A Condition for a Graph in the Disc to Contain the Half Hemisphere of the Dodecahedron —Application to Dehn Surgery Theory—

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§1 Theorems on graphs imbedded in the 2-sphere and the disc

We will summarise the paper [HS] on Dehn surgery on knots in 3-dimensional manifold, where two (perhaps new) theorems on genuine graph theory on graphs imbedded in the 2-dimensional disc are successfully applied. In fact, combinatorics on graphs imbedded in real 2-dimensional surfaces is one of the most important tactics in Dehn surgery theory. Roughly speaking, two graphs arise from two surfaces which are properly imbedded in a 3-manifold and intersecting each other. We cap off the boundary loops of the surfaces with discs, and regard them as “fat” vertices, and the arcs of the intersection as edges. This technique is introduced by R. Litherland in [Li], and developed and successfully applied to many problems on Dehn surgery, like as the knot complement theorem [G-Lu]. See for surveys [Gr1], [Gr2], [Gr3], [Gr4] and [H-M2]. We will introduce two theorems on graph theory in this section, and in the next section summarise the way how these theorems are applied to Dehn surgery theory.

Let Λ be a finite graph imbedded in the 2-sphere S^2 . If Λ has a vertex v as below, then we call v a *cut-vertex*. That is, there is a simple closed curve c on S^2 such that $\Lambda \cap c = v$ and that c divides S^2 into two discs each of which contains an edge of Λ . See Figure 1.1. A *face* of Λ is a component of $S^2 - \Lambda$. A face is called a *q-gon* if it is incident to q edges of Λ . A vertex v of Λ is called of *valency* k if it is an amalgamation of k endpoints of edges of Λ .

Theorem 1. *Let i be 3, 4 or 5. Let Λ be a finite graph imbedded in the 2-sphere S^2 such that Λ does not have a cut-vertex. Suppose that Λ has a face which is a q -gon for some integer $q \geq 2$, and the other faces are i -gons. We separates the vertices of Λ into two classes: The vertices of one of them are called non-singular vertices and have valency 3, and those of the other class are called singular vertices and have valency no less than 2. Suppose that all the singular vertices are on the q -gon, and Λ contains at most $i - 1$ singular vertices.*

Then Λ contains the pattern as described by solid lines in Figure 1.2, 1.3 or 1.4 (i.e., the “hemisphere” of the tetrahedron, the hexahedron or the dodecahedron), according to the value of $i = 3, 4$ or 5 , where none of the vertices of valencies 3 in these figures is singular, and none of the faces of in these figures is the q -gon. In fact, in case $i = 3$, Λ is one of the infinitely many graphs as described in Figure 1.5.

Remark. In the above statement, $q = i$ is possible, and singular vertices with valency 3 are allowable.

The situation of Theorem 1 gives us a ‘boundary value problem’ on graphs imbedded in surfaces. We gave the proof of this theorem in [HS, Appendix] in case $i = 5$. The proof is elementary but very long. In the other cases

Fig. 1.1

Fig. 1.2

Fig. 1.3

Fig. 1.4

$i = 3$ or 4 the proofs are much easier and much shorter. We give a proof of Theorem 1 in case $i = 4$ in Appendix in this paper.

Problems. Determine Λ wholly in cases $i = 4$ and 5 .

What happens when Λ has i or more singular vertices?

For example, when $i = 5$ and Λ has precisely 5 singular vertices, Λ may consist of a cycle of 5 edges which divides S^2 into two 5-gon faces.

Theorem 2. Let Λ be a graph imbedded in the disc D^2 such that each vertex of Λ is of valency 3 or 4, such that the vertices of valency 3 are the vertices of a cycle c in Λ and such that the cycle c is imbedded onto the boundary circle ∂D^2 . Suppose that each edge of Λ is given a label α or β so that every pair of opposite edges at every vertex of valency 4 are separated into one with a label α and the other with a label β and so that every pair of adjacent edges of the cycle c are separated similarly. Then there is a face of Λ such that it is incident only to edges with labels α or only to those with labels β .

Fig. 1.5

Fig. 1.6

See Figure 1.6, where we can find two such faces encircled by edges with labels α . One is incident to ∂D , and the other is not. This theorem is very easily proved. Seiya Negami told the authors a simpler proof which uses smoothing operation and induction on the number of vertices. (The original proof is based on the techniques of crossing changes and Reidemeister moves on link and tangle diagrams.) But before telling the proof, we can obtain a good sense by showing a resemble fact on graphs in the 2-sphere S^2 as below.

Theorem 2'. *Let Λ be a graph imbedded in the 2-sphere S^2 such that all the vertices of Λ are of valency 4. Suppose that each edge of Λ is given a label α or β so that every pair of opposite edges at every vertex of valency 4 are separated into one with a label α and the other with a label β . Then there are two faces of Λ such that each of them is incident only to edges with labels α or only to those with labels β .*

This theorem is rather trivial. The next very simple proof is due to Kouki Taniyama.

