The Solution of the Shifted Wave Equation on Damek-Ricci Space

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The aim of this paper is to give explicitly the solution of the shifted wave equation on Damek-Ricci space which is a semi-direct extension of the generalized Heisenberg groups. As its application, we show that Huygens' principle holds for every odd-dimensional Damek-Ricci space.

KEYWORDS: shifted wave equation, heat kernel, Damek-Ricci space

1. Introduction

In this paper, we give explicitly the solution of the shifted wave equation on Damek-Ricci spaces, which are defined as the semi-direct extensions of Heisenberg-type groups. Let us first explain the motivation of our study. To solve the Cauchy's problem of the wave equation on Riemannian manifolds, one can derive the solution from the heat kernel. The relation between the heat kernel and the solution of the wave equation is classical in some sense. For example, if the initial value of the heat equation is expressed as a Fourier expansion or an eigenfunction expansion, then the solution of the heat equation is given by multiplying $\exp(\lambda_n t)$ on each term of the expansion, where λ_n is the eigenvalue of the corresponding eigenfunction. Whereas one can derive the solution of the wave equation from the solution of the heat equation by replacing the coefficients $\exp(\lambda_n t)$ by $\cos(\sqrt{-\lambda_n} t)$, the behavior of the solution or even the its convergence is not clear from such expressions. The situation is similar for other spaces such as symmetric spaces, though one can still consider a Fourier transform there. Thus the use of Fourier analysis is not always helpful in determining an exact formula of the solution of the wave equation. On the other hand, Helgason [8], [9] obtained the solution of the shifted wave equation on real hyperbolic space $\mathbb{H}^n$, and studied Huygens' principle on $\mathbb{H}^n$. His solution is expressed in terms of spherical meanvalue on $\mathbb{H}^n$. Using the meanvalue theorem, he linked the solution of the shifted wave equation to the spherical meanvalue on $\mathbb{H}^n$. This method can be extended to rank-one symmetric space S of non-compact type. However, similar methods cannot be applied anymore to non-symmetric spaces such as general Damek-Ricci spaces.

In this paper, we give an explicit solution of the shifted wave equation on Damek-Ricci spaces, abbreviated as D-R spaces (cf. Theorem 5). Just as the hyperbolic space case, we give the solution in terms of the integral of the initial value over geodesic sphere. We also give the explicit formula of the integral over geodesic spheres. By giving the solution of the shifted wave equation in such a form, we can show that Huygens' principle holds for every odd-dimensional D-R space (cf. Corollary 12). To deal with the wave equation on D-R space, our method in this paper consists of the following three essential ingredients:

- (i) an explicit formula of the integral operator over geodesic sphere,
- (ii) use of the explicit formula of the heat kernel on D-R space, and
- (iii) a formula in spectral theory to link the heat kernel with the solution of wave equation.

Although the heat kernels on D-R spaces are computed explicitly in [4], the wave equation has not been solved explicitly so far. We can derive the solution of the shifted wave equation from this heat kernel on D-R space which is our main result (cf. Theorem 5).

In Section 2, after recalling briefly the structure of D-R spaces, we derive an explicit formula of the integral operator I' over geodesic sphere on D-R space, which is actually an Abel-type integral operator. Therefore I' has an explicit inversion formula which shall be exploited in Section 5.

In Section 3, we give an explicit relation between the solution $w(x, t)$ of the shifted heat equation (H.1) and the solution $u(x, t)$ of the shifted wave equation.

In Section 4, combining the result of Section 3 with the formula of the integral operator over geodesic sphere in Theorem 2, we describe explicitly the solution of the shifted wave equation in Theorem 5.

In Section 5, in the case of symmetric D-R space, we can obtain directly a solution of the shifted wave equation using the meanvalue theorem of Helgason to link the shifted wave equation to the spherical meanvalue, and then invert the explicit formula of the integral operator I' .

2. Integral Over Geodesic Sphere

Let $\mathfrak{n}$ be a two step nilpotent Lie algebra with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$, $\mathfrak{z}$ the center of $\mathfrak{n}$, and $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. We have then the direct sum: $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Let $J_Z: \mathfrak{v} \rightarrow \mathfrak{v}$ be the linear map defined by

$$\langle J_Z V_1, V_2 \rangle_{\mathfrak{n}} = \langle Z, [V_1, V_2] \rangle_{\mathfrak{n}}, \quad V_1, V_2 \in \mathfrak{v}, Z \in \mathfrak{z}.$$

Then $\mathfrak{n}$ is said to be of *Heisenberg type* if one of the following equivalent conditions is satisfied:

- (i) $\text{ad} V$ is an isometry from $(\text{Ker ad} V)^\perp$ onto $\mathfrak{z}$, for each $V \in \mathfrak{v}$;
- (ii) $J_Z^2 = -|Z|^2 I$ for each $Z \in \mathfrak{z}$.

Let $k = \dim \mathfrak{z}$ and $m = \dim \mathfrak{v}$. See [1] for the possible dimension of $\mathfrak{n}$. We consider the simply connected Lie group N whose Lie algebra is $\mathfrak{n}$. For $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$, let $(V, Z)_n$ be a point in $\mathfrak{n}$. Then we identify points in N with points in $\mathfrak{n}$ via the exponential map $\exp_n$ and denote by $(V, Z)_N$ a point $\exp_n((V, Z))$. The multiplication in N is given, for $(V, Z)_N, (V', Z')_N \in N$, by

$$(V, Z)_N \cdot (V', Z')_N = \left(V + V', Z + Z' + \frac{1}{2} [V, V'] \right)_N, \quad V, V' \in \mathfrak{v}, Z, Z' \in \mathfrak{z}.$$

Consider the semi-direct product $S = N \rtimes \mathbb{R}_+^*$, where N is the above Lie group of Heisenberg type and $\mathbb{R}_+^*$ is the one-dimensional Abelian Lie group whose Lie algebra is $\mathbb{R}$. The Lie algebra of S is given by $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$. For two points $(V, Z, l)_s, (V', Z', l')_s \in \mathfrak{s}$, the Lie bracket is defined by

$$[(V, Z, l)_s, (V', Z', l')_s] = \left(\frac{1}{2} l V' - \frac{1}{2} l' V, l Z' - l' Z + [V, V'], 0 \right)_s.$$

Again we identify points in S with points in $\mathfrak{s}$ via exponential map $\exp_s$ and denote by $(V, Z, l)_S$ a point $\exp_s((V, Z, l)_s)$, $(V \in \mathfrak{v}, Z \in \mathfrak{z}, l \in \mathbb{R})$. Then the product on S is defined, for $(V, Z, a)_S, (V', Z', a')_S \in S$, by

$$(V, Z, a)_S \cdot (V', Z', a')_S = \left(V + a^{1/2} V', Z + a Z' + \frac{1}{2} a^{1/2} [V, V'], a a' \right)_S.$$

The left invariant Riemannian metric g , or simply denoted by $\langle \cdot, \cdot \rangle$, on S is induced by the inner product on $\mathfrak{s}$ defined by

$$\langle (V, Z, l), (V', Z', l') \rangle_s = \langle V, V' \rangle_{\mathfrak{n}} + \langle Z, Z' \rangle_{\mathfrak{n}} + l l'.$$

Then a solvable simply connected Lie group (S, g) equipped the Riemannian metric g is called a *Damek-Ricci* space. The Lie exponential map $\exp_s = \exp_n \times \exp_a$ is a diffeomorphism from $\mathfrak{s}$ to S (cf. [1, p. 81]). Using this diffeomorphism, we can define a global coordinate on S . Let $\{V_1, \dots, V_m, Z_1, \dots, Z_k, A\}$ be an orthonormal basis of the Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ and $\{\tilde{v}_1, \dots, \tilde{v}_m, \tilde{z}_1, \dots, \tilde{z}_k, s\}$ the corresponding coordinate functions on $\mathfrak{s}$. Using the diffeomorphism $\exp_s$ from $\mathfrak{n} \times \mathbb{R}$ to $S = NA$, we obtain the global canonical coordinate $\{v_1, \dots, v_m, z_1, \dots, z_k, s\}$ by

$$(v_1, \dots, v_m, z_1, \dots, z_k, s) \circ (\exp_n \times \exp_a) = (\tilde{v}_1, \dots, \tilde{v}_m, \tilde{z}_1, \dots, \tilde{z}_k, s).$$

