Digital Sum Problems for the p -adic Expansion of Natural Numbers

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Let p be a positive integer greater than 1 and denote the p -adic expansion of $n \in \mathbb{N}$ by $n = \sum_{i \geq 0} \alpha_i(n)p^i$, where $\alpha_i(n) \in \{0, 1, \dots, p-1\}$. We set $s(n, l) = \sum_{i \geq 0} \mathbf{1}_{\{\alpha_i(n)=l\}}$ for $l = 1, 2, \dots, p-1$, and $s(n) = \sum_{l=1}^{p-1} ls(n, l)$. We define the power sum of sum of digits by $S_k(N) = \sum_{n=0}^{N-1} s(n)^k$, $k \in \mathbb{N}$, and the exponential sum of sum of digits by $F(\xi, N) = \sum_{n=0}^{N-1} e^{\xi s(n)}$, $\xi \in \mathbb{R}$ for $N \in \mathbb{N}$. Explicit formulas of these sums have been investigated by many authors for a long time. In this paper, developing our previous results, we shall give explicit formulas of $F(\xi, N)$ and $S_k(N)$.

KEYWORDS: digital sum problem, singular function, multinomial measure, Bernoulli polynomial

1 Introduction

We first give a short historical sketch for explicit formulas of these sums. Trollope [16] obtained an explicit formula of $S_1(N)$ for $p = 2$ and Delange [4] gave its elegant proof. Coquet [3] studied an explicit formula of $S_k(N)$ for $k \geq 2$, $p = 2$ and obtained an explicit one. However his formula contains functions defined by recursive equations, which cannot be solved in general. Osbaldestin [11] constructed these functions for $k = 2$. In Okada, Sekiguchi and Shiota [8], we have obtained an explicit formula of $S_k(N)$ for $p = 2$ by use of the binomial measure and the result obtained in Sekiguchi and Shiota [13]. Our formula also contains functions defined by recursive equations. We emphasize that these equations can be solved completely. For an explicit formula of $F(\xi, N)$, Stein [14] gave a one. In Okada, Sekiguchi and Shiota [9], we have given another type of explicit formula of $F(\xi, N)$ and noticed that we can directly get explicit formulas of $S_k(N)$ by use of the equality

$$\frac{\partial^k}{\partial \xi^k} F(\xi, N) \Big|_{\xi=0} = \sum_{n=0}^{N-1} s(n)^k e^{\xi s(n)} \Big|_{\xi=0} = S_k(N). \quad (1)$$

In this paper, developing this direct method and using the result on multinomial measures obtained in Okada, Sekiguchi and Shiota [10], we give an explicit formula of $F(\xi, N)$ and explicit formulas of $S_k(N)$. We finally remark that the study of explicit formulas of $S_k(N)$ and $F(\xi, N)$ is essentially reduced to that of multinomial measures.

2 Preliminaries

Let $I = I_{0,0} = [0, 1]$ and $I_{n,j} = [j/p^n, (j+1)/p^n]$, $j = 0, 1, \dots, p^n - 2$, $I_{n,p^n-1} = [(p^n - 1)/p^n, 1]$ for $n = 1, 2, 3, \dots$. Let $r = (r_0, r_1, \dots, r_{p-2})$ be a vector such that $0 < r_l < 1$ for $l = 0, 1, \dots, p-2$ and $\sum_{l=0}^{p-2} r_l < 1$ and set $r_{p-1} = 1 - \sum_{l=0}^{p-2} r_l$. The probability measure μ_r on I defined by

$$\mu_r(I_{n+1,pj+l}) = r_l \mu_r(I_{n,j}) \quad (2)$$

for $n = 0, 1, 2, \dots$, $j = 0, 1, \dots, p^n - 1$, $l = 0, 1, \dots, p-1$, is said to be a multinomial measure. If $r_l = 1/p$ for all l , μ_r is the Lebesgue measure. We denote the distribution function of μ_r by $L(r, \cdot)$:

$$L(r, x) = \mu_r([0, x]), x \in I. \quad (3)$$

In [10], we summarized some fundamental properties of $L(r, \cdot)$ and got an exact form of the k -th derivative of $L(r, \cdot)$ with respect to the parameters r_l .

3 An explicit formula of $F(\xi, N)$

For $N \in \mathbb{N}$, set $t = \log_p N$, and denote its integer part by $[t]$ and its decimal part by $\{t\}$. Evidently $p^{[t]} \leq N < p^{[t]+1}$ and $1/p \leq N/p^{[t]+1} = 1/p^{1-\{t\}} < 1$.