Proof of Theorem 2'. We perform smoothing operations on all the vertices of Λ so that very small neighbourhood of each vertex is removed and two short arcs are added, where one of them connects the two edges with labels α and the other does the two edges with labels β . See Figure 1.7. Thus we obtain a disjoint union of loops imbedded in the 2-sphere S^2 . There are at least two innermost loops, that is, they bound disjoint discs which are disjoint from the other loops. Then these two discs give the two faces of Λ as desired. $\square$

Fig. 1.7

Proof of Theorem 2. When Λ does not have a vertex of valency 4, the graph Λ does not have a vertex in the interior of the disc D^2 , and we can obtain the desired result by an easy inductive argument on the number of the edges which intersects the interior of the disc. When Λ has a vertex of valency 4, we perform smoothing operations at all the vertices of valencies 4 as in the proof of Theorem 2'. If we obtain a loop component, then innermost one gives a face of Λ as desired. If we obtain no loop components, then we obtain a graph Λ' with no vertices of valencies 4. We apply the result of the first sentence of this proof to Λ' , and find a face which is incident only to edges with labels α or only to edges with labels β . This face gives a face of Λ as desired. $\square$

§2 Application to Dehn surgery

In this section we show that we can apply Theorems 1 and 2 to a problem on Dehn surgery theory. We summarise the proof of the main theorem in [HS] “Symmetric knots satisfy the cabling conjecture”.

Results on Dehn surgery

First, we recall the cabling conjecture. Let X be an orientable 3-manifold. Let K be a knot in X , i.e., a simple closed curve in the interior of X . We can regard a knot as an imbedding of a circle S^1 in $\text{int } X$. Let $N(K)$ be a regular neighbourhood of K in X . Note that $N(K)$ is homeomorphic to $S^1 \times D^2$ a solid torus. A simple closed curve imbedded in a surface is called *essential* if it does not bound a disc on the surface. We call an isotopy class of essential simple closed curves on the torus $\partial N(K)$ a *slope*. Let W be a solid torus. It is well-known that in the torus ∂W all the essential loops which bound discs in W are isotopic. This gives a slope on ∂W , which we call the *meridian slope*. Such discs in a solid torus are called *meridian discs*. Let r be a slope on $\partial N(K)$. We let $X(K; r)$ denote the 3-manifold obtained from X by Dehn surgery of slope r (r -surgery for short) on K , i.e., the result of attaching a solid torus W to $E(K) = X - \text{int } N(K)$ by identifying ∂W with $\partial E(K) \cap \partial N(K)$ so that r is the meridian slope on ∂W . We use K^* to denote the core circle $S^1 \times (\text{an interior point})$ of $W \cong S^1 \times D^2$ in the 3-manifold $X(K; r)$. We call K^* the dual knot. A Dehn surgery on a 3-manifold yields a 3-manifold which is usually not homeomorphic to the original one. It is very rare case where the resulting 3-manifold is homeomorphic to the original one. A knot $K \subset X$ is said to be *cabled* in X if there is another knot K' in X such that there is a regular neighbourhood $N(K')$ of K' as below. That is, K is contained in the boundary torus $\partial N(K')$, the isotopy class represented by K is not the meridian slope on $\partial N(K')$ and the homology class represented by K is not a generator of the 1-dimensional homology $H_1(N(K'); \mathbf{Z})$ of the solid torus $N(K')$. For example, a torus knot in S^3 is a cabled knot. A 3-manifold is said to be *reducible* if it contains a 2-sphere that does not bound a 3-ball. The cabling conjecture (formally raised in [G-S]) states that a knot K is cabled in S^3 if $S^3(K; r)$ is reducible. The conjecture has been proved for several classes of knots in S^3 including satellite knots [Sch], strongly invertible knots [E], alternating knots [M-T], arborescent knots [W] and knots with essential tangle decomposition [V].

A symmetry for a knot K in X is a finite group G acting nontrivially on X such that $g(K) = K$ for each element g of G . In [L-Z], Luft and Zhang showed that “symmetric knots satisfy the cabling conjecture” except for periodic knots of order 2, 3 or 5. It is confirmed in [H-M] that periodic knots of order 3 or 5 also satisfy the cabling conjecture. In [HS] we considered periodic knots of order 2, and gave a proof of the following result.

Theorem A. *Symmetric knots in S^3 satisfy the cabling conjecture.*

C. McA. Gordon and J. Luecke also proved independently this theorem ([G-Lu3]) by a different method from ours. Our proof of Theorem A uses Theorem B below.