The Riemannian volume element on S is given (cf. [3, p. 645]) by

$$v_g = a^{-(k+m/2)} dV dZ \frac{da}{a} = e^{-(k+m/2)s} dV dZ ds,$$

where $dV = dv_1 \cdots dv_m$, $dZ = dz_1 \cdots dz_k$ and $a = e^s \in \mathbb{R}_+^*$. Note that, by the left translation L_x^* ($x \in S$) defined on $C(S)$ as

$$(L_x^* f)(y) = f(xy), \quad y \in S, \quad f \in C(S),$$

we can shift a domain $U \subset S$ of an integral to the shifted domain $U_x = \{xy: y \in U\}$ as follows:

$$\int_{U_x} f(y) v_g(y) = \int_U (L_x^* f)(y) v_g(y), \quad x \in S, \quad f \in C(S).$$

The Riemannian distance $r = d(o, x)$ from the origin $o = (0, 0, 1)_S$ to the point $x = (V, Z, a)_S$ in S is given (cf. [3, p. 650]) by

$$(2.1) \quad \cosh^2 \frac{r}{2} = \left(\cosh \frac{s}{2} + \frac{1}{8} e^{-s/2} |V|^2 \right)^2 + \frac{1}{4} e^{-s} |Z|^2, \quad s = \log a \in \mathbb{R}.$$

Let $S_r(x) = \{y \in S: d(y, x) = r\}$ be the geodesic sphere centered at $x \in S$ with radius $r > 0$, which is induced from the left invariant Riemannian distance d . Then the total area $A(r)$ of $S_r(o)$ is given ([4, p. 647]) by

$$(2.2) \quad A(r) = \frac{2^n \pi^{n/2}}{\Gamma(n/2)} \sinh^{m+k} \frac{r}{2} \cosh^k \frac{r}{2},$$

where $n = k + m + 1$.

We will describe the volume element in terms of the geodesic polar coordinates so that we can decompose it into the metric on radial direction and the surface element given in (2.8) below. Let us denote by $S_1^v(0)$ and $S_1^z(0)$ the unit spheres in $\mathfrak{v}$ and $\mathfrak{z}$ respectively induced from Euclidean distance on $\mathfrak{v}$ and $\mathfrak{z}$. Then a point $(V, Z, e^s)_s$ in S is represented as $(r_v p, r_z q, e^s)$, where $r_v = |V|$, $r_z = |Z|$, $p \in S_1^v(0)$ and $q \in S_1^z(0)$ are determined by

$$(2.3) \quad \begin{cases} |V|p = V, \\ |Z|q = Z. \end{cases}$$

Obviously (r_v, r_z, s, p, q) is a global coordinate system on S . Let $d\omega^v$ and $d\omega^z$ be the $(m-1)$ -dimensional and $(k-1)$ -dimensional volume elements on $S_1^v(0)$ and $S_1^z(0)$ respectively, induced from the respective Euclidean metric on $\mathfrak{v}$ and $\mathfrak{z}$. Then the $(m+k+1)$ -dimensional volume element v_g with respect to the above left invariant Riemannian metric on S is of the form:

$$(2.4) \quad v_g = e^{-(k+m/2)s} r_v^{m-1} r_z^{k-1} dr_v dr_z d\omega_v d\omega_z ds.$$

In (2.1), we put

$$(2.5) \quad \cosh(\alpha/2) = \cosh(s/2) + (1/8)e^{(-s/2)}|V|^2,$$

then we have

$$(2.6) \quad \begin{cases} |V| = 2\sqrt{2}e^{s/4} \sqrt{\cosh(\alpha/2) - \cosh(s/2)}, \\ |Z| = 2e^{s/2} \sqrt{\cosh^2(r/2) - \cosh^2(\alpha/2)}, \end{cases}$$

where $0 \leq \alpha \leq r$, $-\alpha \leq s \leq \alpha$ and $\text{sh } r := \sinh r$, $\text{ch } r := \cosh r$.

Let us consider the transformation T from $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ into $D = \{(r, \alpha, s) \in \mathbb{R}^3; 0 \leq \alpha \leq r, -\alpha \leq s \leq \alpha\}$ defined by $(r, \alpha, s) \rightarrow (|V|, |Z|, s)$, where $|V|$ and $|Z|$ are defined in (2.6). Then the Jacobian J of T is given as

$$(2.7) \quad J = \frac{1}{\sqrt{2}} e^{3s/4} \frac{\text{sh}(\alpha/2) \text{sh}(r/2) \text{ch}(r/2)}{\sqrt{\{\cosh(\alpha/2) - \cosh(s/2)\} \{\cosh^2(r/2) - \cosh^2(\alpha/2)\}}}.$$

The set of functions $\{(r, \alpha, s, p, q): 0 \leq \alpha \leq r, -\alpha \leq s \leq \alpha, p \in S_1^v(0), q \in S_1^z(0)\}$ defined by (2.3), (2.5) and (2.6) is yet again a global coordinate system on S . Then, by (2.6) and (2.7), the volume element v_g given in (2.4) is of the following form:

$$v_g = 2^{k+3m/2-3} e^{-vs} \left(\text{ch} \frac{\alpha}{2} - \text{ch} \frac{s}{2} \right)^\mu \left(\text{ch}^2 \frac{r}{2} - \text{ch}^2 \frac{\alpha}{2} \right)^\kappa \text{sh} \frac{\alpha}{2} \text{sh} \frac{r}{2} \text{ch} \frac{r}{2} dr d\alpha ds d\omega_v d\omega_z,$$

where $\mu = (m-2)/2$, $\kappa = (k-2)/2$ and $v = (k+m/2)/2$. For fixed $r > 0$, the set of functions $D_r = \{(\alpha, s, p, q): 0 \leq \alpha \leq r, -\alpha \leq s \leq \alpha, p \in S_1^v(0), q \in S_1^z(0)\}$ forms a global coordinate on geodesic sphere $S_r(o)$. Then the $(m+k)$ -dimensional volume element $d\omega'_o$ of $S_r(o)$ is given by

$$(2.8) \quad d\omega'_o = 2^{k+3m/2-3} e^{-vs} \left(\text{ch} \frac{\alpha}{2} - \text{ch} \frac{s}{2} \right)^\mu \left(\text{ch}^2 \frac{r}{2} - \text{ch}^2 \frac{\alpha}{2} \right)^\kappa \text{sh} \frac{\alpha}{2} \text{sh} \frac{r}{2} \text{ch} \frac{r}{2} d\alpha ds d\omega_v d\omega_z.$$

Now we introduce an integral operator over the geodesic sphere $S_r(x)$. The integral operator I' is defined by

$$(I'f)(x) = \int_{S_r(x)} f(z) d\omega'_x(z), \quad f \in C(S), \quad r > 0,$$

where $d\omega'_x$ is the $(m+k)$ -dimensional volume element on $S_r(o)$ induced from the left invariant Riemannian metric g on S , and $C(S)$ is the space of all continuous functions on S .

We express every point in S as $(V, Z, e^s) = (|V|p, |Z|q, e^s)$, where $p \in S_1^v(0)$ and $q \in S_1^z(0)$. For $f \in C(S)$, we define a function $\tilde{f}$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ by

$$(2.9) \quad \tilde{f}(|V|, |Z|, e^s) = \int_{S_1^v(0)} \int_{S_1^z(0)} f(|V|p, |Z|q, e^s) d\omega^v(p) d\omega^z(q),$$

and $\tilde{f}_D$, the corresponding function defined on $D = \{(r, \alpha, s) \in \mathbb{R}^3; 0 \leq \alpha \leq r, -\alpha \leq s \leq \alpha\}$ by

$$(2.10) \quad \begin{aligned} \tilde{f}_D(r, \alpha, s) &= \tilde{f}(|V|, |Z|, e^s) \\ &= \tilde{f}(2\sqrt{2}e^{s/4} \sqrt{\cosh(\alpha/2) - \cosh(s/2)}, 2e^{s/2} \sqrt{\cosh^2(r/2) - \cosh^2(\alpha/2)}, e^s). \end{aligned}$$

Then due to the above formulas (2.1) ~ (2.10), we obtain the following explicit formula of the integral operator

I' :

Proposition 1. Let f be a continuous function on S . The value $(I'f)(o)$ at the origin o is given by

$$(I'f)(o) = 2^{k+3m/2-4} \operatorname{sh} r \int_0^r \left\{ \left(\operatorname{ch}^2 \frac{r}{2} - \operatorname{ch}^2 \frac{\alpha}{2} \right)^\kappa \operatorname{sh} \frac{\alpha}{2} \int_{-\alpha}^\alpha \left(\operatorname{ch} \frac{\alpha}{2} - \operatorname{ch} \frac{s}{2} \right)^\mu e^{-\nu s} \bar{f}_D(r, \alpha, s) ds \right\} d\alpha,$$

where $r > 0$, $\mu = (m-2)/2$, $\kappa = (k-2)/2$, $\nu = (k+m/2)/2$. and the functions $\bar{f}$ and $\bar{f}_D$ are defined by (2.9) and (2.10), respectively.

