Linear Functionals on Certain Martingale Hardy Spaces

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We shall introduce some martingale Hardy spaces associated with Banach function spaces, which are extensions of well-known martingale Hardy spaces H^p , and consider continuous linear functionals on the spaces. Some new duality theorems are established.

KEYWORDS: martingale, Banach function space, Hardy space, rearrangement in variant function space

1 Introduction

Let $1 \leq p < \infty$ and H^p be the linear space of martingales $X = (X_t)_{0 \leq t < \infty}$ such that $X^* = \sup_t |X_t| \in L^p$ equipped with the norm $\|X\|_{H^p} = \|X^*\|_{L^p}$. It is well-known that, if $1 < p < \infty$ and $1/p + 1/p' = 1$, then $H^{p'}$ is the dual space to H^p , and BMO is the dual space to H^1 . More generally, for a Young function Φ , H^Φ denotes the space of martingales X such that $X^* \in L^\Phi$, where L^Φ denotes the Orlicz space as usual. Doob's inequality for Young functions shows that, if Ψ is the conjugate function of Φ and both Φ and Ψ satisfy the Δ_2 -condition, then the dual of H^Φ is H^Ψ . However any representation of the dual space of H^Φ is not known for Φ such as $\Phi(x) = x \log(x+1)$.

On the other hand, Garsia [9] introduced the spaces $\mathcal{H}^p$ ($2 \leq p \leq \infty$) and proved that the dual of H^p is $\mathcal{H}^{p'}$ when $1 \leq p \leq 2$. Thus $\mathcal{H}^p$ and H^p are isomorphic when $2 \leq p < \infty$, while $\mathcal{H}^\infty$ and BMO are isomorphic. These facts suggest that the dual space of H^Φ may be represented by a suitable extension of $\mathcal{H}^p$. In fact, it is possible if every martingale is continuous, see Theorem 4.6.

In this note we shall consider such a problem in much more general setting. We shall introduce a new Hardy space of martingales associated with a Banach function space, which is not necessarily rearrangement invariant, and study linear functionals on the space. In the last section, we restrict ourselves to the case where the Banach function space is rearrangement invariant.

2 Notation and definitions

The reader is assumed to be familiar with the theory of martingales expounded in [7,8], and required some knowledge on the theory of Banach function spaces and rearrangement invariant function spaces. Almost results on function spaces which we use in this note are found in [4,6]. When we use a nontrivial result on function spaces, we shall mention it.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions (see [7, p. 183]) and $\bigvee_{t \geq 0} \mathcal{F}_t = \mathcal{F}$. Throughout this note, we shall deal with only right continuous processes with left-hand limits. Such a process is said to be càdlàg (continu à droite et pourvu de limites à gauche). For each process $X = (X_t)_{t \geq 0}$ we set $X_s^* = \sup_{t \leq s} |X_t|$, $X^* = X_\infty^* = \sup_t |X_t|$ and $X_{0-} = 0$. If T is a stopping time or more generally a random time, the jump of X at T is denoted by ΔX_T .

If f is a random variable on a probability space (S, μ) , such as Ω and the unit interval I with the Lebesgue measure m , then f' denotes the *decreasing rearrangement* of f , that is, the (unique) right continuous decreasing function on I satisfying $\mu\{|f| > \eta\} = m\{|f'| > \eta\}$ for all $\eta > 0$. For random variables f and g on S , we write $f < g$ if

$$\int_0^t f'(s) ds \leq \int_0^t g'(s) ds, \quad t \in I.$$

A Banach space $L^p = (L^p(S), \|\cdot\|_p)$ satisfying the following conditions is called a *Banach function space*:

- i) $L^\infty \hookrightarrow L^p \hookrightarrow L^1$;
- ii) if $|f| \leq |g|$ and $g \in L^p$, then $f \in L^p$ and $\|f\|_p \leq \|g\|_p$;
- iii) if $0 \leq f_n \uparrow f$ and $\sup_n \|f_n\|_p < \infty$, then $f \in L^p$ and $\|f\|_p = \sup_n \|f_n\|_p$.

L^p is said to have *absolutely continuous norm* if $\|f 1_{E_n}\|_p \rightarrow 0$ holds for every $f \in L^p$ and every sequence $\{E_n\}$ satisfying $E_n \downarrow \emptyset$ a.s. ([4, p. 14]). Clearly L^p has absolutely continuous norm if and only if $\|f_n\|_p \downarrow 0$ holds for every sequence $\{f_n\}$ in L^p satisfying $f_n \downarrow 0$ a.s. and $\sup_n |f_n| \in L^p$.

For a Banach function space L^p , its associate space $L^{p'}$ is the Banach function space defined by

$$\|f\|_{p'} = \sup \{E[fg] : g \in L^p, \|g\|_p \leq 1\};$$

$$L^{p'} = \{f : \|f\|_{p'} < \infty\}.$$

Then the associate space of $L^{p'}$ is equal to L^p , that is, $(L^{p'})' = L^p$ [4, p. 10].

A Banach function space L^p is said to be *rearrangement invariant* or briefly r.i. if $f^r = g^r$ and $g \in L^p$, then $f \in L^p$ and $\|f\|_p = \|g\|_p$. Suppose that S has no atom. Then L^p is r.i. if and only if $f < g$, $g \in L^p$ implies $f \in L^p$ and $\|f\|_p \leq \|g\|_p$. Another important characterization of a r.i. space is the Luxemburg representation theorem [10]: $L^p(S)$ is r.i. if and only if there exists a r.i. function space $L^\lambda(I)$ over $I = [0, 1]$ such that

$$\begin{cases} f \in L^p(S) \Leftrightarrow f^r \in L^\lambda(I), \\ \|f\|_p = \|f^r\|_\lambda \quad \text{for all } f \in L^p(S). \end{cases} \quad (2.1)$$

Such an L^λ is unique, provided that Ω has no atom.

Some of our results are described with the *Boyd indices* introduced by Boyd [5]. For each $s > 0$, let D_s denote the *dilation operator* on $L^1(I)$ defined by

$$D_s x(t) = \begin{cases} x(st), & \text{if } st \in I, \\ 0, & \text{otherwise.} \end{cases}$$

If $L^\lambda(I)$ is r.i., then D_s is a bounded operator on L^λ into itself and the norm of D_s , denoted by $\|D_s\|_\lambda$, does not exceed $s^{-1} \vee 1$ [4, p. 148]. The Boyd indices of L^λ are given by

$$\bar{\alpha}_\lambda = \inf_{t>1} \frac{\log \|D_{1/t}\|_\lambda}{\log t} = \lim_{t \rightarrow \infty} \frac{\log \|D_{1/t}\|_\lambda}{\log t};$$

$$\underline{\alpha}_\lambda = \sup_{0<t<1} \frac{\log \|D_{1/t}\|_\lambda}{\log t} = \lim_{t \downarrow 1} \frac{\log \|D_{1/t}\|_\lambda}{\log t}.$$

More generally the Boyd indices of a function space $L^p(S)$ over S are given by $\bar{\alpha}_p = \bar{\alpha}_\lambda$ and $\underline{\alpha}_p = \underline{\alpha}_\lambda$, where, needless to say, $\bar{\alpha}_\lambda$ and $\underline{\alpha}_\lambda$ are indices of $L^\lambda(I)$ satisfying (2.1).

3 Linear functionals on spaces of adapted processes

Let $\mathcal{R}_t^p$ denote the linear space consisting of all càdlàg processes $X = (X_t)_{t \geq 0}$ possessing a finite limit $X_\infty = \lim_{t \rightarrow \infty} X_t$ and satisfying $\|X^*\|_p < \infty$, and $\mathcal{R}^p$ be the linear subspace of $\mathcal{R}_t^p$ consisting of all adapted processes in $\mathcal{R}_t^p$. The norm of $X \in \mathcal{R}_t^p$ is given by $\|X\|_{\mathcal{R}_t^p} = \|X^*\|_p$. Then $\mathcal{R}^p$ is a closed linear subspace of $\mathcal{R}_t^p$. In this section, we shall consider linear functionals on $\mathcal{R}^p$ and $\mathcal{R}_t^p$.

Lemma 3.1 $\mathcal{R}^p$ and $\mathcal{R}_t^p$ are Banach spaces.