Let F be a 2-dimensional surface properly imbedded in a 3-manifold M , where “properly” means that $F \cap \partial M = \partial F$. The surface F is called *separating* if $M - F$ has larger number of connected components than M . The surface F is called *incompressible* if for every imbedded disc D with $D \cap F = \partial D$ there is a disc D' on F with $\partial D' = \partial D$. If F is a 2-sphere or a disc, then it is incompressible. The surface F is called *∂ -incompressible* if for every imbedded disc D with $D \cap F = (\text{a subarc of } \partial D)$ and with $D \cap \partial M = (\text{the complementary subarc of } \partial D)$ there is a disc D' on F with $D \cap F = (\text{a subarc of } \partial D')$ and with $\partial D' - (D \cap F) \subset \partial F$. The surface F is called

Fig. 2.1

boundary parallel if a component obtained by cutting M along F is homeomorphic to the product space $N = F \times [0, 1]$, where F is identified with $F \times \{0\}$ and $\partial N - F \subset \partial M$. The surface F is called *essential* if it is incompressible, ∂ -incompressible and not boundary parallel.

Let $\Delta(r, s)$ denote the minimal number of intersection points up to isotopy between representative simple closed curves of the two slopes r and s on a torus.

Theorem B. *Let V be a solid torus, D a meridian disc of V , and $K \subset V$ a non-cabled knot. Assume that ∂V is incompressible in $V - \text{int } N(K)$. If the manifold $V(K; r)$ contains a separating essential annulus A such that each component of ∂A intersects the loop ∂D transversely in one point, then $\Delta(\mu, r) = 1$, where μ is the meridian slope of $N(K)$.*

Luft and Zhang proved a weaker version of Theorem B ([L-Z, Lemma]), which states $\Delta(\mu, r) \leq 6$. (Note that 2, 3, 5 are prime numbers less than 6. In fact, $\Delta(\mu, r)$ corresponds to the order of a periodic knot in S^3 .) Moreover they pointed out at p. 490 ll. 1–3 in [L-Z] that all symmetric knots will satisfy the cabling conjecture if one can show that $\Delta(\mu, r) = 1$. That is, Theorem B implies Theorem A.

Formation of graphs

In the remainder of this section, we give a sketch of a proof of Theorem B. We take the meridian disc D of V so that it intersects K in minimal number, say $n_D = |D \cap K|$, of points over all meridian discs. Then we can show by a standard cut and paste argument that the planar surface $P_D = D \cap (V - \text{int } N(K))$ is incompressible and ∂ -incompressible in $V - \text{int } N(K)$. (Please believe this fact.)

We are given a boundary slope ∂A on the torus ∂V . We take the annulus A in $V(K; r)$ so that $n_A = |A \cap K^*|$ is minimal over all essential annuli in $V(K; r)$ with such a boundary slope as above, where K^* is the dual knot. Then we can show that the surface $P_A = A \cap (V - \text{int } N(K))$ is incompressible and ∂ -incompressible in $V - \text{int } N(K)$.

A standard cut and paste argument allows us to take the meridian disc D and the annulus A with the same boundary slopes on ∂V as before so that ∂P_D and ∂P_A intersect in minimal points and that the intersection $P_D \cap P_A$ consists of loops and arcs which are essential on both P_D and P_A . (A properly imbedded arc in a surface is called *essential* if it is not parallel to the boundary of the surface.)

As in [G-Lu] we will form graphs G_D and G_A on D and A respectively.

In the following we assume that $\{i, j\} = \{D, A\}$. Assigning arbitrary orientations to P_i allows us to refer to + and – boundary components of $\partial P_i \cap \partial N(K)$, according to the direction of the induced orientation of a boundary component as it lies on $\partial N(K)$.

We orient arbitrarily the knots K and K^* . Number the components of $\partial P_i \cap \partial N(K)$, $\{1, 2, \dots, n_i\}$ in the order as they appear on $\partial N(K)$. We may assign the number “1” to an arbitrary component of ∂P_i . Thus K and K^* are divided into n_D and n_A intervals $[1, 2], [2, 3], \dots, [n_i, 1]$. Note that we distinguish the intervals $[1, 2]$ and $[2, 1]$ when $n_i = 2$.

Since ∂V is incompressible in the exterior of K , we have $n_D \geq 1$. If $n_A = 0$, then we have $A \cap K^* = \emptyset$, and Lemma 2.1 in [H-M] shows that A is ∂ -compressible in $V - \text{int } N(K)$, which contradicts that A is ∂ -incompressible in $V(K; r)$. We assume in the following that $n_A \geq 2$. Note that n_A is even since A is separating in $V(K; r)$.