Proof. For $f \in C(S)$, we have

$$\begin{aligned} (I'f)(o) &= \int_{S(o)} f(z) d\omega_x(z) \\ &= C_{k,m} \int_{D_r} \left\{ \int_{S_1^+(0)} \int_{S_1^-(0)} f(|V|p, |Z|q, s) d\omega^{\mathfrak{v}}(p) d\omega^{\mathfrak{h}}(q) \right\} \\ &\quad \times e^{-\nu s} \left(\operatorname{ch} \frac{\alpha}{2} - \operatorname{ch} \frac{s}{2} \right)^\mu \left(\operatorname{ch}^2 \frac{r}{2} - \operatorname{ch}^2 \frac{\alpha}{2} \right)^\kappa \operatorname{sh} \frac{\alpha}{2} \operatorname{sh} \frac{r}{2} \operatorname{ch} \frac{r}{2} d\alpha ds, \end{aligned}$$

where $|V|$, $|Z|$ are defined in (2.3) and $C_{k,m} = 2^{k+3m/2-3}$. Substituting (2.9), we have

$$\begin{aligned} (I'f)(o) &= C_{k,m} \int_{D_r} \bar{f}(|V|, |Z|, s) e^{-\nu s} \left(\operatorname{ch} \frac{\alpha}{2} - \operatorname{ch} \frac{s}{2} \right)^\mu \left(\operatorname{ch}^2 \frac{r}{2} - \operatorname{ch}^2 \frac{\alpha}{2} \right)^\kappa \operatorname{sh} \frac{\alpha}{2} \operatorname{sh} \frac{r}{2} \operatorname{ch} \frac{r}{2} d\alpha ds \\ &= C_{k,m} \int_{D_r} \bar{f}_D(r, \alpha, s) e^{-\nu s} \left(\operatorname{ch} \frac{\alpha}{2} - \operatorname{ch} \frac{s}{2} \right)^\mu \left(\operatorname{ch}^2 \frac{r}{2} - \operatorname{ch}^2 \frac{\alpha}{2} \right)^\kappa \operatorname{sh} \frac{\alpha}{2} \operatorname{sh} \frac{r}{2} \operatorname{ch} \frac{r}{2} d\alpha ds. \end{aligned}$$

Hence we have the desired formula. $\square$

By the left translation L_x^* ($x \in S$), we can shift the integral over the sphere centered at the origin to the one centered at any point $x \in S$, since the surface area element on the geodesic sphere is a left invariant measure. Hence we have the following theorem:

Theorem 2. Let S be a Damek-Ricci space, $k = \dim \mathfrak{h}$, $m = \dim \mathfrak{v}$ and $f \in C(S)$. The value $(I'f)(x)$ at every point $x \in S$ is given by

$$(2.11) \quad (I'f)(x) = 2^{k+3m/2-4} \operatorname{sh} r \int_0^r \left\{ \left(\operatorname{ch}^2 \frac{r}{2} - \operatorname{ch}^2 \frac{\alpha}{2} \right)^\kappa \operatorname{sh} \frac{\alpha}{2} \int_{-\alpha}^\alpha \left(\operatorname{ch} \frac{\alpha}{2} - \operatorname{ch} \frac{s}{2} \right)^\mu e^{-\nu s} (\bar{L}_x^* f)_D(r, \alpha, s) ds \right\} d\alpha,$$

where $r > 0$, $\mu = (m-2)/2$, $\kappa = (k-2)/2$, $\nu = (k+m/2)/2$, $\bar{f}$ and $(\bar{f})_D$ are defined by (2.9) and (2.10), respectively.

3. The Relation between The Wave Equation and The Heat Kernel

In this section, we consider three differential equations with the same initial data and show the relation among them.

Let $(v_1, \dots, v_m, z_1, \dots, z_k, s)$ be the canonical global coordinate on S , defined in Section 2. Then the vector fields $\{(\partial/\partial v_1), \dots, (\partial/\partial v_m), \dots, (\partial/\partial z_k), (\partial/\partial s)\}$ on S are given (cf. [1], p. 82) as follows:

$$\begin{cases} \frac{\partial}{\partial v_i} = e^{-s/2} V_i + \frac{1}{2} e^{-s} \sum_{j,k} A_{ij}^k v_j Z_k, \\ \frac{\partial}{\partial z_i} = e^{-s} Z_i, \\ \frac{\partial}{\partial s} = A, \end{cases}$$

where $A_{ij}^k = \langle [V_i, V_j], Z_k \rangle$. With the above coordinate and the vector fields, the Laplace-Beltrami operator on S with respect to the left invariant Riemmanian metric $g = \langle, \rangle$ can be expressed as follows:

$$\Delta = e^s \sum_{i=1}^m \frac{\partial^2}{\partial v_i^2} + e^s \left(e^s + \frac{1}{4} \sum_{i=1}^m v_i^2 \right) \sum_{i=1}^k \frac{\partial^2}{\partial z_i^2} + \frac{\partial^2}{\partial s^2} - \left(k + \frac{m}{2} \right) \frac{\partial}{\partial s} + \frac{1}{2} e^s \sum_{i=1}^k \sum_{j=1}^m A_{ij}^i v_i \frac{\partial}{\partial v_j} \frac{\partial}{\partial z_i}.$$

The bottom of the spectrum of $-\Delta$ on $L^2(S)$ is known to be $(k+m/2)^2/4$, and $-(\Delta + (k+m/2)^2/4)$ is a positive self-adjoint operator on $L^2(S)$ (cf. [1, Chapter 4], [4]).

Let $\mathcal{E}_1(S \times \mathbb{R}_+^*)$ be the space of all functions which satisfy following (i) and (ii):

- (i) For every fixed $t \in \mathbb{R}_+^*$, $f(t, x)$ is twice differentiable in $x \in S$ and belongs to $L^2(S)$.
- (ii) For every fixed $x \in S$, $f(t, x)$ is differentiable in $t \in \mathbb{R}_+^*$ and bounded on $\mathbb{R}_+^*$.

Similarly, $\mathcal{E}_2(S \times \mathbb{R}_+^*)$ is the space of all functions satisfying (i) and the following condition:

(iii) For every fixed $x \in S$, $f(t, x)$ is twice differentiable in $t \in \mathbb{R}_+^*$ and bounded on $\mathbb{R}_+^*$.

For given initial value w_0 which belongs to $C_0^\infty(S)$, the set of all compactly supported C^∞ functions on S , let $v \in \mathcal{E}_1(S \times \mathbb{R}_+^*)$ satisfy the heat equation:

$$(H.0) \quad \begin{cases} \Delta v = \frac{\partial v}{\partial t}, & t > 0, \\ v(x, 0) = w_0(x), & x \in S. \end{cases}$$

Similarly, let $w \in \mathcal{E}_1(S \times \mathbb{R}_+^*)$ satisfy the shifted heat equation:

$$(H.1) \quad \begin{cases} \left(\Delta + \frac{(k + m/2)^2}{4} \right) w = \frac{\partial w}{\partial t}, & t > 0, \\ w(x, 0) = w_0(x), & x \in S. \end{cases}$$

Then, $v(x, t)$ and $w(x, t)$ are related clearly as

$$(3.1) \quad v(x, t) = e^{-t(k+m/2)^2/4} w(x, t), \quad x \in S, \quad t > 0.$$

The solution of the shifted wave equation we consider here is a function $u \in \mathcal{E}_2(S \times \mathbb{R}_+^*)$, which satisfies

$$(W.1) \quad \begin{cases} \left(\Delta + \frac{(k + m/2)^2}{4} \right) u = \frac{\partial^2 u}{\partial t^2}, & t > 0, \\ u(x, 0) = w_0(x), \\ \frac{\partial u}{\partial t}(x, 0) = 0, & x \in S. \end{cases}$$

We extend the function $u(x, t)$ on $S \times \mathbb{R}_+^*$ to the function on $S \times \mathbb{R}$ also denoted by $u(x, t)$ which is an even function in t . Then the relation between the solution of (H.1) and that of (W.1) is given by the following proposition:

Proposition 3. *Let u be a function which belongs to $\mathcal{E}_2(S \times \mathbb{R}_+^*)$ and satisfies the shifted wave equation (W.1). And let w be a function which belong to $\mathcal{E}_1(S \times \mathbb{R}_+^*)$ and satisfies the shifted heat equation (H.1). Then the solutions u and w have the following relation:*

$$(3.2) \quad w(x, t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} u(x, s) ds,$$

Moreover the solution u of the shifted wave equation (W.1) is unique.

