12** (1976), 391–425.
- [De] D. M. DeTurck, *Audible and inaudible geometric properties* (to appear).
- [De.G 1] ——— and C. S. Gordon, *Isospectral metrics and finite Riemannian coverings*, Contemp. Math. **64** (1987), 79–92.
- [De.G 2] ———, *Isospectral deformations I: Riemannian structures on two-step nilspaces*, Commun. Pure Appl. Math. **40** (1987), 367–387.
- [De.G 3] ———, *Isospectral deformations II: Trace formulas, metrics and potentials*, Commun. Pure Appl. Math. **42** (1989), 1067–1095.
- [De.G 4] ———, *Isospectral Riemannian metrics and potentials*, Bull. Amer. Math. Soc. **17** (1987), 137–139.
- [D.G.G.W 1] D. DeTurck, H. Gluck, C. Gordon and D. Webb, *You can not hear the mass of a homology class*, Comment. Math. Helvetici **64** (1989), 589–617.
- [D.G.G.W 2] ———, *How can a drum change shape, while sounding the same?*, Differential Geometry, Pitman, 1991, pp. 111–122.
- [D.G.G.W 3] ———, *Conformal isospectral deformations*, Indiana Univ. Math. J. **41** (1992), 99–107.
- [D.G.G.W 4] ———, *The inaudible geometry of nilmanifolds*, Invent. Math. **111** (1993), 271–284.
- [D.G.G.W 5] ———, *How can a drum change shape, while sounding the same? Part II* (to appear).
- [Do 1] J. Dodziuk, *Maximum principle for parabolic inequalities and the heat flow on open manifolds*, Indiana Univ. Math. J. **32** (1983), 703–716.
- [Do 2] ———, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc. **284** (1984), 787–794.
- [Do 3] ———, *Nonexistence of universal upper bounds for the first positive eigenvalue of the Laplace-Beltrami operator*, Contemp. Math. **173** (1994), 109–114.
- [D.Kn] ——— and W. S. Kendall, *Combinatorial Laplacians and isoperimetric inequality*, From Large Times to Global Geometry, Control and Physics, Pitman Research Notes in Math. Series, vol. 150, 1986, pp. 68–74.
- [D.Kr] ——— and L. Karp, *Spectral and function theory for combinatorial Laplacian*, Contemp. Math. **73** (1988), 25–40.
- [Du 1] H. Donnelly, *Eigenvalues embedded in the continuum for negatively curved manifolds*, Michigan Math. J. **28** (1979), 53–62.
- [Du 2] ———, *On the essential spectrum of a complete Riemannian manifold*, Topology **20** (1981), 1–14.

- [Du 3] ———, *The differential form spectrum of hyperbolic space*, Manuscripta Math. **33** (1981), 365–385.
- [Du 4] ———, *Eigenvalue estimates for certain noncompact manifolds*, Michigan Math. J. **31** (1984), 349–357.
- [Du 5] ———, *Positive solutions of the heat and eigenvalue equations on Riemannian manifolds*, Lecture Notes in Math. **1209** (1986), 143–151, Springer.
- [Du 6] ———, *Essential spectrum and heat kernel*, J. Funct. Anal. **75** (1987), 362–381.
- [Du 7] ———, *Uniqueness of positive solutions of the heat equation*, Proc. Amer. Math. Soc. **99** (1987), 353–356.
- [D.L 1] ——— and P. Li, *Pure point spectrum and negative curvature for noncompact manifolds*, Duke Math. J. **46** (1979), 497–503.
- [D.L 2] ———, *Heat equation and compactifications of complete Riemannian manifolds*, Duke Math. J. **51** (1984), 667–673.
- [D.G] ——— and N. Garofalo, *Riemannian manifolds whose Laplacians have purely continuous spectrum*, Math. Ann. **293** (1992), 143–161.