Proof. It is sufficient to prove the completeness of $\mathcal{R}^p$. Let (X^n) be a Cauchy sequence in $\mathcal{R}^p$ and, choosing a subsequence if necessary, we assume that $\sum_n \|X^n - X^{n+1}\|_{\mathcal{R}^p} < \infty$. As $L^p \hookrightarrow L^1$, the series $\sum_n (X^n - X^{n+1})^*$ converges in L^1 , and hence we may assume that the series converges a.s. by extracting a suitable subsequence. If we set $X_t = \sum_n (X_t^n - X_t^{n+1})$ for $t \geq 0$, then we easily obtain that $X \in \mathcal{R}_t^p$ and $X_n \rightarrow X$ in $\mathcal{R}_t^p$. Thus $\mathcal{R}^p$ is complete. ■

Our first result is a representation of linear functionals on $\mathcal{R}^p$.

Theorem 3.2 (a) Let $A^+ = (A_t^+)_{0 \leq t \leq \infty}$ and $A^- = (A_t^-)_{0 \leq t \leq \infty}$ be processes of finite variation which are not necessarily adapted. Suppose that $A_0^- = 0$, $A_\infty^- = A_\infty^+$ and $\int_{[0, \infty[} |dA_s^\pm| \in L^{p'}$. Then the expression

$$J(X) = E \left[\int_{[0, \infty[} X_{s-} dA_s^- + \int_{[0, \infty[} X_s dA_s^+ \right], \quad X \in \mathcal{R}_t^p, \quad (3.1)$$

is meaningful and defines a continuous linear functional on $\mathcal{R}_t^p$. Moreover we have

$$\|J\| \leq \left\| \int_{[0, \infty[} |dA_s^-| + \int_{[0, \infty[} |dA_s^+| \right\|_{p'}.$$

(b) Suppose that L^p has absolutely continuous norm. Then every continuous linear functional J on $\mathcal{R}_t^p$ has a representation of the form (3.1) with $A^\pm$ satisfying

$$\left\| \int_{[0, \infty[} |dA_s^-| + \int_{[0, \infty[} |dA_s^+| \right\|_{p'} \leq 4\|J\|.$$

Note that (a) follows immediately from the inequalities

$$|J(X)| \leq E \left[X^* \left(\int_{[0, \infty[} |dA_s^-| + \int_{[0, \infty[} |dA_s^+| \right) \right] \leq \|X\|_{\mathcal{R}_t^p} \left\| \int_{[0, \infty[} |dA_s^-| + \int_{[0, \infty[} |dA_s^+| \right\|_{p'}.$$

To prove (b), we use the following representation theorem of linear functionals (for the case where $\mathcal{F}_t = \mathcal{F}$, $t \geq 0$), due to Dellacherie and Meyer [8, p. 201].

Theorem 3.3 *Let $\mathcal{D}$ be the linear space of all bounded processes and J be a positive linear functional on $\mathcal{D}$. If $\lim_n J(X^n) = 0$ holds for every sequence (X^n) of elements of $\mathcal{D}$ such that $(X^n)^* \downarrow 0$ a.s., then there exist increasing processes $A^\pm = (A_t^\pm)_{0 \leq t \leq \infty}$, which are not necessarily adapted, satisfying the following conditions:*

- i) $A_{\infty}^+ = A_{\infty}^-$ and $A_0^- = 0$;
- ii) $J(X) = E \left[\int_{[0, \infty]} X_s^- dA_s^- + \int_{[0, \infty[} X_s dA_s^+ \right]$ for all $X \in \mathcal{D}$.

Furthermore we can take A^+ so as to be purely discontinuous; then the representation is unique.

Proof of Theorem 3.2 (b). We first assume that J is positive and continuous. Suppose that (X^n) is a sequence of nonnegative processes in $\mathcal{D}$ such that $(X^n)^* \downarrow 0$ a.s. Since L^p has absolutely continuous norm, we have $\lim_{n \rightarrow \infty} \|X^n\|_{\mathcal{R}^p} = 0$ and hence $\lim_{n \rightarrow \infty} J(X^n) = 0$. By Theorem 3.3, there exist increasing processes $A^\pm$ satisfying i) and ii) of the theorem.

Now we prove that $A_\infty^\pm \in L^{p'}$. To this end, let $\xi \geq 0$ be a bounded random variable with $\|\xi\|_p \leq 1$ and set $X_t = \xi$ for every $t \geq 0$. Since $\|X\|_{\mathcal{R}^p} = \|\xi\|_p \leq 1$, we have

$$E[(A_\infty^+ + A_\infty^-)\xi] = E \left[\int_{[0, \infty]} X_s^- dA_s^- + \int_{[0, \infty[} X_s dA_s^+ \right] = J(X) \leq \|J\|.$$

Clearly this remains valid for every nonnegative $\xi \in L^p$ with $\|\xi\|_p \leq 1$. Thus we see that $A^\pm \in L^{p'}$ and $\|A_\infty^+ + A_\infty^-\|_{p'} \leq \|J\|$. Therefore, by (a) of Theorem 3.2, the functional

$$X \mapsto E \left[\int_{[0, \infty]} X_s^- dA_s^- + \int_{[0, \infty[} X_s dA_s^+ \right]$$

is continuous and linear on $\mathcal{R}^p$. To prove that this functional is equal to J , it remains to show that $\mathcal{D}$ is dense in $\mathcal{R}^p$. For each $X \in \mathcal{R}^p$, the process $X^k = (X_t^k)$ defined by

$$X_t^k = X_t 1_{\{|X_t| \leq k\}} + k 1_{\{|X_t| > k\}}$$

tends to X in $\mathcal{R}^p$, since $(X - X^k)^* \leq 2X^* 1_{\{X^* > k\}} \downarrow 0$ as $k \rightarrow \infty$. Thus $\mathcal{D}$ is dense in $\mathcal{R}^p$.

Next let J be an arbitrary continuous linear functional on $\mathcal{R}^p$. Then there exist positive linear functionals J^+ and J^- such that $J = J^+ - J^-$ and $\|J^\pm\| \leq 2\|J\|$, which are given by

$$J^+(X) = \sup \{J(Y) : Y \in \mathcal{D}, 0 \leq Y \leq X\}; \quad J^-(X) = J(X) - J^+(X).$$

Applying the result for positive functionals just have been proved above, we easily obtain (b) of Theorem 3.2. ■

We now pass to the study of functionals on $\mathcal{R}^p$. In what follows, we frequently use a letter γ to denote a random variable in L^p (or $L^{p'}$). For such a γ we denote by $(\gamma_t)_{t \geq 0}$ the martingale induced by γ : $\gamma_t = E[\gamma | \mathcal{F}_t]$, $t \geq 0$.

Theorem 3.4 (a) *Let $A^+ = (A_t^+)_{0 \leq t \leq \infty}$ and $A^- = (A_t^-)_{0 \leq t \leq \infty}$ be processes of finite variation satisfying the following conditions:*

- i) A^+ is optional, $A_{\infty}^+ = A_{\infty}^-$, and purely discontinuous;
- ii) A^- is predictable and $A_0^- = 0$;
- iii) there exists a $\gamma \in L^{p'}$ such that

$$E \left[\int_{[T, \infty]} |dA_s^-| + \int_{[T, \infty[} |dA_s^+| \middle| \mathcal{F}_T \right] \leq \gamma_T \quad (3.2)$$

holds for every stopping time T .

Then the linear functional defined by

$$J(X) = E \left[\int_{[0, \infty]} X_s^- dA_s^- + \int_{[0, \infty[} X_s dA_s^+ \right], \quad X \in \mathcal{R}^p, \quad (3.3)$$

is continuous on $\mathcal{R}^p$ and $\|J\| \leq \|\gamma\|_{p'}$.