Proof. Let $L = \Delta + (k + m/2)^2/4$. Then the solution w of (H.1) can be represented formally as $w(x, t) = e^{tL} w_0(x)$, whereas the solution u of (W.1) can be represented as $u(x, t) = \cos(t\sqrt{-L}) w_0(x)$.

On the other hand, if A is a self-adjoint negative operator on $L^2(S)$, we have (cf. [10 p. 225]) the following identity from operator calculus:

$$(3.3) \quad e^{tA} = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} \cos(s\sqrt{-A}) ds.$$

We can now take $A = L$, since the spectrum of $-L$ coincides with $(\infty, 0]$. Then we obtain (3.2). Moreover, assume that $z(x, t)$ satisfies

$$e^{tL} w_0(x) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} z(x, s) ds.$$

Then, combining with (3.2), we have

$$\int_{-\infty}^{\infty} e^{-s^2/(4t)} (z(x, s) - u(x, s)) ds = 0, \quad \text{for all } t > 0.$$

Hence,

$$(3.4) \quad \int_{-\infty}^{\infty} e^{-s^2/(4t)} (z(x, s) - u(x, s)) ds = 0, \quad \text{for all } t > 0.$$

We differentiate the both sides of (3.4) in q . Using Lebesgue's dominant convergence theorem, we can exchange the order of the integration and the differentiation because both $z(x, s)$ and $u(x, s)$ are bounded functions in s , and $\exp(-s^2/(4t))$ is a rapidly decreasing function. Therefore we obtain

$$\int_{-\infty}^{\infty} P(s^2) e^{-s^2 q} (z(x, s) - u(x, s)) ds = 0, \quad q > 0,$$

where $P(r)$ is an arbitrary polynomial function in $r = s^2$. Since $u(x, s)$ and $z(x, s)$ are even functions in s , this equality means that $z(x, s) = u(x, s)$ a.e. in s . By the continuity of u and z , we have $z(x, t) = u(x, t)$, for $x \in S$, $t > 0$. $\square$

Proposition 3 and (3.1) yield immediately.

Proposition 4. *Let $z(x, t)$ be a function which belongs to $\mathcal{E}_2(S \times \mathbb{R}_+^*)$. Then $z(x, t)$ is the solution of (W.1) if $z(x, t)$ satisfies the following relation:*

$$(3.5) \quad e^{t(k+m/2)^2/4} v(x, t) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} z(x, s) ds.$$

4. The Shifted Wave Equation

In this section we give an explicit formula of the solution of the shifted wave equation in Theorem 5 which is our main theorem. Before stating the theorem, we introduce two differential operator $D_{1,r}$ and $D_{2,r}$ acting on the space of smooth functions in $r \in \mathbb{R}_+^*$ by

$$D_{1,r} = \frac{-1}{\text{sh}(r/2)} \cdot \frac{\partial}{\partial r} \quad \text{and} \quad D_{2,r} = \frac{-1}{\text{sh } r} \cdot \frac{\partial}{\partial r}.$$

Theorem 5. *Let S be a Damek-Ricci space, $k = \dim \mathfrak{g}$ and $m = \dim \mathfrak{v}$. Then the solution $u(x, t)$ of (W.1) is given as follows:*

(i) *If k is odd,*

$$u(x, t) = C_{k,m} \frac{\partial}{\partial t} (-D_{1,t})^{m/2} \left(\int_0^t \frac{\text{sh } r}{\sqrt{\text{ch } t - \text{ch } r}} (-D_{2,r})^{(k-1)/2} \left\{ \frac{(I^r u_0)(x)}{\text{sh } r} \right\} dr \right)$$

for $x \in S$, $t > 0$, and $C_{k,m} = 2^{-m-k/2-1} \pi^{-(k+m+1)/2}$.

(ii) *If k is even,*

$$u(x, t) = C'_{k,m} \frac{\partial}{\partial t} (-D_{1,t})^{m/2} (-D_{2,t})^{(k-2)/2} \left(\frac{(I^t u_0)(x)}{\text{sh } t} \right),$$

for $x \in S$, $t > 0$ and $C'_{k,m} = 2^{-m-k/2-1} \pi^{-(k+m)/2}$. Here $(I^t u_0)(x)$ is given by (2.11).

To prove Theorem 5, we use the following formula of the heat kernel obtained by the spherical transform on D-R space in [4].

Theorem 6 (cf. (5.7), (5.8) in [4]). *Let S be a Damek-Ricci space, $k = \dim \mathfrak{g}$, $m = \dim \mathfrak{v}$ and $n = k + m + 1$. Then the solution $h(x, t)$ of (H.0) is given as follows:*

(i) *If k is odd,*

$$h(x, t) = 2^{-m-k/2-1} \pi^{-(n+1)/2} t^{-1/2} e^{-tQ^2/4} \int_r^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } r}} (D_{2,s})^{(k+1)/2} (D_{1,s})^{m/2} e^{-s^2/(4t)} ds.$$

(ii) *If k is even,*

$$h(x, t) = 2^{-m-k/2-1} \pi^{-n/2} t^{-1/2} e^{-tQ^2/4} (D_{2,r})^{k/2} (D_{1,r})^{m/2} e^{-r^2/(4t)},$$

where $r = d(x, o)$ is the distance from o to $x \in S$ and $Q = k + m/2$.

We give a proof of Theorem 5 in the remainder of this section. We substitute the formula of the integral kernel $h_1(x, t)$ of (H.1) into (4.1) below, and compare the right hand side of it with that of (3.5) to obtain the solution of (W.1). First of all, from (3.1), the integral kernel of (H.1) is given as follows:

(i) *If k is odd,*

$$h_1(x, t) = 2^{-m-k/2-1} \pi^{-(n+1)/2} t^{-1/2} \int_r^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } r}} (D_{2,s})^{(k+1)/2} (D_{1,s})^{m/2} e^{-s^2/(4t)} ds.$$

(ii) *If k is even,*

$$h_1(x, t) = 2^{-m-k/2-1} \pi^{-n/2} t^{-1/2} (D_{2,r})^{k/2} (D_{1,r})^{m/2} e^{-r^2/(4t)},$$

where $r = d(x, o)$ is the distance between $x \in S$ and o .

From now on, we divide the proof into two cases depending on k .

(i) *The case where k is odd.*

Note that the function $e^{t(k+m/2)^2/4} v(x, t)$ is the solution of (H.1) with initial value u_0 and its integral kernel is given by $h_1(x, t)$. Hence one has

$$(4.1) \quad e^{i(k+m/2)^2/4} v(x, t) = \int_S h_1(z^{-1}x, t) u_0(z) dz.$$

After substituting h_1 into (4.1), we introduce the polar coordinates centered at $x \in S$, then we have

$$(4.2) \quad \begin{aligned} e^{i(k+m/2)^2/4} v(x, t) &= \frac{C_1}{t^{1/2}} \int_S \left\{ \int_{d(z^{-1}x)}^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } \{d(z^{-1}x)\}}} D_{2,s}^{(k+1)/2} D_{1,s}^{m/2} e^{-s^2/(4t)} ds \right\} u_0(z) dz \\ &= \frac{C_1}{t^{1/2}} \int_0^\infty \left\{ \int_r^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } r}} D_{2,s}^{(k+1)/2} D_{1,s}^{m/2} e^{-s^2/(4t)} ds \right\} \left(\int_{S_r(x)} u_0 d\omega \right) dr, \end{aligned}$$

where $C_1 = 2^{-m-k/2-1} \pi^{-(n+1)/2}$.

In the second step, we exchange the order of the integration in s and the differentiation $D_{2,s}$ in (4.2). For this purpose, we make use of the following lemma.

Lemma 7. *The differential operator $D_{2,r}$ commutes with the integral operator K given by*

$$(Kf)(r) = \int_r^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } r}} f(s) ds, \quad f \in S(\mathbb{R}_+^*),$$

where $S(\mathbb{R}_+^*)$ is the set of all rapidly decreasing functions on $\mathbb{R}_+^*$.