- [E] N. Ejiri, *A construction of non-flat compact irreducible Riemannian manifolds which are isospectral but not isometric*, Math. Z. **168** (1979), 207–212.
- [E.F] J. F. Escobar and A. Freire, *The spectrum of the Laplacian of manifolds of positive curvature*, Duke Math. J. **65** (1992), 1–21.
- [Fj 1] K. Fujiwara, *On the bottom of the spectrum of the Laplacian on graphs*, Geometry and its Applications (T. Nagano et al., ed.), World Sci., 1993, pp. 21–27.
- [Fj 2] ———, *Convergence of the eigenvalues of Laplacians in a class of finite graphs*, Contemp. Math. **173** (1994), 115–120.
- [Fj 3] ———, *The Laplacian on rapidly branching trees*, Duke Math. J. **83** (1996), 191–202.
- [Fj 4] ———, *Growth and the spectrum of the Laplacian of an infinite graph*, Tohoku Math. J. **48** (1996), 293–302.
- [Fr 1] J. Friedman, *Some geometric aspects of graphs and their eigenfunctions*, Duke Math. J. **69** (1993), 487–525.
- [Fr 2] ———, *Some graphs with small second eigenvalue*, Combinatorica **15** (1995), 31–42.
- [Fr 3] ———, *Minimum higher eigenvalues of Laplacians on graphs*, Duke Math. J. **83** (1996), 1–18.
- [Ga 1] M. P. Gaffney, *A special Stokes' theorem for complete Riemannian manifolds*, Ann. Math. **60** (1954), 140–145.
- [Ga 2] ———, *The heat equation method of Milgram and Rosenbloom for open Riemannian manifolds*, Ann. Math. **60** (1954), 458–466.
- [Ga 3] ———, *Hilbert space methods in the theory of harmonic integrals*, Trans. Amer. Math. Soc. **78** (1955), 426–444.
- [Gi] P. Gilkey, *Invariant Theory, the Heat Equations and Atiyah-Singer Index Theorem*, Publ. Perish, Inc., 1984.
- [Go 1] C. S. Gordon, *Riemannian manifolds isospectral on functions but not on 1-forms*, J. Differ. Geom. **24** (1986), 79–96.
- [Go 2] ———, *Isospectral closed Riemannian manifolds which are not locally isometric*, J. Differ. Geom. **37** (1993), 639–649.
- [Go 3] ———, *Isospectral closed Riemannian manifolds which are not locally isometric, Part II*, Contemp. Math. **173** (1994), 121–131.
- [G.We] ——— and D. L. Webb, *Isospectral convex domains in Euclidean space*, MSRI preprint series 1994 (to appear).
- [G.W.W 1] ———, D. Webb and S. Wolpert, *You cannot hear the shape of a drum*, Bull. Amer. Math. Soc. **27** (1992), 134–138.
- [G.W.W 2] ———, D. Webb and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. **110** (1992), 1–22.
- [G.Wi 1] ——— and E. N. Wilson, *Isospectral deformations of compact solvmanifolds*, J. Differ. Geom. **19** (1984), 241–256.
- [G.W 2] ——— and E. N. Wilson, *The spectrum of the Laplacian on Riemannian Heisenberg manifolds*, Michigan Math. J. **33** (1986), 253–271.
- [G.W] R. E. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Math., Springer **699** (1979).
- [Gr 1] A. A. Grigor'yan, *The heat equation on noncompact Riemannian manifolds*, Math. USSR Sobornik **72** (1992), 47–77.
- [Gr 2] ———, *Heat kernel on a manifold with a local Harnack inequality*, Commun. Anly. Geom. **2** (1994), 111–138.
- [Gr 3] ———, *Heat kernel upper bounds on a complete non-compact Riemannian manifold*, Revista Math. Iberoamericana **10** (1994), 395–452.
- [Gr 4] ———, *Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold*, J. Func. Anal. **127** (1995), 363–389.