We label the end points of arcs of $P_D \cap P_A$ in P_i with the number assigned to the corresponding boundary components of P_j . Remember that ∂P_D and ∂P_A intersect in minimal number of points on the torus $\partial N(K)$. Thus around each component of $\partial P_i \cap \partial N(K)$ we see the labels $\{1, 2, \dots, n_j\}$ appearing sequentially (either clockwise or anticlockwise according to the sign + or – of this component of ∂P_i) Δ times. See Figure 2.1.

We regard the discs $D \cap N(K)$ and $A \cap N(K^*)$ as forming the “fat vertices” of graphs G_D and G_A . The edges of G_i are corresponding to the arcs of $P_D \cap P_A$ in P_i except for the arcs whose both end points are in ∂i . We ignore loop components of $P_D \cap P_A$. We call the closure of a component of $\partial(\text{fat vertex})$ – (end points of edges) a *corner*. If an edge e connects a vertex to a vertex, then we say e is an *interior edge*, otherwise a *boundary edge*. A boundary edge connects a vertex to a component of ∂i . Since each component of ∂A intersects ∂D in precisely one point, the graphs G_D and G_A have zero or two boundary edges. If an interior edge e has both two end points in the same fat vertex, then we say e is a *loop edge*. The graph G_i contains no *trivial loop edges*, i.e., G_i does not have a 1-sided face (no arc of $P_A \cap P_D$ is boundary parallel in P_i). If $n_D = 1$ and $\Delta(\mu, r) \geq 2$, then the vertex of G_D has valency at least 4, and G_D must contain a trivial loop edge, which is a contradiction. Hence we assume that $n_D \geq 2$ in the following. Two edges e and e' of G_i are *parallel* if there is a disc B in P_i such that $\partial B = e \cup b \cup e' \cup b'$, where b and b' are arcs in ∂P_i . Every fat vertex v is assigned a sign $+$ or $-$ according to that of the loop ∂v . If an interior edge e connects vertices of the same sign, then we say e is a *homo-edge*, otherwise a *hetero-edge*. A loop edge is a homo-edge. Since $V - \text{int } N(K)$ is orientable, we have the *parity rule*: an interior edge of the graph G_i is a homo-edge if and only if the corresponding edge of the other graph G_j is a hetero-edge. We thus obtain two labeled graphs in D and A , whose edges are in one to one correspondence. We call components of $i - G_i$ *faces* of G_i . A face P is called a *disc face* if P is an open disc. For every face P , let ∂P denote its *boundary subgraph*, i.e., the subgraph which consists of vertices and edges intersecting $\bar{P} - P$.

Combinatorics and application of Theorems 1 and 2

Let x be a label of G_i . An x -edge in G_i is an interior edge with label x at an end point. A *Scharlemann cycle for the interval* $[x, x + 1]$ is a cycle σ of G_i such that there is a face F of G_i as below.

- (1) $\partial F = \sigma$.
- (2) All the vertices of σ are of the same sign.
- (3) All the edges are x -edges and also $(x + 1)$ -edges simultaneously.

See Figure 2.2.

We see labels x and $x + 1$ appear alternately on the circle $\bar{F} - F$. We call the subgraph of G_j consisting of the vertices x and $x + 1$ and the edges corresponding to those of σ the *Scharlemann co-cycle* of σ . Note that the edges of the Scharlemann co-cycle are hetero-edges because of the parity rule. A Scharlemann co-cycle in G_j is *inessential* if it is contained in a disc imbedded in the surface j .

We refer to the following two Lemmas 2.1 and 2.2.

Lemma 2.1. ([C-G-L-S, Lemma 2.5.2.(a)]) *The graph G_A does not contain a Scharlemann cycle. The graph G_D does not contain a Scharlemann cycle whose Scharlemann co-cycle is inessential.*

Rough sketch of Proof. Suppose that the graph G_D contains a Scharlemann cycle for the interval $[x, x + 1]$. Let F be the face of the Scharlemann cycle. We remove the two discs of fat vertices numbered x and $x + 1$ from the annulus A , and add the annulus between the boundaries of these discs on the torus $\partial N(K^*)$. Then we obtain a torus with two holes, say $\tilde{A}$. Note that ∂F is contained in $\tilde{A}$. See Figure 2.3. We cut $\tilde{A}$ along the circle ∂F and add two parallel copies of F to the resulting boundaries. Thus we obtain a new annulus A' such that $\partial A' = \partial A$

Fig. 2.2

Fig. 2.3

and that $|A' \cap K^*| = |A \cap K^*| - 2$. We can show by a standard cut and paste argument that the annulus A' is essential if the corresponding Scharlemann co-cycle is inessential. This contradicts the minimality of the number n_A . $\square$

If there are no boundary edges, then there is an arc α of $P_D \cap P_A$ connecting the two boundary components ∂A . At this time we regard the graph G_A as a graph in the disc $A - \alpha$. Hence if $\Delta(\mu, r) \geq 2$, then we can easily find a Scharlemann cycle with inessential co-cycle (see for example [H-M, Theorem 2.1]) on the graph G_D or G_A , which contradicts Lemma 2.1. Hence we assume in the following that G_D and G_A have precisely two boundary edges.