Proof of Lemma 7. By changing the order of integration, we have

$$\begin{aligned} (K^2 f)(t) &= \int_t^\infty \frac{\text{sh } r}{\sqrt{\text{ch } r - \text{ch } t}} \left(\int_r^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } r}} f(s) ds \right) dr \\ &= \int_t^\infty f(s) \text{sh } s \left(\int_t^s \frac{\text{sh } r}{\sqrt{(\text{ch } s - \text{ch } r)(\text{ch } r - \text{ch } t)}} dr \right) ds \\ &= \int_t^\infty f(s) \pi \text{sh } s ds. \end{aligned}$$

Differentiating the both sides in t , we obtain

$$-\frac{1}{\text{sh } t} \frac{\partial}{\partial t} (K^2 f)(t) = \pi f(t) \quad \text{that is} \quad D_{2,t} K^2 f(t) = \pi f(t).$$

which means that $D_{2,t} = \pi K^{-2}$ on $S(\mathbb{R})$ and then $D_{2,t}$ and K commute. $\square$

Applying Lemma 7, $(k-1)/2$ times, to the inner integral of (4.2), we obtain

$$(4.3) \quad e^{i(k+m/2)^2/4} v(x, t) = \frac{C_1}{t^{1/2}} \int_0^\infty \left\{ D_{2,r}^{(k-1)/2} \int_r^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } r}} D_{2,s} D_{1,s}^{m/2} e^{-s^2/(4t)} ds \right\} \left(\int_{S_r(x)} u_0 d\omega \right) dr.$$

In the third step, we use the following two lemmas to compute the right hand side of (4.3).

Lemma 8. *For two positive integers n_1 and n_2 , we have the following two formulas of integration by parts:*

(i) *Let $f_1 \in C^{n_1}(\mathbb{R}_+^*)$ be a function which satisfies that $D_{2,r}^{n_1} f_1$ is bounded on $\mathbb{R}_+^*$, and let $f_2 \in C^{n_1}(\mathbb{R}_+^*)$ be a function which is compactly supported and satisfies*

$$f_2(s) = o(s^{2n_1}) \quad \text{as } s \rightarrow 0.$$

Then we have

$$\int_0^\infty (D_{2,r}^{n_1} f_1)(r) f_2(r) dr = (-1)^{n_1} \int_0^\infty f_1(r) \frac{\partial}{\partial r} D_{2,r}^{n_1-1} \left(\frac{f_2(r)}{\text{sh } r} \right) dr.$$

(ii) *Let g_1 be a rapidly decreasing function on $\mathbb{R}_+^*$ which satisfies*

$$\left(\frac{\partial}{\partial s} D_{1,s}^{n_2} g_1 \right)(s) \rightarrow 0, \quad \text{as } s \rightarrow 0,$$

and let $g_2 \in C^{n_2}(\mathbb{R}_+^)$ be a compactly supported function which satisfies*

$$g_2(s) = o(s^{2n_2+1}).$$

Then we have

$$\int_0^\infty \left(\frac{\partial}{\partial s} D_{1,s}^{n_2} g_1 \right)(s) g_2(s) ds = (-1)^{n_2+1} \int_0^\infty g_1(s) \left(\frac{\partial}{\partial s} D_{1,s}^{n_2} g_2 \right)(s) ds.$$

To apply Lemma 8 to (4.3), we need the behavior of $(I^l u_0)(x)$ around $r = 0$.

Lemma 9. *As a function of r , $I^l u_0$ satisfies $I^l u_0 \in C^{m+k}(\mathbb{R}_+^*)$ and for $l = 0, 1, \dots, k+m$,*

$$\lim_{r \rightarrow 0} \left(\frac{\partial}{\partial r} \right)^l (I^r u_0)(x) = 0.$$

The proof goes by the similar way as [6, p. 647]. □

In order to calculate the right hand side of (4.3), we put

$$f_1(r) = \int_r^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } r}} D_{2,s} D_{1,s}^{m/2} e^{-s^2/(4t)} ds, \quad \text{and} \quad f_2(r) = (I^r u_0)(x).$$

By Lemma 9, $f_2(r) = o(r^{k+m})$ as $r \rightarrow 0$, then we can apply (i) of Lemma 8 with the above f_1 and f_2 , and obtain

$$(4.5) \quad e^{t(k+m/2)^2/4} v(x, t) = (-1)^{(k-1)/2} \frac{C_1}{t^{1/2}} \int_0^\infty \left\{ \int_r^\infty \frac{\text{sh } s}{\sqrt{\text{ch } s - \text{ch } r}} D_{2,s} D_{1,s}^{m/2} e^{-s^2/(4t)} ds \right\} \\ \times \frac{\partial}{\partial r} D_{2,r}^{(k-3)/2} \left(\frac{(I^r u_0)(x)}{\text{sh } r} dr \right) dr.$$

In the fourth step we want to change the order of the integrations of (4.5). For this purpose, we need the following lemma.

Lemma 10 (cf. [4, p. 666]). *For non-negative integers p, q such that $p + q > 0$, we have*

$$D_{2,r}^q D_{1,r}^p e^{-r^2/(4t)} \simeq \frac{1+r}{t} \left(1 + \frac{1+r}{t} \right)^{p+q-1} e^{-(q+p/2)r - r^2/(4t)}, \quad \text{for } t > 0, \quad r > 0.$$

By this lemma with $(q = 1, p = m/2)$, we have

$$-\frac{\partial}{\partial s} D_{1,s}^{m/2} e^{-s^2/(4t)} \simeq \text{sh } s \frac{1+s}{t} \left(1 + \frac{1+s}{t} \right)^{m/2} e^{-(1+m/4)s - s^2/(4t)} \quad \text{for } t > 0, \quad s > 0,$$

which implies that

$$(4.6) \quad \lim_{s \rightarrow 0} \left(-\frac{\partial}{\partial s} D_{1,s}^{m/2} e^{-s^2/(4t)} \right) = \lim_{s \rightarrow \infty} \left(-\frac{\partial}{\partial s} D_{1,s}^{m/2} e^{-s^2/(4t)} \right) = 0.$$

Furthermore, if we define the function $j(r)$ by

$$(4.7) \quad j(r) = D_{2,r}^{(k-1)/2} \left(\frac{(I^r u_0)(x)}{\text{sh } r} \right),$$

then $j(r) = o(r^m)$ as $r \rightarrow 0$, and $j(r)$ vanishes for large enough r because the support of u_0 is compact. The function $h(s)$ defined by

$$(4.8) \quad h(s) = \int_0^s \frac{\text{sh } r}{\sqrt{\text{ch } s - \text{ch } r}} D_{2,r}^{(k-1)/2} \left(\frac{(I^r u_0)(x)}{\text{sh } r} \right) dr$$

is bounded on $\mathbb{R}_+^*$ by mean of (4.7). Thus, together with (4.6),

$$\int_0^\infty \left| -\frac{\partial}{\partial s} D_{1,s}^{m/2} e^{-s^2/(4t)} \right| \left| \int_0^s \frac{\text{sh } r}{\sqrt{\text{ch } s - \text{ch } r}} D_{2,r}^{(k-1)/2} \left(\frac{(I^r u_0)(x)}{\text{sh } r} \right) dr \right| ds$$

converges. Since we may change the order of the integrations in (4.5) by Fubini's theorem, we have

$$(4.9) \quad e^{t(k+m/2)^2/4} v(x, t) = (-1)^{(k-1)/2} \frac{C_1}{t^{1/2}} \int_0^\infty \left\{ -\frac{\partial}{\partial s} D_{1,s}^{m/2} e^{-s^2/(4t)} \right\} \\ \times \left\{ \int_0^s \frac{\text{sh } r}{\sqrt{\text{ch } s - \text{ch } r}} D_{2,r}^{(k-1)/2} \left(\frac{(I^r u_0)(x)}{\text{sh } r} \right) dr \right\} ds.$$

In the final step, we want to apply (ii) of Lemma 8 to (4.9). We need the following lemma.