- [Gr 5] ———, *Heat kernel of a noncompact Riemannian manifold*, Proc. Symp. Pure Math. **57** (1995), 239–263.
- [Gu.K 1] V. Guillemin and D. Kazhdan, *Some inverse spectral results for negatively curved 2-manifolds*, Topology **19** (1980), 153–180.
- [Gu.K 2] ———, *Some inverse spectral results for negatively curved n -manifolds*, Proc. Sympos. Pure Math. **36** (1980), 153–180.
- [Gu.M] V. Guillemin and R. Melrose, *An inverse spectral result for elliptic regions in $\mathbf{R}^2$* , Advances in Math. **32** (1979), 128–148.
- [He] J. Hersch, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C. R. Acad. Sci. Paris **270** (1970), 1645–1648.
- [Hi] Y. Higuchi, *Random walks and isoperimetric inequalities on infinite planar graphs and their duals*, Doctoral Thesis in Tokyo Univ., Math. Sci.

- [H.M] P. Hsu and P. March, *The limiting angle of certain riemannian Brownian motions*, Commun. Pure Appl. Math. **38** (1985), 755–768.
- [Ik 1] A. Ikeda, *On the spectrum of a Riemannian manifold of positive constant curvature, II*, Osaka J. Math. **17** (1980), 75–93, 691–762.
- [Ik 2] ———, *On lens spaces which are isospectral but not isometric*, Ann. Scient. École Norm. Sup. **13** (1980), 303–315.
- [Ik 3] ———, *On spherical space forms which are isospectral but not isometric*, J. Math. Soc. Japan **17** (1983), 437–444.
- [Ik 4] ———, *Riemannian manifolds p -isospectral but not $(p + 1)$ -isospectral*, Geometry of Manifolds (K. Shiohama, ed.), Acad. Press, 1989, pp. 383–417.
- [Ik 5] ———, *On space forms of real Grassmann manifolds which are isospectral but not isometric* (to appear).
- [I.T] ——— and Y. Taniguchi, *Spectra and eigenforms of the Laplacian on S^n and $F^n(C)$* , Osaka J. Math. **15** (1978), 515–546.
- [Ik] S. Ito, *On the existence of Green function and positive superharmonic functions for linear elliptic operators of second order*, J. Math. Soc. Japan **16** (1964), 299–306.
- [Iv] V. J. Ivrii, *Second term of the spectral asymptotic expansion of the Laplace-Beltrami operator on manifolds with boundary*, Funct. Anal. Appl. **14** (1980), 98–106.
- [Kc] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly **73** (1966), 1–23.
- [Kn 1] M. Kanai, *Rough isometries, and combinatorial applications of geometries of non-compact Riemannian manifolds*, J. Math. Soc. Japan **37** (1985), 391–413.
- [Kn 2] ———, *Rough isometries and the parabolicity of Riemannian manifolds*, J. Math. Soc. Japan **38** (1986), 227–238.
- [Kn 3] ———, *Analytic inequalities, and rough isometries between non-compact Riemannian manifolds*, Lecture Notes in Math., vol. 1201, Springer, 1986, pp. 122–137.
- [Ks 1] A. Kasue, *A Laplacian comparison theorem and function theoretic properties of a complete Riemannian manifold*, Japan J. Math. **8** (1982), 309–341.
- [Ks 2] ———, *Applications of Laplacian and Hessian comparison theorems*, Advanced Studies Pure Math. **3** (1984), 333–386.
- [Ks 3] ———, *Harmonic functions of polynomial growth on complete manifolds*, Proc. Symp. Pure Math. **54** (1993), 281–290.
- [Ks 4] ———, *Measured Hausdorff convergence of Riemannian manifolds and Laplace operator*, Osaka J. Math. **30** (1993), 613–651.
- [Ks 5] ———, *Harmonic functions of polynomial growth on complete manifolds II*, J. Math. Soc. Japan **47** (1995), 37–65.