(b) *Assume that L^p has absolutely continuous norm. Then every continuous linear functional J on $\mathcal{R}^p$ has a representation of the form (3.3), with $A^\pm$ and γ satisfying $\|\gamma\|_{p'} \leq 12\|J\|$ and i)–iii) of (a). Moreover the representation is unique.*

Proof. (a) We first point out that, for the proof of (a), A^+ does not need to be purely discontinuous. Suppose that J is given by (3.3). For the sake of simplicity, we set $B_t^\pm = \int_{[0, t]} |dA_s^\pm|$ for each t . Let Z^+ (resp. Z^-) denote the left potential (resp. potential) generated by the process B^+ (resp. B^-). Then by (3.2) we have $Z_t^+ + Z_t^- \leq \gamma_t$ outside of an evanescent set. Using integration by parts, we have

$$\begin{aligned}
|J(X)| &\leq E \left[\int_{[0,\infty]} X_s^* dB_s^- + \int_{[0,\infty[} X_s^* dB_s^+ \right] \\
&= E \left[\int_{[0,\infty[} (B_\infty^- - B_s^-) dX_s^* + \int_{[0,\infty[} (B_\infty^+ - B_s^+) dX_s^* \right] \\
&= \left[\int_{[0,\infty[} (Z_s^- + Z_s^+) dX_s^* \right] \leq \left[\int_{[0,\infty[} \gamma_s dX_s^* \right] \\
&= E[\gamma X^*] \leq \|\gamma\|_{\rho'} \|X\|_{\mathcal{R}^\rho},
\end{aligned}$$

where we have used the fact that the optional projection of the process $Y_t = \gamma$, $t \geq 0$, is equal to (γ_t) . Thus J is well-defined, continuous on $\mathcal{R}^\rho$ and satisfies $\|J\| \leq \|\gamma\|_{\rho'}$.

(b) We shall prove only the existence of the representation: the uniqueness of the representation is proved as in [8, p. 204]. Suppose that L^ρ has absolutely continuous norm and J is a continuous linear functional on $\mathcal{R}^\rho$. By the Hahn-Banach theorem, some $\bar{J} \in (\mathcal{R}_T^\rho)^*$ extends J without increasing norm. According to Theorem 3.2 (b), there exist processes $B^\pm = (B_t^\pm)_{0 \leq t \leq \infty}$ of finite variation such that $B_0^- = 0$, $B_{\infty}^- = B_\infty^+$, $\int_{[0,\infty[} |dB_s^\pm| \in L^{\rho'}$ and

$$\bar{J}(X) = E \left[\int_{[0,\infty]} X_s^- dB_s^- + \int_{[0,\infty[} X_s dB_s^+ \right], \quad X \in \mathcal{R}_T^\rho.$$

In addition, setting $\gamma = \int_{[0,\infty]} |dB_s^-| + \int_{[0,\infty[} |dB_s^+| \in L^{\rho'}$, we have $\|\gamma\|_{\rho'} \leq 4\|J\|$.

Now let A^+ and A^- be the dual optional and dual predictable projections of B^+ and B^- respectively. Then we have $A_{\infty}^+ = A_\infty^-$, $A_0^- = 0$ and (3.3). Moreover we claim that

$$E \left[\int_{[T,\infty]} |dA_s^-| + \int_{[T,\infty[} |dA_s^+| \middle| \mathcal{F}_T \right] \leq \gamma_T. \quad (3.4)$$

To see this, let $H^+ = (H_t^+)_{0 \leq t \leq \infty}$ and $H^- = (H_t^-)_{0 \leq t \leq \infty}$ be densities $dA_s^+ / |dA_s^+|$ and $dA_s^- / |dA_s^-|$ with values in $\{-1, 1\}$ respectively. Since H^- is predictable and H^+ is optional, we have

$$\begin{aligned}
E \left[\int_{[T,\infty]} |dA_s^-| + \int_{[T,\infty[} |dA_s^+| \right] &= E \left[\int_{[T,\infty]} H_s^- dA_s^- + \int_{[T,\infty[} H_s^+ dA_s^+ \right] \\
&= E \left[\int_{[T,\infty]} H_s^- dB_s^- + \int_{[T,\infty[} H_s^+ dB_s^+ \right] \\
&\leq E \left[\int_{[0,\infty]} |dB_s^-| + \int_{[0,\infty[} |dB_s^+| : T < \infty \right]
\end{aligned}$$

for every stopping time T . Taking T_F , $F \in \mathcal{F}_T$ ([7, p. 187]) instead of T , we obtain (3.4).

Thus we have obtained the representation of J , provided that A^+ is not required to be purely discontinuous. To prove the theorem completely, let A^{+c} denote the continuous part of A^+ . Since the set of s such that $\Delta X_s \neq 0$ is at most countable, we have

$$E \left[\int_{[0,\infty[} X_s dA_s^{+c} \right] = E \left[\int_{[0,\infty]} X_s^- dA_s^{+c} \right],$$

and hence

$$J(X) = E \left[\int_{[0,\infty]} X_s^- d(A_s^- + dA_s^{+c}) + \int_{[0,\infty[} X_s dA_s^{+d} \right],$$

where $A^{+d} = A^+ - A^{+c}$. Observe that

$$\begin{aligned}
\int_{[T,\infty]} |d(A_s^- + dA_s^{+c})| + \int_{[T,\infty[} |dA_s^{+d}| &\leq \int_{[T,\infty]} |dA_s^-| + \int_{[T,\infty]} |dA_s^{+c}| + \int_{[T,\infty[} |dA_s^{+d}| \\
&\leq \int_{[T,\infty]} |dA_s^-| + \int_{[T,\infty[} |dA_s^+| + 2 \int_{[T,\infty[} |dA_s^{+d}| \\
&\leq \int_{[T,\infty]} |dA_s^-| + 3 \int_{[T,\infty[} |dA_s^+|.
\end{aligned}$$

From this it follows that

$$E \left[\int_{[T,\infty]} |d(A_s^- + dA_s^{+c})| + \int_{[T,\infty[} |dA_s^{+d}| \middle| \mathcal{F}_T \right] \leq 3\gamma_T. \quad (3.6)$$

From (3.5) and (3.6), we see that the processes $A^- + A^{+c}$ and A^{+d} give a required representation. Thus the theorem is established. ■

4 Linear functionals on martingale spaces

Let H^p denote the linear subspace of $\mathcal{R}^p$ consisting of all martingales in $\mathcal{R}^p$. If $X^n = (X_t^n) \in H^p$ converges to $X = (X_t)$ in $\mathcal{R}^p$, then $\|(X - X^n)^*\|_1 \rightarrow 0$ as $n \rightarrow \infty$, and hence

$$X_t = \lim_n X_t^n = \lim_n E[X_\infty^n | \mathcal{F}_t] = E[X_\infty | \mathcal{F}_t]$$

holds for all $t \geq 0$, where the limits are taken in L^1 . Therefore X is a martingale, and hence H^p is a closed linear subspace of $\mathcal{R}^p$.

In this section we shall consider linear functionals on H^p . Main results of this section are Theorems 4.2 and 4.6. To describe our results, we shall introduce the following spaces of processes.

Definition 4.1 Let L^p be a Banach function space.

(a) $\mathcal{V}^p$ denotes the space of processes $A = (A_t)_{0 \leq t \leq \infty}$ of integrable variation such that there exists a $\gamma \in L^p$ satisfying

$$E \left[\int_{[T, \infty]} |dA_s| \middle| \mathcal{F}_T \right] \leq \gamma_T \quad (4.1)$$

for every stopping time T . For each $A \in \mathcal{V}^p$ we set

$$\|A\|_{\mathcal{V}^p} = \inf \{ \|\gamma\|_p : \gamma \in L^p \text{ satisfies (4.1)} \}.$$

(b) $\mathcal{K}^p$ denotes the space of all uniformly integrable martingales Y such that there exists a $\gamma \in L^p$ satisfying

$$E[|Y_\infty - Y_{T-}| | \mathcal{F}_T] \leq \gamma_T \quad (4.2)$$

for every stopping time T . For each $Y \in \mathcal{K}^p$ we set

$$\|Y\|_{\mathcal{K}^p} = \inf \{ \|\gamma\|_p : \gamma \text{ satisfies (4.2)} \}.$$

Note that (4.2) implies that $|\Delta Y_T| \leq \gamma_T$ for every stopping time T , which will be used frequently.

We say that *the Doob-type inequality holds in H^p* if

$$\|X\|_{H^p} \leq C \|X_\infty\|_p$$

holds for all $X \in H^p$ with some constant $C > 0$ independent of X . Recall that the Doob-type inequality holds in many martingale spaces such as H^p with $p > 1$, H^Φ with a Young function Φ satisfying the ∇_2 -condition, and more generally the space H^p associated with a r.i. function space L^p satisfying $\tilde{\alpha}_p < 1$ (cf. [1]).

