Saturation Problem in Approximation of Functions by Some Operators Associated with the Generalized Jackson's Operators

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In this paper, we shall introduce some operators that are associated with the generalized Jackson's operators, and determine the saturation classes and orders in approximation of functions by those operators.

KEYWORDS: Jackson's operator, saturation in approximation theory

1 Introduction

The purpose of the present paper lies in determining the classes and orders of saturation by some linear operators which are introduced in the section §2. The saturation problem had been solved for some cases of those operators nonsystematically, but for a special one of those operators, its saturation class has been indistinct for a long time. For the other several cases of those operators, their saturation classes and orders are determined newly here.

Let $f(x)$ be periodic with period 2π and belong to the space $X (= C_{2\pi}, \text{ or } L_{2\pi}^p, 1 \leq p < \infty)$, and let its Fourier series be

$$S[f] = \sum_{k=0}^{\infty} A_k(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Let us consider a family of linear operators of convolution type:

$$\begin{aligned} L_n(f; x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) U_n(t) dt \quad (n \rightarrow \infty) \\ &= \sum_{k=0}^{\infty} \rho_k^{(n)} A_k(x), \end{aligned} \tag{1.1}$$

where

$$U_n(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \rho_k^{(n)} \cos kx, \tag{1.2}$$

and

$$\frac{1}{\pi} \int_{-\pi}^{\pi} U_n(t) dt = 1 \quad \text{for all } n. \tag{1.3}$$

The convergence factor $\rho_k^{(n)}$ is given by

$$\begin{aligned} \rho_k^{(n)} &= \frac{2}{\pi} \int_0^{\pi} U_n(t) \cos kt dt \quad (k = 0, 1, \dots, n), \\ &= 0 \quad (k > n). \end{aligned} \tag{1.4}$$

That is to say, it is the real Fourier coefficient of (1.2) and the kernel $U_n(x)$ is a trigonometric polynomial of degree n . If the positivity of $f(x)$ implies the positivity of a linear operator $L_n(f; x)$, the operator is called a linear positive operator. Furthermore, let us suppose that the linear operator (1.1) is a linear approximation process, i.e.,

$$\lim_{n \rightarrow \infty} \|L_n(f) - f\|_X = 0, \quad \text{for all } f \in X. \tag{1.5}$$

Necessary and sufficient conditions for (1.5) are

$$\left. \begin{array}{l} \text{(a) } \exists M(>0): \int_0^\pi |U_n(t)| dt \leq M \quad (n \rightarrow \infty), \\ \text{and} \\ \text{(b) } \lim_{n \rightarrow \infty} \rho_k^{(n)} = 1, \text{ for all } k. \end{array} \right\} \quad (1.6)$$

(It is a well-known fact in the approximation theory. For example, see [4] or [7].)

Also, let us write

$$\text{Lip}(1; X) \equiv \left\{ f(x) \in X \mid \sup_{|h| \leq \delta} \|f(x+h) - f(x)\|_X = O(\delta) \right\},$$

where the norm be taken with respect to the variable x . Denoting the trigonometric moment of order 2σ of $U_n(t)$ by

$$T(2\sigma; n) = \frac{2}{\pi} \int_0^\pi \left(2 \sin \frac{t}{2} \right)^{2\sigma} U_n(t) dt \quad (\sigma = 0, 1, 2, \dots),$$

and the algebraic moment of order 2σ of $U_n(t)$ by

$$M(2\sigma; n) = \frac{2}{\pi} \int_0^\pi t^{2\sigma} U_n(t) dt \quad (\sigma = 0, 1, 2, \dots),$$

we know the following theorems which are applied later.

Theorem A (P. L. Butzer [2] and G. Sunouchi [11], [12])

Suppose that

$$\exists c \text{ (constant): } \lim_{n \rightarrow \infty} \frac{1 - \rho_k^{(n)}}{\varphi(n)\psi(k)} = c \neq 0 \quad (k = 1, 2, \dots), \quad (1.7)$$

where $\varphi(n)$ is positive, non-increasing and $\psi(k)$ is non-vanishing. Then, if the formal trigonometric series

$$f^{[\psi]}(x) \equiv \sum_{k=1}^{\infty} \psi(k) A_k(x) \quad (1.8)$$

represents

$$\begin{array}{ll} \text{the Fourier series of a function of } L^\infty & \text{(for } C\text{-norm),} \\ \text{the Fourier series of a function of } L^p & \text{(for } L^p\text{-norm, } 1 < p < \infty), \\ \text{the Fourier-Stieltjes series of a function of } BV & \text{(for } L\text{-norm)} \end{array}$$

respectively, and if

$$\lambda_k(n) = \frac{1 - \rho_k^{(n)}}{\varphi(n)\psi(k)} \quad (1.9)$$

is a multiplier uniform in n

$$\begin{array}{ll} \text{from } L^\infty \text{ to } L^\infty & \text{(for } C\text{-norm),} \\ \text{from } L^p \text{ to } L^p & \text{(for } L^p\text{-norm, } 1 < p < \infty), \\ \text{from } dV \text{ to } dV & \text{(for } L\text{-norm)} \end{array}$$

respectively, we have

$$\|L_n(f) - f\|_X = O\{\varphi(n)\}.$$

Remark [1.1] Since we know (L^∞, L^∞) -multiplier = (dV, dV) -multiplier and a (L^∞, L^∞) -multiplier is a (L^p, L^p) -multiplier ($1 < p < \infty$), the hypothesis of this theorem can be replaced by the condition that $\lambda_k(n)$ is a (L^∞, L^∞) -multiplier uniform in n for all these norms.

Remark [1.2] If $\lambda_k(n)$ is quasi-convex uniformly in n , that is

$$\sum_{k=1}^{\infty} k |\Delta^2 \lambda_k| = \sum_{k=1}^{\infty} k |\lambda_{k-1}(n) - 2\lambda_k(n) + \lambda_{k+1}(n)|$$

is uniformly bounded, then $\lambda_k(n)$ is a (L^∞, L^∞) -multiplier uniformly in n . (See, A. Zygmund [14].)

Theorem B (K. Azuma and Y. Suzuki [1]) Suppose that

$$\lim_{n \rightarrow \infty} \frac{T(2\sigma; n)}{T(2; n)} = 0 \quad (\sigma = 2, 3, \dots). \quad (1.10)$$

Then for any function $f(x) \in X$, we get the following (1°), (2°) and (3°).

- (1°) $\|L_n(f) - f\|_X = o(1 - \rho_1^{(n)})$ if and only if $f(x)$ is constant.
- (2°) If $\|L_n(f) - f\|_X = O(1 - \rho_1^{(n)})$, then $f'(x)$ exists and belongs to $Lip(1; X)$.
- (3°) If $f'(x)$ exists and belongs to $Lip(1; X)$ with the assumption:

$$\int_0^\pi t^2 |U_n(t)| dt = O(1 - \rho_1^{(n)}), \quad (1.11)$$

then

$$\|L_n(f) - f\|_X = O(1 - \rho_1^{(n)}).$$

In particular, if $U_n(x) \geq 0$, (1.11) is true always.

Hereafter, this result is represented by the notation:

$$Sat[L_n] = [\{f \in X \mid f' \in Lip(1; X)\}, 1 - \rho_1^{(n)}, f = \text{constant}].$$

Remark [1.3] If $U_n(x) \geq 0$, this theorem is reduced to Y. Suzuki's theorem in the local case. (see [3].)

Remark [1.4] The statements (2°) and (3°) of the theorem are designated as the inverse part and the direct part of saturation, respectively.

Theorem C (E. L. Stark [7], [8], [9]) For arbitrary kernel $U_n(x)$ with the first convergence factor $\rho_1^{(n)} \neq 1$, the following two assertions are equivalent:

$$(a) \quad \lim_{n \rightarrow \infty} \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2 \quad (k = 2, 3, \dots). \quad (1.12)$$

$$(b) \quad \lim_{n \rightarrow \infty} \frac{T(2\sigma; n)}{T(2; n)} = 0 \quad (\sigma = 2, 3, \dots). \quad (1.13)$$

Remark [1.5] Here it is by no means necessary that the kernel $U_n(x)$ is positive. In the particular case of positive kernel, we have the following enlarged version of the Theorem C.