Lemma 11. *Let $h(s)$ be defined by (4.8), then $h(s) = o(s^{m+1})$ as $s \rightarrow 0$.*

Proof of Lemma 11. Indeed, for small $s > 0$, we have by integration by parts,

$$\begin{aligned} h(s) &= \int_0^s \frac{\text{sh } r}{\sqrt{\text{ch } s - \text{ch } r}} j(r) dr \\ &= [-\sqrt{\text{ch } s - \text{ch } r} j(r)]_{r=0}^s + \int_0^s \sqrt{\text{ch } s - \text{ch } r} j'(r) dr \\ &= \sqrt{2} \text{sh } \frac{s}{2} \int_0^s \sqrt{1 - \left(\frac{\text{sh } (r/2)}{\text{sh } (s/2)} \right)^2} j'(r) dr \end{aligned}$$

$$\leq \sqrt{2} \operatorname{sh} \frac{s}{2} \int_0^s |j'(s)| ds.$$

Combining with (4.7), we have the Lemma 11. □

Applying (ii) of Lemma 8 to

$$g_1(s) = e^{-s^2/(4t)} \quad \text{and} \quad g_2(s) = h(s),$$

the right hand side of (4.9) coincides with

$$\frac{(-1)^{(k+m+3)/2} C_1}{t^{1/2}} \int_0^\infty e^{-s^2/(4t)} \frac{\partial}{\partial s} D_{1,s}^{m/2} \left\{ \int_0^s \frac{\operatorname{sh} r}{\sqrt{\operatorname{ch} s - \operatorname{ch} r}} D_{2,r}^{(k-1)/2} \left(\frac{(I^r u_0)(x)}{\operatorname{sh} r} \right) dr \right\} ds.$$

Thus we have

$$e^{t(k+m/2)^2/4} v(x, t) = \frac{C_2}{(4\pi t)^{1/2}} \int_0^\infty e^{-s^2/(4t)} U(x, s) ds,$$

where

$$U(x, s) = \frac{\partial}{\partial s} D_{1,s}^{m/2} \int_0^s \frac{\operatorname{sh} r}{\sqrt{\operatorname{ch} s - \operatorname{ch} r}} D_{2,r}^{(k-1)/2} \left(\frac{(I^r u_0)(x)}{\operatorname{sh} r} \right) dr,$$

and $C_2 = (-1)^{(k+m+3)/2} 2^{-m-k/2-1} \pi^{-n/2}$. For fixed $t > 0$, the support of $U(x, t)$ in S is compact. Since $v(x, t)$ is a solution of (H.1), $C_2 U(x, t)$ is the solution of (W.1) due to Proposition 4. We have Theorem 5 in the case (i). □

(ii) The case where k is even. The computation goes in a way similar to the case (i). Substituting $h_1(x, t)$ into (4.1) and introducing the polar coordinates centered at $x \in S$, we have

$$\begin{aligned} e^{t(k+m/2)^2/4} v(x, t) &= \int_S h_1(z^{-1}x, t) u_0(z) dz \\ &= \frac{C_3}{t^{1/2}} \int_0^\infty D_{2,r}^{k/2} D_{1,r}^{m/2} e^{-r^2/(4t)} \left(\int_{S_t(x)} u_0(z) d\omega(z) \right) dr, \end{aligned}$$

where $C_3 = 2^{-m-k/2-1} \pi^{-(k+m+1)/2}$. Again using Lemma 8 successively, with

$$f_1(r) = D_{1,r}^{m/2} e^{-r^2/(4t)}, \quad f_2(r) = (I^r u_0)(x),$$

and

$$g_1(r) = e^{-r^2/(4t)}, \quad g_2(r) = D_{1,r} D_{2,r}^{(k-2)/2} \left(\frac{(I^r u_0)(x)}{\operatorname{sh} r} \right),$$

we obtain

$$\begin{aligned} e^{t(k+m/2)^2/4} v(x, t) &= \frac{(-1)^{(m+k)/2-1} C_3}{t^{1/2}} \int_0^\infty e^{-r^2/(4t)} \frac{\partial}{\partial r} D_{1,r}^{m/2} D_{2,r}^{(k-2)/2} \left(\frac{(I^r u_0)(x)}{\operatorname{sh} r} \right) dr \\ &= \frac{(-1)^{(m+k)/2-1} C_3}{(4\pi t)^{1/2}} \int_0^\infty e^{-r^2/(4t)} \frac{\partial}{\partial r} D_{1,r}^{m/2} D_{2,r}^{(k-2)/2} \left(\frac{(I^r u_0)(x)}{\operatorname{sh} r} \right) dr \\ &= \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^\infty e^{-r^2/(4t)} V(x, r) dr, \end{aligned}$$

where

$$V(x, r) = (-1)^{(m+k)/2-1} \pi^{1/2} C_3 \frac{\partial}{\partial r} D_{1,r}^{m/2} D_{2,r}^{(k-2)/2} \left(\frac{(I^r u_0)(x)}{\operatorname{sh} r} \right).$$

For fixed $t > 0$, the support of $V(x, t)$ in S is compact. $V(x, t)$ is the solution of (W.1) by Proposition 4. Thus, we have Theorem 5 in the case (ii). □

Notice that, if $k = \dim \mathfrak{g}$ is an even integer, the function $u(x, t)$ is a derivative in t of the function $(I^r u_0)(x)/\operatorname{sh} t$. Hence the value $u(x, t)$ is determined only by the value of initial data $u_0(x)$ in the thin shell $H_t(x) = \{y \in S : y \in S_{t+\varepsilon}(x), y \notin S_{t-\varepsilon}(x)\}$ around $S_t(x)$ for every $\varepsilon > 0$, which is so called the *Huygens' principle* as given in the following corollary.

Corollary 12. *If $\dim S$ is odd, the Huygens' principle holds for (W.1), that is, $u(x, t)$ is determined only by the values of the initial data in an arbitrarily thin shell around $S_t(x)$.*

Remark. Although the Huygens' principle for symmetric spaces was studied in [9], non-symmetric cases have not been done so far. In fact Corollary 12 is the first example of non-symmetric spaces where the Huygens' principle holds.

5. Symmetric D-R Space Case

Every symmetric D-R space is a non-compact symmetric space of rank one and every rank-one symmetric space G/K of non-compact type can be realized as $S = NA$ where N and A are the nilpotent and the Abelian subgroup of the Iwasawa decomposition $G = NAK$. The dimensions for the rank-one symmetric space of non-compact types are given in the following table.

Types of spaces	$k = \dim \mathfrak{g}$	$m = \dim \mathfrak{v}$
$H^N(\mathbb{R})$	0	$N - 1$
$H^N(\mathbb{C})$	1	$2(N - 1)$
$H^N(\mathbb{H})$	3	$4(N - 1)$
$H^2(\mathbb{O})$	7	8

In this section, we give the solution of the shifted wave equation on symmetric D-R spaces making use of the meanvalue theorem and the explicit formula of the integral operator in Proposition 1.

Let us consider the shifted wave equation of a bounded continuous function $u \in C(S \times \mathbb{R}_+^*)$, which is C^2 in both $x \in S$ and $t \in \mathbb{R}_+^*$ given by

$$(W.2) \quad \begin{cases} \left(\Delta + \frac{(k+m/2)^2}{4} \right) u = \frac{\partial^2 u}{\partial t^2}, & t > 0, \\ u(x, 0) = u_0(x), \\ \left(\frac{\partial u}{\partial t} \right)(x, 0) = u_1(x), & x \in S, \end{cases}$$

for two given initial values $u_0(x)$ and $u_1(x) \in C_0^\infty(S)$. Then we have:

Theorem 13. *Let S be a symmetric D-R space. Let $k = \dim \mathfrak{g}$ and $m = \dim \mathfrak{v}$. Then the solution $u(x, t)$ of (W.2) is given explicitly as follows:*

(i) *If k is odd,*

$$u(x, t) = \frac{1}{2^{m+k/2+1}\pi^{(k+m+1)/2}} \left\{ \frac{\partial}{\partial t} (-D_{1,t})^{m/2} U_0(x, t) + (-D_{1,t})^{m/2} U_1(x, t) \right\},$$

for $x \in S$, $t > 0$, and $U_i(x, t)$, ($i = 0, 1$) are defined by

$$U_i(x, t) = \int_0^t \frac{\text{sh } r}{\sqrt{\text{ch } t - \text{ch } r}} (-D_{2,r})^{(k-1)/2} \left\{ \frac{1}{\text{sh } r} (I' u_i)(x) \right\} dr.$$

(ii) *If k is even,*

$$u(x, t) = \frac{1}{2^{m+k/2+1}\pi^{(k+m)/2}} \left\{ \frac{\partial}{\partial t} (-D_{1,t})^{m/2} (-D_{2,t})^{(k-2)/2} V_0(x, t) + (-D_{1,t})^{m/2} (-D_{2,t})^{(k-2)/2} V_1(x, t) \right\},$$

for $x \in S$ and $t > 0$. Here $V_i(x, t)$ ($i = 0, 1$) are defined by

$$V_i(x, t) = \frac{1}{\text{sh } t} (I' u_i)(x).$$

Remark. In [9], Helgason discussed the energy equipartition theorem with respect to the shifted wave equation on non-compact symmetric spaces. There, without solving (W.2), he obtained the Fourier transform $\tilde{u}$ of the solution u as follows:

$$\tilde{u}(\lambda, b, t) = \tilde{u}_0(\lambda, b) \cos |\lambda|t + \tilde{u}_1(\lambda, b) \frac{\sin |\lambda|t}{|\lambda|}, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*, \quad b \in B.$$

Here, $\mathfrak{a}_\mathbb{C}^*$ is the complex dual of the Lie algebra $\mathfrak{a}$ of the abelian subgroup A which is associated with the Iwasawa decomposition $G = NAK$, and B is defined by $B = K/M$ where M is the centralizer of A in K . In rank one case, our Theorem 13 implies that the Fourier inverse transform of the first and the second term of $\tilde{u}(\lambda, b, t)$ are equivalent respectively to the first and the second terms in the formula in Theorem 13.