Suppose that the Doob-type inequality holds in H^p and γ is a random variable in L^p . Then, setting $\tilde{\gamma} = \gamma^*$, we have

$$\gamma_T^* \leq \tilde{\gamma}_T \quad (4.3)$$

for all stopping times T and

$$\|\tilde{\gamma}\|_p \leq C \|\gamma\|_p,$$

where $(\tilde{\gamma}_t)$ stands for the martingale induced by $\tilde{\gamma}$.

Theorem 4.2 (a) If $A = (A_t)_{0 \leq t \leq \infty} \in \mathcal{V}^{p'}$ then the expression

$$J(X) = E \left[\int_{[0, \infty]} X_s dA_s \right], \quad X \in H^p, \quad (4.4)$$

defines a continuous linear functional on H^p , and satisfies $\|J\| \leq 2\|A\|_{\mathcal{V}^{p'}}$.

(b) Assume that L^p has absolutely continuous norm and that the Doob-type inequality holds in $H^{p'}$. If J is a continuous linear functional on H^p , then there exists a process $A = (A_t)_{0 \leq t \leq \infty} \in \mathcal{V}^{p'}$ satisfying (4.4) and $\|A\|_{\mathcal{V}^{p'}} \leq 24C\|J\|$. In addition, if X is bounded, then $J(X) = E[X_\infty A_\infty]$.

Furthermore if every $(\mathcal{F}_t)$ -martingale is continuous, then we can remove the hypothesis that the Doob-type inequality holds in $H^{p'}$, and 12 replaces 24C.

Proof. (a) Assume that $\gamma \in L^{p'}$ satisfies (4.1). Since (4.1) is reduced to the inequality $|\Delta A_\infty| \leq \gamma$ a.s. when $T = \infty$, the functional $H^p \ni X \mapsto E[X_\infty \Delta A_\infty] \in \mathbb{R}$ is continuous on H^p . On the other hand, by virtue of Theorem 3.4 (a), the functional j given by

$$j(X) = E \left[\int_{[0, \infty]} X_s dA_s \right], \quad X \in H^p$$

is continuous and $\|J\| \leq \|\gamma\|_{\rho'}$. Therefore J is a continuous linear functional on H^ρ and satisfies $\|J\| \leq 2\|\gamma\|_{\rho'}$, which implies that $\|J\| \leq 2\|A\|_{\gamma\rho'}$.

(b) We assume the validity of Doob-type inequality in $H^{\rho'}$ and nothing on the continuity of $(\mathcal{F}_t)$ -martingale: if every $(\mathcal{F}_t)$ -martingale is continuous, then the proof is more simple. Let J be a continuous linear functional on H^ρ . We may extend J to a linear functional $\bar{J}$ on $\mathcal{R}^\rho$ so that $\|J\| = \|\bar{J}\|$ by the Hahn-Banach theorem. According to Theorem 3.4 (b), there exist processes $A^\pm = (A_t^\pm)_{0 \leq t \leq \infty}$ of integrable variation satisfying i)-iii) of Theorem 3.4 and

$$\bar{J}(X) = E \left[\int_{[0, \infty[} X_s - dA_s^- + \int_{[0, \infty[} X_s dA_s^+ \right], \quad X \in \mathcal{R}^\rho,$$

with some $\gamma \in L^{\rho'}$ such that $\|\gamma\|_{\rho'} \leq 12\|J\|$.

We claim that $|\Delta A_T^-| \leq \gamma_{T-}$ holds for every stopping time T . If T is predictable, this follows immediately from the inequality

$$E[|\Delta A_{T_n}^-| | \mathcal{F}_{T_n}] \leq E \left[\int_{[T_n, \infty[} |dA_s^-| \middle| \mathcal{F}_{T_n} \right] \leq \gamma_{T_n},$$

where (T_n) is an announcing sequence of T . Next let T be an arbitrary stopping time. Since A^- is predictable, the set $\{\Delta A^- \neq 0\}$ is indistinguishable from the countable union of a sequence of disjoint graphs of predictable stopping time S_n (cf. [7, p. 221, p. 261]). Hence we obtain

$$|\Delta A_T^-| = \sum_{n=1}^{\infty} |\Delta A_{S_n}^-| 1_{\{T=S_n\}} \leq \sum_{n=1}^{\infty} \gamma_{S_n} 1_{\{T=S_n\}} \leq \gamma_{T-}.$$

Since the Doob-type inequality holds in $H^{\rho'}$, there exists a $\bar{\gamma} \in L^\rho$ satisfying (4.3) and $\|\bar{\gamma}\|_{\rho'} \leq C\|\gamma\|_{\rho'}$. Hence, if we set $A = A^+ + A^-$, then

$$E \left[\int_{[T, \infty[} |dA_s| \middle| \mathcal{F}_T \right] \leq E \left[\int_{[T, \infty[} |dA_s^+| \middle| \mathcal{F}_T \right] + \left[\int_{[T, \infty[} |dA_s^-| \middle| \mathcal{F}_T \right] + \gamma_{T-} \leq 2\bar{\gamma}_T \equiv \xi_T.$$

Since $\|\bar{\gamma}\|_{\rho'} \leq C\|\gamma\|_{\rho'}$, the above inequality gives that $\|A\|_{\gamma\rho'} \leq \|\xi\|_{\rho'} \leq 24C\|J\|$.

Now it remains to show that J is written as (4.4). As in the proof of Theorem 3.4 (a), we can show that

$$E \left[\int_{[0, \infty[} X_s^* |dA_s^-| \right] \leq \|\xi\|_{\rho'} \|X\|_{H^\rho} < \infty. \quad (4.5)$$

Since the process $X_- = (X_{t-})$ is the predictable projection of X , we have

$$E \left[\int_{[0, \infty[} X_{s-} dA_s^- \right] = E \left[\int_{[0, \infty[} X_s dA_s^- \right],$$

where these expectations are certain to exist by (4.5). This implies (4.4).

Finally suppose that X is bounded. Then we see that $|X_\infty| \int_{[0, \infty[} |dA_s| \in L^1$ and hence

$$J(X) = E \left[\int_{[0, \infty[} X_s dA_s \right] = E[X_\infty A_\infty],$$

since X is the optional projection of the process Y defined by $Y_t = X_\infty$. Thus the theorem is established. ■

When L^ρ is equal to an Orlicz space L^Φ , H^ρ (resp. $\mathcal{H}^\rho$, $\mathcal{V}^\rho$) is denoted by H^Φ (resp. $\mathcal{H}^\Phi$, $\mathcal{V}^\Phi$).

Corollary 4.3 (a) Let L^ρ be a r.i. function space which has absolutely continuous norm. If every $(\mathcal{F}_t)$ -martingale is continuous, then the dual of H^ρ is $\mathcal{V}^{\rho'}$.

(b) Let L^ρ be as in (a). If $\underline{\alpha}_\rho > 0$, then the dual of H^ρ is $\mathcal{V}^{\rho'}$.

(c) Let L^Φ be an Orlicz space. If Φ satisfies the Δ_2 -condition, i.e., $\Phi(2x) \leq K\Phi(x)$ ($x \geq 0$) with some constant $K > 0$, then the dual of H^Φ is $\mathcal{V}^\Psi$, where Ψ stands for the conjugate function of Φ .

Proof. (a) This is an immediate consequence of Theorem 4.2.

(b) Let L^ρ be a r.i. function space with absolutely continuous norm. Antipa [1] proved that the Doob-type inequality holds in $H^{\rho'}$ if and only if $\bar{\alpha}_{\rho'} < 1$. (Note that in his paper the definition of Boyd indices is different from ours.) As $\underline{\alpha}_\rho = 1 - \bar{\alpha}_{\rho'}$, we obtain (b) from the preceding theorem.