Theorem D (P. P. Korovkin [5], E. L. Stark [7], [8], [9], K. Ikeno-Y. Suzuki [3], and T. Nishishiraho-O. Takenouchi [6])

For kernel $U_n(x) \geq 0$, the following assertions are equivalent:

$$(a) \quad \lim_{n \rightarrow \infty} \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2 \quad (k = 2, 3, \dots).$$

$$(b) \quad \lim_{n \rightarrow \infty} \frac{1 - \rho_2^{(n)}}{1 - \rho_1^{(n)}} = 4.$$

$$(c) \quad \lim_{n \rightarrow \infty} \frac{T(4; n)}{T(2; n)} = 0.$$

$$(d) \quad \lim_{n \rightarrow \infty} \frac{T(2\sigma; n)}{T(2; n)} = 0 \quad (k = 2, 3, \dots).$$

$$(e) \quad \int_0^\pi U_n(t) \sin^4 \frac{t}{2} dt = o(1 - \rho_1^{(n)}) \quad (n \rightarrow \infty).$$

$$(f) \quad \text{For an arbitrary } \delta (0 < \delta < \pi), \int_\delta^\pi U_n(t) dt = o(1 - \rho_1^{(n)}) \quad (n \rightarrow \infty).$$

Lemma A (E. L. Stark [7], [8], [9]) The following relations are effected

$$(i) \quad T(2\sigma; n) = 2 \sum_{k=1}^{\sigma} (-1)^{k+1} \binom{2\sigma}{\sigma-k} (1 - \rho_k^{(n)}) \quad (\sigma = 1, 2, \dots).$$

$$(ii) \quad 1 - \rho_k^{(n)} = \sum_{\sigma=1}^k (-1)^{\sigma+1} \frac{k}{k+\sigma} \binom{k+\sigma}{2\sigma} T(2\sigma; n) \quad (k = 1, 2, \dots).$$

Lemma B If $U_n(x) \geq 0$, then we get

$$\left(\frac{2}{\pi}\right)^{2\sigma} M(2\sigma; n) \leq T(2\sigma; n) \leq M(2\sigma; n) \quad (\sigma = 0, 1, \dots).$$

Proof Since

$$\frac{x}{\pi} \leq \sin \frac{x}{2} \leq \frac{x}{2} \quad (0 \leq x \leq \pi),$$

it follows that

$$\begin{aligned} T(2\sigma; n) &\leq \frac{2}{\pi} \int_0^\pi \left(2 \cdot \frac{t}{2}\right)^{2\sigma} U_n(x) dx = M(2\sigma; n), \\ T(2\sigma; n) &\geq \frac{2}{\pi} \int_0^\pi \left(2 \cdot \frac{t}{\pi}\right)^{2\sigma} U_n(x) dx = \left(\frac{2}{\pi}\right)^{2\sigma} M(2\sigma; n). \end{aligned}$$

Corollary If $U_n(x) \geq 0$, then we obtain

$$M(2; n) = O(1 - \rho_1^{(n)}).$$

Proof From Lemma A and Lemma B, we get

$$M(2; n) \leq \frac{\pi^2}{4} T(2; n) = \frac{\pi^2}{2} (1 - \rho_1^{(n)}).$$

At the end of this section, in order to make sure, we shall reproduce the proof of Theorem B from the paper [1].

(Proof of Theorem B)

(1°) (→) From the hypothesis (1.10) and Theorem C,

$$\lim_{n \rightarrow \infty} \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2 \quad (k = 2, 3, \dots). \quad (1.14)$$

If $\|L_n(f; x) - f(x)\|_X = o(1 - \rho_1^{(n)})$, we have for every fixed $k \geq 1$,

$$a_k(1 - \rho_k^{(n)}) = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x) - L_n(f; x)) \cos kx dx = o(1 - \rho_1^{(n)}).$$

Furthermore, comparing this with (1.14), we have

$$a_k = 0 \quad (k = 1, 2, \dots).$$

Similarly, we obtain

$$b_k = 0 \quad (k = 1, 2, \dots).$$

Thus, $f(x)$ is constant.

(←) If $f(x)$ is constant, it is obvious that

$$\|L_n(f; x) - f(x)\|_X = o(1 - \rho_1^{(n)}),$$

because of the assumption of the kernel $U_n(t)$:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} U_n(t) dt = 1.$$

(2°) Let us write

$$\Delta_n(x) = f(x) - L_n(f; x)$$

and taking the N -th arithmetic mean $\sigma_N(\Delta_n; x)$ of the series:

$$\Delta_n(x) = \sum_{k=0}^{\infty} (1 - \rho_k^{(n)}) A_k(x),$$

we have

$$\sigma_N(\Delta_n; x) = \sum_{k=0}^N (1 - \rho_k^{(n)}) \left(1 - \frac{k}{N+1}\right) A_k(x).$$

Since it is well-known that

$$\|\Delta_n\|_X \geq \|\sigma_N(\Delta_n; x)\|_X$$

and

$$\|L_n(f; x) - f(x)\|_X = O(1 - \rho_1^{(n)}),$$

we get

$$\left\| \sum_{k=0}^N (1 - \rho_k^{(n)}) \left(1 - \frac{k}{N+1}\right) A_k(x) \right\|_X = O(1 - \rho_1^{(n)}),$$

or, equivalently

$$\left\| \sum_{k=0}^N \frac{(1 - \rho_k^{(n)})}{(1 - \rho_1^{(n)})} \left(1 - \frac{k}{N+1}\right) A_k(x) \right\|_X = O(1).$$

Hence, by the condition (1.10) and Theorem C, we obtain

$$\left\| \sum_{k=1}^N k^2 \left(1 - \frac{k}{N+1}\right) A_k(x) \right\|_X = O(1).$$

This is equivalent to the assertion that the formal trigonometric series $g(x) = \sum_{k=1}^{\infty} k^2 A_k(x)$ is the Fourier series of a bounded function (if $X = C$), the Fourier series of a function in L^p (if $X = L^p$ ($1 < p < \infty$)), the Fourier-Stieltjes series of a function of bounded variation (if $X = L$). (For example, see [15].)

These classes are also characterized by the property that the indefinite integral of $g(x)$ belongs to the class $Lip(1; X)$. Consequently, it effects that $f'(x)$ belongs to the class $Lip(1; X)$. (For instance, see [15].)
(3°) Since $f'(x)$ belongs to the class $Lip(1; X)$, we have

$$\begin{aligned} \|L_n(f; x) - f(x)\|_X &\leq \frac{1}{\pi} \int_0^\pi \|f(x+t) - 2f(x) + f(x-t)\|_X |U_n(t)| dt \\ &\leq \frac{M}{\pi} \int_0^\pi t^2 |U_n(t)| dt, \end{aligned}$$

where M is the Lipschitz's constant. By the assumption of our theorem, we get

$$\|L_n(f; x) - f(x)\|_X = O(1 - \rho_1^{(n)}).$$

■

2 Definition of Some Operators

We introduce two kinds of linear operators which are associated with the generalized Jackson's operators. Those operators are defined in the following manner.

$$(a) \quad J_n^{(q)}(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} U_n^{(q)}(t) f(x-t) dt = \sum_{k=0}^{\infty} \rho_k^{(n)} A_k(x) \quad (q = 2, 3, 4, \dots),$$

where

$$U_n^{(q)}(t) = \frac{1}{A_q(n)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q, \quad \frac{1}{\pi} \int_{-\pi}^{\pi} U_n^{(q)}(t) dt = 1,$$

and

$$(b) \quad \bar{J}_n^{(q)}(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \bar{U}_n^{(q)}(t) f(x-t) dt = \sum_{k=0}^{\infty} \bar{\rho}_k^{(n)} A_k(x), \quad \text{for a real number } q > 1,$$

where

$$\bar{U}_n^{(q)}(t) = \frac{1}{\bar{A}_q(n)} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q, \quad \frac{1}{\pi} \int_{-\pi}^{\pi} \bar{U}_n^{(q)}(t) dt = 1,$$

and

$$\bar{\rho}_0^{(n)} = 1, \quad \bar{\rho}_k^{(n)} = 0 (k > n); \quad \bar{\rho}_k^{(n)} = \frac{2}{\pi} \int_0^\pi \bar{U}_n^{(q)}(t) \cos kt dt.$$

Let us represent these kernels: $U_n^{(q)}(t)$ and $\bar{U}_n^{(q)}(t)$, graphically in several cases of q for $n = 10$. Now, we set $f(t) = A_q(n) U_n^{(q)}(t)$, or $\bar{A}_q(n) \bar{U}_n^{(q)}(t)$.