- [K Wa] ——— and T. Washio, *Growth of equivariant harmonic maps and harmonic morphisms*, Osaka J. Math. **27** (1990), 899–928.
- [K.K] ——— and H. Kumura, *Spectral convergence of Riemannian manifolds, II*, Tohoku Math. J. **46**, **48** (1994, 1996), 147–179, 71–120.
- [K.K.O] ———, H. Kumura and Y. Ogura, *Convergence of heat kernels on a compact manifold*, Kyushu J. Math. **51** (1997), 453–524.
- [K.U] A. Katsuda and H. Urakawa, *The Faber-Krahn type isoperimetric inequalities for a graph* (to appear).
- [Ki] Y. Kitaoka, *Positive definite quadratic forms with the same representation numbers*, Arch. Math. **28** (1977), 495–497.
- [Ky] K. Kiyohara, *Riemannian manifolds with periodic geodesic flows on projective spaces*, Japan J. Math. **13** (1987), 209–234.
- [Kn] M. Kneser, *Lineare Relationen zwischen Darstellungszahlen quadratischer Formen*, Math. Ann. **168** (1967), 31–39.
- [K.O.S] T. Kobayashi, K. Ono and T. Sunada, *Periodic Schrödinger operators on a manifold*, Forum Math. **1** (1989), 69–79.
- [Kt] M. Kotani, *The first eigenvalue of homogeneous minimal hypersurfaces in a unit sphere S^{n+1}* , Tohoku Math. J. **37** (1985), 523–532.
- [Ku 1] H. Kumura, *The Dirichlet problem at infinity on Hadamard manifolds*, Nagoya Math. J. **138** (1995), 1–18.
- [Ku 2] ———, *On the essential spectrum of the Laplacian on complete manifolds*, J. Math. Soc. Japan **49** (1997), 1–14.
- [Kur] Y. Kurylev, *Inverse boundary problems on Riemannian manifolds*, Contemp. Math. **173** (1994), 181–192.
- [Kw 1] R. Kuwabara, *On isospectral deformations of Riemannian metrics, II*, Compositio Math. **40**, **47** (1980, 1982), 319–324, 195–205.
- [Kw 2] ———, *On the characterization of flat metrics by the spectrum*, Comment. Math. Helvetici **55** (1980), 427–444.
- [Kw 3] ———, *Spectra and geodesic flows on nilmanifolds: Reductions of hamiltonian systems and differential operators*, J. Math. Soc. Japan **46** (1994), 119–137.
- [L.J 1] J. Y. Li, *On the Harnack inequality for harmonic functions on complete Riemannian manifolds*, Chin. Ann. Math. **14B** (1993), 1–12.
- [L.J 2] ———, *Spectrum of the Laplacian on a complete Riemannian manifold with nonnegative Ricci curvature which possess a pole*, J. Math. Soc. Japan **46** (1994), 213–216.
- [L 1] P. Li, *Uniqueness of L^1 solutions for the Laplace equation and the heat equation on Riemannian manifolds*, J. Differ. Geom. **20** (1984), 447–457.
- [L 2] P. Li, *Large time behavior of the heat equation on complete manifolds with non-negative Ricci curvature*, Ann. Math. **124** (1986), 1–21.
- [L 3] ———, *Harmonic functions on complete manifolds*, IXth Intern. Congr. on Math. Phys., IOP Publishing, 1989,

pp. 365–368.

- [L.Ta 1] P. Li and L. F. Tam, *Symmetric Green's functions on complete manifolds*, Amer. J. Math. **109** (1987), 1129–1154.
- [L.Ta 2] ———, *Positive harmonic functions on complete manifolds with non-negative curvature outside a compact set*, Ann. Math. **125** (1987), 171–207.
- [L.Ta 3] ———, *Harmonic functions and the structure of complete manifolds*, J. Differ. Geom. **35** (1992), 359–383.
- [L.Ta 4] ———, *Green's functions, harmonic functions, and volume comparison*, J. Differ. Geom. **41** (1995), 277–318.