To show Theorem 13, we use the following meanvalue theorem by Helgason which plays an essential role to link the spherical meanvalue to the solution.

Theorem 14. (cf. [7], [8]) *Let S be a symmetric space of rank one and let the function $v \in C(S \times S)$ satisfy the differential equation*

$$\Delta_x(v(x, y)) = \Delta_y(v(x, y))$$

on $S \times S$, where Δ_x and Δ_y are the Laplace-Beltrami operators of S on the first and second valuable respectively. Then for each $r \geq 0$ and each $(x_0, y_0) \in S \times S$, we have

$$\int_{S_r(x_0)} v(x, y_0) d\omega(x) = \int_{S_r(y_0)} v(x_0, y) d\omega(y).$$

Now we prove Theorem 13. *First step.* We rewrite the Laplacian on S in a more useful form as follows.

$$(5.1) \quad \Delta = e^s \sum_i \frac{\partial^2}{\partial v_i^2} + e^s \left(e^s + \frac{1}{4} \sum_i v_i^2 \right) \sum_i \frac{\partial^2}{\partial z_i^2} + \frac{1}{2} e^s \sum_{i,j,k} A_{ij}^k v_i \frac{\partial}{\partial v_j} \frac{\partial}{\partial z_k} + e^{vs} \left(\frac{\partial^2}{\partial s^2} - v^2 \right) \circ e^{-vs},$$

where $v = (k + m/2)/2$, and e^{-vs} is the multiplier operator defined by $e^{-vs}: f \rightarrow e^{-vs}f, f \in C(S)$.

It is known (cf. [9], p. 491) that the existence of the solution u of (W.2). Therefore we can define a function v on $S \times S$ by

$$v(x, y) = v(x_1, \dots, x_n, y_1, \dots, y_n) = e^{vs} u(x_1, \dots, x_n, s),$$

where $x = (x_1, \dots, x_n) \in S, y = (y_1, \dots, y_n) \in S, n = m + k + 1$, and u is the solution of the shifted wave equation (W.2). Then, it is easy to see that v satisfies

$$(5.2) \quad \Delta_x(v(x, y)) = e^{vs} \left(\frac{\partial^2 u}{\partial s^2} - v^2 u \right) = \Delta_y(v(x, y)),$$

because the first identity comes from the wave equation (W.2), and the second one is due to the definition of v and the explicit expression (5.1) of the Laplacian.

We can now apply Theorem 14 to the function v taking $(x_0, y_0) = (x, o)$, and we have

$$(5.3) \quad \int_{S_r(x)} u(z, o) d\omega(z) = \int_{S_r(o)} v(x, y) d\omega(y),$$

because $v(x, 0) = e^{0v} \cdot u(x, 0) = u(x, 0)$. By Proposition 1 and the definition of I' , (5.3) yields

$$\frac{1}{\text{sh } r} (I' u_0)(x) = C_4 \int_0^r d\alpha \left(\text{ch}^2 \frac{r}{2} - \text{ch}^2 \frac{\alpha}{2} \right)^\kappa \text{sh} \frac{\alpha}{2} \int_{-\alpha}^\alpha ds \left(\text{ch} \frac{\alpha}{2} - \text{ch} \frac{s}{2} \right)^\mu u(x, s),$$

where $\mu = (m - 2)/2, \kappa = (k - 2)/2$ and $C_4 = 2^{(3m/2) + k - 2} \pi^{(k+m)/2} / \{\Gamma(k/2)\Gamma(m/2)\}$.

Therefore, we obtain

$$(5.4) \quad \frac{1}{\text{sh } r} (I' u_0)(x) = C_4 \int_0^r d\alpha \left(\text{ch}^2 \frac{r}{2} - \text{ch}^2 \frac{\alpha}{2} \right)^\kappa \text{sh} \frac{\alpha}{2} \int_0^\alpha ds \left(\text{ch} \frac{\alpha}{2} - \text{ch} \frac{s}{2} \right)^\mu \{u(x, s) + u(x, -s)\}.$$

Second step. We will solve the integral equation (5.4). Notice that $m = \dim \mathfrak{h}$ is an even integer except only the real hyperbolic space $\mathbb{H}^n(\mathbb{R})$. See [8, p. 343] in the case of real hyperbolic space $\mathbb{H}^n(\mathbb{R})$. As in the proof of Theorem 5, we divide the proof into two cases depending on the value of k .

(i) The case where k is odd. For $0 < \gamma \leq \kappa$, let us define the function $k(r)$ by

$$k(r, \alpha) = \left(\text{ch}^2 \frac{r}{2} - \text{ch}^2 \frac{\alpha}{2} \right)^\gamma \text{sh} \frac{\alpha}{2} \int_0^\alpha \left(\text{ch} \frac{\alpha}{2} - \text{ch} \frac{s}{2} \right)^\mu \{u(x, s) + u(x, -s)\} ds.$$

For any $\varepsilon > 0$, we have

$$(5.5) \quad \frac{1}{\varepsilon} \left\{ \int_0^{r+\varepsilon} k(r + \varepsilon, \alpha) d\alpha - \int_0^r k(r, \alpha) d\alpha \right\} = \frac{1}{\varepsilon} \int_r^{r+\varepsilon} k(r + \varepsilon, \alpha) d\alpha + \int_0^r \frac{1}{\varepsilon} (k(r + \varepsilon, \alpha) - k(r, \alpha)) d\alpha.$$

For each $r > 0$, $(\partial k / \partial r)(r, \alpha)$ is a bounded continuous function in α on $[0, r]$ if $1 \leq \gamma \leq \kappa$. So, by Lebesgue's convergence theorem, we can change the order of $\lim_{\varepsilon \rightarrow 0}$ and the integral over α in the second term of the right hand side of (5.5), since we have, for $0 < \gamma < 1$,

$$\frac{k(r + \varepsilon, \alpha) - k(r, \alpha)}{\varepsilon} \leq C \frac{\text{sh } r \cdot \text{sh} \frac{\alpha}{2}}{(\text{ch } r - \text{ch } \alpha)^{1-\gamma}} \int_0^\alpha \left(\text{ch} \frac{\alpha}{2} - \text{ch} \frac{s}{2} \right)^\mu \{u(x, s) + u(x, -s)\} ds.$$

Therefore, we obtain

$$\begin{aligned} \frac{\partial}{\partial r} \int_0^r k(r, \alpha) d\alpha &= k(r, r) + \int_0^r \frac{\partial k}{\partial r}(r, \alpha) d\alpha \\ &= \int_0^r \frac{\partial k}{\partial r}(r, \alpha) d\alpha, \quad (0 < r < R), \end{aligned}$$

which implies

$$(5.6) \quad D_{2,r} \left(\int_0^r k(r, \alpha) d\alpha \right) = \int_0^r (D_{2,r} k)(r, \alpha) d\alpha.$$

Applying the differential operator $-D_{2,r}$ to the both sides of (5.4), $(k-1)/2$ times, and using (5.6), we have

$$(5.7) \quad C_4 \frac{k-2}{2} \frac{k-4}{2} \cdots \frac{1}{2} \int_0^r d\alpha \left(\text{ch}^2 \frac{r}{2} - \text{ch}^2 \frac{\alpha}{2} \right)^{-1/2} \text{sh} \frac{\alpha}{2} \int_0^\alpha ds \left(\text{ch} \frac{\alpha}{2} - \text{ch} \frac{s}{2} \right)^\mu \{u(x, s) + u(x, -s)\} \\ = 2^{(k-1)/2} \cdot (-D_{2,r})^{(k-1)/2} \left\{ \frac{1}{\text{sh } r} (I' u_0)(x) \right\}.$$