Theorem 3.1 We have

$$F(\xi, N) = \frac{1}{r_0^{[t]+1}} L\left(r, \frac{1}{p^{1-\{t\}}}\right), \quad (4)$$

where

$$r_l = \frac{e^{l\xi}}{1 + e^\xi + \dots + e^{(p-1)\xi}} \quad (5)$$

for $l = 0, 1, \dots, p-1$.

Proof. By definitions of L , μ_r and $s(n, l)$, we have

$$\begin{aligned} L\left(r, \frac{1}{p^{1-\{t\}}}\right) &= L\left(r, \frac{N}{p^{[t]+1}}\right) = \sum_{n=0}^{N-1} \left\{ L\left(r, \frac{n+1}{p^{[t]+1}}\right) - L\left(r, \frac{n}{p^{[t]+1}}\right) \right\} \\ &= \sum_{n=0}^{N-1} \mu_r(I_{[t]+1}, n) = \sum_{n=0}^{N-1} r_0^{[t]+1 - \sum_{l=1}^p s(n, l)} r_1^{s(n, 1)} \dots r_{p-1}^{s(n, p-1)} \\ &= \sum_{n=0}^{N-1} r_0^{[t]+1} \left(\frac{r_1}{r_0}\right)^{s(n, 1)} \dots \left(\frac{r_{p-1}}{r_0}\right)^{s(n, p-1)} \\ &= r_0^{[t]+1} \sum_{n=0}^{N-1} e^{\xi(s(n, 1) + \dots + (p-1)s(n, p-1))} \\ &= r_0^{[t]+1} \sum_{n=0}^{N-1} e^{\xi s(n)} = r_0^{[t]+1} F(\xi, N). \end{aligned}$$

■

4 An explicit formula of $S_k(N)$

To obtain an explicit formula of $S_k(N)$, we have only to calculate the left-hand side of (1). We now set

$$a(m, x, \xi_0) = \frac{\partial^m}{\partial \xi^m} \left(\frac{1 + e^\xi + \dots + e^{(p-1)\xi}}{1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0}} \right)^x \Big|_{\xi = \xi_0} \quad (6)$$

for $x \in \mathbf{R}$ and denote $(\partial^l / \partial x^l) a(m, x, \xi_0)$ by $a^{(l)}(m, x, \xi_0)$.

Theorem 4.1 *We have*

$$\frac{\partial^k}{\partial \xi^k} F(\xi, N) \Big|_{\xi = \xi_0} = (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^t \sum_{l=0}^k H_{k,l}(t, \xi_0) \left(\frac{t}{1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0}} \right)^l \quad (7)$$

for $k \geq 0$, where $H_{k,l}(x, \xi)$ is a continuous periodic function of period 1 with respect to x defined by

$$\begin{aligned} H_{k,l}(x, \xi_0) &= (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^l \sum_{j=0}^{k-l} \binom{k}{j} (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^{1-\{x\}} \\ &\quad \times \frac{1}{l!} a^{(l)}(k-j, 1-\{x\}, \xi_0) \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-\{x\}}}\right) \Big|_{\xi = \xi_0}. \end{aligned} \quad (8)$$

Proof. The assertion for $k = 0$ is that of Theorem 3.1. We show for $k > 0$. We have, by Leibniz's formula,

$$\begin{aligned} \frac{\partial^k}{\partial \xi^k} F(\xi, N) \Big|_{\xi = \xi_0} &= (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^t \sum_{j=0}^k \binom{k}{j} (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^{1-\{t\}} \\ &\quad \times \frac{\partial^{k-j}}{\partial \xi^{k-j}} \left(\frac{1 + e^\xi + \dots + e^{(p-1)\xi}}{1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0}} \right)^{[t]+1} \Big|_{\xi = \xi_0} \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-\{t\}}}\right) \Big|_{\xi = \xi_0} \\ &= (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^t \sum_{j=0}^k \binom{k}{j} (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^{1-\{t\}} \\ &\quad \times a(k-j, [t]+1, \xi_0) \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-\{t\}}}\right) \Big|_{\xi = \xi_0} \\ &= (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^t \sum_{j=0}^k \binom{k}{j} (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^{1-\{t\}} \\ &\quad \times a(k-j, t+(1-\{t\}), \xi_0) \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-\{t\}}}\right) \Big|_{\xi = \xi_0}. \end{aligned}$$