- [L.Ti] ——— and G. Tian, *On the heat kernel of the Bergmann metric on algebraic varieties*, J. Amer. Math. Soc. **8** (1995), 857–877.
- [L.Y 1] ——— and S. T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Commun. Math. Phys. **88** (1983), 309–318.
- [L.Y 2] ——— and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.
- [Lt] J. Lott, *The zero-in-the-spectrum question*, L'Enseignement Mathématique **42** (1996), 341–376.
- [M.M 1] S. Marvizi and R. Melrose, *Some spectrally isolated convex planar regions*, Proc. Nat. Acad. Sci. U.S.A. **79** (1982), 7066–7067.
- [M.M 2] ———, *Spectral invariants of convex planar regions*, J. Differ. Geom. **17** (1982), 475–502.
- [Mc 1] H. P. McKean, *An upper bound for the spectrum of Δ on a manifold of negative curvature*, J. Differ. Geom. **4** (1970), 359–366.
- [Mc 2] ———, *Selberg's trace formula as applied to compact Riemann surface*, Commun. Pure Appl. Math. **25** (1972), 225–246.
- [M.S] ——— and I. M. Singer, *Curvature and eigenvalues of the Laplacian*, J. Differ. Geom. **1** (1967), 43–69.
- [Me 1] R. B. Melrose, *Weyl's conjecture for manifolds with concave boundary*, Proc. Sympos. Pure Math. **36** (1980), 257–274.
- [Me 2] ———, *Isospectral sets of drumheads are compact in C^∞ , a preprint 1984*.
- [Mi] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 542.
- [M.P] S. Minakshisundaram and A. Pleijel, *Some properties of the Laplace operator on riemannian manifolds*, Canad. J. Math. **1** (1949), 242–256.
- [MiO] M. Min-Oo, *Spectral rigidity for manifolds with negative curvature operator*, Contemp. Math. **51** (1986), 99–103.
- [Mo 1] B. Mohar, *The spectrum of an infinite graph*, Linear Alg. Appl. **48** (1982), 245–256.
- [Mo 2] ———, *Isoperimetric inequalities, growth, and the spectrum of graphs*, Linear Alg. Appl. **103** (1988), 119–131.
- [Mo 3] ———, *Isoperimetric numbers of graphs*, J. Combinatorial Theory **B47** (1989), 274–291.
- [Mo 4] ———, *The Laplacian spectrum of graphs*, Graph Theory, Combinatorics, and Applications (Y. Alavi et al., ed.), John Wiley.
- [M.W] ——— and W. Woess, *A survey on spectra of infinite graphs*, Bull. London Math. Soc. **21** (1989), 209–234.
- [Mu 1] H. Muto, *The first eigenvalue of the Laplacian on even dimensional spheres*, Tohoku Math. J. **32** (1980), 427–432.
- [Mu 2] ———, *The first eigenvalue of the Laplacian of an isoparametric minimal hypersurface in a unit sphere*, Math. Z. **197** (1988), 531–549.
- [M.U] ——— and H. Urakawa, *On the least positive eigenvalue of Laplacian for compact homogeneous spaces*, Osaka J. Math. **17** (1980), 471–484.
- [M.O.U] ———, Y. Ohnita and H. Urakawa, *Homogeneous minimal hypersurfaces in the unit spheres and the first eigenvalues of their Laplacian*, Tohoku Math. J. **36** (1984), 253–267.
- [N.U] S. Nayatani and H. Urakawa, *Spectrum of the Schrödinger operator on a complete Riemannian manifold*, J. Funct. Anal. **112** (1993), 459–479.
- [O.U] Y. Ohno and H. Urakawa, *On the first eigenvalue of the combinatorial Laplacian for a graph*, Interdisciplinary Information Sciences **1** (1994), 33–46.