Lemma 2.2. ([G-Li, Proposition 1.3]) *If G_A contains a parallel family of more than $n_D - 1$ interior edges, then the knot K is a cabled knot in V .*

Lemma 2.3. *Let σ_1 and σ_2 be Scharlemann cycles of G_D for the same intervals. Then the length of σ_1 is equal to that of σ_2 .*

Proof. Let Q_1 and Q_2 be the faces of the Scharlemann cycles σ_1 and σ_2 respectively. Without loss of generality we assume that these two Scharlemann cycles are for the interval $[1, 2]$. The two vertices of G_A divides the solid torus $N(K^*)$ into two balls. Let B be the one which corresponds to the interval $[1, 2]$. We take a meridian loop l of the solid torus $N(K^*)$ on ∂B . Let Y be the torus with two holes $(A - (A \cap B)) \cup (\partial B - \text{int}(B \cap A))$. Note that l is still an essential loop if we capping off the two holes of Y with discs to obtain an abstract torus. The two loops $\partial \bar{Q}_1$ and $\partial \bar{Q}_2$ are disjoint simple essential loops on Y' , and they intersect l minimally up to isotopy on Y' . Hence $|l \cap \partial \bar{Q}_1| = |l \cap \partial \bar{Q}_2|$. $\square$

When $n_A \geq 4$, Theorem B follows from the technique of “representing all types” as in [G-Lu2] (see also [Gr4]) and a result in [Gd]. We omit the proof in this case. See [HS] for details. (But the authors are very interested in how graph theorists think about the definition of “representing all types”. We can find it in the summary [Gr4].)

Hence we concentrate on case $n_A = 2$ in the remain of this article. Thus K^* is divided into $n_A = 2$ intervals $[1, 2]$ and $[2, 1]$. Moreover we assume $\Delta(\mu, r) = 2$ by means of Theorem 1.2 in [H-M] where $\Delta(\mu, r) \leq 2$ is shown. This assumption will lead us to a contradiction.

We form a graph Γ_D by connecting endpoints of the two boundary edges of G_D along a subarc of ∂D . Then the boundary edges are amalgamated into an edge, say e_0 . Thus Γ_D is a graph imbedded in the interior of the disc D such that all the vertices are of valencies 4. A disc face of Γ_D is called a fake disc face if it is incident to the edge e_0 and disjoint from ∂D . A fake disc face is not a disc face of the original graph G_D . Since we are assuming that $n_D \geq 2$, we can form Γ_D so that the edge e_0 is not a trivial loop edge by adequately selecting the subarc of ∂D connecting the two boundary edges. Then, again applying the technique of [G-Lu2] and a result in [Gd], we can show that Γ_D contains a non-empty connected subgraph Σ satisfying the following conditions (1), (2) and (3). We omit the proof.

- (1) All the faces of the graph Σ are Scharlemann cycles except for one, say U , incident to ∂D .
- (2) Either all the vertices of Σ are of valency 4, or so except for two vertices of valency 3 which are on the subgraph ∂U .
- (3) For $x = 1$ or 2 , for any vertex v of Σ other than a vertex, say v_0 , there is an edge of Σ incident to v at each occurrence of the label x at v .

A vertex v of Σ is called a *cut-vertex* if D contains a simple closed curve C such that $C \cap \Sigma = v$ and Σ has edges in both sides of C . See Figure 1.1.

We say that Σ is *composite* if D contains a simple closed curve C such that $C \cap \Sigma = C \cap \{\text{interior of edges of } \Sigma\}$, $|C \cap \Sigma| = 2$ and Σ has vertices on both sides of C . Otherwise Σ is *prime*. It is clear that we can retake Σ to be prime and without cut-vertices.

Except for the fake disc face, for some integers a and b larger than 1, all the faces of Scharlemann cycles for the interval $[1, 2]$ are a -gons, and all the faces of Scharlemann cycles for the interval $[2, 1]$ are b -gons by Lemma 2.3. An elementary calculation on Euler characteristic shows that $\{a, b\} = \{2, n\}, \{3, 3\}, \{3, 4\}$ or $\{3, 5\}$, where n is any integer larger than 1. When $\{a, b\} = \{2, n\}$, we do not use Theorems 1 and 2 in section 1. Hence we omit the proof in this case.

We consider the other cases in the rest of this section. We can assume $a = 3$ without loss of generality. We assume for simplicity of arguments that a fake disc face is that of a Scharlemann cycle for the interval $[1, 2]$ rather than $[2, 1]$ if Σ has a fake disc face of Γ_D as a face of it. The fake disc face is a c -gon for some integer c larger than 1.