(c) Let Φ be a Young function satisfying Δ_2 -condition and Ψ be the conjugate function of Φ . Note that the associate space of L^Φ is isomorphic to L^Ψ and that the Doob-type inequality holds in H^Ψ (cf. [8, p. 186]). On the other hand, since Φ satisfies the Δ_2 -condition, L^Φ has absolutely continuous norm, and hence (c) follows from the preceding theorem. ■

Remark. (a) Note that a Banach function space L^ρ does not always have absolutely continuous norm when $0 < \underline{\alpha}_\rho \leq \bar{\alpha}_{\rho'} < 1$. In fact, let $1 < p < \infty$ and L^ρ be the r.i. space consisting of all $f \in L^1(\Omega)$ such that $f' \in M(\varphi)$, where $\varphi(t) = t^{1/p}$ and $M(\varphi)$ denotes the Lorentz M -space. That is to say, $f \in L^\rho$ if and only if

$$\|f\|_p = \|f'\|_{M(\varphi)} = \sup_{0 < t \leq 1} \frac{\varphi(t)}{t} \int_0^t f'(s) ds = \sup_{0 < t \leq 1} t^{-1/p'} \int_0^t f'(s) ds < \infty,$$

where $p' = p/(p-1)$. Assume that Ω has no atom. Then there exists a random variable f such that $f'(s) = s^{-1/p}$, $s \in I$ [6, p. 44]. It is easy to see that

$$(|f| 1_{\{|f| > \alpha\}})'(s) = f'(s) 1_{\{0 \leq s \leq \varepsilon_\alpha\}}, \quad \varepsilon_\alpha = P(|f| > \alpha).$$

It follows that for each $\alpha > 0$

$$\|f 1_{\{|f| > \alpha\}}\|_p = \sup_{0 < t \leq 1} t^{-1/p'} \int_0^{t \wedge \varepsilon_\alpha} f'(s) ds = p'.$$

This means that L^p does not have absolutely continuous norm. However we have $\underline{\alpha}_p = \bar{\alpha}_p = 1/p$, since $\|D_\alpha\|_{M(\varphi)} = (1/\alpha)^{1/p}$ for all $\alpha > 0$.

(b) In (b) of Corollary 4.3 we cannot remove the hypothesis of absolute continuity of the norm $\|\cdot\|_p$; it is necessary in a trivial case. Suppose that $\mathcal{F}_t = \mathcal{F}$ for every $t \geq 0$. Then H^p is reduced to L^p , and $\mathcal{V}^{p'}$ is reduced to $L^{p'}$: hence $(H^p)^* = \mathcal{V}^{p'}$ if and only if $(L^p)^* = L^{p'}$. Furthermore $(L^p)^* = L^{p'}$ implies that L^p has absolutely continuous norm (see [4, p. 23]). As the preceding example shows, however, the condition $\alpha_p > 0$ does not imply that L^p has absolutely continuous norm, and the hypothesis of absolute continuity is essential in Corollary 4.3 (b).

We have established the duality between H^p and $\mathcal{V}^{p'}$, however, it is natural to ask whether $(H^p)^*$ is isomorphic to a martingale space. As stated before, the answer is affirmative if every $(\mathcal{F}_t)$ -martingale is continuous. We begin with the following lemma.

Lemma 4.4 Suppose that $Y = (Y_t) \in \mathcal{H}^{p'}$ and $\gamma \in L^{p'}$ satisfies (4.2). If $\gamma_{T-} \leq K\gamma_T$ holds for every stopping time T , then

$$|E[X_\infty Y_\infty]| \leq (3 + 2\sqrt{2})KE[\gamma X^*]. \quad (4.6)$$

holds for every bounded martingale $X = (X_t)$, where $K \geq 1$ is a constant.

Proof. Without loss of generality, we may assume that $\gamma \geq \varepsilon$ a.s. for some constant $\varepsilon > 0$. Let $\delta > 1$ be an arbitrary constant and define a sequence of stopping times (T_n) inductively by

$$T_0 = 0; \quad T_{n+1} = \inf \{t \geq T_n : |Y_t - Y_{T_n-}| > \delta\gamma_t\}, \quad n \geq 0.$$

Since $\gamma \geq \varepsilon$ a.s. and $|\Delta Y_{T_n}| \leq \gamma_{T_n}$ by (4.2), we have $T_n < T_{n+1}$ on $\{T_n < \infty\}$. Moreover we see that $T_n(\omega)$ strictly increases to ∞ , if $T_n(\omega) < \infty$ for all n . We set

$$A_t = \sum_{n=0}^{\infty} \gamma_{T_n} 1_{\{T_n \leq t\}}, \quad 0 \leq t < \infty; \quad A_\infty = \lim_{t \rightarrow \infty} A_t. \quad (4.7)$$

Then the increasing process $A = (A_t)$ belongs to $\mathcal{V}^{p'}$ and, in particular, A_∞ is integrable. To see this, observe first that

$$\delta E[\gamma_{T_{n+1}} 1_{\{T_{n+1} < \infty\}} | \mathcal{F}_{T_n}] \leq E[|Y_{T_{n+1}} - Y_{T_n-}| | \mathcal{F}_{T_n}] 1_{\{T_n < \infty\}} \leq \gamma_{T_n} 1_{\{T_n < \infty\}}$$

holds for each n . Let T be an arbitrary stopping time and set $N = \inf \{n \geq 0 : T_n \geq T\}$; then N is $\mathcal{F}_T$ -measurable. Hence, by the above inequality, we have

$$\begin{aligned} E[A_\infty - A_T | \mathcal{F}_T] &= \sum_{0 \leq k \leq N} E[\gamma_{T_k} 1_{\{T_k < \infty\}} | \mathcal{F}_T] 1_{\{N=k\}} \\ &\leq \sum_{0 \leq k \leq N} \delta^{-n+k} E[\gamma_{T_k} 1_{\{T_k < \infty\}} | \mathcal{F}_T] 1_{\{N=k\}} \\ &\leq \sum_{0 \leq k \leq N} \delta^{-n+k} \gamma_T 1_{\{T < \infty\}} 1_{\{N=k\}} \\ &= \frac{\delta}{\delta - 1} \gamma_T 1_{\{T < \infty\}}, \end{aligned} \quad (4.8)$$

which implies that $A \in \mathcal{V}^{p'}$. Now let U be a random variable defined by $U = \sup \{T_n : T_n < \infty\}$, which is not necessarily a stopping time. Then we have

$$Y_\infty = \sum_{n=1}^{\infty} (Y_{T_{n+1}-} - Y_{T_n-}) 1_{\{T_{n+1} < \infty\}} + Y_\infty - Y_{U-}, \quad (4.9)$$

where $Y_{U-} = Y_\infty$ if $U = \infty$. Note that the series converges in L^1 : for we have $|Y_{T_{n+1}-} - Y_{T_n-}| \leq \delta\gamma_{T_{n+1}-} \leq \delta K\gamma_{T_{n+1}}$ and hence

$$\sum_{n=0}^{\infty} \|(Y_{T_{n+1}-} - Y_{T_n-})1_{\{T_{n+1} < \infty\}}\|_1 \leq \|A_{\infty}\|_1 < \infty.$$

For each $t \geq 0$, put $B_t = \sum_{n=0}^{\infty} (Y_{T_{n+1}-} - Y_{T_n-})1_{\{T_{n+1} \leq t\}}$. Then $B = (B_t)$ is a process of integrable variation and

$$E[X_{\infty} Y_{\infty}] = E[X_{\infty} B_{\infty}] + E[X_{\infty} (Y_{\infty} - Y_{U-})]. \quad (4.10)$$

We shall calculate the right-hand side. On one hand, using (4.8) we have

$$\begin{aligned} |E[X_{\infty} B_{\infty}]| &= \left| \sum_{n=0}^{\infty} E[X_{\infty} (Y_{T_{n+1}-} - Y_{T_n-})1_{\{T_{n+1} < \infty\}}] \right| \\ &\leq \sum_{n=0}^{\infty} |E[X_{T_{n+1}} (Y_{T_{n+1}-} - Y_{T_n-})1_{\{T_{n+1} < \infty\}}]| \\ &\leq \delta K \sum_{n=0}^{\infty} E[X_{T_{n+1}}^* \gamma_{T_{n+1}} 1_{\{T_{n+1} < \infty\}}] \\ &\leq \delta KE \left[X_s^* dA_s \right] = \delta KE \left[(A_{\infty} - A_{s-}) \right] \\ &\leq \frac{\delta^2 K}{\delta - 1} E \left[\gamma_s \right] = \frac{\delta^2 K}{\delta - 1} E[\gamma X^*]. \end{aligned} \quad (4.11)$$

On the other hand, we have $|Y_{\infty} - Y_{U-}| \leq \delta \gamma$ and therefore

$$|E[X_{\infty} (Y_{\infty} - Y_{U-})]| \leq \delta E[\gamma X^*].$$

From (4.10), (4.11) and the above inequality, it follows that

$$|E[X_{\infty} Y_{\infty}]| \leq \frac{2\delta^2 - \delta}{\delta - 1} KE[\gamma X^*].$$

Since the smallest value of $(2\delta^2 - \delta)/(\delta - 1)$ for $\delta > 1$ is equal to $3 + 2\sqrt{2}$, we obtain (4.6) and the lemma is proved. $\blacksquare$

Lemma 4.5 Suppose that L^p has absolutely continuous norm and the Doob-type inequality holds in $H^{p'}$. Then for each $J \in (H^p)^*$, there exists a martingale $Y = (Y_t) \in \mathcal{K}^{p'}$ such that $\|Y\|_{\mathcal{K}^{p'}} \leq 48C^2\|J\|$ and

$$J(X) = E[X_{\infty} Y_{\infty}] \quad (4.12)$$

holds for every bounded martingale X , where $C > 0$ is a constant appearing in the Doob-type inequality. Moreover if every $(\mathcal{F}_t)$ -martingale is continuous, then the hypothesis that the Doob-type inequality holds in $H^{p'}$ can be removed, and 24 replaces $48C^2$.