We apply the operator P defined by

$$(Pf)(t) = \int_0^t \frac{\text{sh}(r/2) \text{ch}(r/2)}{\sqrt{\text{ch}^2(t/2) - \text{ch}^2(r/2)}} f(r) dr \quad \text{for } f \in C(\mathbb{R}^*),$$

on the both sides of (5.7), and change the order of integrations in the left hand side, and then use the identity

$$\int_\alpha^t \frac{\text{ch}(r/2) \text{sh}(r/2) dr}{\{\text{ch}^2(t/2) - \text{ch}^2(r/2)\}^{1/2} \{\text{ch}^2(r/2) - \text{ch}^2(\alpha/2)\}^{1/2}} = \pi,$$

we obtain

$$(5.8) \quad C_4 \frac{k-2}{2} \frac{k-4}{2} \cdots \frac{1}{2} \pi \int_0^t d\alpha \text{sh} \frac{\alpha}{2} \int_0^\alpha ds \left(\text{ch} \frac{\alpha}{2} - \text{ch} \frac{s}{2} \right)^\mu \{u(x, s) + u(x, -s)\} \\ = 2^{(k-1)/2} \int_0^t \left(\text{ch}^2 \frac{t}{2} - \text{ch}^2 \frac{r}{2} \right)^{-1/2} \text{sh} \frac{r}{2} \text{ch} \frac{r}{2} (-D_{2,r})^{(k-1)/2} \left\{ \frac{1}{\text{sh } r} (I' u_0)(x) \right\} dr.$$

Differentiating the both sides of (5.8) in t , we have

$$C_4 \frac{k-2}{2} \frac{k-4}{2} \cdots \frac{1}{2} \pi \cdot \text{sh} \frac{t}{2} \int_0^t ds \left(\text{ch} \frac{t}{2} - \text{ch} \frac{s}{2} \right)^\mu \{u(x, s) + u(x, -s)\} \\ = 2^{(k-1)/2} \frac{\partial}{\partial t} \int_0^t \left(\text{ch}^2 \frac{t}{2} - \text{ch}^2 \frac{r}{2} \right)^{-1/2} \text{sh} \frac{r}{2} \text{ch} \frac{r}{2} (-D_{2,r})^{(k-1)/2} \left\{ \frac{1}{\text{sh } r} (I' u_0)(x) \right\} dr.$$

Therefore, we obtain

$$(5.9) \quad \int_0^t ds \left(\text{ch} \frac{t}{2} - \text{ch} \frac{s}{2} \right)^\mu \{u(x, s) + u(x, -s)\} \\ = 2^{(k-1)/2} C_5 \frac{1}{\text{sh}(t/2)} \frac{\partial}{\partial t} \int_0^t \left(\text{ch}^2 \frac{t}{2} - \text{ch}^2 \frac{r}{2} \right)^{-1/2} \text{sh} \frac{r}{2} \text{ch} \frac{r}{2} (-D_{2,r})^{(k-1)/2} \left\{ \frac{1}{\text{sh } r} (I' u_0)(x) \right\} dr,$$

where $C_5 = 2^{-(3m/2)-k+2} \pi^{-(k+m+1)/2} \Gamma(m/2)$.

For $1 \leq \delta \leq \mu$, let us define the function $l(t, s)$ by

$$l(t, s) = \left(\text{ch} \frac{t}{2} - \text{ch} \frac{s}{2} \right)^\delta \{u(x, s) + u(x, -s)\}.$$

For each $t > 0$, $(\partial l / \partial t)(t, s)$ is a bounded continuous function in s on $[0, t]$. So, by Lebesgue's convergence theorem, we can change the order of the integration and the differentiation in t , and have

$$\frac{\partial}{\partial t} \int_0^t l(t, s) ds = l(t, t) + \int_0^t \frac{\partial l}{\partial t}(t, s) ds. \\ = \int_0^t \frac{\partial l}{\partial t}(t, s) ds, \quad (0 < t < T),$$

which implies

$$(5.10) \quad D_{1,t} \left(\int_0^t l(t, s) ds \right) = \int_0^t (D_{1,t} l)(t, s) ds.$$

Applying the differential operator $-D_{1,t}$ to the both sides of (5.9), $(m-2)/2$ times, and using (5.10), we have

$$(5.11) \quad \int_0^t (u(x, s) + u(x, -s)) ds$$

$$= 2^{(k+m-3)/2} \frac{C_5}{\mu!} (-D_{1,t})^{m/2} \int_0^t \frac{\operatorname{sh}(r/2) \operatorname{ch}(r/2)}{\{\operatorname{ch}^2(t/2) - \operatorname{ch}^2(r/2)\}^{1/2}} (-D_{2,r})^{(k-1)/2} \\ \times \left\{ \frac{1}{\operatorname{sh} r} (I^r u_0)(x) \right\} dr.$$

Differentiating the both sides of (5.11) in t , we have

$$(5.12) \quad u(x, t) + u(x, -t) = C_6 \frac{\partial}{\partial t} (-D_{1,t})^{m/2} \int_0^t \frac{\operatorname{sh} r}{\sqrt{\operatorname{ch} t - \operatorname{ch} r}} (-D_{2,r})^{(k-1)/2} \left\{ \frac{1}{\operatorname{sh} r} (I^r u_0)(x) \right\} dr,$$

where $C_6 = 2^{-m-k/2} \pi^{-(k+m+1)/2}$.

On the other hand, the function v_1 on $S \times S$ defined by

$$v_1(x, y) = v_1(x_1, \dots, x_n, y_1, \dots, y_n) = e^{v_s} \cdot \frac{\partial u}{\partial s}(x_1, \dots, x_n, s),$$

satisfies (5.2). Hence, we can replace u and u_0 in (5.11) by $u_s = \partial u / \partial s$ and u_1 respectively. Then we have

$$(5.13) \quad u(x, t) - u(x, -t) = \int_0^t (u_s(x, s) + u_s(x, -s)) ds \\ = 2^{(k+m-3)/2} \frac{C_5}{\mu!} (-D_{1,t})^{m/2} \int_0^t \frac{\operatorname{sh}(r/2) \operatorname{ch}(r/2)}{\{\operatorname{ch}^2(t/2) - \operatorname{ch}^2(r/2)\}^{1/2}} (-D_{2,r})^{(k-1)/2} \left\{ \frac{1}{\operatorname{sh} r} (I^r u_1)(x) \right\} dr.$$

By (5.12) and (5.13), we have the desired result for the case where k is odd.

(ii) The case where k is even. As in the case (i), applying the differential operator $-D_{2,r}$ to the both sides of (5.4), κ times, and using (5.6), we have

$$2^{(k-2)/2} (-D_{2,r})^\kappa \left(\frac{1}{\operatorname{sh} r} (I^r u_0)(x) \right) = C_4 \kappa! \int_0^r d\alpha \operatorname{sh} \frac{\alpha}{2} \int_0^\alpha \left(\operatorname{ch} \frac{\alpha}{2} - \operatorname{ch} \frac{s}{2} \right)^\mu (u(x, s) + u(x, -s)) ds.$$

Then, differentiating the above equation in r and dividing by $\operatorname{sh}(r/2)$, we obtain

$$(5.14) \quad 2^{(k-2)/2} \frac{1}{\operatorname{sh}(r/2)} \frac{\partial}{\partial r} (-D_{2,r})^\kappa \left(\frac{1}{\operatorname{sh} r} (I^r u_0)(x) \right) = C_4 \kappa! \int_0^r \left(\operatorname{ch} \frac{r}{2} - \operatorname{ch} \frac{s}{2} \right)^\mu (u(x, s) + u(x, -s)) ds.$$

Applying the differential operator $D_{1,r}$ to the both sides of (5.14), μ times, and using (5.10) as in the case (i), we can solve the integral equation (5.14), as follows.

$$(5.15) \quad 2^{(k+m)/2-2} \frac{\partial}{\partial r} (-D_{1,r})^{m/2} (-D_{2,r})^\kappa \left(\frac{1}{\operatorname{sh} r} (I^r u_0)(x) \right) = \kappa! \cdot \mu! \cdot C_4 \{u(x, r) + u(x, -r)\}.$$

Since we can replace u and u_0 in (5.14) by u_s and u_1 respectively as before, we have

$$(5.16) \quad 2^{(k+m)/2-2} (-D_{1,r})^{m/2} (-D_{2,r})^\kappa \left(\frac{1}{\operatorname{sh} r} (I^r u_1)(x) \right) = \kappa! \cdot \mu! \cdot C_4 \{u(x, r) - u(x, -r)\}.$$

The two equations (5.15) and (5.16) lead to the desired formula of $u(x, t)$. Hence, we have Theorem 13. $\square$

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