(1) $q = 2$ (2) $q = 3$ (3) $q = 4$ (4) $q = 5$

Fig. 1 $f(t) = \left(\frac{\sin \frac{n}{2}t}{\sin \frac{t}{2}} \right)^q$.

Remark [2.1] For $q = 2, 4, 6, \dots$, $U_n^{(q)}(t) = \bar{U}_n^{(q)}(t)$, $A_q(n) = \bar{A}_q(n)$ and $J_n^{(q)}(f)(x) = \bar{J}_n^{(q)}(f)(x)$.

Remark [2.2] Generally, we have for $q = 2, 3, 4, \dots$,

$$A_q(n) \leq \bar{A}_q(n).$$

Remark [2.3] $J_n^{(2)}(f)(x)$ is the Fejér's operator and $J_n^{(4)}(f)(x)$ is the Jackson's operator. If $q(\geq 4)$ is even, $J_n^{(q)}(f)(x)$ is the generalized Jackson's operator.

Remark [2.4] For $q > 1$, $J_n^{(q)}(f)(x)$ and $\bar{J}_n^{(q)}(f)(x)$ are the linear approximation processes. (See, Theorem 2.)

3 Auxiliary Lemmas and Theorems

As a rule, the constants used in different arguments are not identical, even if their notations are similar.

(5) $q = 1.5$ (6) $q = 2.5$ (7) $q = 3.0$ (8) $q = 3.5$

Fig. 2 $f(t) = \left| \frac{\sin \frac{n}{2} t}{\sin \frac{t}{2}} \right|^q$.

Lemma 1 (S. Watanabe and Y. Suzuki [14]) There are positive constants c_1, c_2 such that

$$c_1(2n+1)^{q-1} \leq A_q(n) \leq c_2(2n+1)^{q-1}, \quad \text{for } q = 2, 3, 4, 5, \dots$$

Proof For the sake of convenience, we shall reproduce the proof of Lemma 1 from the previous paper [14].

(i) If q is even, we have

$$\begin{aligned} A_q(n) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q dt \\ &= \frac{2}{\pi} \left\{ \int_0^{\pi/(2n+1)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q dt + \int_{\pi/(2n+1)}^{\pi} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q dt \right\} \\ &= \frac{2}{\pi} (I_1 + I_2), \text{ say.} \end{aligned}$$

Since $(2/\pi)u \leq \sin u \leq u$ ($0 \leq u \leq (\pi/2)$), we get

$$\begin{aligned}
I_1 &\leq \left(\frac{\pi}{2}\right)^q (2n+1)^q \int_0^{\pi/(2n+1)} dt = \pi \left(\frac{\pi}{2}\right)^q (2n+1)^{q-1}, \\
I_2 &\leq \int_{\pi/(2n+1)}^{\pi} \frac{1}{\left(\sin \frac{t}{2}\right)^q} dt \leq \left(\frac{\pi}{2}\right)^q \int_{\pi/(2n+1)}^{\pi} \frac{1}{\left(\frac{t}{2}\right)^q} dt \\
&\leq \pi^q \int_{\pi/(2n+1)}^{\infty} \frac{1}{t^q} dt = \frac{\pi}{q-1} (2n+1)^{q-1},
\end{aligned}$$

and

$$I_1 \geq \left(\frac{2}{\pi}\right)^q (2n+1)^q \int_0^{\pi/(2n+1)} dt = \left(\frac{2}{\pi}\right)^q (2n+1)^{q-1} \pi.$$

Hence we have

$$2 \left(\frac{2}{\pi}\right)^q (2n+1)^{q-1} < A_q(n) \leq 2 \left\{ \left(\frac{2}{\pi}\right)^q + \frac{1}{q-1} \right\} (2n+1)^{q-1}.$$

(ii) If q is odd, we have

$$\begin{aligned}
A_q(n) &= \frac{2}{\pi} \left\{ \sum_{j=0}^{n-1} \int_{2\pi j/(2n+1)}^{2\pi(j+1)/(2n+1)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q dt + \int_{2\pi n/(2n+1)}^{\pi} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q dt \right\} \\
&= \frac{2}{\pi} \left\{ \sum_{j=0}^{n-1} (-1)^j \int_0^{2\pi/(2n+1)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} j \right)} \right)^q dt + (-1)^n \int_0^{\pi/(2n+1)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} n \right)} \right)^q dt \right\} \\
&= \frac{2}{\pi} \left(\sum_{j=0}^n (-1)^j I_j \right), \text{ say.}
\end{aligned}$$

Since I_j ($j = 0, 1, \dots, n$) are positive and decreasing with j , we obtain

$$\frac{2}{\pi} (I_0 - I_1) < A_q(n) < \frac{2}{\pi} I_0,$$

and

$$\begin{aligned}
I_0 &= \int_0^{2\pi/(2n+1)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q dt \leq \left(\frac{\pi}{2}\right)^q (2n+1)^q \frac{2\pi}{2n+1} = 2\pi (2n+1)^{q-1} \left(\frac{\pi}{2}\right)^q, \\
I_0 - I_1 &= \int_0^{2\pi/(2n+1)} \left\{ \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q - \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} \right)} \right)^q \right\} dt \\
&\geq \int_0^{\pi/(2n+1)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q dt - \int_0^{\pi/(2n+1)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} \right)} \right)^q dt = J_0 - J_1, \text{ say.}
\end{aligned}$$

Then we get

$$\begin{aligned}
J_1 &\leq \int_0^{\pi/(2n+1)} \left(\frac{\sin \frac{\pi}{2}}{\sin \left(t + \frac{2\pi}{2n+1} \right)} \right)^q dt \leq \int_0^{\pi/(2n+1)} \left(\frac{1}{\sin \frac{n}{2n+1}} \right)^q dt \\
&\leq \left(\frac{\pi}{2}\right)^q \left(\frac{2n+1}{\pi}\right)^q \frac{\pi}{2n+1} = \left(\frac{1}{2}\right)^q (2n+1)^{q-1} \pi.
\end{aligned}$$

Thus we have

$$I_0 - I_1 \geq J_0 - J_1 \geq \pi \left(\frac{2}{\pi} \right)^q (2n+1)^{q-1} - \left(\frac{1}{2} \right)^q (2n+1)^{q-1} \pi = \pi (2n+1)^{q-1} \left\{ \left(\frac{2}{\pi} \right)^q - \left(\frac{1}{2} \right)^q \right\} > 0.$$

Consequently, Lemma 1 is proved completely. $\blacksquare$

Lemma 2 There are positive constants $\bar{c}_1, \bar{c}_2$ such that

$$\bar{c}_1(2n+1)^{q-1} \leq \bar{A}_q(n) \leq \bar{c}_2(2n+1)^{q-1} \quad \text{for a real number } q > 1.$$

Proof This is verified by the same calculation method that is used in the proof of Lemma 1. $\blacksquare$

Let $\{k_n(t)\}$ ($n = 1, 2, \dots$) be a sequence of functions of $L^1_{2\pi}$. In general, if $k_n(t)$ satisfies the following three conditions, we say that it is a summation kernel (or an approximation unit).

$$(A_1) \quad \sup_n \|k_n\|_1 < \infty,$$

$$(A_2) \quad \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(t) dt = 1 \quad (\text{i.e., } \lim_{n \rightarrow \infty} \hat{k}_n(0) = 1),$$

$$(A_3) \quad \text{For an arbitrarily fixed } \delta \ (0 < \delta < \pi),$$

$$\lim_{n \rightarrow \infty} \int_{\delta \leq |t| \leq \pi} |k_n(t)| dt = 0.$$

We know the next statements which are used in this section. (For example, see [6].)

Theorem E If $k_n(t)$ is a summation kernel, then

$$\lim_{n \rightarrow \infty} \|k_n * f - f\|_X = 0 \quad (f \in X),$$

where

$$(k_n * f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) k_n(t) dt.$$

In other words, the operator $(k_n * f)(x)$ is a linear approximation process.

Lemma C If $k_n(t)$ satisfies the condition (A_1) and the next two conditions, $k_n(t)$ is a summation kernel.

$$(B_1) \quad |k_n(t)| \leq \alpha n \quad (\alpha (>0) \text{ is constant}),$$

$$(B_2) \quad |k_n(t)| \leq \frac{\beta}{n^\lambda |t|^{1+\lambda}} \quad (\beta (>0) \text{ and } \lambda (>0) \text{ are constant}).$$

Theorem 1 For $q > 1$, we have the following assertions.