Proof. According to Theorem 4.2, there exists a process $A = (A_t) \in \mathcal{V}^{p'}$ such that $\|A\|_{\mathcal{V}^{p'}} \leq 24C\|J\|$ and $J(X) = E[X_{\infty} A_{\infty}]$ holds for every bounded martingale X . Let $Y = (Y_t)$ be the martingale induced by A_{∞} ; then (4.12) is obvious. If γ satisfies (4.1), then

$$|Y_T - A_{T-}| \leq E[|A_{\infty} - A_{T-}| | \mathcal{F}_T] \leq \gamma_T$$

holds for every stopping time T . The optional section theorem ([7, p. 220]) therefore yields that $|Y_{\cdot} - A_{\cdot-}| \leq \gamma_{\cdot}$ outside an evanescent set. Taking the left-hand limit of this inequality, we also have $|Y_{\cdot-} - A_{\cdot-}| \leq \gamma_{\cdot-}$. Hence we see that

$$E[|Y_{\infty} - Y_{T-}| | \mathcal{F}_T] \leq E[|A_{\infty} - A_{T-}| | \mathcal{F}_T] + |A_{T-} - Y_{T-}| \leq \gamma_T + \gamma_{T-} \leq 2\bar{\gamma}_T,$$

where $\bar{\gamma} \in L^{p'}$ satisfies (4.3) and $\|\bar{\gamma}\|_{p'} \leq C\|\gamma\|_{p'}$. From this it follows that $\|Y\|_{\mathcal{K}^{p'}} \leq 2C\|A\|_{\mathcal{V}^{p'}} \leq 48C^2\|J\|$. Thus Y has the required properties.

If every $(\mathcal{F}_t)$ -martingale is continuous, then we can obtain $\|Y\|_{p'} \leq 2\|A\|_{\mathcal{V}^{p'}} \leq 24\|J\|$ without the hypothesis of Doob-type inequality in $H^{p'}$, and the lemma is established. $\blacksquare$

Combining the above two lemmas, we obtain the other duality theorem.

Theorem 4.6 Assume that every $(\mathcal{F}_t)$ -martingale is continuous. If L^p has absolutely continuous norm, then $(H^p)^* = \mathcal{K}^{p'}$.

Proof. Let $Y \in \mathcal{K}^{p'}$ and set $J_Y(X) = E[X_{\infty} Y_{\infty}]$ for bounded martingales X . It is clear from Lemma 4.4 that

$$|J_Y(X)| \leq (3 + 2\sqrt{2})\|X\|_{H^p}\|Y\|_{\mathcal{K}^{p'}},$$

since (γ_t) is continuous. Now note that the set of bounded martingales is dense in $H^{p'}$. To see this, for each $X \in H^p$, put $T_n = \inf \{t \geq 0: |X_t - X_0| > n\}$ and $X_t^n = X_t^n 1_{\{|X_0| \leq n\}}$. Then each X^n is a bounded martingale,

and $L^p \ni 2X^* \geq (X^n - X)^* \rightarrow 0$ a.s. as $n \rightarrow \infty$. Since L^p has absolutely continuous norm, we have $\|X^n - X\|_{H^p} \rightarrow 0$ as $n \rightarrow \infty$. Thus the set of bounded martingales is dense in H^p . Therefore J_Y extends uniquely to a continuous linear functional on $H^{p'}$, which is denoted by J_Y again, and $\|J_Y\| \leq (3 + 2\sqrt{2})\|Y\|_{\mathcal{H}^{p'}}$.

On the other hand, Lemma 4.5 shows that if $J \in (H^p)^*$, then $J = J_Y$ for some $Y \in \mathcal{H}^{p'}$ and $\|Y\|_{\mathcal{H}^{p'}} \leq 24\|J\|$. This completes the proof. ■

Remark. In the case where $L^p = L^1$, Theorem 4.6 reduces to the well-known H^1 -BMO duality theorem, if we can remove the extrahypothesis of the continuity of $(\mathcal{F}_t)$ -martingales. In fact, this is possible, since we may take a positive constant in place of γ in Lemmas 4.4 and 4.5.

5 Martingale space and rearrangement invariant function spaces

In this section we shall consider the linear functionals on the space H^p associated with a r.i. function space L^p . Throughout this section we assume that Ω has no atom and L^p is r.i.

In the preceding section we have proved that $(H^p)^* = \mathcal{H}^{p'}$ if every $(\mathcal{F}_t)$ -martingale is continuous. On the other hand, Doob's inequality applied to the L^p -norm gives that, if $1 < p < \infty$ and $1/p + 1/p' = 1$, then $(H^p)^* = (L^p)^* = L^{p'} = H^{p'}$. Hence it is natural to ask when $H^{p'}$ and $\mathcal{H}^{p'}$ are equal. We shall study this problem in more general setting.

Definition 5.1 Let $1 \leq p < \infty$. $\mathcal{H}_p^p$ denotes the space of uniformly integrable martingales $X = (X_t)$ such that there exists a $\gamma \in L^p$ satisfying

$$E[|X_\infty - X_{T-}|^p | \mathcal{F}_T] \leq E[\gamma^p | \mathcal{F}_T] \quad (5.1)$$

for all stopping times T . The norm of $X \in \mathcal{H}_p^p$ is given by

$$\|X\|_{\mathcal{H}_p^p} = \inf \{ \|\gamma\|_p : \gamma \text{ satisfies (5.1)} \}.$$

It is obvious that $\mathcal{H}^p = \mathcal{H}_2^p$. Garsia [9] studied the spaces $\mathcal{H}_2^p$, i.e. the case where $L^p = L^2$ and $p = 2$, in detail: he proved that if $1 \leq p \leq 2$, then $(H^p)^* = \mathcal{H}_2^{p'}$, where $1/p + 1/p' = 1$.

Suppose that $X \in H^p$. Setting $\gamma = 2X^*$, we have (5.1) and hence $\|X\|_{\mathcal{H}_p^p} \leq 2\|X^*\|_p = 2\|X\|_{H^p}$. Thus $H^p \hookrightarrow \mathcal{H}_p^p$ holds for every finite $p \geq 1$. To obtain more relations between H^p and $\mathcal{H}_p^p$, we need some preliminaries.

Lemma 5.2 Let $p > 1$ and $1/p + 1/p' = 1$. If x is a nonnegative decreasing function on the interval $I = [0, 1]$, then

$$\left\{ \int_0^t x(s)^p ds \right\}^{1/p} \leq p^{-1} \int_0^t x(s) s^{-1/p'} ds$$

holds for every $t \in I$.