We form a graph $\tilde{\Sigma}$ from Σ as below. We take a vertex of $\tilde{\Sigma}$ in each a -gon disc face of Σ , and take a vertex w of $\tilde{\Sigma}$ in the face U containing ∂D . We take a vertex v of $\tilde{\Sigma}$ in the fake disc face. We take an edge of $\tilde{\Sigma}$ corresponding to each vertex of Σ as shown in Figure 2.4. We cap off the boundary loop ∂D with a disc D' and obtain an abstract 2-sphere S . Thus $\tilde{\Sigma}$ is a graph on the sphere S satisfying the following conditions.

- (1) The valency of w is p for some integer p at least 2. The valency of v is c . The other vertices are of valency $a = 3$.

Fig. 2.4

- (2) For some integer q at least 2, the graph $\tilde{\Sigma}$ has a q -gon face incident to w . This q -gon face contains the disc D' . It is incident to the vertex v and the additional edge e_0 if Σ has a fake disc face. The other faces of $\tilde{\Sigma}$ are b -gons.

We call the vertices w and v *singular* vertices. Note that this graph $\tilde{\Sigma}$ does not have a cut-vertex since the graph Σ is prime. However $\tilde{\Sigma}$ may or may not be prime.

Then we can apply Theorem 1 in section 1 to the graph $\tilde{\Sigma}$, to conclude that $\tilde{\Sigma}$ is one of the Figures 1.2, 1.3 and 1.4. (To prove Theorem 1 only for graphs with at most two singular vertices is much more difficult than to prove Theorem 1 entirely.) Hence the graph Σ contains the subgraph Λ described by broken lines in these figures. Theorem 1 assures that the edges of Λ are not the additional edge e_0 .

Thus we obtained the subgraph Λ of G_D as in the form described in Theorem 2. We give a labeling on edges of Λ as below. Recall that G_A has precisely two vertices since $n_A = 2$. The graph G_A on the annulus A contains at most two parallel families of hetero-edges connecting the two vertices. We call this possible two families α and β where β may be empty.

Lemma 2.4. *Assume that K is not a cabled knot in V . Let e and f be homo-edges of G_D (rather than Γ_D) which are incident to the same vertex, say z and have the same label, say y , at z . If e is contained in α , then f is contained in β .*

Proof. If the edge f is also contained in α , then α would contain two edges which have the label z at the vertex y in G_A , which implies that the graph G_A would contain a parallel family of more than n_D edges. Then K is a cabled knot in V by Lemma 2.2, which contradicts the assumption of this lemma. $\square$

We label the edges of the graph Λ with α or β according as the corresponding edges of G_A are contained in the parallel classes α or β . Then applying Theorem 2 in section 1, we can find a face F of Λ which is encircled by edges with the label α , say. This implies that the edges incident to F forms a Scharlemann cycle such that all the edges of the corresponding Scharlemann co-cycle are contained in the class α , and hence is inessential on A . This contradicts Lemma 2.1, and the proof of Theorem B is completed.

Appendix: The proof of Theorem 1

We show Theorem 1 for $i = 4$. In case of $i = 3$, the theorem is trivial. In case of $i = 5$, the proof is much harder, and is included in [HS, Appendix].

We compute the graph Λ beginning with the q -gon. We take the q -gon to be an ‘outermost’ one in the following sense. The graph Λ may contain a q' -gon Q' satisfying the conditions (1) and (2) below for some $q' \geq 2$. (1) Q' is an embedded disc composed of several faces of Λ containing the q -gon. (2) Q' has at most $4 - 1 = 3$ singular vertices and vertices with valency not equal to 3 in Λ' , where $\Lambda' = \Lambda \cap (S^2 - \text{int } Q')$. If Λ contains such a q' -gon, then the graph Λ' satisfies the conditions of the theorem by regarding formally the vertices with valency not equal to 3 as singular vertices. Hence we can assume that Λ does not contain such a q' -gon other than the

original q -gon.

There are edges incident to the vertices of the q -gon such that we do not know yet where their other endpoints are. We call such edges *free edges*.

If three singular vertices of valency 2 appear sequentially on the q -gon, then we find a 3-gon when $q = 3$, and we find a more than 4-gon when $q > 3$, which are contradictions. If the q -gon contains two singular vertices of valency 2 appear sequentially on the q -gon, then free edges adjacent to these singular vertices are amalgamated into an edge to form a 4-gon when $q \geq 4$, which contradicts the outermost property of the q -gon. When $q = 3$, we find a cut-vertex, and when $q = 2$, we find a 2-gon, which are also contradictions.