(1) The kernel $U_n^{(q)}(t)$ is a summation kernel, for $q = 2, 3, 4, \dots$

(2) The kernel $\bar{U}_n^{(q)}(t)$ is a summation kernel, for $q > 1$.

Proof For (1), we have only to prove the conditions (B_1) and (B_2) in Lemma C, because the condition (A_1) is satisfied by the assumption (1.3):

$$2\hat{U}_n^{(q)}(0) = \frac{1}{\pi} \int_{-\pi}^{\pi} U_n^{(q)}(t) dt = 1.$$

Since $U_n^{(q)}(t)$ is an even kernel, we can take the variable t in the interval $[0, \pi]$. Then we get

$$\begin{aligned} |U_n^{(q)}(t)| &= \frac{1}{A_q(n)} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q \leq \frac{1}{c_1} (2n+1)^{1-q} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q \quad (0 \leq t \leq \pi) \\ &\leq \frac{1}{c_1} (2n+1)^{1-q} \frac{\left(\frac{2n+1}{2} t \right)^q}{\left(\frac{2}{\pi} \cdot \frac{t}{2} \right)^q} \leq \frac{1}{c_1} \left(\frac{\pi}{2} \right)^q (2n+1) \leq \alpha n, \end{aligned}$$

where $\alpha (>0)$ is constant. Thus, the condition (B_1) is satisfied. On the other hand, we have

$$|U_n^{(q)}(t)| = \frac{1}{A_q(n)} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q \leq \frac{1}{c_1} (2n+1)^{1-q} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q \quad (0 \leq t \leq \pi)$$

$$\leq \frac{1}{c_1} (2n+1)^{1-q} \frac{\pi^q}{|t|^q} \leq \frac{\beta}{n^{q-1}|t|^q} = \frac{\beta}{n^\lambda |t|^{1+\lambda}} \quad (\beta > 0, \lambda > 0).$$

Hence, the condition (B_2) is proved. For (2), we are able to verify the conditions (B_1) and (B_2) similarly to the case of (1). ■

Theorem 2 Linear operators $J_n^{(q)}(f)$ ($q = 2, 3, \dots$) and $\bar{J}_n^{(q)}(f)$ ($q > 1$, real) are linear approximation processes.

Proof From Theorem E and Theorem 1, these are trivial. ■

Lemma 3 There are positive constants $\bar{c}_1, \bar{c}_2$ such that

$$\bar{c}_1(2n+1)^{1-q} \leq 1 - \bar{\rho}_1^{(n)} \leq \bar{c}_2(2n+1)^{1-q}, \quad \text{for a real number } q \ (1 < q < 3).$$

Lemma 4 There are positive constants $\bar{c}_1, \bar{c}_2$ such that

$$\bar{c}_1(2n+1)^{-2} \leq 1 - \bar{\rho}_1^{(n)} \leq \bar{c}_2(2n+1)^{-2}, \quad \text{for a real number } q \ (> 3).$$

Remark [3.1] For $q = 2$, we know already that $\exists \bar{c}_1, \bar{c}_2 (> 0)$

$$(2n+1)^{-1} \leq 1 - \bar{\rho}_1^{(n)} = 1 - \rho_1^{(n)} \leq \bar{c}_2(2n+1)^{-1}. \quad (\text{For instance, see [6].})$$

In order to prove those lemmas, we need the following estimation of $1 - \bar{\rho}_1^{(n)}$.

$$\begin{aligned} 1 - \bar{\rho}_1^{(n)} &= \frac{1}{\bar{A}_q(n)} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} (1 - \cos t) \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q dt \\ &= \frac{1}{\bar{A}_q(n)} \cdot \frac{4}{\pi} \int_0^{\pi} \sin^2 \frac{t}{2} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q dt \\ &= \frac{1}{\bar{A}_q(n)} \cdot \frac{4}{\pi} \left[\int_0^{\pi/(2n+1)} \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\left(\sin \frac{t}{2} \right)^{q-2}} dt + \int_{\pi/(2n+1)}^{\pi} \frac{\left| \sin \frac{2n+1}{2} t \right|^q}{\left(\sin \frac{t}{2} \right)^{q-2}} dt \right] \\ &= \frac{1}{\bar{A}_q(n)} \cdot \frac{4}{\pi} [I_1 + I_2], \text{ say.} \end{aligned} \quad (3.1)$$

(A) For I_1 , we get that if $q \geq 2$,

$$I_1 \leq \int_0^{\pi/(2n+1)} \frac{\left(\frac{2n+1}{2} t \right)^q}{\left(\frac{2}{\pi} \cdot \frac{t}{2} \right)^{q-2}} dt = \left(\frac{\pi}{2} \right)^{q-2} \frac{(2n+1)^q}{4} \int_0^{\pi/(2n+1)} t^2 dt = \frac{\pi^{q+1}}{3 \cdot 2^q} (2n+1)^{q-3}, \quad (3.2)$$

$$I_1 \geq \int_0^{\pi/(2n+1)} \frac{\left(\frac{2}{\pi} \cdot \frac{2n+1}{2} t \right)^q}{\left(\frac{t}{2} \right)^{q-2}} dt = \frac{2^{q-2}}{3\pi^{q-3}} (2n+1)^{q-3}, \quad (3.3)$$

and if $1 < q \leq 2$,

$$I_1 \leq \int_0^{\pi/(2n+1)} \left(\frac{2n+1}{2} t \right)^q \left(\frac{t}{2} \right)^{2-q} dt = \frac{\pi^3}{12} (2n+1)^{q-3}, \quad (3.4)$$

$$I_1 \geq \int_0^{\pi/(2n+1)} \left(\frac{2}{\pi} \cdot \frac{2n+1}{2} t \right)^q \left(\frac{2}{\pi} \cdot \frac{t}{2} \right)^{2-q} dt = \frac{\pi}{3} (2n+1)^{q-3}. \quad (3.5)$$

(B) For I_2 , we get that if $q > 3$,

$$I_2 \leq \int_{\pi/(2n+1)}^{\pi} \frac{1}{\left(\sin \frac{t}{2} \right)^{q-2}} dt \leq \int_{\pi/(2n+1)}^{\infty} \frac{1}{\left(\frac{2}{\pi} \cdot \frac{t}{2} \right)^{q-2}} dt = \frac{\pi}{q-3} (2n+1)^{q-3} \quad (\because q \neq 3), \quad (3.6)$$

$$I_2 = \frac{2}{2n+1} \int_{\pi/2}^{(2n+1)\pi/2} \frac{|\sin t|^q}{\left(\sin \frac{1}{2n+1} t \right)^{q-2}} dt \geq \frac{2}{2n+1} \sum_{k=1}^n \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} \frac{|\sin t|^q}{\left(\frac{1}{2n+1} t \right)^{q-2}} dt$$

Fig. 3 Example: $y = |\sin x|^{5/2}$.

$$\begin{aligned}
 &= 2 \cdot (2n+1)^{q-3} \sum_{k=1}^n \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} \frac{|\sin t|^q}{t^{q-2}} dt > \frac{2^{q-1}}{\pi^{q-2}} (2n+1)^{-1} \sum_{k=1}^n \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} |\sin t|^q dt \\
 &\geq \frac{2^{q-1}}{\pi^{q-2}} (2n+1)^{-1} n \cdot \frac{\pi}{q+1} = \frac{2^{q-1}}{(q+1)\pi^{q-3}} \cdot \frac{n}{2n+1} \geq \frac{1}{3} \cdot \frac{2^{q-1}}{(q+1)\pi^{q-3}}. \\
 &\left(\because \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} |\sin t|^q dt = 2 \int_0^{\pi/2} |\sin t|^q dt \geq 2 \int_0^{\pi/2} \left(\frac{2}{\pi} \cdot t \right)^q dt = \frac{\pi}{q+1} \right)
 \end{aligned} \tag{3.7}$$

and if $1 < q \leq 2$,

$$I_2 = \int_{\pi/(2n+1)}^{\pi} \left| \sin \frac{2n+1}{2} t \right|^q \left(\sin \frac{t}{2} \right)^{2-q} dt \leq \frac{2n\pi}{2n+1} < \pi, \tag{3.8}$$