Proof. Suppose first that x is of the form

$$x(t) = \sum_{k=1}^n a_k 1_{[0, t_k]}(t), \quad t \in I, \quad (5.2)$$

where $a_k \geq 0$ and $0 = t_0 < t_1 < \dots < t_n \leq 1$. Using Minkowski's inequality we have

$$\begin{aligned} \left\{ \int_0^t x(s)^p ds \right\}^{1/p} &\leq \sum_{k=1}^n a_k \left\{ \int_0^t 1_{[0, t_k]}(s) ds \right\}^{1/p} = \sum_{k=1}^n a_k (t \wedge t_k)^{1/p} \\ &= p^{-1} \sum_{k=1}^n a_k \int_0^{t \wedge t_k} s^{-1/p'} ds = p^{-1} \int_0^t x(s) s^{-1/p'} ds. \end{aligned}$$

For an arbitrary decreasing function x , we can choose a sequence of the functions (x_n) of the form (5.2) such that $0 \leq x_n \uparrow x$ a.e. to obtain the lemma. ■

Now let $x \in L^1(I)$ and set

$$\begin{aligned} \mathcal{P}_p x(t) &= t^{-1/p} \int_0^t x(s) s^{-1/p'} ds, \quad 0 < t < 1; \\ \mathcal{Q}_p x(t) &= t^{-1/p} \int_t^1 x(s) s^{-1/p'} ds, \quad 0 < t < 1, \end{aligned}$$

where $p \geq 1$, and p' stands for the exponent conjugate to p . For the sake of simplicity, $\mathcal{P}_1$ (resp. $\mathcal{Q}_\infty$) will be denoted by $\mathcal{P}$ (resp. $\mathcal{Q}$). The following lemma is due to Boyd. For the proof see [5].

Lemma 5.3 Let L^λ be a r.i. function space over the unit interval I and $p \geq 1$. Then:

- i) the operator $\mathcal{P}_p: L^\lambda \rightarrow L^\lambda$ is bounded if and only if $\bar{\alpha}_\lambda < 1/p$;
- ii) the operator $\mathcal{Q}_p: L^\lambda \rightarrow L^\lambda$ is bounded if and only if $\underline{\alpha}_\lambda > 1/p$.

Using the above lemmas, we have:

Lemma 5.4 *Let $1 \leq p < \infty$ and $\eta(\Omega)$ be a r.i. function space such that $0 < \underline{\alpha}_p \leq \bar{\alpha}_p < 1/p$. If $0 \leq f \in L^1$, $0 \leq g \in \eta$ and*

$$(\eta_1 - \eta_2)P(f \geq \eta_1) \leq \int_{\{f > \eta_2\}} g^p dP \quad (5.3)$$

holds for all η_1 and η_2 such that $\eta_1 \geq \eta_2 \geq 0$, then $f \in L^p$ and $\|f\|_p \leq C_{p,p} \|g\|_p$, where $C_{p,p} > 0$ is a constant depending only on p and the norm in the space L^p .

Proof. We may assume that f is bounded and hence in L^p , since $f \wedge n$ satisfies (5.3) if f satisfies (5.3). Let $0 < t \leq 1$, $s > 1$, $\eta_1 = f^r(t)$ and $\eta_2 = D_s f^r(t)$: then $\eta_1 \geq \eta_2$ and (5.3) gives that

$$(f^r(t) - D_s f^r(t))^p P(f \geq f^r(t)) \leq \int_{\{f > D_s f^r(t)\}} g^p dP \leq \int_0^{P(f > D_s f^r(t))} \{g^r(u)\}^p du.$$

Since $P(f \geq f^r(t)) \geq t$ and $P(f > D_s f^r(t)) \leq st \wedge 1$, we have

$$\begin{aligned} f^r(t) &\leq D_s f^r(t) + \left[\frac{1}{t} \int_0^{st \wedge 1} \{g^r(u)\}^p du \right]^{1/p} \\ &\leq D_s f^r(t) + \left[\frac{1}{t} \int_0^t \{g^r(u)\}^p du \right]^{1/p} + \left[\frac{1}{t} \int_t^{st \wedge 1} \{g^r(u)\}^p du \right]^{1/p} \\ &\leq D_s f^r(t) + p^{-1} \mathcal{P}_p g^r(t) + (s-1)^{1/p} g^r(t), \end{aligned}$$

where the last inequality follows from Lemma 5.2. Let $L^\lambda(I)$ be a r.i. function space such that $\|h\|_\rho = \|h'\|_\lambda$ holds for all $h \in L^p(\Omega)$. As $\bar{\alpha}_\lambda = \bar{\alpha}_p < 1/p$, Lemma 5.3 gives that the operator $\mathcal{P}_p: L^\lambda \rightarrow L^\lambda$ is bounded. Hence it follows that

$$\|f\|_\rho = \|f^r\|_\lambda \leq \|D_s\|_\lambda \|f\|_\rho + p^{-1} \|\mathcal{P}_p\|_\lambda \|g\|_\rho + (s-1)^{1/p} \|g\|_\rho,$$

where $\|D_s\|_\lambda$ and $\|\mathcal{P}_p\|_\lambda$ denote the norm of operators D_s and $\mathcal{P}_p$ on L^λ into itself, respectively. Since $\underline{\alpha}_\lambda = \underline{\alpha}_p > 0$ implies that $\lim_{s \rightarrow \infty} \|D_s\|_\lambda = 0$, we can find an $s > 1$ such that $\|D_s\|_\lambda < 1$. It then follows that

$$\|f\|_\rho \leq \frac{1}{1 - \|D_s\|_\lambda} (p^{-1} \|\mathcal{P}_p\|_\lambda + (s-1)^{1/p}) \|g\|_\rho.$$

This proves the theorem. ■

Now we can prove the following theorem.

Theorem 5.5 *If $1 \leq p < \infty$ and L^p is a r.i. function space such that $0 < \underline{\alpha}_p \leq \bar{\alpha}_p < 1/p$, then $\mathcal{H}_p^\rho = H^p$.*

Recall that $H^p \hookrightarrow \mathcal{H}_p^\rho$. Therefore, to prove the theorem, it is sufficient to show that $\mathcal{H}_p^\rho \hookrightarrow H^p$. This is a straightforward consequence of the following lemma, because (5.1) implies (5.4).

Lemma 5.6 *Let L^p be as in Theorem 5.5, $\gamma \in L^p$ and $X = (X_t)$ be a càdlàg process. If*

$$E[|X_T - X_{S-}|^p | \mathcal{F}_S] \leq E[\gamma^p | \mathcal{F}_S] \quad (5.4)$$

holds for all stopping times S and T such that $S \leq T$, then $\|X^\|_\rho \leq C_{p,p} \|\gamma\|_\rho$.*

Proof. Let $\eta_1 > \eta_2 > 0$, and define the stopping times S and T by $S = \inf \{t \geq 0: |X_t| > \eta_2\}$ and $T = \inf \{t \geq 0: |X_t| > \eta_1\}$. Then $|X_T| \geq \eta_1$ a.s. on $\{T < \infty\}$, $|X_{S-}| \leq \eta_2$ and $S \leq T$ a.s. on Ω . Therefore we have $(\eta_1 - \eta_2)^p \leq |X_T - X_{S-}|^p$ on $\{T < \infty\}$ and

$$\begin{aligned} (\eta_1 - \eta_2)^p P(X^* > \eta_1) &\leq E[|X_T - X_{S-}|^p 1_{\{T < \infty\}}] \\ &\leq E[|X_T - X_{S-}|^p 1_{\{S < \infty\}}] \\ &\leq E[\gamma^p 1_{\{S < \infty\}}] = \int_{\{X^* > \eta_2\}} \gamma^p dP. \end{aligned}$$

This means that (5.3) holds for $f = X^*$ and $g = \gamma$; hence we obtain the result by Lemma 5.4. ■

By Theorem 5.5 we have $\mathcal{H}^\rho = H^p$ if $0 < \underline{\alpha}_p \leq \bar{\alpha}_p < 1$. Furthermore, as the following theorem shows, the converse is also true. The rest of this note is devoted to the proof of this result.