- [O.P.S 1] B. Osgood, R. Phillips and P. Sarnak, *Extremals of determinants of Laplacian*, J. Funct. Anal. **80** (1988), 148–211.
- [O.P.S 2] ———, *Compact isospectral sets of surfaces*, J. Funct. Anal. **80** (1988), 212–234.
- [O.P.S 3] ———, *Compact isospectral sets of plane domains*, Proc. Nat. Acad. Sci. U.S.A. **85** (1988), 5359–5361.
- [O.P.S 4] ———, *Moduli space, heights and isospectral sets of plane domains*, Ann. Math. **129** (1989), 293–362.
- [Pe 1] H. Pesce, *Déformations isospectrales sur certaines nilvariétés et finitude spectrale des variétés du Heisenberg*, Ann. Scient. École Norm. Sup. **25** (1992), 515–538.
- [Pe 2] ———, *Déformations isospectrales sur les nilvariétés de rang deux*, C. R. Acad. Sci. Paris **314** (1992), 621–623.
- [Pe 3] ———, *Calcul du spectre d'une nilvariété de rang deux et applications*, Trans. Amer. Math. Soc. **339** (1993), 433–461.
- [Pe 4] ———, *Représentations relativement équivalentes et variétés riemannienne isospectrales*, C. R. Acad. Sci. Paris **318** (1994), 657–659.
- [Pe 5] ———, *Une formule de Poisson pour les variétés du Heisenberg*, Duke Math. J. **73** (1994), 79–95.
- [Pe 6] ———, *Variétés isospectrales et représentations de groupes*, Contemp. Math. **173** (1994), 231–240.
- [Pe 7] ———, *Variétés hyperboliques et elliptiques fortement isospectrales*, J. Funct. Anal. **134** (1995), 363–391.
- [Pe 8] ———, *Représentations relativement équivalentes et variétés riemanniennes isospectrales*, Comment. Math. Helvetici **71** (1996), 243–268.
- [Pe 9] ———, *Compacité de l'ensemble des réseaux isospectraux et conséquence*, Topology, **36** (1996), 695–710.
- [Sa] T. Sakai, *On eigenvalues of Laplacian and curvature of Riemannian manifolds*, Tohoku Math. J. **23** (1971), 589–603.
- [Sc] A. Schiemann, *Ein Beispiel positive definiter quadratischer Formen der Dimension 4 mit gleichen Darstellungszahlen*,

- Arch. Math. **54** (1990), 372–375.
- [Sh 1] T. Shirai, *The spectrum of the infinite regular line graphs* (to appear).
- [Sh 2] ———, *A trace formula for discrete Schrödinger operators* (to appear).
- [St] R. S. Strichartz, *Analysis of the Laplacian on the complete Riemannian manifold*, J. Funct. Anal. **52** (1983), 48–79.
- [SI] D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold*, J. Differ. Geom. **18** (1983), 723–732.
- [Su 1] T. Sunada, *Spectrum of a compact flat manifold*, Comment. Math. Helvetici **53** (1978), 613–621.
- [Su 2] ———, *Riemannian coverings and isospectral manifolds*, Ann. Math. **121** (1985), 169–186.
- [Su 3] ———, *Gelfand's problem on unitary representations associated with discrete subgroups of $PSL_2(\mathbb{R})$* , Bull. Amer. Math. Soc. **12** (1985), 237–238.
- [Su 4] ———, *Fundamental groups and Laplacian*, Lecture Notes in Math., Springer **1339** (1988), 248–277.
- [Su 5] ———, *Unitary representations of fundamental groups and the spectrum of twisted Laplacians*, Topology **28** (1989), 125–132.
- [Su 6] ———, *A discrete analogue of periodic magnetic Schrödinger operators*, Contemp. Math. **173** (1994), 284–299.
- [Sz] G. Szegő, *Inequalities of certain eigenvalues of a membrane of given area*, J. Rat. Mech. Anal. **3** (1954), 342–356.