$$\begin{aligned}
 I_2 &= \frac{2}{2n+1} \int_{\pi/2}^{(2n+1)\pi/2} |\sin t|^q \left(\sin \frac{1}{2n+1} t \right)^{2-q} dt \geq \frac{2}{2n+1} \sum_{k=1}^n \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} |\sin t|^q \left(\frac{2}{\pi} \cdot \frac{1}{2n+1} t \right)^{2-q} dt \\
 &= \frac{2^{3-q}}{\pi^{2-q}} \cdot \frac{1}{(2n+1)^{3-q}} \sum_{k=1}^n \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} |\sin t|^q t^{2-q} dt \\
 &\geq \frac{2^{3-q}}{\pi^{2-q}} \cdot \frac{1}{(2n+1)^{3-q}} \sum_{k=1}^n \left(\frac{2k-1}{2} \pi \right)^{2-q} \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} |\sin t|^q dt \\
 &= \frac{2}{(2n+1)^{3-q}} \sum_{k=1}^n (2k-1)^{2-q} \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} |\sin t|^q dt \\
 &\geq \frac{2}{(2n+1)^{3-q}} \cdot \frac{\pi}{q+1} \{1^{2-q} + 3^{2-q} + \dots + (2n-1)^{2-q}\} \\
 &\geq \frac{2}{(2n+1)^{3-q}} \cdot \frac{\pi}{q+1} \cdot \frac{1}{2} \cdot \frac{1}{3-q} (2n-1)^{3-q}
 \end{aligned}$$

$$\begin{aligned}
 &\left(\because S = 1^{2-q} + 3^{2-q} + \dots + (2n-1)^{2-q} \right. \\
 &\geq \int_{1/2}^n (2x-1)^{\alpha} dx = \frac{1}{2} \cdot \frac{1}{\alpha+1} \cdot (2n-1)^{\alpha+1} = \frac{1}{2} \cdot \frac{1}{3-q} \cdot (2n-1)^{3-q} \Big) \\
 &= \frac{\pi}{q+1} \cdot \frac{1}{3-q} \left(\frac{2n-1}{2n+1} \right)^{3-q} \\
 &= \frac{\pi}{q+1} \cdot \frac{1}{3-q} \left(1 - \frac{2}{2n+1} \right)^{3-q} \\
 &\geq \frac{1}{3^{3-q}} \cdot \frac{\pi}{q+1} \cdot \frac{1}{3-q}.
 \end{aligned} \tag{3.9}$$

and if $2 \leq q < 3$,

$$\begin{aligned} I_2 &= \frac{2}{2n+1} \int_{\pi/2}^{(2n+1)\pi/2} \frac{|\sin t|^q}{\left(\sin \frac{1}{2n+1} t\right)^{q-2}} dt \leq \frac{2}{2n+1} \int_{\pi/2}^{(2n+1)\pi/2} \frac{|\sin t|^q}{\left(\frac{2}{\pi} \cdot \frac{1}{2n+1} t\right)^{q-2}} dt \\ &\leq \frac{\pi^{q-2}}{2^{q-3}} (2n+1)^{q-3} \int_{\pi/2}^{(2n+1)\pi/2} t^{2-q} dt \quad (\because q \neq 3) \\ &= \frac{\pi}{3-q} (2n+1)^{q-3} \{(2n+1)^{3-q} - 1\} \leq \frac{\pi}{3-q}, \end{aligned} \quad (3.10)$$

$$I_2 \geq \frac{1}{3} \cdot \frac{2^{q-1}}{(q+1)\pi^{q-3}}. \quad (3.11)$$

Because, this lower bound is equivalent to the lower estimate of the case $q > 3$. (See, (3.7).)

In the next proofs of lemmas, c, c', c_i ($i = 1, 2$), k and k_i ($i = 1, 2$) are positive constants.

(Proof of Lemma 3) We obtain that if $2 \leq q < 3$,

$$\begin{aligned} 1 - \bar{\rho}_1^{(n)} &\leq \frac{4}{\pi} \cdot \frac{1}{c(2n+1)^{q-1}} \left\{ \frac{\pi^{q+1}}{3 \cdot 2^q} (2n+1)^{q-3} + \frac{\pi}{3-q} \right\} = k_1(2n+1)^{-2} + k_2(2n+1)^{1-q} \\ &\leq \bar{c}_2(2n+1)^{1-q} \quad (q \neq 3) \quad (\because (3.2) \text{ and } (3.10)), \\ 1 - \bar{\rho}_1^{(n)} &\geq \frac{4}{\pi} \cdot \frac{1}{c(2n+1)^{q-1}} \left\{ \frac{2^{q-2}}{3 \cdot \pi^{q-3}} (2n+1)^{q-3} + \frac{1}{3} \cdot \frac{2^{q-1}}{(q+1)\pi^{q-3}} \right\} \\ &= k_1(2n+1)^{-2} + k_2(2n+1)^{1-q} \geq \bar{c}_1(2n+1)^{1-q} \quad (\because (3.3) \text{ and } (3.11)), \end{aligned}$$

and if $1 < q \leq 2$,

$$\begin{aligned} 1 - \bar{\rho}_1^{(n)} &\leq \frac{4}{\pi} \cdot \frac{1}{c(2n+1)^{q-1}} \left\{ \frac{\pi^3}{12} (2n+1)^{q-3} + \pi \right\} \leq c'(2n+1)^{1-q} \quad (\because (3.4) \text{ and } (3.8)), \\ 1 - \bar{\rho}_1^{(n)} &\geq \frac{4}{\pi} \cdot \frac{1}{c(2n+1)^{q-1}} \left\{ \frac{\pi}{3} (2n+1)^{q-3} + \frac{1}{3^{3-q}} \cdot \frac{\pi}{q+1} \cdot \frac{1}{3-q} \right\} = k_1(2n+1)^{-2} + k_2(2n+1)^{1-q} \\ &\geq \bar{c}_1(2n+1)^{1-q} \quad (\because (3.5) \text{ and } (3.9)). \end{aligned}$$

Hence, the proof of Lemma 3 is completed. ■

(Proof of Lemma 4) It follows that

$$1 - \bar{\rho}_1^{(n)} \leq \frac{1}{c(2n+1)^{q-1}} \cdot \frac{4}{\pi} (2n+1)^{q-3} \left\{ \frac{\pi^{q+1}}{3 \cdot 2^q} + \frac{\pi}{q-3} \right\} \leq \bar{c}_2(2n+1)^{-2} \quad (\because (3.2) \text{ and } (3.6)),$$

and

$$\begin{aligned} 1 - \bar{\rho}_1^{(n)} &\geq \frac{4}{\pi} \cdot \frac{1}{c(2n+1)^{q-1}} \left\{ \frac{2^{q-2}}{3 \cdot \pi^{q-3}} (2n+1)^{q-3} + \frac{1}{3} \cdot \frac{2^{q-1}}{(q+1)\pi^{q-3}} \right\} = k_1(2n+1)^{-2} + k_2(2n+1)^{1-q} \\ &\geq \bar{c}_1(2n+1)^{-2} \quad (\because (3.3) \text{ and } (3.7)). \end{aligned}$$

Therefore, Lemma 4 is proved completely. ■

Lemma 5 For $q = 3$, we have that if n is large sufficiently,

$$\exists \quad \bar{c}_1, \bar{c}_2(>0) | \bar{c}_1 \cdot \frac{\log n}{n^2} \leq 1 - \bar{\rho}_1^{(n)} \leq \bar{c}_2 \cdot \frac{\log n}{n^2}.$$

Proof

$$1 - \bar{\rho}_1^{(n)} = \frac{4}{\pi} \cdot \frac{1}{\bar{A}_3(n)} \left[\int_0^{\pi/(2n+1)} \frac{\left(\sin \frac{2n+1}{2} t\right)^3}{\sin \frac{t}{2}} dt + \int_{\pi/(2n+1)}^{\pi} \frac{\left|\sin \frac{2n+1}{2} t\right|^3}{\sin \frac{t}{2}} dt \right] = \frac{4}{\pi} \cdot \frac{1}{\bar{A}_3(n)} (I_1 + I_2), \text{ say.}$$