Since $a(m, x, \xi_0)$ is a polynomial in x of degree m , we obtain by Taylor's expansion

$$\frac{\partial^k}{\partial \xi^k} F(\xi, N) \Big|_{\xi = \xi_0} = (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^t \sum_{j=0}^k \binom{k}{j} (1 + e^{\xi_0} + \dots + e^{(p-1)\xi_0})^{1-\{t\}}$$

$$\begin{aligned}
& \times \sum_{l=0}^{k-j} \frac{1}{l!} a^{(l)}(k-j, 1-\{t\}, \xi_0) t^l \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-\{t\}}}\right) \Big|_{\xi=\xi_0} \\
& = (1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^t \sum_{l=0}^k (1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^l \\
& \quad \times \sum_{j=0}^{k-l} \binom{k}{j} (1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^{1-\{t\}} \frac{1}{l!} a^{(l)}(k-j, 1-\{t\}, \xi_0) \\
& \quad \times \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p^{1-\{t\}}}\right) \Big|_{\xi=\xi_0} \left(\frac{t}{1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0}} \right)^l.
\end{aligned}$$

The periodicity of $H_{k,l}(x, \xi_0)$ is evident. It suffices to show the continuity of $H_{k,l}(x, \xi_0)$ at $x \in \mathbb{Z}$. As $L(r, 1) = 1$,

$$\begin{aligned}
H_{k,l}(1, \xi_0) &= (1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^l \sum_{j=0}^{k-l} \binom{k}{j} \frac{1}{l!} a^{(l)}(k-j, 0, \xi_0) \frac{\partial^j}{\partial \xi^j} L(r, 1) \Big|_{\xi=\xi_0} \\
&= \frac{(1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^l}{l!} a^{(l)}(k, 0, \xi_0)
\end{aligned}$$

for $k \geq l \geq 0$. Since $a^{(l)}(k-j, 1, \xi_0) = 0$ for $k-l < j$ and $L(r, 1/p) = 1/(1 + e^\xi + \cdots + e^{(p-1)\xi})$,

$$\begin{aligned}
H_{k,l}(0, \xi_0) &= (1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^l \\
& \quad \times \sum_{j=0}^k \binom{k}{j} (1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^l \frac{1}{l!} a^{(l)}(k-j, 1, \xi_0) \frac{\partial^j}{\partial \xi^j} L\left(r, \frac{1}{p}\right) \Big|_{\xi=\xi_0} \\
&= \frac{(1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^l}{l!} \frac{\partial^l}{\partial x^l} \left[\sum_{j=0}^k \binom{k}{j} \frac{\partial^{k-j}}{\partial \xi^{k-j}} \left(\frac{1 + e^\xi + \cdots + e^{(p-1)\xi}}{1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0}} \right)^x \right. \\
& \quad \times \left. \frac{\partial^j}{\partial \xi^j} \left(\frac{1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0}}{1 + e^\xi + \cdots + e^{(p-1)\xi}} \right) \right] \Big|_{\xi=\xi_0, x=1} \\
&= \frac{(1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^l}{l!} \frac{\partial^l}{\partial x^l} \frac{\partial^k}{\partial \xi^k} \left(\frac{1 + e^\xi + \cdots + e^{(p-1)\xi}}{1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0}} \right)^{x-1} \Big|_{\xi=\xi_0, x=1}
\end{aligned}$$

for $k \geq l \geq 0$. Hence $H_{k,l}(x, \xi_0)$ is continuous at $x \in \mathbb{Z}$. ■

To calculate the function $a(m, x, \xi_0)$, we use the following well-known lemma in calculus.

Lemma 4.1 *Let f, g be infinitely differentiable functions. Then we have*

$$(f \circ g)^{(m)} = \sum_{v=1}^m f^{(v)} \circ g \sum_v \frac{m!}{v_1! \cdots v_m!} \prod_{i=1}^m \left(\frac{g^{(i)}}{i!} \right)^{v_i} \quad (9)$$

where the summation $\sum_v$ is taken over $v_1, v_2, \dots, v_m \geq 0, v_1 + v_2 + \cdots + v_m = v$ and $v_1 + 2v_2 + \cdots + mv_m = m$.

Proposition 4.1 *We have*

$$\begin{aligned}
a(m, x, \xi_0) &= \sum_{v=1}^m \frac{1}{(1 + e^{\xi_0} + \cdots + e^{(p-1)\xi_0})^v} x(x-1) \cdots (x-v+1) \\
& \quad \times \sum_v \frac{m!}{v_1! \cdots v_m!} \prod_{i=1}^m \left(\frac{1}{i!} \sum_{l=0}^\infty \frac{\phi_{l+i+1}(p)}{(l+i+1)!} \xi_0^l \right)^{v_i} \quad (10)
\end{aligned}$$

for $m > 0$. Here ϕ_i is Bernoulli's polynomial of degree i , that is,

$$\frac{t(e^{xt} - 1)}{e^t - 1} = \sum_{i=1}^\infty \frac{\phi_i(x)}{i!} t^i \quad (11)$$

and the summation $\sum_v$ is taken over $v_1, v_2, \dots, v_m \geq 0, v_1 + v_2 + \cdots + v_m = v$ and $v_1 + 2v_2 + \cdots + mv_m = m$.