- [Tm] L. F. Tam, *Harmonic functions on connected sums of manifolds*, Math. Z. **211** (1992), 315–322.
- [Tn 1] S. Tanno, *Eigenvalues of the Laplacian of the Laplacian of Riemannian manifolds*, Tohoku Math. J. **25** (1973), 391–403.
- [Tn 2] ———, *The first eigenvalue of the Laplacian on spheres*, Tohoku Math. J. **31** (1979), 179–185.
- [Tn 3] ———, *A characterization of the canonical spheres*, Math. Z. **175** (1980), 267–274.
- [U 1] H. Urakawa, *On the least positive eigenvalue of the Laplacian for compact group manifolds*, J. Math. Soc. Japan **31** (1979), 209–226.
- [U 2] ———, *Bounded domains which are isospectral but not congruent*, Ann. Scient. École Norm. Sup. **15** (1982), 441–456.
- [U 3] ———, *Reflection groups and the eigenvalue problems of vibrating membranes with mixed boundary conditions*, Tohoku Math. J. **36** (1984), 175–183.
- [U 4] ———, *Lower bounds for the eigenvalues of the fixed vibrating membrane problems*, Tohoku Math. J. **36** (1984), 185–189.
- [U 5] ———, *Numerical computations of the spectra of the Laplacian on 7-dimensional homogeneous manifolds $SU(3)/T(k, l)$* , SIAM J. Math. Anal. **15** (1984), 979–987.
- [U 6] ———, *The first eigenvalue of the Laplacian for a positively curved homogeneous Riemannian manifold*, Compositio Math. **59** (1986), 57–71.
- [U 7] ———, *Spectra of Riemannian manifolds without focal points*, Geometry of Manifolds (K. Shiohama, ed.), Academic Press, 1989, pp. 435–443.
- [U 8] ———, *Geometry of Laplace-Beltrami operator on a complete Riemannian manifold*, Advanced Studies Pure Math. **22** (1993), 347–406.
- [U 9] ———, *Heat kernel and Green kernel comparison theorems for infinite graphs*, J. Funct. Anal. **146** (1997) 206–235.
- [U 10] ———, *Eigenvalue comparison theorems of the discrete Laplacian for a graph* (to appear).
- [U 11] ———, *Laplacian and Networks*, Shokabo, 1996. (Japanese)
- [V.S.C] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Univ. Press, 1992.
- [Vg] F. Vignéras, *Variétés riemanniennes isospectrales et non isométriques*, Ann. Math. **112** (1980), 21–32.
- [W] W. Woess, *Random walks on infinite graphs and graphs—A survey on selected topics*, Bull. London Math. Soc. **26** (1994), 1–60.
- [X] X. Xu, *On compactness of isospectral conformal metrics of 4-manifolds*, Nagoya Math. J. **140** (1995), 77–99.
- [Y.Y] P. C. Yang and S. T. Yau, *Eigenvalues of the Laplacian of compact Riemannian surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa **7** (1980), 55–63.
- [Y 1] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Commun. Pure Appl. Math. **28** (1975), 201–228.
- [Y 2] ———, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), 659–670.
- [Ze 1] S. Zelditch, *On the generic spectrum of a Riemannian cover*, Ann. Inst. Fourier, Grenoble **40** (1990), 407–422.
- [Ze 2] ———, *Isospectrality in the FIO category*, J. Differ. Geom. **35** (1992), 689–710.
- [Ze 3] ———, *Maximally degenerate Laplacians*, Ann. Inst. Fourier, Grenoble **46** (1996), 547–587.
- [Ze 4] ———, *Maximally degenerate Laplacians I* (to appear).
- [Zh 1] D. Zhou, *Essential spectrum of the Laplacian on manifolds of nonnegative curvature*, International Math. Res. Notice **4** (1994), 209–214.
- [Zh 2] ———, *Spectrum of Laplacian on noncompact Riemannian manifolds* (to appear).