From (3.3), we get $I_1 \geq (2/3)$, and also, it follows that

$$I_2 = \int_{\pi/(2n+1)}^{\pi} \frac{\left|\sin \frac{2n+1}{2} t\right|^3}{\sin \frac{t}{2}} dt = \frac{2}{2n+1} \int_{\pi/2}^{(2n+1)\pi/2} \frac{|\sin y|^3}{\sin \frac{1}{2n+1} y} dy$$

$$\begin{aligned}
&\geq \frac{4}{\pi} \sum_{k=1}^n \frac{1}{2k+1} \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} |\sin y|^3 dy \geq \sum_{k=1}^n \frac{1}{2k+1} \left(\because \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} |\sin y|^3 dy \geq \frac{\pi}{4} \right) \\
&> \frac{1}{2} \log n - 1. \\
&\left(\because \sum_{k=1}^n \frac{1}{2k+1} > \frac{1}{2} \sum_{k=1}^n \frac{1}{k} - 1, \sum_{k=1}^n \frac{1}{k} > \int_1^n \frac{1}{x} dx. \right)
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
1 - \bar{\rho}_1^{(n)} &\geq \frac{1}{\bar{A}_3(n)} \cdot \frac{4}{\pi} \left(\frac{2}{3} + \frac{1}{2} \log n - 1 \right) = \frac{1}{\bar{A}_3(n)} \cdot \frac{4}{\pi} \cdot \frac{1}{6} (3 \log n - 2) \\
&\geq \frac{1}{6cn^2} \cdot \frac{4}{\pi} (3 \log n - 2) \geq \bar{c}_1 \cdot \frac{\log n}{n^2} \quad (\text{for } n \geq e.)
\end{aligned}$$

and if n is large sufficiently,

$$1 - \bar{\rho}_1^{(n)} = \frac{1}{\bar{A}_3(n)} \cdot \frac{4}{\pi} \int_0^\pi \frac{\left| \sin \frac{2n+1}{2} t \right|^3}{\sin \frac{t}{2}} dt \leq \frac{1}{cn^2} \cdot \frac{4}{\pi} \int_0^\pi \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right| dt \leq \bar{c}_2 \frac{\log n}{n^2} \quad (\because \text{Fejér's theorem.})$$

Consequently, the Lemma 5 is verified perfectly. ■

Lemma 6 For $q = 3, 4, 5, \dots$, it follows that

$$\exists \quad c_1, c_2 > 0 \quad | \quad c_1(2n+1)^{-2} \leq 1 - \rho_1^{(n)} \leq c_2(2n+1)^{-2} \quad (n > 2).$$

Proof (i) If q is even, the proof is similar to Lemma 4.

(ii) If q is odd, we get

$$\begin{aligned}
1 - \rho_1^{(n)} &= \frac{1}{A_q(n)} \cdot \frac{4}{\pi} \left\{ \sum_{j=0}^{n-1} (-1)^j \int_0^{2\pi/(2n+1)} \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\sin \frac{1}{2} \left(t + \frac{2\pi j}{2n+1} \right)^{q-2}} dt \right. \\
&\quad \left. + (-1)^n \int_0^{\pi/(2n+1)} \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\sin \frac{1}{2} \left(t + \frac{2\pi n}{2n+1} \right)^{q-2}} dt \right\} = \frac{1}{A_q(n)} \cdot \frac{4}{\pi} \left(\sum_{j=0}^n (-1)^j I_j \right), \text{ say.}
\end{aligned}$$

Since I_j ($j = 0, 1, \dots, n$) are positive and decreasing with j , we have

$$I_0 - I_1 < \sum_{j=0}^n (-1)^j I_j < I_0.$$

On the other hand, we obtain

$$\begin{aligned}
I_0 &= \int_0^{2\pi/(2n+1)} \left(\frac{1}{\sin \frac{t}{2}} \right)^{q-2} \left(\sin \frac{2n+1}{2} t \right)^q dt \leq \int_0^{2\pi/(2n+1)} \left(\frac{1}{\frac{2}{\pi} \cdot \frac{t}{2}} \right)^{q-2} \left(\frac{2n+1}{2} t \right)^q dt \\
&= \frac{\pi^{q-2}}{2^q} \cdot (2n+1)^q \int_0^{2\pi/(2n+1)} t^2 dt = \frac{\pi^{q+1}}{3 \cdot 2^{q-3}} (2n+1)^{q-3},
\end{aligned}$$

and

$$I_0 - I_1 = \int_0^{2\pi/(2n+1)} \left\{ \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\left(\sin \frac{t}{2} \right)^{q-2}} - \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\left\{ \sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} \right) \right\}^{q-2}} \right\} dt,$$

where

$$\left\{ \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\left(\sin \frac{t}{2} \right)^{q-2}} - \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\left\{ \sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} \right) \right\}^{q-2}} \right\} > 0 \quad (n > 2).$$

Hence we get

$$\begin{aligned} I_0 - I_1 &> \int_0^{\pi/(2n+1)} \left\{ \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\left(\sin \frac{t}{2} \right)^{q-2}} - \frac{\left(\sin \frac{2n+1}{2} t \right)^q}{\left\{ \sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} \right) \right\}^{q-2}} \right\} dt \\ &= \int_0^{\pi/(2n+1)} \left(\sin \frac{2n+1}{2} t \right)^q \left\{ \frac{1}{\left(\sin \frac{t}{2} \right)^{q-2}} - \frac{1}{\left\{ \sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} \right) \right\}^{q-2}} \right\} dt \\ &= \int_0^{\pi/(2n+1)} \left(\sin \frac{2n+1}{2} t \right)^q \left\{ \frac{\left\{ \sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} \right) \right\}^{q-2} - \left(\sin \frac{t}{2} \right)^{q-2}}{\left(\sin \frac{t}{2} \right)^{q-2} \left\{ \sin \frac{1}{2} \left(t + \frac{2\pi}{2n+1} \right) \right\}^{q-2}} \right\} dt \\ &\geq \int_0^{\pi/(2n+1)} \left(\frac{2}{\pi} \cdot \frac{2n+1}{2} t \right)^q \frac{\left(\sin \frac{\pi}{2n+1} \right)^{q-2} - \left(\sin \frac{\pi}{2(2n+1)} \right)^{q-2}}{\left(\frac{t}{2} \cdot \frac{3}{2} \cdot \frac{\pi}{2n+1} \right)^{q-2}} dt \\ &\geq \int_0^{\pi/(2n+1)} \left(\frac{2n+1}{\pi} t \right) \frac{\left(\frac{2}{\pi} \cdot \frac{\pi}{2n+1} \right)^{q-2} - \left(\frac{\pi}{2} \cdot \frac{1}{2n+1} \right)^{q-2}}{\left(\frac{3}{4} \cdot \frac{\pi t}{2n+1} \right)^{q-2}} dt \\ &= \frac{1}{\pi^{2q-5}} \cdot \frac{4^{q-2}}{3^{q-1}} \left\{ 2^{q-2} - \left(\frac{\pi}{2} \right)^{q-2} \right\} (2n+1)^{q-3} = \alpha (2n+1)^{q-3}. \quad (\alpha(>0) \text{ is constant.}) \end{aligned}$$

Therefore, we have

$$\exists \quad c_1(>0) \quad | \quad 1 - \rho_1^{(n)} > \frac{4}{\pi} \cdot \frac{1}{A_q(n)} \cdot \alpha (2n+1)^{q-3} > c_1 (2n+1)^{-2}.$$

On the other hand, we get

$$\exists \quad c_2(>0) \quad | \quad 1 - \rho_1^{(n)} < \frac{4}{\pi} \cdot \frac{1}{A_q(n)} \cdot \frac{\pi^{q+1}}{3 \cdot 2^{q-3}} (2n+1)^{q-3} < c_2 (2n+1)^{-2}.$$

Consequently, the Lemma 6 is proved completely. ■

Remark [3.2] If $q = 2$, we know

$$1 - \rho_1^{(n)} = \frac{1}{2n+1}. \quad (\text{For instance, see [6].})$$

Remark [3.3] If $q = 3$, by the following Lemma 8, we have the equation:

$$1 - \rho_1^{(n)} = \frac{1}{3n^2 + 3n + 1}.$$

Lemma 7 For a trigonometric polynomial:

$$\left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^3 = \frac{a_0}{2} + \sum_{m=1}^{3n} a_m \cos mt,$$

it is obtained that

$$a_m = \begin{cases} 6n^2 + 6n + 2 - 2m^2 & (0 \leq m \leq n), \\ 9n^2 + 9n + 2 + m^2 - 3m - 6mn & (n+1 \leq m \leq 3n). \end{cases}$$