Claim 1: Each free edge incident to a vertex of the q -gon must be incident to a new vertex which is not contained in the q -gon.

Proof: Suppose that there is an edge e which violates the conclusion of this claim. If e has both two endpoints at the same singular vertex, then this vertex is a cut-vertex, which is a contradiction. Hence we assume that e has two endpoints at distinct vertices s and t . These two endpoints divide the boundary of the q -gon into two arcs, one of which, say α contains at most one singular vertex in its interior. Then we find a ' q' -gon' bounded by $e \cup \alpha$, which contradicts the outermost property of the q -gon regarding the vertices s and t as singular vertices formally. See Figure A.1. $\square$ (Claim 1)

Thus each *parent* vertex on the q -gon bears *child* vertices of the second *generation*. Suppose, for a contradiction, that the q -gon contains a singular vertex s of valency 2. If $q = 2$, then the other vertex of the q -gon is a cut-vertex. This is a contradiction. Hence we can assume that $q \geq 3$. Let t_1, t_2 be vertices adjacent to s on the q -gon. They are not of valency 2. The new vertices, which are children of t_1 and t_2 , are amalgamated into a vertex, say u , to form a 4-gon, say R , with vertices s, t_1 and t_2 . See Figure A.2. If $q = 3$ or 4 , then this contradicts the outermost property of the q -gon. Hence we can assume that $q \geq 5$. If t_1 or t_2 is a singular vertex of $\mathcal{A}$, then this contradicts the outermost property of the q -gon since the q -gon contains at most three singular vertices. Hence we can assume that t_1 and t_2 are non-singular vertices. Let r_i be the vertex adjacent to t_i on the q -gon other than s for

Fig. A.1

Fig. A.2

Fig. A.3

Fig. A.4

$i = 1$ and 2 . Note that the vertex u has a single free edge incident to it. This edge must be amalgamated with a free edge incident to a child vertex, say v_i , of the vertex r_i to form a 4-gon, for both $i = 1$ and 2 . This is impossible if $v_1 \neq v_2$. Hence $v_1 = v_2$, and we call this vertex v . A free edge f' is incident to v , and f' is amalgamated with the edge f to form two 4-gons R_1 and R_2 , where the vertices of R_i are u , t_i , r_i and v . Then the three 4-gons R , R_1 and R_2 give a pattern as desired since the vertices u , t_1 , t_2 and v are not singular. See Figure A.3. Hence there are no singular vertices of valency 2.

Now two free edges are incident to every new vertex of the second generation. Let Q be the disc face bounded by the q -gon, and R be the disc $S^2 - \text{int } Q$.

Claim 2: Among these new vertices of the second generation, no pair is amalgamated into a vertex.

Proof: Suppose, for a contradiction, that there is a pair of new vertices violating this claim. Let u be the vertex obtained by amalgamating these new vertices, and e_1 and e_2 the edges connecting the q -gon and u , and e the free edge incident to u . If e_1 and e_2 are incident to the same singular vertex on the q -gon, then this singular vertex is a cut-vertex or we find a 2-gon, which are contradictions. Hence the arc $\alpha = e_1 \cup e_2$ has two endpoints at distinct vertices, say s and t , on the q -gon. This arc α divides the disc R into two discs R_1 and R_2 where R_1 contains the free edge e . For $j = 1, 2$ let $\alpha_j = R_j \cap Q$ the arc on the q -gon. If the arc α_2 contains no singular vertices in its interior, then the disc $Q \cup R_1$ forms a ' q' -gon', and have a contradiction by formally regarding the vertices s , t and u as singular vertices. See Figure A.4. Hence the arc α_2 contains a singular vertex in its interior. If both s and t are non-singular vertices of Λ , then the vertices s , t and u form vertices of valency 2 of the graph $\Lambda \cap R_2$. Hence Λ has a l -gon for some $l > 4$ since Λ has no vertex of valency 2. This is a contradiction. Thus s or t , say s , is a singular vertex of Λ . Then the arc α_1 contains at most one singular vertex in its interior since q -gon contains

Fig. A.5

Fig. A.6

Fig. A.7

Fig. A.8

at most three singular vertices. The disc $Q \cup R_2$ forms a ' q' -gon', and have a contradiction by formally regarding the vertices s and t as singular vertices. $\square$ (Claim 2)

Claim 3: Suppose that a pair of free edges f_1 and f_2 , which are incident to new vertices u_1 and u_2 of the second generation respectively, are amalgamated into an edge f of $\mathcal{A}$. For $i = 1$ and 2 , let e_i be the edge of $\mathcal{A}$ which connects u_i to a vertex, say r_i , of the q -gon. Then r_1 and r_2 are connected by an edge, say e , of the q -gon, and $f \cup e_1 \cup e_2 \cup e$ bounds a 4-gonal face of $\mathcal{A}$. See Figure A.5.