Theorem 5.7 *There exists a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)$ which satisfies the following conditions: if L^p is a r.i. function space and $\mathcal{H}^\rho \hookrightarrow H^p$, then $0 < \underline{\alpha}_p < \bar{\alpha}_p < 1$.*

To prove this theorem, we use an excellent theorem due to Azéma and Yor. Let $\varphi \in L^1(I)$ be a decreasing function such that $\int_I \varphi(s) ds = 0$, and $\psi_\varphi(s)$ be the function defined on R by

$$\psi_\varphi(s) = \begin{cases} \frac{1}{m(\varphi \geq s)} \int_{\{\varphi \geq s\}} \varphi dm, & \text{if } m(\varphi \geq s) > 0, \\ s, & \text{otherwise,} \end{cases}$$

where m denotes the Lebesgue measure. We have:

Theorem 5.8 (Azéma and Yor [2,3]) *Let $X = (X_t)$ be a continuous martingale with $X_0 = 0$, φ as above, and T be the stopping time defined by*

$$T = \inf \{t \geq 0 : S_t \geq \psi_\varphi(X_t)\}, \quad S_t = \sup_{s \leq t} X_s.$$

Then:

- i) *the stopped martingale $(X_{T \wedge t})$ is uniformly integrable;*
- ii) *the distribution of X_T is equal to the distribution of φ with respect to the Lebesgue measure;*
- iii) *the distribution of S_T is equal to the distribution of $\mathcal{P}\varphi$ with respect to the Lebesgue measure.*

Proof of Theorem 5.7. Let $B = (B_t)$ be a one dimensional standard Brownian motion on a probability space $(\Omega', \mathcal{F}', P')$, and $(\mathcal{P}_t')$ be the filtration generated by B . Besides this, we take the probability space $I = [0, 1]$ with the σ -field $\mathcal{G}$ of Lebesgue measurable subsets of I and the Lebesgue measure m . For each $t \in I$, let $\mathcal{G}_t$ denote the sub- σ -field of $\mathcal{G}$ generated by the Lebesgue measurable subsets of $[1 - t, 1]$, and put $\mathcal{G}_t = \mathcal{G}_1$ for $t \geq 1$. We set

$$\Omega = \Omega' \times I, \quad \mathcal{F}_t^\circ = \mathcal{F}_t' \otimes \mathcal{G}_t \quad (0 \leq t < \infty), \quad P = P' \otimes m.$$

Let $(\mathcal{F}_t)$ be the augmentation of $(\mathcal{F}_t^\circ)$: $\mathcal{F}_t = \bigcap_{s > t} \sigma(\mathcal{F}_s \cup \mathcal{N})$, where $\mathcal{N}$ denotes the collection of all P -negligible sets. Then $(\mathcal{F}_t)$ satisfies the usual conditions. We prove here that, if

$$\|X\|_{H^p} \leq C \|X\|_{\mathcal{H}^p} \quad (5.5)$$

holds for every $((\mathcal{F}_t), P)$ -martingale X , then $0 < \underline{\alpha}_p \leq \bar{\alpha}_p < 1$.

Let $X = (X_t)$ be the process on Ω defined by

$$X_t(\omega', s) = B_{T \wedge t}(\omega'), \quad (\omega', s) \in \Omega = \Omega' \times I.$$

Then X is a continuous $((\mathcal{F}_t), P)$ -martingale.

Let $L^\lambda(I)$ be a r.i. function space satisfying $\|h\|_\rho = \|h'\|_\lambda$ for all $h \in L^\rho(\Omega)$, and $\varphi \in L^\lambda(I)$ be a decreasing function such that $\int_I \varphi(s) ds = 0$. Define the stopping time T and the process $S = (S_t)$ as in Theorem 5.8. Since the inequality

$$E[|X_\infty^T - X_\tau^T| | \mathcal{F}_\tau] \leq 2E[|X_T| | \mathcal{F}_\tau]$$

holds for all stopping times τ , we have $\|X^T\|_{\mathcal{H}^p} \leq 2\|X_T\|_\rho$, where X^T denotes the stopped martingale $(X_{T \wedge t})$ as usual. This, together with (5.5), implies that $\|S_T\|_\rho \leq 2C\|X_T\|_\rho$. Since the distribution of S_T (resp. X_T) is equal to the distribution of $\mathcal{P}\varphi$ (resp. φ) by Theorem 5.8, we obtain

$$\|\mathcal{P}\varphi\|_\lambda \leq 2C\|\varphi\|_\lambda.$$

Now let $\varphi \in L^\lambda$ be an arbitrary decreasing function. Then the above inequality rewritten as

$$\|\mathcal{P}\varphi - \varphi_I\|_\lambda \leq 2C\|\varphi - \varphi_I\|_\lambda,$$

where φ_I denotes the integral of φ over I . As $L^\lambda \hookrightarrow L^1$, we have

$$\|\mathcal{P}\varphi\|_\lambda \leq 2C\|\varphi\|_\lambda + (2C + 1)\|\varphi\|_1\|1\|_\lambda \leq K\|\varphi\|_\lambda,$$

with $K = 2C + (1 + 2C)\|1\|_\lambda$. Thus we see that $\|\mathcal{P}\varphi\|_\lambda \leq K\|\varphi\|_\lambda$ holds for every decreasing function $\varphi \in L^\lambda$. According to [11], this implies that $\mathcal{P}$ is a bounded operator on L^λ into itself. Hence we have $\bar{\alpha}_p = \bar{\alpha}_\lambda < 1$ by Lemma 5.3.

We now prove that $\underline{\alpha}_p = \underline{\alpha}_\lambda > 0$, or equivalently that the operator $\mathcal{Q}: L^\lambda \rightarrow L^\lambda$ is bounded. Let $\varphi \in L^1(I)$ be an arbitrary function, and $\bar{X} = (X_t)$ be the process defined by

$$X_t(\omega', s) = \begin{cases} \mathcal{P}\varphi(1 - t) & \text{if } s < 1 - t, \\ \varphi(s) & \text{if } 1 - t \leq s \leq 1. \end{cases} \quad (5.6)$$

Then $X = (X_t)$ is a uniformly integrable $((\mathcal{F}_t), P)$ -martingale such that $X^*(\omega', s) \geq |\mathcal{P}\varphi(s)|$. Note that almost every path of X is of bounded variation; thus the continuous (local) martingale part of X vanishes. If we put $T(\omega', s) = 1 - s$ for each $(\omega', s) \in \Omega$, then T is an $(\mathcal{P}_t)$ -stopping time, and the each path of X jumps at T only. This implies that

$$[X, X]_\infty^{1/2}(\omega', s) \leq |\Delta X_T(\omega', s)| + \|\varphi\|_{L^1} \leq |\varphi(s) - \mathcal{P}\varphi(s)| + \|\varphi\|_{L^1}. \quad (5.7)$$

Now note that the inequality

$$E[|X_\infty - X_{\tau-}| | \mathcal{F}_\tau] \leq E[\sup_{t \geq \tau} |X_t - X_{\tau-}| | \mathcal{F}_\tau]$$

$$\begin{aligned} &\leq KE([X, X]_\infty - [X, X]_{\tau-})^{1/2} |\mathcal{F}_\tau| \\ &\leq KE([X, X]_\infty^{1/2} |\mathcal{F}_\tau|) \end{aligned}$$

holds for every stopping time τ , where the middle inequality follows from the conditional form of Davis' inequality. Hence we have

$$\|X\|_{\mathcal{H}^p} \leq K \| [X, X]_\infty^{1/2} \|_p. \quad (5.8)$$

Form (5.5), (5.7), (5.8) and the fact that $X^*(\omega', s) \geq |\mathcal{P}\phi(s)|$, it follows that

$$\|\mathcal{P}\phi\|_\lambda \leq \|X\|_{H^p} \leq C \|X\|_{\mathcal{H}^p} \leq CK \| [X, X]_\infty^{1/2} \|_p \leq C' (\|\phi - \mathcal{P}\phi\|_\lambda + \|\phi\|_1). \quad (5.9)$$

Now let ψ be an arbitrary integrable function on I , and set $\phi = \mathcal{Q}\psi' - \psi'$. Then we have $\phi \in L^1(I)$ and $\mathcal{P}\phi = \mathcal{Q}\psi' = \phi + \psi'$, because $\mathcal{P}\mathcal{Q} = \mathcal{P} + \mathcal{Q}$. Form the above inequality we obtain

$$\|\mathcal{Q}\psi'\|_\lambda = \|\mathcal{P}\phi\|_\lambda \leq C' (\|\psi'\|_\lambda + \|\phi\|_1) \leq C'' \|\psi\|_\lambda. \quad (5.10)$$

On the other hand, for every $t \in I$ we have

$$\int_0^t \mathcal{Q}\psi(s) ds = \int_0^t \psi(s) \left(\frac{s \wedge t}{s} \right) ds \leq \int_0^t \psi(s) \left(\frac{s \wedge t}{s} \right) ds = \int_0^t \mathcal{Q}\psi(s) ds,$$

where we have used the Hardy's inequality ([4, p. 56]). This, together with (5.10), implies that

$$\|\mathcal{Q}\psi\|_\lambda \leq C'' \|\psi\|_\lambda.$$

Thus $\mathcal{Q}: L^\lambda \rightarrow L^\lambda$ is bounded and the theorem is established. $\blacksquare$

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