Proof Since

$$\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} = 1 + 2 \sum_{k=1}^n \cos kt$$

and

$$\left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^2 = (2n+1) \left(1 + 2 \sum_{j=1}^{2n} \left(1 - \frac{j}{2n+1} \right) \cos jt \right),$$

it follows that

$$\begin{aligned} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^3 &= (2n+1) \left\{ 1 + 2 \sum_{j=1}^{2n} \left(1 - \frac{j}{2n+1} \right) \cos jt + 2 \sum_{k=1}^n \cos kt \right. \\ &\quad \left. + 4 \sum_{k=1}^n \sum_{j=1}^{2n} \left(1 - \frac{j}{2n+1} \right) \cos kt \cos jt \right\} \\ &= (2n+1) \left\{ 1 + 2 \sum_{j=1}^{2n} \left(1 - \frac{j}{2n+1} \right) \cos jt + 2 \sum_{k=1}^n \cos kt \right. \\ &\quad \left. + 2 \sum_{k=1}^n \sum_{j=1}^{2n} \left(1 - \frac{j}{2n+1} \right) \cos (j+k)t + 2 \sum_{k=1}^n \sum_{j=1}^{2n} \left(1 - \frac{j}{2n+1} \right) \cos (j-k)t \right\}. \end{aligned}$$

Also, we have that for $0 \leq m \leq n$,

$$\begin{aligned} a_m &= (2n+1) \left\{ 2 \left(1 - \frac{m}{2n+1} \right) + 2 + 2 \sum_{k=1}^{m-1} \left(1 - \frac{m-k}{2n+1} \right) + 2 \sum_{j=1}^{n-m} \left(1 - \frac{j}{2n+1} \right) \right. \\ &\quad \left. + 2 \sum_{j=m+1}^{m+n} \left(1 - \frac{j}{2n+1} \right) \right\} = 6n^2 + 6n + 2 - 2m^2; \end{aligned}$$

for $n+1 \leq m \leq 2n$,

$$\begin{aligned} a_m &= (2n+1) \left\{ 2 \left(1 - \frac{m}{2n+1} \right) + 2 \sum_{k=1}^n \left(1 - \frac{m-k}{2n+1} \right) + 2 \sum_{j=m+1}^{2n} \left(1 - \frac{j}{2n+1} \right) \right\} \\ &= 9n^2 + 9n + 2 + m^2 - 3m - 6mn; \end{aligned}$$

and for $2n+1 \leq m \leq 3n$,

$$a_m = 2(2n+1) \sum_{j=m-n}^{2n} \left(1 - \frac{j}{2n+1} \right) = 9n^2 + 9n + 2 + m^2 - 3m - 6mn. \quad \blacksquare$$

Lemma 8 For $q = 3$, the following three assertions are effected respectively.

$$A_3(n) = 6n^2 + 6n + 2, \quad 1 - \rho_k^{(n)} = \frac{k^2}{3n^2 + 3n + 1} \quad (k = 0, 1, \dots, n).$$

Proof By Lemma 7, we have that

$$A_3(n) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^3 dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\sum_{k=1}^{3n} a_k \cos kt + \frac{a_0}{2} \right) dt = \frac{a_0}{2\pi} \cdot 2\pi = a_0 = 6n^2 + 6n + 2,$$

and for $0 \leq k \leq n$,

$$\begin{aligned}
\rho_k^{(n)} &= \frac{1}{A_3(n)} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kt \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^3 dt = \frac{1}{A_3(n)} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kt \left(\frac{a_0}{2} + \sum_{k=1}^{3n} a_k \cos kt \right) dt \\
&= a_k \cdot \frac{1}{A_3(n)} = \frac{a_k}{6n^2 + 6n + 2}. \\
\therefore 1 - \rho_k^{(n)} &= 1 - \frac{6n^2 + 6n + 2 - 2k^2}{6n^2 + 6n + 2} = \frac{k^2}{3n^2 + 3n + 1}.
\end{aligned}$$

Lemma 9 For $q > 3$, it follows that

$$\lim_{n \rightarrow \infty} \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2 \quad (k = 2, \dots).$$

Proof Let us take $\varepsilon (> 0)$ satisfying that

$$4 - (2 + \varepsilon) > 0 \quad \text{and} \quad q - (2 + \varepsilon) > 1.$$

Then, we have

$$\begin{aligned}
|T(4; n)| &= \left| \frac{2}{\pi} \int_0^\pi \left(2 \sin \frac{t}{2} \right)^4 \frac{1}{A_q(n)} \left(\frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right)^q dt \right| \\
&\leq \frac{32}{|A_q(n)|} \cdot \frac{1}{\pi} \int_0^\pi \left| \sin \frac{t}{2} \right|^{4 - (2 + \varepsilon)} \frac{\left| \sin \frac{2n+1}{2} t \right|^{q - (2 + \varepsilon)} \left| \sin \frac{2n+1}{2} t \right|^{2 + \varepsilon}}{\left| \sin \frac{t}{2} \right|^{q - (2 + \varepsilon)}} dt \\
&\leq \frac{16}{c_1(2n+1)^{q-1}} \cdot \frac{2}{\pi} \int_0^\pi \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^{q - (2 + \varepsilon)} dt \quad (\because \text{Lemma 1}) \\
&= \frac{16}{c_1(2n+1)^{q-1}} \cdot \bar{A}_{q-(2+\varepsilon)}(n) \\
&\leq \frac{16}{c_1(2n+1)^{q-1}} \cdot \bar{c}_2(2n+1)^{q-(2+\varepsilon)-1} \quad (\because \text{Lemma 2}) \\
&= c(2n+1)^{-(2+\varepsilon)}, \\
\therefore \left| \frac{T(4; n)}{T(2; n)} \right| &\leq \frac{c(2n+1)^{-(2+\varepsilon)}}{2(1 - \rho_1^{(n)})} \quad (\because \text{Lemma A}) \\
&\leq \frac{c}{2} \cdot \frac{(2n+1)^{-(2+\varepsilon)}}{c_1(2n+1)^{-2}} \quad (\because \text{Lemma 6}) \\
&= K(2n+1)^{-\varepsilon} \rightarrow 0 \quad (n \rightarrow \infty).
\end{aligned} \tag{3.12}$$

For $\sigma \geq 3$, we get

$$\begin{aligned}
|T(2\sigma; n)| &= \left| \frac{2}{\pi} \int_0^\pi \left(2 \sin \frac{t}{2} \right)^{2\sigma-4} \left(2 \sin \frac{t}{2} \right)^4 U_n^{(q)}(t) dt \right| \\
&\leq \frac{1}{|A_q(n)|} \cdot \frac{2}{\pi} \int_0^\pi \left(2 \sin \frac{t}{2} \right)^{2\sigma-4} \left(2 \sin \frac{t}{2} \right)^4 \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q dt \\
&\leq 2^{2\sigma-4} \cdot \frac{1}{|A_q(n)|} \cdot \frac{2}{\pi} \int_0^\pi \left(2 \sin \frac{t}{2} \right)^4 \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q dt.
\end{aligned}$$

In the same way as (3.12), we get

$$\left| \frac{T(2\sigma; n)}{T(2; n)} \right| \leq K'(2n+1)^{-\varepsilon} \rightarrow 0 \quad (n \rightarrow \infty).$$

Consequently, we have

$$\lim_{n \rightarrow \infty} \frac{T(2\sigma; n)}{T(2; n)} = 0 \quad (\sigma = 2, 3, \dots).$$

Hence, by Theorem C, we obtain the conclusion of Lemma 9. ■

Lemma 10 For $q > 3$, it follows that

$$\lim_{n \rightarrow \infty} \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}} = k^2 \quad (k = 2, \dots).$$

Proof Analogously with the proof of Lemma 9, we get

(1) $k = 2$

(2) $k = 3$

Fig. 4 $f(n, k) = \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}}$, for $q = 3$.

$$\begin{aligned}
\bar{T}(4; n) &= \frac{2}{\pi} \int_0^\pi \left(2 \sin \frac{t}{2} \right)^4 \cdot \frac{1}{\bar{A}_q(n)} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^q dt \\
&\leq \frac{16}{\bar{c}_1(2n+1)^{q-1}} \cdot \bar{A}_{q-(2+\varepsilon)}(n) \quad (\because \text{Lemma 2}) \\
&\leq \frac{16}{\bar{c}_1(2n+1)^{q-1}} \cdot \bar{c}_2(2n+1)^{q-(2+\varepsilon)-1} \quad (\because \text{Lemma 2}) \\
&= c(2n+1)^{-(2+\varepsilon)}. \\
\therefore \frac{\bar{T}(4; n)}{\bar{T}(2; n)} &\leq \frac{c(2n+1)^{-(2+\varepsilon)}}{2(1-\bar{\rho}_1^{(n)})} \leq \frac{c}{2} \cdot \frac{(2n+1)^{-(2+\varepsilon)}}{\bar{c}_1(2n+1)^{-2}} = K(2n+1)^{-\varepsilon} \rightarrow 0 \quad (n \rightarrow \infty),
\end{aligned}$$

(3) $k = 2$ (4) $k = 3$

Fig. 5 $f(n, k) = \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}}$, for $q = 3$.