Proof: If $u_1 = u_2$, then this is a cut-vertex, which is a contradiction. Let $\alpha = e_1 \cup e_2 \cup f$. If $r_1 = r_2$, then α forms a loop bounding a trigonal disc containing Q , and we regard this disc as a ' q' -gon' and obtain a contradiction to the outermost property of the q -gon. Hence $r_1 \neq r_2$ and α divides the disc R into two discs R_1 and R_2 . For $j = 1, 2$, let $\alpha_j = R_j \cap Q$, the arc on the q -gon.

First we suppose that the two free edges g_1 and g_2 , which are incident to the new vertices u_1 and u_2 respectively, are contained in the same region, say R_1 . If $\text{int } R_2$ contains a new vertex of the second generation, then we find in R_2 an l -gon ($l > 4$) which contains f as an edge. This is a contradiction. See Figure A.6. Hence $\text{int } R_2$ contains no new vertex of the second generation, and the cycle $f \cup e_1 \cup e_2 \cup \alpha_2$ bounds a face of $\mathcal{A}$. This face must be 4-gon, and the interior of the arc α_2 contains no vertex of the q -gon, which is the conclusion of this claim.

Next we suppose that the two free edges g_1 and g_2 are separated into the two regions, say $g_1 \subset R_1$ and $g_2 \subset R_2$. For $(i, j) = (1, 2)$ or $(2, 1)$, if r_i is non-singular, then we find in R_j an l -gonal face ($l > 4$) which is adjacent to the edges g_j, f and e_i . This is a contradiction. See Figure A.7. Hence both r_1 and r_2 are singular vertices, and one of the arcs α_1 or α_2 , say α_1 contains no singular vertex in its interior since the q -gon contains at most three singular vertices. Then $Q \cup R_2$ forms a ' q' -gon' and gives a contradiction to the outermost property of the q -gon. $\square$ (Claim 3)

Conversely, every such pair of free edges f_1 and f_2 must be amalgamated to form a 4-gon. (That is, let r_1 and r_2 be vertices of the q -gon such that they are adjacent. Let e_1 and e_2 be edges incident to r_1 and r_2 respectively such that they are not edges of the q -gon. Let u_1 and u_2 be child vertices of r_1 and r_2 respectively such that they are incident to the edges e_1 and e_2 respectively. Suppose that e_1 and e_2 are adjacent so that the vertices u_1, u_2, r_1 and r_2 are vertices of the same face of $\mathcal{A}$. Then two free edges f_1 and f_2 incident to u_1 and u_2 respectively must be amalgamated to form a 4-gon.)

Suppose, for a contradiction, that the q -gon contains a pair of adjacent singular vertices with valencies more than 2. Then union of the q -gon and the 4-gon containing these two singular vertices gives a ' q' -gon', contradicting the outermost property of the q -gon.

If all the vertices of the q -gon are of valency 3, then the graph $\mathcal{A}$ is completed by the operation of amalgamating free edges described right after the proof of Claim 3. We can find a new q -gon such that its vertices are new vertices of the second generation. This new q -gon must be a 4-gon. Hence the original q -gon is a 4-gon. We can find a pattern as desired since the original q -gon contains at most three singular vertices.

Hence we can assume that q -gon has a vertex with valency not equal to 3. By Claim 3, each free edge is incident to a new vertex of the third generation if it is not in the pattern of the conclusion of Claim 3. Two free edges are incident to every new vertex of the third generation.

For a pair of new vertices of the third generation, if their grandparent vertices coincide, if their parent vertices do not coincide and if the edges connecting the parent vertices and the new vertices are adjacent, then the new vertices must coincide to form a 4-gon as shown in Figure A.8.

If the q -gon contains a sequence of 2 or more vertices of valency 3, then we can find an l -gon ($l > 4$) which contains their children vertices. If the q -gon contains a vertex of valency 3 which is adjacent to two singular vertices of valencies 4 or larger, then two grandchildren vertices of the singular vertices are amalgamated into a single vertex to form a 4-gon. This amalgamated vertex is of valency 4 or more, which is a contradiction.

Thus the proof of Theorem 1 is completed in case of $i = 4$.

ERRATUM. The authors found a logical mistake in the proof of Claim 3 in Appendix in [HS]. There the amalgamated edge f is assumed to be an innermost one at the beginning of the proof. However, under such assumption, Claim 3 is shown only for such innermost amalgamated edges. We delete this assumption, and go in the same way as described in the proof of Claim 3 in Appendix in [HS], to reach "See Figure 23" in the last line in the page 520. At this time, we can find a ' q' -gon' which is the union of the q -gon, R_2 and the 5-gon. Thus we obtain a contradiction to the outermost property of the q -gon, and complete the proof.

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