Taking $\xi_0 = 0$, we have the following corollaries.

Corollary 4.1 *We have*

$$S_k(N) = N \sum_{l=0}^k H_{k,l}(t, 0) \left(\frac{t}{p} \right)^l. \quad (12)$$

Corollary 4.2 *We have*

$$a(m, x, 0) = \sum_{v=1}^m \frac{1}{p^v} x(x-1) \cdots (x-v+1) \sum_v \frac{m!}{v_1! \cdots v_m!} \prod_{i=1}^m \left(\frac{\phi_{i+1}(p)}{(i+1)!} \right)^{v_i} \quad (13)$$

for $m > 0$.

5 Remarks

5.1 Asymptotic behaviors

We can investigate various types of asymptotic behavior of $(\partial^k / \partial \xi^k) F(\xi, N)$ by use of Theorem 4.1. For example, since

$$\begin{aligned} \frac{\partial^k}{\partial \xi^k} F(\xi, N) & \sim \frac{H_{k,k}(\log_p N, \xi)}{N^{\log_p(1 + e^\xi + \dots + e^{(p-1)\xi})} (\log_p N)^k} \\ & = \left(\frac{\sum_{l=0}^{\infty} (l+1) \frac{\phi_{l+2}(p)}{(l+2)!} \xi^l}{1 + e^\xi + \dots + e^{(p-1)\xi}} \right)^k (1 + e^\xi + \dots + e^{(p-1)\xi})^{1-\{l\}} L\left(r, \frac{1}{p^{1-\{l\}}}\right) \end{aligned}$$

as $N \rightarrow \infty$, it follows that

$$\limsup_{N \rightarrow \infty} \frac{\partial^k}{\partial \xi^k} F(\xi, N) = \left(\frac{\sum_{l=0}^{\infty} (l+1) \frac{\phi_{l+2}(p)}{(l+2)!} \xi^l}{1 + e^\xi + \dots + e^{(p-1)\xi}} \right)^k \max_{1/p \leq x \leq 1} x^{\log_p r_0} L(r, x), \quad (14)$$

$$\liminf_{N \rightarrow \infty} \frac{\partial^k}{\partial \xi^k} F(\xi, N) = \left(\frac{\sum_{l=0}^{\infty} (l+1) \frac{\phi_{l+2}(p)}{(l+2)!} \xi^l}{1 + e^\xi + \dots + e^{(p-1)\xi}} \right)^k \min_{1/p \leq x \leq 1} x^{\log_p r_0} L(r, x). \quad (15)$$

To obtain the precise values of the left-hand sides, it suffices to estimate the function $x^{\log_p r_0} L(r, x)$. However it is hard, because $L(r, x)$ is a singular function for $\xi \neq 0$.

5.2 Inductive formula of $H_{k,l}$

We notice that $H_{k,l}$ satisfies the next inductive formula.

$$H_{0,0}(x, \xi) = (1 + e^\xi + \dots + e^{(p-1)\xi})^{1-\{x\}} L\left(r, \frac{1}{p^{1-\{x\}}}\right), \quad (16)$$

$$\begin{aligned} H_{k,0}(x, \xi) &= (1 + e^\xi + \dots + e^{(p-1)\xi})^{1-\{x\}} \frac{\partial^k}{\partial \xi^k} L\left(r, \frac{1}{p^{1-\{x\}}}\right) \\ &\quad - \sum_{j=0}^{k-1} \binom{k}{j} a(k-j, \{x\}-1, \xi) H_{j,0}(x, \xi), \end{aligned} \quad (17)$$

$$\begin{aligned} H_{k,l}(x, \xi) &= - \sum_{j=0}^{k-1} \sum_{q=0 \vee (l-j)}^{l \wedge (k-j)} (-1)^q \binom{k}{j} \frac{(1 + e^\xi + \dots + e^{(p-1)\xi})^q}{q!} \\ &\quad \times a^{(q)}(k-j, \{x\}-1, \xi) H_{j,l-q}(x, \xi) \end{aligned} \quad (18)$$

for $k \geq 1$ and $1 \leq l \leq k$. This kind of formula has been treated in [3] and [8].

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