Hence, we have

$$\lim_{n \rightarrow \infty} \frac{\bar{T}(2\sigma; n)}{\bar{T}(2; n)} = 0 \quad (\sigma = 2, 3, \dots).$$

According, by Theorem C, we come to the conclusion of Lemma 10. ■

Lemma 11 The following identities are materialized for $q = 3$.

$$(1^\circ) \quad \lim_{n \rightarrow \infty} \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}} = k^2 \quad (k = 2, 3, \dots).$$

$$(2^\circ) \quad \lim_{n \rightarrow \infty} \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2 \quad (k = 2, 3, \dots).$$

Proof (1°) Using (e) of Theorem D, we have only to verify that

(5) $k = 2$

(6) $k = 3$

$$\text{Fig. 6} \quad f(n, k) = \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}} = \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}}, \quad \text{for } q = 4.$$

$$\frac{1}{\bar{A}_3(n)} \int_0^\pi \sin^4 \frac{t}{2} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^3 dt = o(1 - \bar{\rho}_1^{(n)}),$$

that is to say,

$$\frac{1}{\bar{A}_3(n)(1 - \bar{\rho}_1^{(n)})} \int_0^\pi \sin^4 \frac{t}{2} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^3 dt = o(1).$$

Since

$$\int_0^\pi \sin^4 \frac{t}{2} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^3 dt = \int_0^\pi \sin \frac{t}{2} \left| \sin \frac{2n+1}{2} t \right|^3 dt \leq \pi,$$

we have

$$\frac{1}{\bar{A}_3(n)(1 - \bar{\rho}_1^{(n)})} \int_0^\pi \sin^4 \frac{t}{2} \left| \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \right|^3 dt \leq \frac{\pi}{c_1 n^2 c_2 \frac{\log n}{n^2}} = \frac{c}{\log n} \rightarrow 0 \quad (n \rightarrow \infty),$$

Hence, we have the required result.

(2°) From Lemma 8, we know

$$\frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2 \quad (k = 2, 3, \dots). \quad \blacksquare$$

Remark [3.4] The statement (2°) can be verified by Theorem C, but its calculation is complicated. To comprehend those above-mentioned results easily, let us represent them graphically.

4 Saturation Theorems

Theorem 3 For $q > 3$, it follows that

$$\text{Sat}[\bar{J}_n^{(q)}] = [\{f \in X; f' \in \text{Lip}(1; X)\}, n^{-2}, \text{constant}].$$

Proof From Lemma 10, we know

$$\lim_{n \rightarrow \infty} \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}} = k^2 \quad (k=2, 3, \dots),$$

and since $\bar{U}_n^{(q)}(t)$ is nonnegative, it holds that

$$\int_0^\pi t^2 |\bar{U}_n^{(q)}(t)| dt = \int_0^\pi t^2 \bar{U}_n^{(q)}(t) dt = O(1 - \bar{\rho}_1^{(n)}) = O(n^{-2}) \quad (q > 3).$$

Consequently, it is effected that

$$\text{Sat}[\bar{J}_n^{(q)}] = [\{f \in X; f' \in \text{Lip}(1; X)\}, n^{-2}, \text{constant}]. \quad \blacksquare$$

Theorem 4 For $q > 3$, it follows that

$$\text{Sat}[J_n^{(q)}] = [\{f \in X; f' \in \text{Lip}(1; X)\}, n^{-2}, \text{constant}].$$

Proof From (2°) of Lemma 11, we know

$$\lim_{n \rightarrow \infty} \frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2 \quad (k=2, 3, \dots),$$

and it holds that

$$\int_0^\pi t^2 |U_n^{(q)}(t)| dt \leq \frac{\bar{A}_q(n)}{|A_q(n)|} \int_0^\pi t^2 \bar{U}_n^{(q)}(t) dt \leq O(1) \cdot O(1 - \bar{\rho}_1^{(n)}) = O(1 - \rho_1^{(n)}) = O(n^{-2}).$$

Hence, applying Theorem B here, this theorem is proved completely. ■

Remark [4.1] Though we had obtained Theorem 4 already in the paper [14], the method which was used there is not systematic and more difficult than not. The present proof of Theorem 4 is systematic and easy. Also, in [14], there is not the perfect proof of the saturation theorem for $q = 3$, and Theorem 5 that is described later is new.

Theorem 5

$$\text{Sat}[\bar{J}_n^{(3)}] = \left[\{f \in X; f' \in \text{Lip}(1; X)\}, \frac{\log n}{n^2}, \text{constant} \right].$$

Proof From Lemma 5 and (1°) of Lemma 11, we know

$$1 - \bar{\rho}_1^{(n)} = O\left(\frac{\log n}{n^2}\right),$$

and

$$\lim_{n \rightarrow \infty} \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}} = k^2 \quad (k = 2, 3, \dots).$$

Also, from (a) and (d) of Theorem D, we have

$$\lim_{n \rightarrow \infty} \frac{T(2\sigma; n)}{T(2; n)} = 0 \quad (\sigma = 2, 3, \dots).$$

Fig. 7 $f(n, k) = \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}}$, for $q = 1.5$.

Applying Theorem B, we get Theorem 5. ■

Theorem 6

$$\text{Sat}[J_n^{(3)}] = \left[\{f \in X; f' \in \text{Lip}(1; X)\}, \frac{1}{n^2}, \text{constant} \right].$$

Proof From Lemma 8 and (2°) of Lemma 11, we know

$$1 - \rho_1^{(n)} = \frac{1}{3n^2 + 3n + 1},$$

and for all n ,

$$\frac{1 - \rho_k^{(n)}}{1 - \rho_1^{(n)}} = k^2 \quad (k = 2, 3, \dots).$$

Hence, we have only to verify the direct part of saturation. On the other side, we are able to calculate that

Fig. 8 $f(n, k) = \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}}$, for $q = 2.5$.

$$\lambda_k(n) = \frac{1 - \rho_k^{(n)}}{\varphi(n)\psi(k)} = \frac{\frac{k^2}{3n^2 + 3n + 1}}{\frac{k^2}{3n^2 + 3n + 1}} = 1,$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} k |\Delta^2 \lambda_k(n)| &= \lim_{n \rightarrow \infty} \sum_{k=1}^n k |\Delta^2 \lambda_k(n)| = \lim_{n \rightarrow \infty} \sum_{k=1}^n k |\lambda_{k-1}(n) - 2\lambda_k(n) + \lambda_{k+1}(n)| \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n k |1 - 2 + 1| = 0 = O(1). \end{aligned}$$

Hence, since $\lambda_k(n)$ is quasi-convex uniformly in n , it is a (L^∞, L^∞) -multiplier uniformly in n . Applying Theorem A, we have that if $f' \in Lip(1; X)$, then

(5) $k = 2$

(6) $k = 3$

Fig. 9 $f(n, k) = \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}}$, for $q = 3.5$.

$$\|f - J_n^{(3)}(f)\|_X = O(n^{-2}).$$

Thus, Theorem 6 is proved perfectly. ■

Remark [4.2] What are the saturation classes and orders of $\bar{J}_n^{(q)}(f)$ for $1 < q < 2$ and $2 < q < 3$? Those problems are open. In connection with them, from Lemma 3, we know

$$1 - \bar{\rho}_1^{(n)} = O(n^{1-q}), \quad \text{for } 1 < q < 3.$$

and want to calculate the limitation:

$$\lim_{n \rightarrow \infty} \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}}.$$

For your information, let us simulate the extreme behavior of the quantity:

$$f(n, k) = \frac{1 - \bar{\rho}_k^{(n)}}{1 - \bar{\rho}_1^{(n)}} \quad (n \rightarrow \infty).$$

in several cases of q , graphically.

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